Modeling in State Space, Equilibrium, and Linearization

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1 Motivation

Classical control describes a system through the relationship between a single input and a single output, typically via a transfer function. This description is compact, but it discards information. Two systems with identical input-output behavior can have entirely different internal structure, different numbers of energy-storing elements, and different sensitivity to initial conditions. A transfer function also assumes linearity, time-invariance from the outset and largely initially relaxed behaviour, which excludes most physical systems before any analysis begins.

State space modeling addresses both limitations. It describes a system through a set of internal variables, the state, whose evolution in time is governed by a set of first-order differential equations. This description applies equally to linear and nonlinear systems, to single-input and multi-input systems, and it retains the internal structure that a transfer function discards. The vocabulary built in the previous note, vector, vector space, linear map, matrix, eigenvalue, exists precisely to describe this structure. In this note we define the state space model, identify its equilibrium points, and linearize it about an equilibrium to recover a linear model of the form already familiar from linear algebra.

2 State, State Vector, State Space

Definition 2.1. *The* ***state*** *of a system at time t is the smallest set of variables that, together with the input for $t' \geq t$, uniquely determines the behavior of the system for all $t' \geq t$.*

The state is collected into a **state vector** $x(t) \in \mathbb{R}^n$, where n is the number of state variables, referred to as the **order** of the system. The vector space $\mathbb{R}^n$ in which $x(t)$ evolves is called the **state space**.

Remark 2.1. *The defining property of the state is sufficiency: knowledge of $x(t)$ at a single instant, together with the input applied afterward, determines the system's behavior afterward. Past values of the input or output are not needed once the state at time t is known. This is what makes the state vector a complete description of the system's condition at time t.*

Example 2.1 (Mass–spring–damper system)**.** *Consider a mass attached to a spring and a damper. Let $q(t)$ denote the displacement of the mass from its equilibrium position. To predict the future motion of the mass, knowing $q(t)$ alone is not sufficient: we must also know its velocity $\dot{q}(t)$. A natural choice of state variables is therefore*

$$x_1(t) = q(t), \qquad x_2(t) = \dot{q}(t),$$

so that the state vector is

$$x(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} \in \mathbb{R}^2.$$

Thus the system has two state variables and its state space is $\mathbb{R}^2$. Each point $(q, \dot{q})$ in this space represents one possible instantaneous condition of the mechanical system. Once $q(t)$ and $\dot{q}(t)$ are known, together with the subsequently applied force, the future motion is uniquely determined.

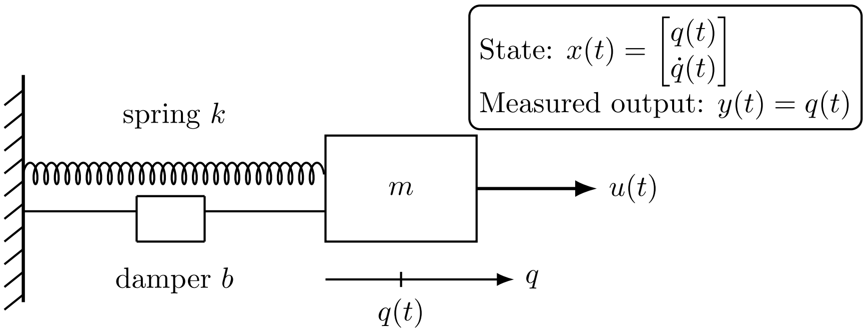

Figure 1: Mass–spring–damper system illustrating the distinction between input, state, and output. The applied force $u(t)$ is the input, the displacement and velocity form the state $x(t) = [q(t)\ \dot{q}(t)]^T$, and the displacement $q(t)$ is assumed to be the measured output.

3 Inputs and Outputs

Definition 3.1. *The **input** $u(t) \in \mathbb{R}^m$ is the set of variables through which external influence is applied to the system. The **output** $y(t) \in \mathbb{R}^p$ is the set of variables that are measured or observed.*

The state, the input, and the output are conceptually distinct. The state describes the internal condition of the system and need not be directly measurable. The input is what we are free to choose. The output is what we are able to observe, and it is generally some function of the state and the input, not the state itself.

Example 3.1 (Input and output of a mechanical system)**.** *For the mass–spring–damper system, suppose an external force $u(t)$ is applied to the mass. Then $u(t)$ is the input because it is the external quantity through which we influence the system.*

The state remains

$$x(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}.$$

Suppose, however, that the available sensor measures only the displacement of the mass. The output is then

$$y(t) = q(t) = x_1(t).$$

Thus the system has a two-dimensional state but only a one-dimensional output. The velocity $\dot{q}(t)$ is part of the internal state even though it is not directly measured.

4 State Space Representation

Definition 4.1. *A system in **state space form** is described by*

$$\dot{x}(t) = f(x(t), u(t), t), \qquad y(t) = h(x(t), u(t), t),$$

*where $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^p$. The first equation is the **state equation**, and the second is the **output equation**.*

Definition 4.2. *A system is **time-invariant** if f and h do not depend explicitly on t, that is,*

$$\dot{x}(t) = f(x(t), u(t)), \qquad y(t) = h(x(t), u(t)).$$

*A system for which f or h depends explicitly on t is called **time-varying**.*

The functions f and h are unrestricted in general, so this description applies to nonlinear systems as well as linear ones, and to time-varying systems as well as time-invariant ones. Time dependence in f or h typically arises from a parameter of the physical system changing with time, such as a mass that depletes fuel, or from an external quantity, such as a reference trajectory, entering the dynamics explicitly.

Example 4.1 (State-space model of a mass–spring–damper system)**.** *Consider a mass m connected to a spring of stiffness k and a viscous damper with damping coefficient b. Let $q(t)$ denote the displacement of the mass from its equilibrium position, and let $u(t)$ be the externally applied force.*

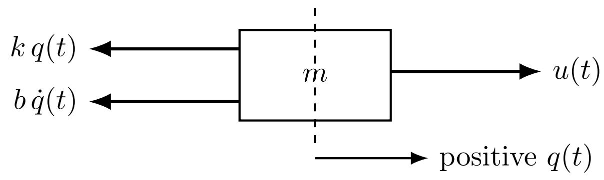

Figure 2: Force balance on the mass. Taking rightward displacement as positive, the applied force is $u(t)$, while the spring force $kq(t)$ and damping force $b\dot{q}(t)$ oppose the motion.

From Newton's second law, the sum of the forces acting on the mass is

$$m\ddot{q}(t) = u(t) - b\dot{q}(t) - kq(t). \tag{1}$$

Equivalently,

$$m\ddot{q}(t) + b\dot{q}(t) + kq(t) = u(t). \tag{2}$$

This is a second-order differential equation. To express it as a set of first-order differential equations, choose the state variables

$$x_1(t) = q(t), \qquad x_2(t) = \dot{q}(t). \tag{3}$$

The first state equation follows immediately:

$$\dot{x}_1(t) = x_2(t). \tag{4}$$

For the second state equation,

$$\dot{x}_2(t) = \ddot{q}(t). \tag{5}$$

Using the equation of motion,

$$\dot{x}_2(t) = -\frac{k}{m}x_1(t) - \frac{b}{m}x_2(t) + \frac{1}{m}u(t). \tag{6}$$

Hence, the state equations are

$$\dot{x}_1 = x_2, \tag{7}$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u. \tag{8}$$

If only the displacement is measured, then

$$y(t) = q(t) = x_1(t). \tag{9}$$

5 Equilibrium Points

Definition 5.1. *For a fixed input u_e, a state $x_e \in \mathbb{R}^n$ is an* ***equilibrium point*** *of the system if*

$$f(x_e, u_e) = 0.$$

At an equilibrium point, the state derivative is zero, so a system started at x_e with input held at u_e remains at x_e for all future time.

Remark 5.1. *A nonlinear system can have more than one equilibrium point, or none, or a continuum of them, depending on f. This is a genuine difference from linear systems, which we will see shortly always have $x_e = 0$ as an equilibrium when $u_e = 0$, and generally have no other isolated equilibrium unless the system matrix is singular.*

Remark 5.2. *An equilibrium point is defined for a time-invariant system, since $f(x_e, u_e) = 0$ must hold for all t for the state to remain fixed. A time-varying system generally has no state at which it remains at rest, because f itself keeps changing. However, we often use the terminology 'steady-state' to define behaviour that is post-transient.*

6 Linearization

Away from an equilibrium point, the behavior of a nonlinear system can be arbitrarily complicated. Near an equilibrium point, its behavior can be approximated by a linear system, obtained from the first-order Taylor expansion of f and h. This section first treats linearization about an equilibrium, which applies to time-invariant systems, and then extends the same idea to a nominal trajectory, which applies to time-varying systems as well.

6.1 Linearization about an Equilibrium

Definition 6.1. *Let (x_e, u_e) be an equilibrium pair, $f(x_e, u_e) = 0$. Define the deviation variables*

$$\delta x(t) = x(t) - x_e, \qquad \delta u(t) = u(t) - u_e, \qquad \delta y(t) = y(t) - y_e,$$

where $y_e = h(x_e, u_e)$. The ***linearization*** *of the system about (x_e, u_e) is*

$$\dot{\delta x} = A\,\delta x + B\,\delta u, \qquad \delta y = C\,\delta x + D\,\delta u,$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)}, \quad C = \left.\frac{\partial h}{\partial x}\right|_{(x_e,u_e)}, \quad D = \left.\frac{\partial h}{\partial u}\right|_{(x_e,u_e)},$$

each a matrix of partial derivatives, a Jacobian, evaluated at the equilibrium.

Remark 6.1. *This linearization is a first-order approximation and is accurate only for δx and δu small enough that higher-order terms in the Taylor expansion remain negligible. The matrices A, B, C, D depend on which equilibrium point (x_e, u_e) is chosen. A system with multiple equilibria generally has a different linearization at each one.*

Example 6.1 (Pendulum)**.** *Consider a rigid pendulum of length l and mass m, with viscous friction coefficient b at the pivot, and an applied torque u. Let $x_1 = \theta$ be the angular displacement from the vertical and $x_2 = \dot{\theta}$ the angular velocity. The equation of motion is*

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 + \frac{1}{ml^2}u.$$

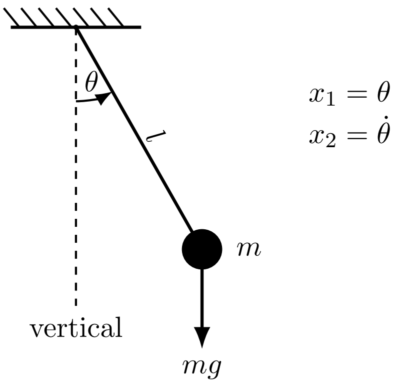

Figure 3: Rigid pendulum of length l and mass m with viscous friction b at the pivot and applied torque u. The angular displacement θ is measured from the downward vertical.

With no applied torque, $u_e = 0$, the equilibria satisfy $x_2 = 0$ and $\sin x_1 = 0$, giving $x_1 = 0$ or $x_1 = \pi$ within one period. These are two physically distinct configurations: the pendulum hanging downward, and the pendulum balanced upward.

The Jacobian of $f = (x_2,\ -\frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 + \frac{1}{ml^2}u)$ with respect to x is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{l}\cos x_1 & -\dfrac{b}{ml^2} \end{bmatrix}, \qquad \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ \dfrac{1}{ml^2} \end{bmatrix}.$$

At the downward equilibrium, $x_1 = 0$, $\cos x_1 = 1$, and the linearization gives

$$A = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{l} & -\dfrac{b}{ml^2} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \dfrac{1}{ml^2} \end{bmatrix}.$$

At the upward equilibrium, $x_1 = \pi$, $\cos x_1 = -1$, and the linearization instead gives

$$A = \begin{bmatrix} 0 & 1 \\ \dfrac{g}{l} & -\dfrac{b}{ml^2} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \dfrac{1}{ml^2} \end{bmatrix}.$$

The two matrices differ only in the sign of the $(2,1)$ entry, yet this sign determines the eigenvalues of A, and therefore the local behavior near each equilibrium: one matrix admits eigenvalues with negative real part, the other does not. Recognizing and computing this distinction is exactly the eigenvalue calculation from the previous note, now applied to a physical system.

6.2 Linearization about a Trajectory

A time-varying system, or a time-invariant system driven away from any equilibrium, does not stay at a fixed point, so linearization about a point does not apply. The same first-order idea extends by linearizing about a **nominal trajectory** instead of a fixed point.

Definition 6.2. *A pair of time functions $(x_0(t), u_0(t))$ is a **nominal trajectory** of the system if it satisfies the state equation exactly,*

$$\dot{x}_0(t) = f(x_0(t), u_0(t), t)$$

for all t of interest. Given a nominal trajectory, define the deviation variables

$$\delta x(t) = x(t) - x_0(t), \qquad \delta u(t) = u(t) - u_0(t), \qquad \delta y(t) = y(t) - y_0(t),$$

where $y_0(t) = h(x_0(t), u_0(t), t)$. The linearization of the system about this trajectory is

$$\dot{\delta x} = A(t)\,\delta x + B(t)\,\delta u, \qquad \delta y = C(t)\,\delta x + D(t)\,\delta u,$$

where

$$A(t) = \left.\frac{\partial f}{\partial x}\right|_{(x_0(t),u_0(t),t)}, \quad B(t) = \left.\frac{\partial f}{\partial u}\right|_{(x_0(t),u_0(t),t)}, \quad C(t) = \left.\frac{\partial h}{\partial x}\right|_{(x_0(t),u_0(t),t)}, \quad D(t) = \left.\frac{\partial h}{\partial u}\right|_{(x_0(t),u_0(t),t)}.$$

Remark 6.2. *The matrices $A(t)$, $B(t)$, $C(t)$, $D(t)$ now depend on time, both because the nominal trajectory $(x_0(t), u_0(t))$ moves and because f or h may depend explicitly on t. An equilibrium point is the special case $x_0(t) \equiv x_e$, $u_0(t) \equiv u_e$, constant in time, together with f and h having no explicit time dependence. In that special case the Jacobians are evaluated at the same point for all t, so $A(t)$, $B(t)$, $C(t)$, $D(t)$ reduce to the constant matrices A, B, C, D of Section 6.1.*

7 Linear State Space Model

Definition 7.1. *A* ***linear time-varying (LTV) state space model*** *is*

$$\dot{x} = A(t)x + B(t)u, \qquad y = C(t)x + D(t)u,$$

with $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $C(t) \in \mathbb{R}^{p\times n}$, $D(t) \in \mathbb{R}^{p\times m}$, matrices whose entries may depend on t.

Definition 7.2. *A* ***linear time-invariant (LTI) state space model*** *is the special case*

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du,$$

in which A, B, C, D are constant, not depending on t.

Every matrix in these models is a linear map between the vector spaces already defined, at each fixed instant t. $A(t)$ maps the state space to itself, so at each t its eigenvalues and eigenvectors, as defined earlier, describe the instantaneous unforced behavior of the system. $B(t)$ maps the input space to the state space, describing how an input redirects the state. $C(t)$ maps the state space to the output space, describing what portion of the state is visible at the output. $D(t)$ maps the input space directly to the output space, bypassing the state entirely. In the LTI case, these maps do not change with time.

Remark 7.1. *The linearization of a nonlinear system about a nominal trajectory, and the LTV model, are the same object; linearization about an equilibrium point of a time-invariant system is the special case that produces an LTI model. In both cases, deviations δx, δu, δy obey the linear model exactly to first order, and every tool developed later in this course for the LTI model, eigenvalue-based stability, controllability, observability, has a corresponding time-varying version for the LTV model, generally stated in terms of $A(t)$, $B(t)$, $C(t)$, $D(t)$ rather than fixed eigenvalues. In either case, the linear model's conclusions hold only in a neighborhood of the trajectory or equilibrium about which the linearization was taken.*

8 Discrete-Time State-Space Representation

The state-space note wrote every model in continuous time, $\dot{x} = f(x, u)$, because physical laws are naturally stated as instantaneous rates of change. Data, however, does not arrive continuously. A sensor reports a reading every Δt seconds. Before we can fit anything to data, we need a state-space representation that speaks in these terms.

Definition 8.1. *A discrete-time state equation describes how the state evolves from one sampling instant to the next:*

$$x_{k+1} = f_d(x_k, u_k), \qquad k = 0, 1, 2, \ldots$$

where $x_k := x(k\Delta t)$ and $u_k := u(k\Delta t)$ for a fixed sampling interval Δt, and f_d plays the role that f played in the continuous-time state equation.

This is not a different kind of system, only a different bookkeeping of the same one, and for a system already given in continuous time, f_d can be obtained directly.

8.1 From Continuous Time to Discrete Time

A direct way to move from a continuous-time state equation to a discrete-time one is to approximate the derivative itself, rather than solve for it exactly. Recall that the derivative is defined as a limit of a difference quotient,

$$\dot{x}(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}.$$

For a fixed, small sampling interval Δt, we drop the limit and approximate the derivative at time $k\Delta t$ by the forward difference

$$\dot{x}(k\Delta t) \approx \frac{x_{k+1} - x_k}{\Delta t}.$$

Substituting this into the continuous-time state equation $\dot{x} = f(x, u)$ and solving for x_{k+1} gives the forward Euler discretization,

$$x_{k+1} \approx x_k + \Delta t\, f(x_k, u_k).$$

For the linear time-invariant model $\dot{x} = Ax + Bu$, obtained by linearizing about an equilibrium as in the earlier note, this becomes

$$x_{k+1} \approx (I + \Delta t A)\, x_k + \Delta t B\, u_k = A_d x_k + B_d u_k, \qquad A_d \approx I + \Delta t A, \quad B_d \approx \Delta t B.$$

Remark 8.1. *This is only an approximation, and its accuracy depends on Δt being small relative to the time scales of the system: the smaller Δt, the closer the forward difference is to the true derivative. An exact discretization can be obtained once we know how to solve the state equation explicitly, a topic in the course such as EE630; for now, the Euler approximation is enough to move between continuous and discrete time.*

Example 8.1 (Discrete-time model of a moving vehicle)**.** *Consider a vehicle moving along a straight line. Let $p(t)$ denote its position, $v(t)$ its velocity, and $u(t)$ its acceleration. The continuous-time dynamics are*

$$\dot{p}(t) = v(t), \tag{10}$$
$$\dot{v}(t) = u(t). \tag{11}$$

Choose the state vector

$$x(t) = \begin{bmatrix} p(t) \\ v(t) \end{bmatrix}. \tag{12}$$

The continuous-time state-space model is therefore

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \tag{13}$$

Suppose the position and velocity are measured every Δt seconds. Using the forward Euler approximation,

$$x_{k+1} \approx x_k + \Delta t\, \dot{x}_k. \tag{14}$$

For the position state,

$$p_{k+1} \approx p_k + \Delta t\, v_k, \tag{15}$$

while for the velocity state,

$$v_{k+1} \approx v_k + \Delta t\, u_k. \tag{16}$$

Hence,

$$\begin{bmatrix} p_{k+1} \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_k \\ v_k \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta t \end{bmatrix} u_k. \tag{17}$$

Thus the discrete-time state-space model has the form

$$x_{k+1} = A_d x_k + B_d u_k, \tag{18}$$

with

$$A_d = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}, \qquad B_d = \begin{bmatrix} 0 \\ \Delta t \end{bmatrix}. \tag{19}$$

For example, if the sampling interval is $\Delta t = 0.1$ s, then

$$A_d = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \qquad B_d = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}. \tag{20}$$

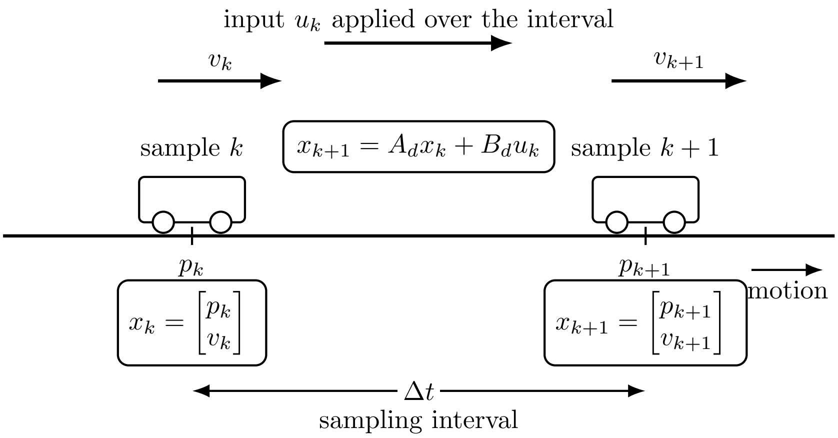

Figure 4: Discrete-time description of vehicle motion. The state is sampled at instants k and $k+1$, separated by the sampling interval Δt. The input u_k acts during the interval and changes the state from x_k to x_{k+1}.

Remark 8.2. *Discretizing with the forward Euler approximation gives $A_d \approx I + \Delta t A$. A useful consequence is how eigenvalues transform: if λ is an eigenvalue of A, the corresponding eigenvalue of A_d is approximately $\lambda_d \approx 1 + \lambda \Delta t$. Unlike an exact discretization, this approximate map does not automatically preserve stability: for λ_d to land inside the unit circle, $|\lambda_d| < 1$, the sampling interval Δt must be small enough relative to $|\lambda|$. This is a genuine limitation of Euler discretization worth keeping in mind whenever a sampling rate is chosen, and we will use the approximate relation $\lambda_d \approx 1 + \lambda \Delta t$, valid for suitably small Δt.*

9 Summary

A state space model describes a system through an internal state vector, governed by a first-order differential equation driven by the input, together with an output equation. The state represents the information required to determine the future evolution of the system, while the input represents the external influence and the output represents the measured or observed variables. The model is time-varying if f or h depends explicitly on time, and time-invariant otherwise. An equilibrium point is a state at which a time-invariant system remains at rest under a constant input, and a nonlinear system may have several such points. A nominal trajectory generalizes this to a state and input pair that changes with time while satisfying the state equation exactly, which is the natural object to linearize about when no fixed equilibrium exists. Linearization about an equilibrium produces constant matrices A, B, C, D and an LTI model, while linearization about a nominal trajectory produces time-varying matrices $A(t)$, $B(t)$, $C(t)$, $D(t)$ and an LTV model. These matrices are precisely the linear maps studied in the previous note, allowing the linear algebra vocabulary already developed to be applied directly to dynamical systems. When a system is observed or controlled at discrete sampling instants, the same state-space description can be written in discrete time, where the state at one sampling instant determines the state at the next. A continuous-time model can be converted approximately to this form using a numerical discretization such as the forward Euler method. Together, these ideas provide the modeling framework on which the analysis and design tools developed later in the course are built.