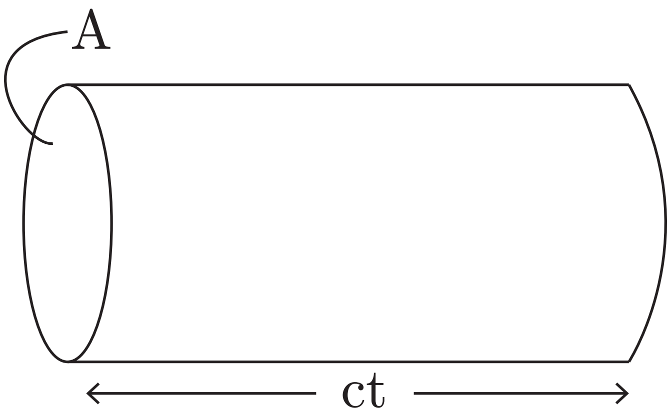
PSO201A: Quantum Physics

Solution # 3

Solution 3.1:



Intensity of a beam

$$I = \frac{Energy}{At} \tag{1}$$

where A is the cross-sectional area of the beam and t is time for which radiation falls on to the area A. If the beam has n photons per unit cross-sectional area of the beam, per second, then

$$I = \frac{nhc \times Atc}{\lambda At} = \frac{nhc^2}{\lambda} \tag{2}$$

Therefore,

$$I_1 = \frac{n_1 hc^2}{\lambda_1} \tag{3}$$

$$I_2 = \frac{n_2 hc^2}{\lambda_2} \tag{4}$$

Again,

$$I_1 = I_2 \tag{5}$$

$$\frac{n_1}{n_2} = \frac{\lambda_1}{\lambda_2} \tag{6}$$

Solution 3.2: Compton Effect

The conservation of momentum requires that

$$P_0 = P_1 cos\theta + P cos\phi \tag{7}$$

$$P sin\theta = P sin\phi \tag{8}$$

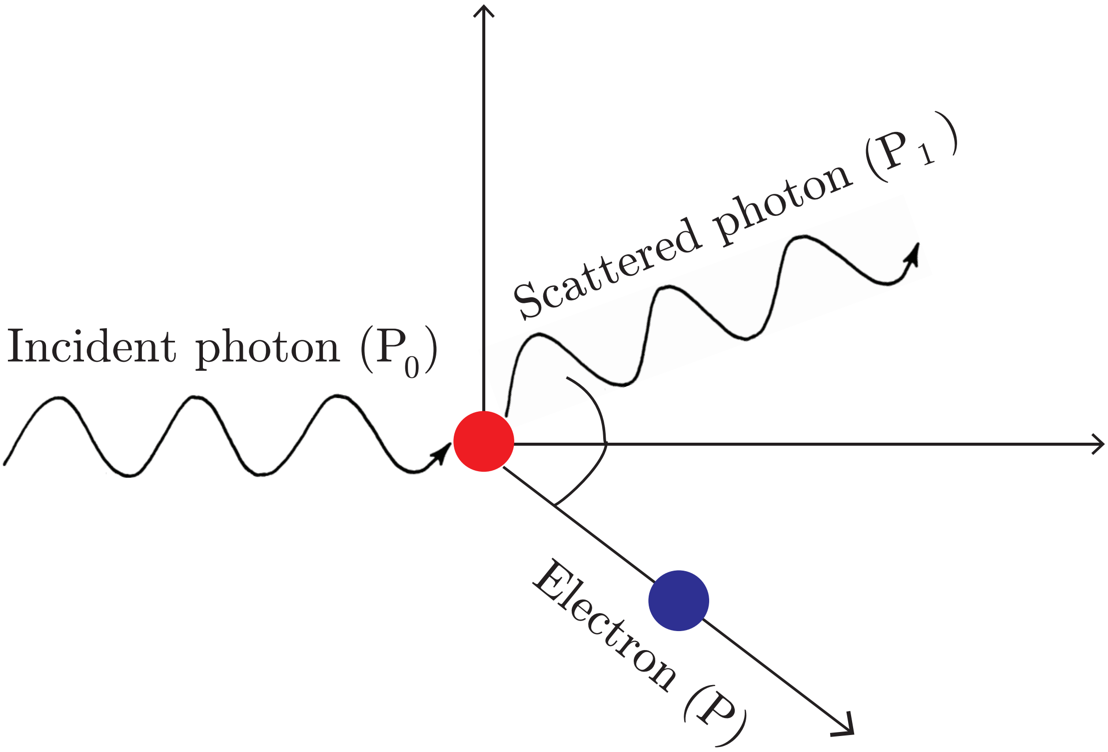


FIG. 1: Compton Sacttering

From equation (7) and (8) we get

$$\frac{P_0 - P_1 cos\theta}{cos\phi} = \frac{P_1 sin\theta}{sin\phi} \tag{9}$$

$$\frac{h}{\lambda_0} - \frac{h}{\lambda_1} cos\theta = \frac{h}{\lambda_1}\frac{sin\theta}{tan\phi} \tag{10}$$

$$\frac{1}{\lambda_0} = \frac{1}{\lambda_1}\left(cos\theta + \frac{sin\theta}{tan\phi}\right) \tag{11}$$

$$\frac{\lambda_1}{\lambda_0} = cos\theta + \frac{sin\theta}{tan\phi} \tag{12}$$

From the homework 2.1 (a) we already have that

$$\lambda_1 - \lambda_0 = \frac{h}{m_0 c}(1 - cos\theta) \tag{13}$$

$$\frac{\lambda_1}{\lambda_0} = 1 + \frac{h}{m_0 c \lambda_0}(1 - cos\theta) \tag{14}$$

From equation (12) and (14) we can write

$$cos\theta + \frac{sin\theta}{tan\phi} = 1 + \frac{h}{\lambda_0 m_0 c}(1 - cos\theta) \tag{15}$$

$$\frac{sin\theta}{tan\phi} = (1 - cos\theta)[1 + \frac{h}{\lambda_0 m_0 c}] \tag{16}$$

$$\frac{sin(\frac{\theta}{2})cos(\frac{\theta}{2})}{tan\phi} = 2sin^2(\frac{\theta}{2})[1 + \frac{h}{\lambda_0 m_0 c}] \tag{17}$$

$$cot(\frac{\theta}{2}) = tan\phi[1 + \frac{h}{\lambda_0 m_0 c}] \tag{18}$$

Solution 3.3:de-Broglie wavelength

The de-Broglie wavelength is given by

$$\lambda = \frac{h}{p} \tag{19}$$

where p is the momentum of the particle. For a free particle of mass m_0 and kinetic energy K, we have the total relativistic energy E given by

$$E = \sqrt{p^2c^2 + m_0^2c^4} = K + m_0c^2 \tag{20}$$

$$\implies p^2c^2 + m_0^2c^4 = (K + m_0c^2)^2 \tag{21}$$

$$\implies p = \sqrt{\frac{K^2 + 2Km_0c^2}{c^2}} \tag{22}$$

Substituting equation (22) in equation (19)

$$\lambda = \frac{hc}{\sqrt{K^2 + 2Km_0c^2}} \tag{23}$$

In the non-relativistic limit, $m_0c^2 >> K$,

$$K^2 + 2Km_0c^2 \approx 2Km_0c^2 \tag{24}$$

Using the above equation the de-Broglie wavelength for non-relativistic particle

$$\lambda = \frac{h}{\sqrt{2m_0K}} = \frac{h}{\sqrt{m_0v}} \tag{25}$$

Solution 3.4

Since the smallest feature size is $0.1\AA$, the required de-Broglie wavelength should be $0.1\AA$. So, the required momentum is $p = \frac{h}{\lambda}$.

(a) For an electron the total relativistic energy

$$E = \sqrt{p^2c^2 + m_0^2c^4} Joule$$
$$m_0c^2 = 0.511 \times 10^6 eV$$

Therefore,

$$E = \sqrt{\frac{hc}{\lambda}(\frac{1}{1.602 \times 10^{-19}})^2 + (0.511 \times 10^6)^2} = 0.5258 \times 10^6 eV$$

So, the required energy is

$$K = E - m_0c^2 = (0.5258 - 0.511) \times 10^6 = 14.8KeV$$

(b) For a photon

$$E = \frac{hc}{\lambda}\frac{1}{1.602 \times 10^{-19}} = 124KeV$$

(c) Electrons would be prefered. They are less energetic and so a milder shielding would be required to the γ-ray photon.

Solution 3.5: Linear Algebra

$$|V_1 >= \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad |V_2 >= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad |V_3 >= \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$$

(a) We have the matrix

$$U = \begin{pmatrix} 1 & 0 & 1 \\ i & 0 & i \\ 0 & 1 & 1 \end{pmatrix}$$

consist of three vectors

$$detU = 2i \neq 0 \implies detU \neq 0$$

Therefore, the vectors are linearly independent.

(b)

$$< V_1|V_2 >= \begin{pmatrix} 1 & -i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1$$

$$< V_2|V_3 >= \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} = 1$$

$$< V_3|V_1 >= \begin{pmatrix} 0 & -i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = 1$$

Therefore, the three vectors are not orthogonal

(c) The above states are not orthogonal but they are linearly independent.So, the basis vectors can be constructed through Gram-Schimdt procedure.

The first basis vector $|x_1 >$ is

$$|x_1 >= \frac{|V_1 >}{\sqrt{< V_1|V_1 >}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

The second basis vector $|x_2 >$ can be calculated as

$$|x_2 >= \frac{|x_2' >}{\sqrt{< x_2'|x_2' >}}$$

where,

$$|x_2' >= |V_2 > -|x_1 >< x_1|V_2 >= \begin{pmatrix} 1/2 \\ -i/2 \\ 1 \end{pmatrix}$$

Therefore,

$$|x_2 >= \frac{1}{\sqrt{6}} \begin{pmatrix} 1/2 \\ -i/2 \\ 1 \end{pmatrix}$$

The third basis vector

$$|x_3 >= \frac{|x_3' >}{\sqrt{< x_3'|x_3' >}}$$

where,

$$|x_3' >= |V_3 > -|x_1 >< x_1|V_3 > -|x_2 >< x_2|V_3 >= \frac{2}{3} \begin{pmatrix} -1 \\ i \\ 1 \end{pmatrix}$$

Therefore,

$$|x_3 >= \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ i \\ 1 \end{pmatrix}$$

(d) There can many other sets of orthogonal vectors. For example

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$