PSO201A: Quantum Physics **Semester II, 2016-17; IIT Kanpur**

Mid-Semester Examination (Solutions)

March 3rd, 2017 Time: 1:00-3:00 pm Maximum Marks: 100

(Answer all 6 questions. Calculators are not allowed. Some important constants are provided below.)

---

1. Stefan's-Boltzmann's constant: $\sigma = 5.67 \times 10^{-8}$ W/m^2–K^4.
2. Wein's constant $= 2.898 \times 10^{-3}$ m-K
3. Planck's constant: $h = 6.626 \times 10^{-34}$ joule-sec

Problem 1:

(a) In a nuclear reaction the temperature of the rising fireball reaches 10^7 K at some instant. What is the wavelength at which the emitted radiation is a maximum at that instant. **(3 marks)**

(b) If the surface temperature of the Sun is 5000 K, calculate the approximate energy lost to radiation per unit time per unit area of the Sun's surface. **(3 marks)**

(c) Write the following in the position basis: $\langle\psi_1|\hat{\Omega}|\psi_2\rangle$. **(3 marks)**

(d) Find the adjoint of the equation: $|\psi\rangle = a|\psi_1\rangle + b|\psi_2\rangle\langle\psi_3|\psi_4\rangle + \sum_i |\psi_5\rangle\langle\psi_6|\hat{\Omega}|v_i\rangle\langle w_i|\psi_7\rangle$. **(3 marks)**

(e) Show that the uncertainty in the energy ΔE of a particle in a stationary state is zero. **(3 marks)**

Solution 1:

(a) Wein's displacement law says $\lambda_{\max}T = 2.898 \times 10^{-3}$ m-K. Therefore, we have $\lambda_{\max} = \dfrac{2.898 \times 10^{-3}}{10^7}$ m $= 2.898$ Å.

(b) The energy lost to radiation per unit time per unit area of the of the Sun's surface is given by Stefan's law. The energy lost$= \sigma T^4 = 5.67 \times 10^{-8} \times (5000)^4 = 5.67 \times 625 \times 10^4 \approx 3.54 \times 10^7$ W/m^2.

(c)

$$\begin{aligned}\langle\psi_1|\hat{\Omega}|\psi_2\rangle &= \iint \langle\psi_1|x\rangle\langle x|\hat{\Omega}|x'\rangle\langle x'|\psi_2\rangle dx dx' = \iint \psi_1^*(x)\Omega_{xx'}\psi_2(x) dx dx' \\ &= \iint \psi_1^*(x)\delta(x-x')\Omega(x', -i\hbar\frac{\partial}{\partial x'})\psi_2(x) dx dx' = \int \psi_1^*(x)\Omega\left(x, -i\hbar\frac{\partial}{\partial x}\right)\psi_2(x) dx\end{aligned}$$

(d) $\langle\psi| = \langle\psi_1|a^* + \langle\psi_4|\psi_3\rangle\langle\psi_2|b^* + \sum_i \langle\psi_7|w_i\rangle\langle v_i|\hat{\Omega}^\dagger|\psi_6\rangle\langle\psi_5|$

(e) A stationary state wave-function can be written as $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$. We can calculate the energy expectation value as follow:

$$\langle E\rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\left(i\hbar\frac{\partial}{\partial t}\right)\Psi(x,t)dx = E\int_{-\infty}^{\infty}\psi^*(x)\psi(x)dx = E \quad \text{(normalization condition)}$$

$$\langle E^2\rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\left(i\hbar\frac{\partial}{\partial t}\right)^2\Psi(x,t)dx = E^2\int_{-\infty}^{\infty}\psi^*(x)\psi(x)dx = E^2$$

$$\Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2} = E^2 - E^2 = 0$$

Thus we see that the energy uncertainty of a stationary state is zero.

Problem 2:

(a) Explicitly work out the commutator: $[\hat{\text{X}}, \hat{\text{P}}]$, where $\hat{\text{X}}$ is the position operator and $\hat{\text{P}}$ is the momentum operator. **(10 marks)**

(b) Assuming no degeneracy and using Dirac notations, prove that the eigenvectors of a Hermitian operator are mutually orthogonal. **(10 marks)**

Solutions 2:

(a) Let $|\psi\rangle$ be an arbitrary state. So, we can write

$$\begin{aligned}
[\hat{\text{X}}, \hat{\text{P}}]|\psi\rangle &= (\hat{\text{X}}\hat{\text{P}} - \hat{\text{P}}\hat{\text{X}})|\psi\rangle \\
&= \iiint \left(|x\rangle\langle x|\hat{\text{X}}|x'\rangle\langle x'|\hat{\text{P}}|x''\rangle\langle x''| - |x\rangle\langle x|\hat{\text{P}}|x'\rangle\langle x'|\hat{\text{X}}|x''\rangle\langle x''|\right)|\psi\rangle dx dx' dx'' \\
&= \iiint \left[|x\rangle\delta(x - x')x'\delta(x' - x'')\left(-i\hbar\frac{d}{dx''}\right)\langle x''| - |x\rangle\delta(x - x')\left(-i\hbar\frac{d}{dx'}\right)\delta(x' - x'')x''\langle x''|\right]|\psi\rangle dx dx' dx'' \\
&= \int \left[|x\rangle x\left(-i\hbar\frac{d}{dx}\right)\langle x| - |x\rangle\left(-i\hbar\frac{d}{dx}\right)x\langle x|\right]|\psi\rangle dx \\
&= -i\hbar\int |x\rangle\langle x|\left[x\left(\frac{d}{dx}\right) - \left(\frac{d}{dx}\right)x\right]|\psi\rangle dx
\end{aligned}$$

We have $\left(\frac{d}{dx}\right)xf(x) = f(x) + x\left(\frac{d}{dx}\right)f(x) = \left(1 + x\frac{d}{dx}\right)f(x)$. Therefore, we have that $\left(\frac{d}{dx}\right)x = \left(1 + x\frac{d}{dx}\right)$. Using this result we can now write the above commutator as

$$\begin{aligned}
[\hat{\text{X}}, \hat{\text{P}}]|\psi\rangle &= -i\hbar\int |x\rangle\langle x|\left[x\left(\frac{d}{dx}\right) - 1 - x\left(\frac{d}{dx}\right)\right]|\psi\rangle dx \\
&= i\hbar\int |x\rangle\langle x|dx|\psi\rangle \\
&= i\hbar\text{I}|\psi\rangle.
\end{aligned}$$

Here we have used the completeness condition $\int |x\rangle\langle x|dx = \text{I}$. Thus we have

$$[\hat{\text{X}}, \hat{\text{P}}] = i\hbar.$$

(b) Let $|\psi_m\rangle$ and $|\psi_n\rangle$ be two eigenvectors of a Hermitian operator $\hat{\text{A}}$ with eigenvalues a_m and a_n, respectively. Thus

$$\hat{\text{A}}|\psi_m\rangle = a_m|\psi_m\rangle \tag{1}$$

$$\hat{\text{A}}|\psi_n\rangle = a_n|\psi_n\rangle \tag{2}$$

Taking the inner product on each side of the first equation with $\langle\psi_n|$ we get

$$\langle\psi_n|\hat{\text{A}}|\psi_m\rangle = a_m\langle\psi_n|\psi_m\rangle \tag{3}$$

Now we take the adjoint of the second equation and write is as

$$\begin{aligned}
&\langle\psi_n|\hat{\text{A}}^\dagger = a_n^*\langle\psi_n| \\
\text{or,}\quad &\langle\psi_n|\hat{\text{A}} = a_n\langle\psi_n|
\end{aligned}$$

Here, we have used the fact that $\hat{\text{A}}$ is Hermitian, that is, $\hat{\text{A}}^\dagger = \hat{\text{A}}$ and $a_n^* = a_n$, since the eigenvalues of a Hermitian operator are real. Taking the inner product on each side of the above equation with $|\psi_m\rangle$ we get

$$\langle\psi_n|\hat{\mathrm{A}}|\psi_m\rangle = a_n\langle\psi_n|\psi_m\rangle \tag{4}$$

Now, subtracting Eq. 3 for Eq. 4, we get

$$(a_n - a_m)\langle\psi_n|\psi_m\rangle = 0$$

Since, we have assumed that there is no degeneracy, $a_n \neq a_m$. Therefore,

$$\langle\psi_n|\psi_m\rangle = 0$$

Thus, we have that the eigenvectors of a Hermitian operator are orthogonal.

Problem 3: If $|x\rangle$ and $|p\rangle$ are the position and momentum eigetkets with eigenvalues x and p, respectively.

(a) Work out the inner product $\langle x|p\rangle$. **(8 marks)**

(b) For a quantum state $|\psi\rangle$, express the momentum-basis wave function $\psi(p)$ in terms of the position-basis wave function $\psi(x)$. **(3 marks)**

(c) write the following completeness relationship in the position basis: $\int_{\infty}^{\infty} |p\rangle\langle p|dp = \text{I}$. **(4 marks)**

Solutions 3:

(a) Since $|p\rangle$ is the eigenket of the momentum operator $\hat{P}$, we have

$$\hat{P}|p\rangle = p|p\rangle$$
$$\text{Or,} \qquad \langle x|\hat{P}|p\rangle = p\langle x|p\rangle$$

Expanding in the $|x'\rangle$, we get

$$\int \langle x|\hat{P}|x'\rangle\langle x'|p\rangle dx' = p\langle x|p\rangle$$
$$\text{Or,} \qquad \int (-i\hbar)\delta(x-x')\frac{d}{dx'}\langle x'|p\rangle dx' = p\langle x|p\rangle$$
$$\text{Or,} \qquad (-i\hbar)\frac{d}{dx}\langle x|p\rangle = p\langle x|p\rangle$$

Solving the above first-order differential equation, we get

$$\langle x|p\rangle = Ae^{ipx/\hbar}$$
$$\text{Or,} \qquad \langle p|x\rangle = A^* e^{-ipx/\hbar}$$

Now, we need to find the constant A. Using the relation $\langle x|x'\rangle = \delta(x-x')$ and the completeness condition for the momentum basis, we write

$$\langle x|x'\rangle = \delta(x-x')$$
$$\text{Or,} \qquad \int \langle x|p\rangle\langle p|x'\rangle dp = \delta(x-x')$$
$$\text{Or,} \qquad \int |A|^2 e^{ipx/\hbar} \times e^{-ipx'/\hbar} dp = \delta(x-x')$$
$$\text{Or,} \qquad |A|^2 \int e^{i(x-x')p/\hbar} dp = \delta(x-x')$$
$$\text{Or,} \qquad |A|^2 2\pi\hbar\delta(x-x') = \delta(x-x')$$
$$\Rightarrow \qquad A = \frac{1}{\sqrt{2\pi\hbar}}$$

Thus we have the required inner product as

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

(b) We have

$$
\begin{aligned}
&\psi(p) = \langle p|\psi\rangle \\
\text{Or,}\qquad &\psi(p) = \int \langle p|x\rangle\langle x|\psi\rangle dx \\
\text{Or,}\qquad &\psi(p) = \int \langle p|x\rangle\psi(x) dx \\
\text{Or,}\qquad &\psi(p) = A\int e^{-ipx/\hbar}\psi(x) dx
\end{aligned}
$$

(c) The completeness condition for the momentum eigenfunctions is :

$$\int_{-\infty}^{\infty} |p\rangle\langle p| dp = \mathrm{I}$$

Taking the inner product of the above equation with $\langle x|$ from left and with $|x'\rangle$ from right, we get

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \langle x|p\rangle\langle p|x'\rangle dp = \langle x|\mathrm{I}|x'\rangle \\
\text{Or,}\qquad &\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{ip(x-x')/\hbar} dp = \langle x|x'\rangle \\
\text{Or,}\qquad &\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{ip(x-x')/\hbar} dp = \delta(x-x')
\end{aligned}
$$

This is the required position representation.

Problem 4: Suppose the position-basis wave function $\Psi(x,t)$ of a particle is given by $\Psi(x,t) = \frac{1}{\sqrt{2}}[\Psi_1(x,t)+\Psi_2(x,t)]$, where $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are the two normalized stationary-state solutions to the Schrödinger equation with energies E_1 and E_2, respectively, with $\Psi_1(x,0) = \psi_1(x)$ and $\Psi_2(x,0) = \psi_2(x)$.

(a) Find out the position probability density of the particle at time t in terms of ψ_1, ψ_2, E_1, and E_2. **(7 marks)**

(b) What is the expectation value $\langle E \rangle$ for energy? **(8 marks)**

Solutions 4

(a) $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are the two normalized stationary-state solutions, we have $\Psi_1(x,t) = \psi_1(x)e^{-iE_1 t/\hbar}$ and $\Psi_2(x,t) = \psi_2(x)e^{-iE_2 t/\hbar}$. The required probability density is then given by

$$\begin{aligned} P(x,t) &= \Psi^*(x,t)\Psi(x,t) \\ &= \frac{1}{2}[\Psi_1^*(x,t) + \Psi_2^*(x,t)][\Psi_1(x,t) + \Psi_2(x,t)] \\ &= \frac{1}{2}\left[|\psi_1(x)|^2 + |\psi_2(x)|^2 + \psi_1^*(x)\psi_2(x)e^{-i(E_2-E_1)t/\hbar} + \psi_2^*(x)\psi_1(x)e^{i(E_2-E_1)t/\hbar}\right] \end{aligned}$$

(b) The expectation value of energy $\langle E \rangle$ is given as

$$\begin{aligned} \langle E \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)(i\hbar\frac{\partial}{\partial t})\Psi(x,t)dx &= \frac{1}{2}\Big[\int_{-\infty}^{\infty} \Psi_1^*(x,t)(i\hbar\frac{\partial}{\partial t})\Psi_1(x,t)dx + \int_{-\infty}^{\infty} \Psi_2^*(x,t)(i\hbar\frac{\partial}{\partial t})\Psi_2(x,t)dx \\ &+ \int_{-\infty}^{\infty} \Psi_1^*(x,t)(i\hbar\frac{\partial}{\partial t})\Psi_2(x,t)dx + \int_{-\infty}^{\infty} \Psi_2^*(x,t)(i\hbar\frac{\partial}{\partial t})\Psi_1(x,t)dx\Big] \\ &= \frac{1}{2}\Big[E_1 \int_{-\infty}^{\infty} \Psi_1^*(x,t)\Psi_1(x,t)dx + E_2 \int_{-\infty}^{\infty} \Psi_2^*(x,t)\Psi_2(x,t)dx \\ &+ E_2 \int_{-\infty}^{\infty} \Psi_1^*(x,t)\Psi_2(x,t)dx + E_1 \int_{-\infty}^{\infty} \Psi_2^*(x,t)\Psi_1(x,t)dx\Big] \end{aligned}$$

The first two integrals are unity because of the fact that the wave-functions are normalized. The last two integrals are zero since they essentially are the inner products of two orthogonal functions. Thus we have

$$\langle E(t) \rangle = \frac{E_1 + E_2}{2}$$

Problem 5: The position-basis wave function $\psi(x)$ of a particle is given by

$$\psi(x) = A; \quad \text{if} \quad -\left(\frac{R+d}{2}\right) < x < -\left(\frac{R-d}{2}\right) \quad \text{and} \quad \left(\frac{R-d}{2}\right) < x < \left(\frac{R+d}{2}\right)$$
$$= 0; \quad \text{otherwise.}$$

(Take $R \gg 2d$)

(a) Find the normalization constant A. **(2 marks)**

(b) Calculate and plot the position probability density that a particle is found at position x. **(2 marks)**

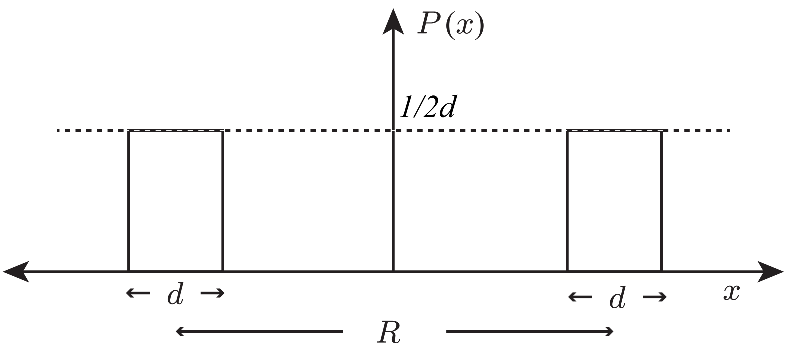
(c) Calculate and plot the momentum probability density that the particle is found with momentum p. **(6 marks)**

Solutions 5:

(a) Normalizing the wave function, we get,

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = 1$$
$$\text{or,} \quad \int_{-(\frac{R+d}{2})}^{-(\frac{R-d}{2})} A^2 dx + \int_{(\frac{R-d}{2})}^{(\frac{R+d}{2})} A^2 dx = 1$$
$$\text{or,} \quad A^2 \times 2d = 1$$
$$\text{or,} \quad A = \sqrt{\frac{1}{2d}}$$

(b) The plot of the probability density is as shown below:



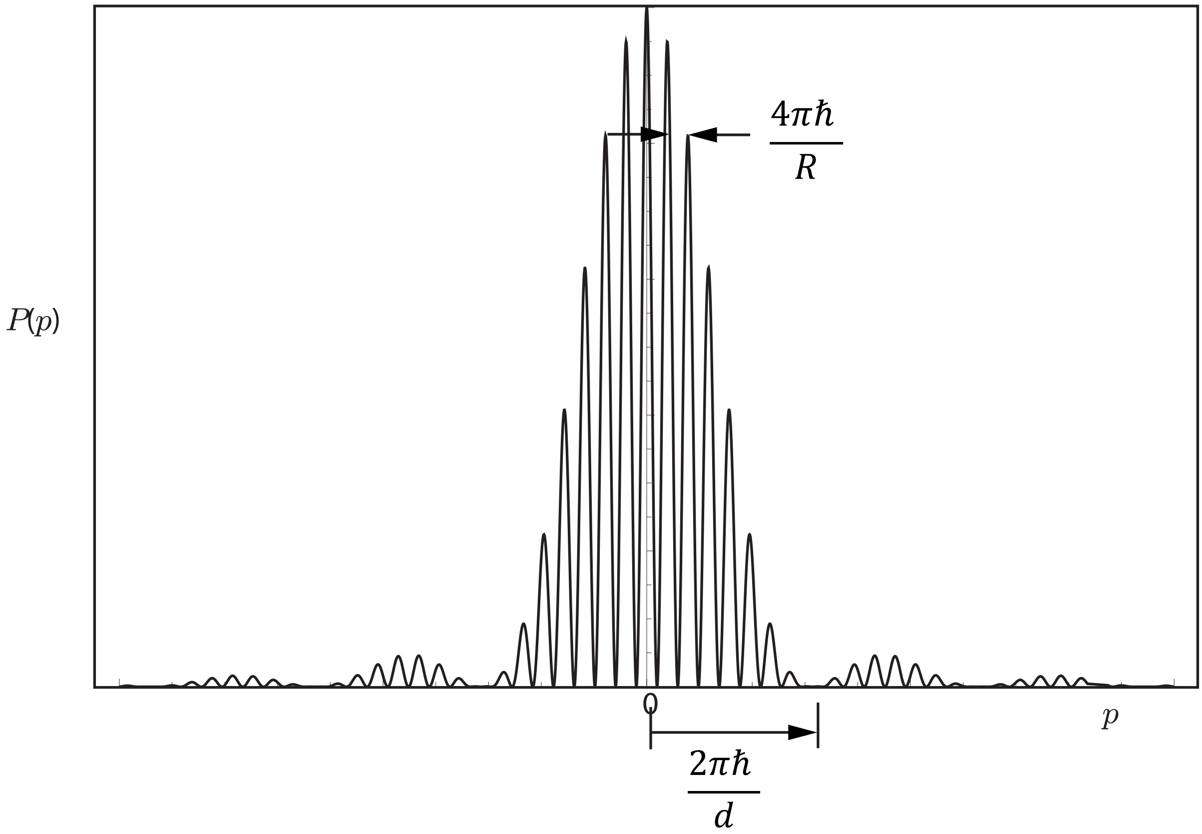
(c) In order to calculate the momentum probability density, we need to first calculate the momentum-space wave-function $\psi(p)$, which is:

$$\psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\psi(x)dx$$
$$= \frac{A}{\sqrt{2\pi\hbar}} \left[\int_{-(\frac{R+d}{2})}^{-(\frac{R-d}{2})} e^{-ipx/\hbar}dx + \int_{(\frac{R-d}{2})}^{(\frac{R+d}{2})} e^{-ipx/\hbar}dx \right]$$
$$= \frac{A}{\sqrt{2\pi\hbar}} d\text{sinc}\left(\frac{d}{2\hbar}p\right) 2\cos\left(\frac{R}{2\hbar}p\right)$$

Therefore, the momentum probability density is

$$P(p) = \psi^*(p)\psi(p) = \frac{d}{\pi\hbar}\text{sinc}^2\left(\frac{d}{2\hbar}p\right)\cos^2\left(\frac{R}{2\hbar}p\right)$$

Below is the plot for the momentum probability density as a function of p. For the plot, we have taken $R/d = 8$. The fringe period is equal to $\frac{4\pi\hbar}{R}$ and the first zero of the sinc function will appear at $x = \pm\frac{2\pi\hbar}{d}$



Problem 6: Consider an infinite potential well of width a centered at $x = 0$. The position-basis wave function of the n^{th} stationary state $\psi_n(x)$ is given by

$$\begin{aligned}
&\text{For} \quad n = 1, 3, 5, ... \qquad &\psi_n(x) &= \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi x}{a}\right); \qquad &\text{if} \quad -\frac{a}{2} \leq x \leq \frac{a}{2} \\
& &&= 0 &\text{else.} \\
&\text{For} \quad n = 2, 4, 6, ... \qquad &\psi_n(x) &= \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right); \qquad &\text{if} \quad -\frac{a}{2} \leq x \leq \frac{a}{2} \\
& &&= 0 &\text{else.}
\end{aligned}$$

Suppose that the particle is in the ground state ($n = 1$) of the potential,

(a) Find the probability that the particle is found between $x = -a/4$ and $x = a/4$. **(5 marks)**

(b) What is the expectation value $\langle x \rangle$ for position? **(3 marks)**

(c) What is the expectation value $\langle p \rangle$ for momentum? **(3 marks)**

(d) Now, suppose that the potential well suddenly expands symmetrically to twice its size. Calculate the probability of finding the particle in the ground state of the new potential well. **(14 marks)**

Solutions 6

(a) The probability that the particle is found between $x = -a/4$ and $x = a/4$ is given by

$$\begin{aligned}
\int_{-a/4}^{a/4} \psi^*(x)\psi(x)dx &= \frac{2}{a} \int_{-a/4}^{a/4} \cos^2\left(\frac{\pi x}{a}\right) dx \\
&= \frac{1}{a} \int_{-a/4}^{a/4} 1 + \cos\left(\frac{2\pi x}{a}\right) dx \\
&= \frac{1}{a} \left[x + \frac{a}{2\pi} \sin\left(\frac{2\pi x}{a}\right)\right]_{-a/4}^{a/4} \\
&= \frac{1}{a} \left[\frac{a}{2} + \frac{a}{\pi}\right] \\
&= \frac{1}{2} + \frac{1}{\pi}
\end{aligned}$$

(b) The position expectation value is $\langle x \rangle$ is

$$\begin{aligned}
\langle x \rangle &= \int_{-a/2}^{a/2} \psi^*(x)x\psi(x)dx \\
&= \frac{2}{a} \int_{-a/2}^{a/2} x \cos^2\left(\frac{\pi x}{a}\right) dx \\
&= 0
\end{aligned}$$

In evaluating the integral we have used the fact that the integrand is an odd function.

(c) The momentum expectation value is $\langle p \rangle$ is

$$\begin{aligned}
\langle p \rangle &= \int_{-a/2}^{a/2} \psi^*(x) \left(-i\hbar \frac{d}{dx}\right) \psi(x) dx \\
&= \frac{2}{a} \int_{-a/2}^{a/2} \cos\left(\frac{\pi x}{a}\right) \left(-i\hbar \frac{d}{dx}\right) \cos\left(\frac{\pi x}{a}\right) dx \\
&= \frac{2}{a}(i\hbar) \int_{-a/2}^{a/2} \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi x}{a}\right) \frac{\pi}{a} dx \\
&= 0
\end{aligned}$$

In evaluating the integral we have again used the fact that the integrand is an odd function.

(d) The state of the particle before the expansion of the well is $|\psi\rangle = |\psi_1^{(\text{old})}\rangle$, where $|\psi_1^{(\text{old})}\rangle$ is the ground state solution of the infinite-potential well of width a. The position-space wave-function of the ground state is given by

$$\begin{aligned}
\langle x|\psi_1^{(\text{old})}\rangle = \psi_1^{(\text{old})}(x) &= \sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right) \quad \text{if} \quad \frac{-a}{2} < x < \frac{a}{2} \\
&= 0 \quad \text{otherwise}
\end{aligned}$$

The width of the well then expands to $2a$. We write the new stationary state solutions as $|\psi_n^{(\text{new})}\rangle$. The position-space wave-function of the ground state can therefore be written as

$$\begin{aligned}
\langle x|\psi_1^{(\text{new})}\rangle = \psi_1^{(\text{new})}(x) &= \sqrt{\frac{2}{2a}} \cos\left(\frac{\pi x}{2a}\right) \quad \text{if} \quad -a < x < a \\
&= 0 \quad \text{otherwise}
\end{aligned}$$

In order to calculate the required probability, we need to write our state in the new basis:

$$|\psi\rangle = |\psi_1^{(\text{old})}\rangle = \sum_{n=0}^{\infty} c_n |\psi_n^{(\text{new})}\rangle,$$

where $c_n = \langle \psi_n^{(\text{new})}|\psi\rangle = \langle \psi_n^{(\text{new})}|\psi_1^{(\text{old})}\rangle$. The required probability is $|c_1|^2$, which we calculate as follows:

$$c_1 = \langle \psi_1^{(\text{new})}|\psi_1^{(\text{old})}\rangle = \int_{-a}^{a} \psi_1^{(\text{new})}(x)\psi_1^{(\text{old})}(x) dx$$

However, the wave-function $\psi_1^{(\text{old})}(x)$ is defined only over the range $(-a/2) < x < (a/2)$. Thus we effectively have,

$$\begin{aligned}
c_1 &= \int_{-a/2}^{a/2} \psi_1^{(\text{new})}(x)\psi_1^{(\text{old})}(x) dx \\
&= \int_{-a/2}^{a/2} \sqrt{\frac{2}{2a}} \cos\left(\frac{\pi x}{2a}\right) \sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right) dx \\
&= \frac{\sqrt{2}}{a} \int_{-a/2}^{a/2} \cos\left(\frac{\pi x}{2a}\right) \cos\left(\frac{\pi x}{a}\right) dx \\
&= \frac{\sqrt{2}}{2a} \int_{-a/2}^{a/2} \left[\cos\left(\frac{\pi x}{2a}\right) + \cos\left(\frac{3\pi x}{2a}\right)\right] dx \\
&= \frac{\sqrt{2}}{2a} \left[\frac{2}{\sqrt{2}} \times \frac{2a}{\pi} + \frac{2}{\sqrt{2}} \times \frac{2a}{3\pi}\right] = \frac{2}{\pi} \times \frac{4}{3} = \frac{8}{3\pi}
\end{aligned}$$

Therefore, the required probability is $|c_1|^2 = \left(\frac{8}{3\pi}\right)^2$