Chapter 5

Ordinary Differential Equations

5.1 Linear and non-linear equations

Assuming x and y to be independent and dependent variable, respectively, a linear differential equation of *order* n is given by

$$a_0 y + a_1 \frac{dy}{dx} + a_2 \frac{d^2y}{dx^2} + a_3 \frac{d^3y}{dx^3} + \cdots + a_n \frac{d^n y}{dx^n} = b, \tag{5.1}$$

where a's and b are functions of x (or constants). Some examples of linear equations are,

$$\begin{aligned} xy' + x^2 y &= e^x, (\text{order} \;\; 1) \\ x^3 y'' + e^x y' + \ln xy &= \cos x, (\text{order} \;\; 2) \\ y''' - 2y'' + y' &= 2\sin x (\text{order} \;\; 3). \end{aligned} \tag{5.2}$$

Note that, in each of the above equation, dependent variable y and all its derivative occur linearly. The order of the differential equation is decided according to the order of the highest derivative included in the equation. General solution of a linear differential equation of order n has n independent arbitrary constants and we can get a particular solution by assigning particular values to the constants, based on *boundary* condition or *initial* condition.

Some examples of the non-linear equations are,

$$\begin{aligned} y' - \ln y &= 0, (\text{order} \;\; 1) \\ x^3 y'' + y' - y^3 &= \sin x, (\text{order} \;\; 2) \\ y''' - 2y'' + y'^2 + x^2 y &= 2\sin y (\text{order} \;\; 3). \end{aligned} \tag{5.3}$$

Note that, in each of the above equation, either the dependent variable y or some of its derivative does not occur linearly.

5.2 First order differential equations

Differential equations of first order contain only first derivative of y, i.e., y'. We are going to discuss different types of first order equations in this section.

5.2.1 Separable equations (optional)

Separable equations are of the form $y' = f(x)/g(y)$, such that all the terms containing y can be written on one side of the equation and all the terms containing x can be written on the other,

$$g(y)dy = f(x)dx. \tag{5.4}$$

Note that, we can solve linear, as well as non-linear equations using this method.

Example 1: Solve $xy' = y$, given boundary condition $y = 3$, when $x = 2$.
We can write,

$$\int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \ln(y) = \ln(ax) \Rightarrow y = ax.$$

Using the boundary condition, we get $3/2$.

Example 2: Solve $x\sqrt{1-y^2}dx + y\sqrt{1-x^2}dy = 0$, given boundary condition $y = 0.5$, when $x = 0.5$.
We can write,

$$\int \frac{ydy}{\sqrt{1-y^2}} = \int \frac{-xdx}{\sqrt{1-x^2}}.$$

Let us put $1 - y^2 = u^2$ and $1 - x^2 = v^2$, such that the above equation is converted to

$$-\int du = \int dv \Rightarrow (1-y^2)^{1/2} + (1-x^2)^{1/2} = c,$$

where $c = \sqrt{3}$.

Example 3: Solve $y' \sin x = y \ln y$, given boundary condition $y = e$, when $x = \pi/3$.
We can write,

$$\int \frac{dy}{y \ln y} = \int \frac{dx}{\sin x} \Rightarrow \int \frac{dy}{y \ln y} = \int \csc x dx.$$

Left hand side is easy to integrate, if we substitute $\ln y = u$, such that,

$$\int \frac{dy}{y \ln y} = \int \frac{du}{u} = \ln u = \ln \ln y.$$

In order to integrate the right hand side, we multiply and divide by $(\csc x - \cot x)$, and substitute $(\csc x - \cot x) = v$, such that,

$$\int \csc x dx = \int \frac{\csc x(\csc x - \cot x)}{(\csc x - \cot x)} dx = \int \frac{dv}{v} = \ln v = \ln(\csc x - \cot x).$$

Using the boundary condition, the answer is $\ln y = c(\csc x - \cot x)$, where $c = \sqrt{3}$.

5.2.2 Exact equations

Say we have to solve a first order differential equation of the form,

$$P(x,y)dx + Q(x,y)dy = 0. \tag{5.5}$$

We know that, if the above expression is an exact differential, then we can define a function $F(x,y)$, such that $P = \partial F/\partial x$ and $Q = \partial F/\partial y$.[1] Thus, we can write

$$Pdx + Qdy = dF = 0 \Rightarrow F(x,y) = c. \tag{5.6}$$

Often, an inexact differential can be converted to an exact equation by multiplying it by an appropriate *integrating factor*. For example, $xdy - ydx = 0$ is not an exact differential, because $P = -y$, $Q = x$ and $\partial P/\partial y \neq \partial Q/\partial x$. But we can make it exact by dividing it with x^2 and thus, $1/x^2$ is the integrating factor. Let us verify this,

$$\frac{xdy - ydx}{x^2} = 0 \Rightarrow \frac{1}{x}dy - \frac{y}{x^2}dx = 0 \Rightarrow P_1(x,y) = -\frac{y}{x^2} \;\;\&\;\; Q_1(x,y) = \frac{1}{x}.$$

Now, we satisfy the condition $\partial P_1/\partial y = \partial Q_1/\partial x = -1/x^2$.

In general, by multiplying the given inexact equation with the integrating factor $U(x,y)$, we get an equation of the form $\underbrace{U(x,y)P(x,y)}_{P_1(x,y)}dx + \underbrace{U(x,y)Q(x,y)}_{Q_1(x,y)}dy = 0$, which is an exact equation, i.e., $\frac{\partial P_1(x,y)}{\partial y} = \frac{\partial Q_1(x,y)}{\partial x}$. However, it might not be a trivial exercise to find the integrating factor by inspection. We will learn a few tricks to do this via some examples.

Example 1: Solve $(3x^2y^3 - 5x^4)dx + (y + 3x^3y^2)dy = 0$

Since $P(x,y) = 3x^2y^3 - 5x^4$ and $Q(x,y) = y + 3x^3y^2$, you can easily verify that, $\partial P/\partial y = \partial Q/\partial x = 9x^2y^2$ and thus, this is an exact equation. Now, we have to find a function $F(x,y)$, such that $P = \partial F/\partial x$ and $Q = \partial F/\partial y$. We can write,

$$F(x,y) = \int P(x,y)dx = \int (3x^2y^3 - 5x^4)dx = x^3y^3 - x^5 + f(y).$$

Thus, we have to find $f(y)$ to get the solution. Using the other equation, we can write,

$$y + 3x^3y^2 = \frac{\partial F}{\partial y} = 3x^3y^2 + f'(y) \Rightarrow f(y) = \frac{y^2}{2} + c_1.$$

Thus, $F(x,y) = x^3y^3 - x^5 + \frac{y^2}{2} + c_1$ and the general solution of the given differential equation is

$$F(x,y) = c_2 \Rightarrow \boxed{x^3y^3 - x^5 + \frac{y^2}{2} = c}$$

where the constant c replaces both c_1 and c_2. You should differentiate the answer

[1] Check for an exact differential: $\partial P/\partial y = \partial Q/\partial x$, because $\partial^2 F/\partial y\partial x = \partial^2 F/\partial x\partial y$.

and check whether you get the equation given in the question.

Example 2: Solve $(2xe^{3y} + e^x)dx + (3x^2e^{3y} - y^2)dy = 0$

Since $P(x, y) = (2xe^{3y} + e^x)$ and $Q(x, y) = (3x^2e^{3y} - y^2)$, you can easily verify that $\partial P/\partial y = \partial Q/\partial x = 6xe^{3y}$ and thus, this is an exact equation. Now, we have to find a function $F(x, y)$, such that $P = \partial F/\partial x$ and $Q = \partial F/\partial y$. We can write,

$$F(x, y) = \int P(x, y)dx = \int (2xe^{3y} + e^x)dx = x^2e^{3y} + e^x + f(y).$$

Thus, we have to find $f(y)$ to get the solution. Using the other equation, we can write,

$$3x^2e^{3y} - y^2 = \frac{\partial F}{\partial y} = 3x^2e^{3y} + f'(y) \Rightarrow f(y) = \frac{y^3}{3} + c_1$$

Thus, $F(x, y) = x^2e^{3y}+e^x-y^3/3+c_1$ and the general solution for the given differential equation is

$$F(x, y) = c_2 \Rightarrow \boxed{x^2e^{3y} + e^x - y^3/3 = c}$$

where the constant c replaces both c_1 and c_2.

Example 3: Solve $(x - y)dy + (y + x + 1)dx = 0$

Since $P(x, y) = y + x + 1$ and $Q(x, y) = x - y$, you can easily verify that $\partial P/\partial y = \partial Q/\partial x = 1$, and thus, this is an exact equation. Now, we have to find a function $F(x, y)$, such that $P = \partial F/\partial x$ and $Q = \partial F/\partial y$. We can write,

$$F(x, y) = \int P(x, y)dx = \int (y + x + 1)dx = xy + \frac{x^2}{2} + x + f(y).$$

Thus, we have to find $f(y)$ to get the solution. Using the other equation, we can write,

$$x - y = \frac{\partial F}{\partial y} = x + f'(y) \Rightarrow f(y) = -\frac{y^2}{2} + c_1$$

Thus, $F(x, y) = xy + x^2/2 + x - y^2/2 + c_1$ and the general solution for the given differential equation is

$$F(x, y) = c_2 \Rightarrow \boxed{xy + \frac{x^2}{2} + x - \frac{y^2}{2} = c}$$

where the constant c replaces both c_1 and c_2.

Example 4: Solve $(y^2 + 3xy^3)dx + (1 - xy)dy = 0$.

Since $P(x, y) = y^2+3xy^3$ and $\partial P/\partial y = 2y+9xy^2$; $Q(x, y) = 1-xy$ and $\partial P/\partial x = -y$, the equation is not an exact equation. We are going to use an **integrating factor**

$\boxed{U(x, y) = x^m y^n}$. Then, the above equation is converted to

$$(x^m y^{n+2} + 3x^{m+1}y^{n+3})dx + (x^m y^n - x^{m+1}y^{n+1})dy = 0.$$

For the above equation to be exact, we must have,

$$(n+2)x^m y^{n+1} + 3(n+3)x^{m+1}y^{n+2} = mx^{m-1}y^n - (m+1)x^m y^{n+1}.$$

Rearranging, we can write,

$$[(n+2) + (m+1)]x^m y^{n+1} + 3(n+3)x^{m+1}y^{n+2} - mx^{m-1}y^n = 0.$$

Since the right hand side is zero, every coefficient must be equal to zero,

$$\begin{aligned} (n+2) + (m+1) &= 0, \\ (n+3) &= 0, \\ m &= 0. \end{aligned}$$

Thus, the solution is $m = 0$ and $n = -3$ and the integrating factor is $U(x, y) = y^{-3}$. Given equation is converted to,

$$\underbrace{\frac{y^2 + 3xy^3}{y^3}}_{P_1(x,y)} dx + \underbrace{\frac{1 - xy}{y^3}}_{Q_1(x,y)} dy = 0.$$

We can easily verify that the above equation is exact, $\partial P_1/\partial y = \partial Q_1/\partial x = -1/y^2$. Now, we have to find a function $F(x, y)$, such that $P_1 = \partial F/\partial x$ and $Q_1 = \partial F/\partial y$. We can write,

$$F(x, y) = \int P_1(x, y)dx = \int \left(\frac{1}{y} + 3x\right) dx = \frac{x}{y} + \frac{3x^2}{2} + f(y).$$

Thus, we have to find $f(y)$ to get the solution. Using the other equation, we can write,

$$\frac{1}{y^3} - \frac{x}{y^2} = \frac{\partial F}{\partial y} = -\frac{x}{y^2} + f'(y) \Rightarrow f(y) = -\frac{1}{2y^2} + c_1$$

Thus, $F(x, y) = x/y + 3x^2/2 - 1/2y^2 + c_1$ and the general solution for the given differential equation is

$$F(x, y) = c_2 \Rightarrow \boxed{\frac{x}{y} + \frac{3x^2}{2} - \frac{1}{2y^2} = c}$$

where the constant c replaces both c_1 and c_2.

Example 5: Solve $(3xy - y^2)dx + x(x - y)dy = 0.$
Since $P(x, y) = (3xy - y^2)$ and $\partial P/\partial y = 3x - 2y$; $Q(x, y) = x(x - y)$ and $\partial Q/\partial x =$

$2x - y$, the given equation is not an exact equation. We further see that,

$$\frac{1}{Q}\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right] = \frac{1}{x} = f(x).$$

When such condition (left hand side is a function of x only) is satisfied, I claim that the **integrating factor** is $\boxed{U(x,y) = U(x) = e^{\int f(x)dx} = e^I}$ (see problem set). In this particular case, the integrating factor is,

$$U(x) = e^{\int dx/x} = e^{\ln x} = x. \tag{5.7}$$

Multiplying the given equation with the integrating factor, we obtain,

$$\underbrace{(3x^2y - y^2x)}_{P_1(x,y)}\,dx + \underbrace{(x^3 - x^2y)}_{Q_1(x,y)}\,dy = 0. \tag{5.8}$$

We can verify that, the above equation is exact as $\partial P_1/\partial y = \partial Q_1/\partial x = (3x^2 - 2xy)$. Now, we have to find a function $F(x,y)$, such that $P_1 = \partial F/\partial x$ and $Q_1 = \partial F/\partial y$. We can write,

$$F(x,y) = \int P_1(x,y)dx = \int (3x^2y - y^2x)dx = x^3y - \frac{y^2x^2}{2} + f(y).$$

Thus, we have to find $f(y)$ to get the solution. Using the other equation, we can write,

$$x^3 - x^2y = \frac{\partial F}{\partial y} = x^3 - x^2y + f'(y) \Rightarrow f(y) = c_1$$

Thus, $F(x,y) = x^3y - \frac{y^2x^2}{2} + c_1$ and the general solution for the given differential equation is

$$F(x,y) = c_2 \Rightarrow \boxed{x^3y - \frac{y^2x^2}{2} = c}$$

where the constant c replaces both c_1 and c_2.

5.2.3 Homogeneous equations (optional)

A first order differential equation of the form

$$P(x,y)dx + Q(x,y)dy = 0, \tag{5.9}$$

is *homogeneous* if both P and Q are homogeneous functions of the same degree.[2] Note that, a n^{th} degree of homogeneous function of x and y can be expressed as

[2]Homogeneous function has *multiplicative scaling* behavior. That means, if we multiply each variable by same *factor*, then the function is multiplied by some integral power of this *factor*. For example, $f(x,y)$ is a homogeneous function of degree n, if $f(tx,ty) = t^n f(x,y)$. Example: $x^2 + xy + y^2 + y^3/x$ is a homogeneous function of degree 2. Example: $x^2y + x^4/y + y^3$ is a homogeneous function of degree 3. Example: $x^2y + xy + x$ is not a homogeneous function.

$x^n f(y/x)$.[3] Since P and Q are homogeneous functions of same degree, the factor x^n gets canceled and we can write,

$$y' = \frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)} = f\left(\frac{y}{x}\right). \tag{5.10}$$

Thus, a homogeneous can always be expressed in the form of $y' = f(y/x)$. Now, we can solve this equation by substituting $y = xv$, which gives us a separable equation in x and v.[4] Some examples are given below.

Example 1: Solve $x^2dy - (3y^2 + xy)dx = 0$.

$Q(x,y) = x^2$ and $P(x,y) = -(3y^2 + xy) = -x^2\left(\frac{3y^2}{x^2} + \frac{y}{x}\right)$ are homogeneous functions of degree 2. We can write,

$$\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)} = \frac{3y^2}{x^2} + \frac{y}{x} = f\left(\frac{y}{x}\right).$$

In order to solve the above equation, we substitute, $y = xv$ and get a separable equation in x and v,

$$v + x\frac{dv}{dx} = 3v^2 + v \Rightarrow \frac{dv}{3v^2} = \frac{dx}{x}.$$

Integrating both sides, we get

$$-\frac{1}{3v} = \ln|x| + \ln|c| \Rightarrow \frac{x}{y} = -3\ln|cx| \Rightarrow \boxed{y = \frac{-x}{3\ln|xc|}}$$

In order to verify, you can differentiate the last equation and check whether you get the differential equation given in the question.

Example 2: Solve $x^2dy + (y^2 - xy)dx = 0$.

$Q(x,y) = x^2$ and $P(x,y) = y^2 - xy = x^2\left(\frac{y^2}{x^2} - \frac{y}{x}\right)$ are homogeneous functions of degree 2. We can write,

$$\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)} = \frac{y}{x} - \frac{y^2}{x^2} = f\left(\frac{y}{x}\right).$$

In order to solve the above equation, we substitute, $y = xv$ and get a separable equation in x and v,

$$v + x\frac{dv}{dx} = v - v^2 \Rightarrow \frac{-dv}{v^2} = \frac{dx}{x}.$$

[3]Example: a homogeneous function of degree 2, x^2+xy can be expressed as $x^2(1+y/x) = x^2 f(y/x)$.

[4]There is an alternate method of solving homogeneous differential equations. We can prove that $1/(Px + Qy)$ is an integrating factor for Eq. 5.9.

Integrating both sides, we get

$$\frac{1}{v} = \ln|x| + \ln|c| \Rightarrow \frac{x}{y} = \ln|cx| \Rightarrow \boxed{y = \frac{x}{\ln|cx|}} \tag{5.11}$$

Example 3: Solve $(y^2 - xy)dx + (x^2 + xy)dy = 0$.

$P(x,y) = (y^2 - xy) = x^2(y^2/x^2 - y/x)$ and $Q(x,y) = (x^2 + xy) = x^2(1 + y/x)$ are homogeneous functions of degree 2. We can write,

$$\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)} = \frac{x^2(y/x - y^2/x^2)}{x^2(1+y/x)} = \frac{(y/x - y^2/x^2)}{(1+y/x)} = f\left(\frac{y}{x}\right).$$

In order to solve the above equation, we substitute, $y = xv$ and get a separable equation in x and v,

$$v + x\frac{dv}{dx} = \frac{v - v^2}{1+v} \Rightarrow x\frac{dv}{dx} = \frac{-2v^2}{1+v} \Rightarrow \left[-\frac{1}{v^2} - \frac{1}{v}\right] dv = 2\frac{dx}{x}.$$

Integrating both sides, we get

$$\frac{1}{v} - \ln|v| = \ln|cx^2| \Rightarrow \frac{x}{y} = \ln|cxy| \Rightarrow \boxed{e^{x/y} = cxy}.$$

5.2.4 Linear first order equations

A linear first order equation can be written in the form of

$$y' + P(x)y = Q(x), \tag{5.12}$$

where P and Q are functions of x (or can be constants). If $Q = 0$, then we can easily separate the variables and write,

$$\frac{dy}{y} = -Pdx \Rightarrow \ln y = -\int Pdx + c. \tag{5.13}$$

Assuming $I = \int Pdx$ (equivalently, $dI/dx = P$), we can write the solution in the form

$$y = ce^{-I}. \tag{5.14}$$

Now, let us solve for non-zero Q. In order to do this, first let us calculate the first derivative of ye^I:

$$\frac{d}{dx}(ye^I) = y'e^I + ye^I\frac{dI}{dx} = y'e^I + ye^IP = e^I(y' + yP) = e^IQ. \tag{5.15}$$

Since both e^I and Q are function of x only, we can integrate to get,[5]

$$\boxed{ye^I = \int e^IQdx + c \Rightarrow y = \underbrace{e^{-I}\int e^IQdx}_{y_p} + \underbrace{e^{-I}c}_{y_c}, \quad \text{where} \quad I = \int Pdx}. \tag{5.16}$$

[5]There is an alternate way to solve linear equations, to be shown in the examples.

Note that, we have only one arbitrary constant, as expected for a linear first order equation. Also, y_c is the solution of Eq. 5.12 with $Q = 0$ and y_p is known as the particular solution.[6][7] Some examples are given below.

Example 1: Solve $y' - \frac{y}{x} = 1$.
Method 1:
This is a linear equation, with $P(x) = -1/x$ and $Q(x) = 1$. Thus,

$$I = \int P(x)dx = -\int \frac{1}{x}dx = -\ln x \Rightarrow e^I = e^{-\ln x} = \frac{1}{x}$$

$$ye^I = \int e^I Q(x)dx + \ln c \Rightarrow \frac{y}{x} = \int \frac{1}{x}dx + \ln c = \ln(cx) \Rightarrow \boxed{y = x\ln(xc)}$$

Method 2:
Let $y = uv$ and $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$. Thus, the above equation is converted to

$$u\frac{dv}{dx} + v\frac{du}{dx} - \frac{uv}{x} = 1 \Rightarrow u\frac{dv}{dx} + v\underbrace{\left(\frac{du}{dx} - \frac{u}{x}\right)}_{=0} = 1.$$

Setting the term involving v equal to zero, we get,

$$\frac{du}{dx} = \frac{u}{x} \Rightarrow \ln u = \ln(cx) \Rightarrow u = c_1 x.$$

Let us replace $u = c_1 x$ in the equation above (term involving v is still equal to zero),

$$c_1 x\frac{dv}{dx} = 1 \Rightarrow c_1 v = \ln(cx).$$

Finally, using $y = uv$, we get,

$$y = c_1 x\frac{1}{c_1}\ln(cx) \Rightarrow \boxed{y = x\ln(cx)}.$$

Example 2: Solve $y' + y = e^x$.
Method 1:
This is a linear equation with $P(x) = 1$ and $Q(x) = e^x$. Thus,

$$I = \int P dx = x \Rightarrow e^I = e^x.$$

$$ye^I = \int e^I Q dx + c = \int e^{2x}dx + c = \frac{e^{2x}}{2} + c \Rightarrow \boxed{y = \frac{e^x}{2} + ce^{-x}}.$$

Method 2:

[6] From Eq. 5.16, $y_p e^I = \int e^I Q dx$ and $y_c e^I = c$, such that $(y_p + y_c)e^I = ye^I = \int e^I Q dx + c$.
[7] Note that, the general solution of Eq. 5.12, i.e., $y = y_p + y_c$ is not unique.

Let $y = uv$ and $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$. Thus, the above equation is converted to

$$u\frac{dv}{dx} + v\frac{du}{dx} + uv = e^x \Rightarrow u\frac{dv}{dx} + v\underbrace{\left(\frac{du}{dx} + u\right)}_{=0} = e^x.$$

Setting the term involving v equal to zero, we get,

$$\frac{du}{dx} = -u \Rightarrow \ln u = -x + c_1 \Rightarrow u = c_2 e^{-x}.$$

Let us replace $u = c_2 e^{-x}$ in the equation above (term involving v is still equal to zero),

$$c_2 e^{-x}\frac{dv}{dx} = e^x \Rightarrow c_2 v = \frac{e^{2x}}{2} + c_3.$$

Finally, using $y = uv$, we get,

$$y = c_2 e^{-x}\left(\frac{e^{2x}}{2c_2} + \frac{c_3}{c_2}\right) \Rightarrow \boxed{y = \left(\frac{e^x}{2} + ce^{-x}\right)}.$$

Example 3: Solve $x^2y' + 3xy = 1$.

Method 1:

We can rewrite the given equation as $y' + \frac{3}{x}y = \frac{1}{x^2}$. This is a linear equation with $P(x) = 3/x$ and $Q(x) = 1/x^2$. Thus,

$$I = \int P dx = 3\ln x \Rightarrow e^I = x^3.$$

$$ye^I = \int e^I Q dx + c = \int x dx + c = \frac{x^2}{2} + c \Rightarrow \boxed{y = \frac{1}{2x} + cx^{-3}}.$$

Method 2:

Let $y = uv$ and $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$. Thus, the above equation is converted to

$$u\frac{dv}{dx} + v\frac{du}{dx} + \frac{3uv}{x} = \frac{1}{x^2} \Rightarrow u\frac{dv}{dx} + v\underbrace{\left(\frac{du}{dx} + \frac{3u}{x}\right)}_{=0} = \frac{1}{x^2}.$$

Setting the term involving v equal to zero, we get,

$$\frac{du}{dx} = -\frac{3u}{x} \Rightarrow \ln u = -3\ln x + \ln c_1 \Rightarrow u = c_1 x^{-3}.$$

Let us replace $u = c_1 x^{-3}$ in the equation above (term involving v is still equal to zero),

$$c_1 x^{-3}\frac{dv}{dx} = \frac{1}{x^2} \Rightarrow c_1 v = \frac{x^2}{2} + c_2.$$

Finally, using $y = uv$, we get,

$$y = c_1 x^{-3}\left(\frac{x^2}{2c_1} + \frac{c_2}{c_1}\right) \Rightarrow \boxed{\frac{1}{2x} + cx^{-3}}.$$

5.2.5 Bernoulli equation

Bernoulli equations can be written in the form of

$$y' + Py = Qy^n, \tag{5.17}$$

where P and Q are functions of x (or can be constants). Clearly, it is not a linear equation, but can easily be converted to a linear equation, by making a change of variable,

$$\boxed{z = y^{1-n}} \Rightarrow z' = (1-n)y^{-n}y'. \tag{5.18}$$

Multiplying Eq. 5.17 with $(1-n)y^{-n}$ and then making the above substitution, we get

$$(1-n)y^{-n}y' + (1-n)Py^{1-n} = (1-n)Q \tag{5.19}$$
$$\boxed{z' + (1-n)Pz = (1-n)Q}.$$

Thus, we have converted the non-linear equation 5.17 to a linear equation and we already know how to solve this. Some examples are given below.

Example 1: Solve $y' + y = xy^{2/3}$.
Substitute $z = y^{1/3} \Rightarrow z' = \frac{1}{3}y^{-2/3}y'$. Multiplying both sides of the given equation with $\frac{1}{3}y^{-2/3}$, we get,

$$\frac{1}{3}y^{-2/3}y' + \frac{1}{3}y^{1/3} = \frac{1}{3}x \Rightarrow z' + \frac{1}{3}z = \frac{1}{3}x.$$

Thus, we have converted the non-linear equation to a linear equation in x and z, with $P(x) = 1/3$ and $Q(x) = x/3$. Thus,

$$I = \int P dx = \frac{x}{3} \Rightarrow e^I = e^{x/3}$$

$$ze^I = \int e^I Q dx + c = \int e^{x/3}\frac{x}{3}dx + c = xe^{x/3} - 3e^{x/3} + c \Rightarrow z = x - 3 + ce^{-x/3}.$$

Replacing $z = y^{1/3}$, the answer is $\boxed{y^{1/3} = x - 3 + ce^{-x/3}}$.

Example 2: Solve $y' + \frac{1}{x}y = 2x^{3/2}y^{1/2}$.
Substitute $z = y^{1/2} \Rightarrow z' = \frac{1}{2}y^{-1/2}y'$. Multiplying both sides of the given equation with $\frac{1}{2}y^{-1/2}$, we get,

$$\frac{1}{2}y^{-1/2}y' + \frac{1}{2x}y^{1/2} = x^{3/2} \Rightarrow z' + \frac{1}{2x}z = x^{3/2}.$$

Thus, we have converted the non-linear equation to a linear equation in x and z, with $P(x) = 1/2x$ and $Q(x) = x^{3/2}$. Thus,

$$I = \int P dx = \frac{1}{2}\ln x \Rightarrow e^I = x^{1/2}$$

$$ze^I = \int e^I Q dx + c = \int x^{1/2}x^{3/2}dx + c = \frac{x^3}{3} + c \Rightarrow z = \frac{x^{5/2}}{3} + cx^{-1/2}$$

Replacing $z = y^{1/2}$, the answer is $\boxed{y^{1/2} = \dfrac{x^{5/2}}{3} + cx^{-1/2}}$.

5.2.6 Exercise

Separable equations (optional)

1. Solve $(1 + y^2)dx + xydy = 0$, given the boundary condition $y = 0$, when $x = 5$.

   Answer: $x^2(1 + y^2) = c, c = 25$

2. Solve $xy' - xy = y$, given the boundary condition $y = 1$, when $x = 1$.

   Answer: $y = cxe^x$, where $c = 1/e$

3. Solve $(y - x^2y)dy + (2xy^2 + x)dx = 0$, given the boundary condition $y = 0$, when $x = \sqrt{2}$.

   Answer: $(2y^2 + 1) = c(x^2 - 1)^2$, where $c = 1$

4. Solve $ydy + (xy^2 - 8x)dx = 0$, given the boundary condition $y = 3$, when $x = 1$.

   Answer: $y^2 = 8 + e^{c-x^2}$, where $c = 1$

5. Solve $y' = \cos(x + y)$. [Hint: substitute $u = x + y$]

   Answer: $\tan\frac{1}{2}(x + y) = x + c$

6. Solve $xy' + y = e^{xy}$. [Hint: substitute $u = xy$]

   Answer: $y = -x^{-1}\ln(c - x)$

Exact equations

7. $(\cos x \cos y + \sin^2 x)dx - (\sin x \sin y + \cos^2 y)dy = 0$.

   Answer: $4\sin x\cos y + 2x - \sin 2x - 2y - \sin 2y = c$

8. $(1 + y^2)dx + xydy = 0$.

   Answer: $\frac{x^2}{2} + \frac{x^2y^2}{2} + c = 0$

9. $(x - \cos y)dx - \sin ydy = 0$.

   Answer: $e^{-x}(\cos y - x - 1) = c$

10. $(xy^2 - 2y^3)dx + (3 - 2xy^2)dy = 0$.

    Answer: $xe^{xy} + c = 0$

11. $ydx + (x^2 + y^2 - x)dy = 0.$

    Answer: $\tan^{-1}\frac{x}{y} + y = c$

12. $(x-1)y' + y - x^{-2} + 2x^{-3} = 0.$

    Answer: $y^2 = -1/x^2 + c/(x-1)$

13. For an inexact equation, $P(x,y)dx + Q(x,y)dy = 0$, it is given that $\frac{1}{Q}\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right] = f(x)$. Prove that e^I is an integrating factor for the given equation, where $I = \int f(x)dx$. [Hint: note that $dI/dx = f(x)$. You have to prove that, $e^I P(x,y)dx + e^I Q(x,y)dy = 0$ is an exact differential equation.]

14. For an inexact equation, $P(x,y)dx + Q(x,y)dy = 0$, it is given that $\frac{1}{P}\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] = f(y)$. Prove that e^I is an integrating factor for the given equation, where $I = \int f(y)dy$. [Hint: note that $dI/dy = f(y)$. You have to prove that, $e^I P(x,y)dx + e^I Q(x,y)dy = 0$ is an exact differential equation.]

Homogeneous equations (optional)

15. Check whether following functions are homogeneous and if yes, find the degree.

    (a) $4x^2 + y^2$

    (b) $x^2 - 5xy + y^3/x$

    (c) $xy\sin(x/y)$

    (d) $(y^4 - x^3y)/x - xy^2\sin(x/y)$

    (e) $x\sin(xy)$

    (f) $x^2y^3 + x^5\ln(y/x) - y^6/\sqrt{x^2+y^2}$

    (g) $x^3 + x^2y + xy^2 + y^3$

    (h) $x^2 + y$

    (i) $x^2 + xy + y^3$

    (j) $x + \cos y$

16. Solve $ydy = (-x + \sqrt{x^2+y^2})dx$.

    Answer: $y^2 = 2cx + c^2$

17. Solve $xydx + (y^2 - x^2)dy = 0$.

    Answer: $y^2 = ce^{-x^2/y^2}$

18. Solve $(x^2 + y^2)dx - xydy = 0$.

    Answer: $y = \pm\sqrt{2\ln(cx)}$.

19. Solve $(y - x)dx + (x + y)dy = 0$.

    Answer: $y^2 + 2xy - x^2 = c$, which can be further written as $y = \pm\sqrt{2x^2 + c} - x$.

20. Solve $y' = y/x - \tan(y/x)$.

    Answer: $x \sin(y/x) = c$

21. Prove that, $1/(Px + Qy)$ is an integrating factor for Eq. 5.9. [Hint: you have to prove that $(Pdx + Qdy)/(Px + Qy)$ is an exact differential, provided P and Q are homogeneous functions of same degree.]

Linear equations

22. Prove that e^I is the integrating factor for Eq. 5.12, i.e., $e^I(Py-Q)dx+e^I dy = 0$ is an exact equation. Following the technique of solving an exact equation, prove that $ye^I = \int e^I Q dx + c$.

23. Solve $dy + (2xy - xe^{-x^2})dx = 0$.

    Answer: $y = \frac{x^2}{2}e^{-x^2} + ce^{-x^2}$

24. Solve $2xy' + y = 2x^{5/2}$.

    Answer: $y = \frac{1}{3}x^{5/2} + cx^{-1/2}$

25. Solve $y' \cos x + y = \cos^2 x$.

    Answer: $y(\sec x + \tan x) = x - \cos x + c$

26. Solve $y' + \frac{y}{\sqrt{x^2+1}} = \frac{1}{(x+\sqrt{x^2+1})}$.

    Answer: $y = \frac{(x+c)}{x+\sqrt{x^2+1}}$

Bernoulli equations

27. $3xy^2y' + 3y^3 = 1$.

    Answer: $y^3 = \frac{1}{3} + cx^{-3}$

28. $yy' - 2y^2 \cot x = \sin x \cos x$.

    Answer: $y^2 = \sin^2 x(-1 + c \sin^2 x)$

5.3 Second order linear differential equations

Differential equations of second order contain only first and second derivative of y, i.e., y' and y''. We are going to discuss two types of second order equations in this section.

5.3.1 Constant coefficients and zero right hand side

Equations of the form

$$a_2y'' + a_1y' + a_0y = 0, \tag{5.20}$$

where a_2, a_1, a_0 are constants, are known as *homogeneous* equations. Expressing $D = d/dx$, the above equation is converted to

$$\underbrace{(a_2 D^2 + a_1 D + a_0)}_{\text{auxiliary equation}} y = 0. \tag{5.21}$$

We could also have substituted $y = e^{cx}$ in Eq. 5.20 and get the same auxiliary equation,

$$a_2 c^2 + a_1 c + a_0 = 0. \tag{5.22}$$

Now, let us consider three possible cases.

Case 1: auxiliary equation having two distinct real roots

Expressing the auxiliary equation as $(D - c_1)(D - c_2)$,[8] we can rewrite Eq. 5.21 as,

$$(D - c_1)(D - c_2)y = 0. \tag{5.23}$$

Thus, in order to solve Eq. 5.21, we need to solve two first order equations,

$$(D - c_1)y = 0 \;\;\&\;\; (D - c_2)y = 0. \tag{5.24}$$

These are separable equations, with solutions $y_1 = e^{c_1 x}$ and $y_2 = e^{c_2 x}$ and the general solution is a linear combination of the two.[9] Thus, if c_1 and c_2 are two roots of the auxiliary equation, the general solution is,

$$\boxed{y = Ae^{c_1 x} + Be^{c_2 x}}. \tag{5.25}$$

Case 2: auxiliary equation having complex conjugate roots

Let the roots of the auxiliary equation be, $c_1 = \alpha + \iota\beta$ and $c_2 = \alpha - \iota\beta$. Thus, we have two solutions: $y_1 = e^{(\alpha+\iota\beta)x}$ and $y_2 = e^{(\alpha-\iota\beta)x}$, which are also complex conjugates of each other. By taking linear combination, we get a complex solution of the form,

$$\boxed{y = e^{\alpha x}\left(Ae^{\iota\beta x} + Be^{-\iota\beta x}\right)}, \tag{5.26}$$

where A and B are arbitrary *complex* constants. Since $e^{\pm\iota\beta x} = \cos\beta x \pm \iota \sin\beta x$, we can rewrite the above equation as, $y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$, where $C_1 = (A+B)$ and $C_2 = \iota(A - B)$. Note that, by selecting appropriate constants, we can get real, as well as imaginary solutions. For example, if we take $A = B = 1/2$, we get a real solution $y = e^{\alpha x}\cos\beta x$. Similarly, if we take $A = 1/2\iota$ and $B = -1/2\iota$, we get another real solution $y = e^{\alpha x}\sin\beta x$. Interestingly, since $\cos\beta x$ and $\sin\beta x$ are linearly independent functions, we can get a series of *real* solutions by taking

[8] c_1 and c_2 are the roots of the auxiliary equation $a_2D^2 + a_1D + a_0 = 0$, given by $\frac{-a_1 \pm \sqrt{a_1^2 - 4a_2a_0}}{2a_2}$.

[9] We can do this as two solutions are linearly independent. Two functions $f_1(x)$ and $f_2(x)$ are linearly independent if the *Wronskian* is *not* equal to zero. *Wronskian* is given by: $W = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$.

linear combination of them, i.e.,

$$\boxed{y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)}, \tag{5.27}$$

where C_1 and C_2 are *real* arbitrary constants. We can further express this as,

$$\boxed{y = Ce^{\alpha x} \sin(\beta x + \gamma)}, \tag{5.28}$$

where C and γ are arbitrary constants.

Case 3: auxiliary equation having same roots

There exist one more possibility, that both the roots of the auxiliary equation are same. Then, Eq. 5.20 takes the following form:

$$(D - c)\underbrace{(D - c)y}_{u} = 0. \tag{5.29}$$

Obviously, solving $(D - c)y = 0$, we get one solution to be $y = Ae^{cx}$. In order to get the other solution, we note that $(D - c)y$ is going to be some function $u(x)$, such that we can write the above equation as,

$$(D - c)u = 0 \Rightarrow u = Ae^{cx}. \tag{5.30}$$

Finally, we can solve for y from,

$$(D - c)y = Ae^{cx} \Rightarrow y' - cy = Ae^{cx}. \tag{5.31}$$

This is a linear first order equation, having non-zero right hand side, with $P = -c$ and $Q = Ae^{cx}$. The solution is given by Eq. 5.16, where $I = \int P dx = -cx$. We can write the solution as,

$$ye^I = \int e^I Q dx = \int e^{-cx} B e^{cx} dx = Ax + B \Rightarrow \boxed{y = (Ax + B)e^{cx}}. \tag{5.32}$$

Example 1: Simple harmonic motion.

Such periodic or oscillatory motion happens when the restoring force is proportional to the displacement and acts in the opposite direction. There are several examples, like spring-mass system, pendulum, vibration of a structure (like a bridge), vibration of atoms in a crystal etc.

Let us consider a spring-mass system. Assuming no friction, we can write the Newton's second law of motion as $m\frac{d^2x}{dt^2} = -kx = -m\omega^2 x$.[10] Thus, we have to solve a differential equation of the form $\frac{d^2x}{dt^2} + \omega^2 x = 0$. Writing $D = d/dt$, we get,

$$(D^2 + \omega^2)x = 0.$$

[10] Potential energy of a spring is given by $U(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$ and the force $= -\frac{dU}{dx}$. Such a force is known as conservative force and we already know that work done is independent of the path in such a force field.

Thus, the auxiliary equation we have to solve is,

$$D^2 + \omega^2 = 0,$$

and the roots are $D = \pm\iota\omega$. Thus, the general solution can be expressed in any of the three forms given in Eq. 5.26, Eq. 5.27 or Eq. 5.28,

$$x(t) = Ae^{\iota\omega t} + Be^{-\iota\omega t},$$
$$x(t) = C_1 \cos\omega t + C_2 \sin\omega t,$$
$$x(t) = C \sin(\omega t + \gamma).$$

Example 2: Damped harmonic motion.

If energy of an oscillator is dissipated, leading to gradual decrease of amplitude or preventing it from oscillating, such a motion is termed as damped harmonic motion. Damping happens because of various different reasons like presence of friction or viscous drag etc.[11]

Again, let us consider a spring-mass system. The damping force (due to friction) is linearly dependent on velocity and it acts opposite to the direction of the velocity, i.e., $F_d = -c\frac{dx}{dt}$. Thus, the equation of motion is given by,

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} \Rightarrow \boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0}.$$

The auxiliary equation is $mD^2 + cD + k = 0$, having roots $D = \frac{-c\pm\sqrt{c^2-4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\frac{c^2-4mk}{4m^2}} = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$. Note that, $\gamma = \frac{c}{2m}$ is known as the *damping coefficient* and the reason is going to be obvious when we discuss the solutions of the equation of motion. Let us discuss three possible cases.

Case 1: Overdamped oscillator: $c^2 - 4mk > 0$

In this case, the roots of the auxiliary equation are $-\gamma \pm \beta$, where $\beta = \sqrt{\gamma^2 - \omega^2}$. The general solution is given by $x(t) = e^{-\gamma t}(Ae^{\beta t} + Be^{-\beta t})$.

Case 2: Critically damped oscillator: $c^2 - 4mk = 0$

In this case, both the roots are equal to γ. Thus, the general solution is $x(t) = e^{-\gamma t}(At + B)$.

Case 3: Underdamped oscillator: $c^2 - 4mk < 0$

In this case, we have complex roots $-\gamma \pm \iota\beta$, where $\beta = \sqrt{\gamma^2 - \omega^2}$. The general solution is given by $x(t) = e^{-\gamma t}(C_1 \cos\beta t + C_2 \sin\beta t)$.

Displacement is plotted as a function of time for all three cases in Fig. 5.1. Note that, there are other systems for which we need to solve a similar differential equation and not surprisingly, we will get similar solutions. One famous example

[11] Consider a child playing on a swing. If we do not apply any force, the amplitude of oscillation of the swing decreases gradually.

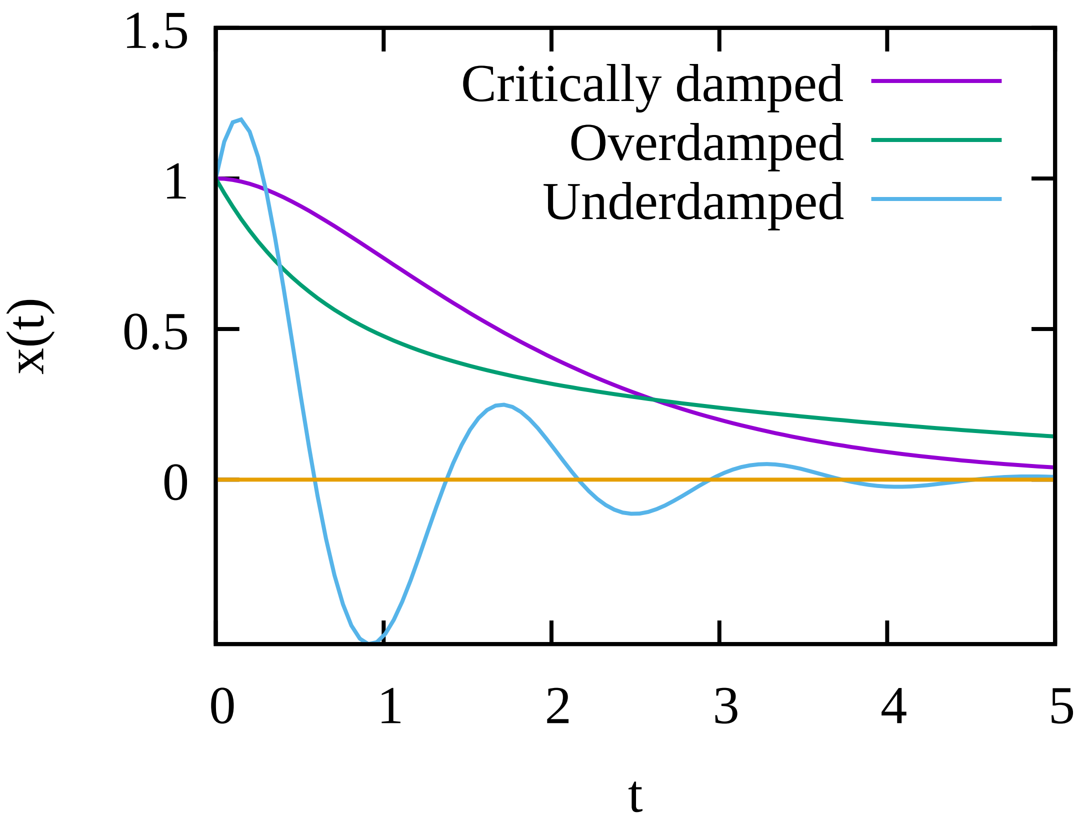

Figure 5.1: Displacement as a function of time for damped harmonic motion.

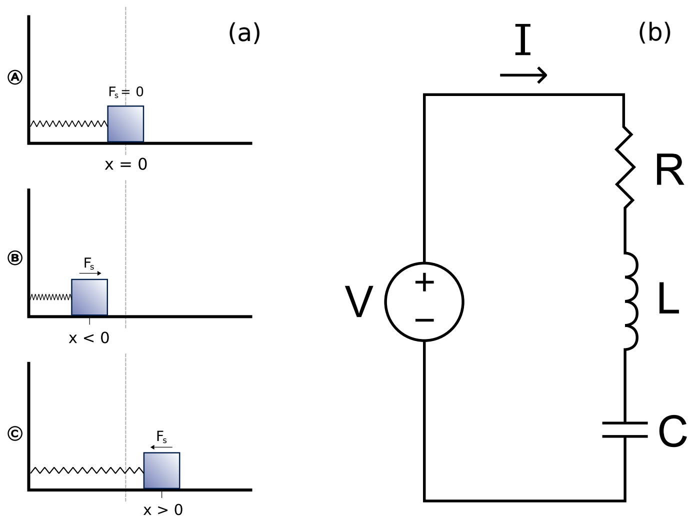

Figure 5.2: Different systems having similar differential equation: (a) spring-mass and (b) RLC circuit connected in series. Images are take from Wikipedia.

is a RLC circuit, where the components are connected in series (see Fig. 5.2). In this case, the governing equation is[12]

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dV}{dt},$$

and if we set right hand side equal to zero, we solve an equation similar to damped harmonic motion. In this case, resistance has a similar role as played by friction in case of spring-mass system.

5.3.2 Exercise

1. Re-derive Eq. 5.25: write auxiliary equation $(D - c_1)\underbrace{(D - c_2)y}_{u(x)} = 0$. $(D - c_2)y$ must be some function of x, say $u(x)$. Now, first solve for $u(x)$ from $(D-c_1)u = 0$. Then, solve for $(D - c_2)y = u$ and check whether you get the same answer as Eq. 5.25.

2. Consider the solution in case of overdamped harmonic motion. Can β and γ take any value for the general solution to be stable [i.e., $x(t)$ does not go to $\pm\infty$ with increasing time] or there is some restriction?

Solve the differential equations

3. $y'' + y' - 2y = 0$

   Answer: $y = Ae^{x} + Be^{-2x}$

4. $y'' + 9y = 0$

   Answer: $y = Ae^{3\iota x} + Be^{-3\iota x}$

5. $y'' - 2y' + y = 0$

   Answer: $y = (Ax + B)e^{x}$

6. $y'' - 5y' + 6y = 0$

   Answer: $y = Ae^{3x} + Be^{2x}$

7. $y'' - 4y' + 13y = 0$

   Answer: $y = Ae^{2x}\sin(2x + \gamma)$

8. $4y'' + 12y' + 9y = 0$

   Answer: $y = (A + Bx)e^{-3x/2}$

[12]We know that $V = RI, V = Q/C, V = L(dI/dt)$. Combining, we get $L\frac{dI}{dt} + RI + \frac{Q}{C} = V$. Taking time derivative and noting that $I = \frac{dQ}{dt}$, we get $L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dV}{dt}$.

Check for linear independence (by calculating Wronskian)

9. e^{-x}, e^{-4x}
10. $e^{ax}, e^{bx}, a \neq b$ (a, b real or imaginary)
11. e^{ax}, xe^{ax}
12. $\sin \beta x, \cos \beta x$
13. $1, x, x^2$
14. $e^{ax}, xe^{ax}, x^2 e^{ax}$

5.3.3 Constant coefficients and non-zero right hand side

Equations of the form

$$a_2 y'' + a_1 y' + a_0 y = f(x), \tag{5.33}$$

where a_2, a_1, a_0 are constants, are known as *inhomogeneous* equations. The function $f(x)$ is often termed as a forcing function, which represents an applied force or emf (electromotive force). If we set the right hand side equal to zero, we get a *complementary function* y_c, which is the solution of the homogeneous equation. For non-zero right hand side, we get a *particular solution* y_p and the general solution is given by,

$$y = y_c + y_p. \tag{5.34}$$

We already know how to solve for y_c. Let us learn a few tricks to solve for y_p.

Method 1: Via inspection

Let us consider the equation $y'' + y' - 2y = -3$. You can check that the complementary function is $y_c = Ae^x + Be^{-2x}$. Via inspection, it is easy to find the particular solution to be $y_p = 3/2$ and the general solution is $y = Ae^x + Be^{-2x} + 3/2$.

Method 2: Solve two successive first order linear equations

Instead of a constant, if the right hand side is some function, then method of inspection to find y_p is most likely going to fail. For example, let us consider the equation $y'' + y' - 2y = e^x$. The complimentary function is same as the previous problem. First we write the differential equation as,

$$(D-1)\underbrace{(D+2)y}_{u} = e^x.$$

Now, let $(D+2)y = u$, such that we get a first order linear differential equation,

$$(D-1)u = e^x \Rightarrow u' - u = e^x,$$

with $P = -1$ and $Q = e^x$. The solution is,

$$I = \int P dx = -x$$
$$ue^I = \int e^I Q dx + c = \int dx + c = x + c \Rightarrow u = xe^x + ce^x.$$

Thus, the first order linear differential equation for y is,

$$(D+2)y = xe^x + ce^x \Rightarrow y' + 2y = xe^x + ce^x,$$

where $P = 2$ and $Q = xe^x + ce^x$. The solution is,

$$I = \int P dx = 2x$$
$$ye^I = \int e^I Q dx + c_1 = \int (xe^{3x} + ce^{3x}) dx + c_1 = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + \frac{1}{3}ce^{3x} + c_1$$
$$y = \underbrace{\frac{1}{3}xe^x - \frac{1}{9}e^x}_{y_p} + \underbrace{\frac{1}{3}ce^x + c_1 e^{-2x}}_{y_c}.$$

I would like to draw attention to the fact that, we have obtained y_c from the arbitrary constants at every step. If we omit the arbitrary constants, we can quickly get the particular solution. Finally, we can beautify the final answer by writing $\frac{1}{3}c - \frac{1}{9} = c_2$, such that the general solution is $y = \frac{1}{3}xe^x + c_2 e^x + c_1 e^{-2x}$.

5.3.4 Exercise

1. $y'' - 4y = 10$

   Answer: $y = Ae^{2x} + Be^{-2x} - \frac{5}{2}$

2. $y'' + y' - 2y = e^{2x}$

   Answer: $Ae^x + Be^{-2x} + \frac{1}{4}e^{2x}$

3. $y'' + y = 2e^x$

   Answer: $y = Ae^{\iota x} + Be^{-\iota x} + e^x$

4. $y'' - y' - 2y = 3e^{2x}$

   Answer: $y = Ae^{-x} + Be^{2x} + xe^{2x}$

5. $y'' + 2y' + y = 2e^{-x}$

   Answer: $y = (Ax + B + x^2)e^{-x}$

5.4 Coupled first order differential equations

This I add as an application of eigenvalues and eigenvectors. Let $y_1(t)$ and $y_2(t)$ are both functions of t, having first derivatives $y_1' = dy_1/dt$ and $y_1' = dy_2/dt$. We have to solve for y_1 and y_2 by solving two differential equations, given by

$$\begin{aligned} y_1' &= ay_1 + by_2, \\ y_2' &= cy_1 + dy_2. \end{aligned} \tag{5.35}$$

Note that, we can express the above equation in the matrix form as,

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{5.36}$$

Two column vectors $\vec{y'}$ and $\vec{y}$ are related by the matrix A, such that $\vec{y'} = A\vec{y}$. Now, let us assume that $\vec{y} = B\vec{x}$, such that $\vec{y'} = AB\vec{x}$ and we get,

$$B^{-1}\vec{y'} = B^{-1}AB\vec{x} \Rightarrow \vec{x'} = B^{-1}AB\vec{x}. \tag{5.37}$$

If matrix B is made of eigenvectors of A, then we know that $B^{-1}AB$ is a diagonal matrix D, such that

$$\boxed{\vec{x'} = D\vec{x}}. \tag{5.38}$$

It is easy to solve for the vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ from the above equation, because D is a diagonal matrix, made of eigenvalues of A, say λ_1 and λ_2. Thus, we solve for

$$\begin{aligned} x_1' &= \lambda_1 x_1 \Rightarrow x_1 = c_1 e^{\lambda_1 t}, \\ x_2' &= \lambda_2 x_2 \Rightarrow x_2 = c_2 e^{\lambda_2 t}. \end{aligned} \tag{5.39}$$

Finally, we use $\vec{y} = B\vec{x}$ to get the solution for $y_1(t)$ and $y_2(t)$.

Let us solve for,

$$\begin{aligned} y_1' &= 3y_1 + 2y_2, \\ y_2' &= 6y_1 - y_2. \end{aligned} \tag{5.40}$$

The eigenvalues and eigenvectors for $A = \begin{pmatrix} 3 & 2 \\ 6 & -1 \end{pmatrix}$ are,

$$\lambda_1 = -3, \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad \& \quad \lambda_2 = 5, \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.41}$$

Let us construct the matrices from the eigenvalues and eigenvectors: $D = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}$. Now, using Eq. 5.38, we can write,

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5.42}$$

and the solutions are, $x_1 = c_1 e^{-3t}$ and $x_2 = c_2 e^{5t}$. Now, we get the final solution from $\vec{y} = B\vec{x} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

$$\begin{aligned} y_1(t) &= c_1 e^{-3t} + c_2 e^{5t}, \\ y_2(t) &= -3c_1 e^{-3t} + c_2 e^{5t}. \end{aligned} \tag{5.43}$$

5.4.1 Exercise

1. Solve $y_1' = y_1 + y_2, y_2' = 4y_1 + y_2$.
   Answer: $y_1 = c_1 e^{3t} + c_2 e^{-t}, y_2 = 2c_1 e^{3t} - 2c_2 e^{-t}$

5.5 Converting higher order to 1^{st} order equations

We can convert a linear differential equation of order n to n first order linear differential equations and then use the above technique to get a solution. Let us solve,

$$y'' + a_1 y' + a_0 y = 0 \Rightarrow y'' = -a_0 y - a_1 y'. \tag{5.44}$$

We do the following change of variables,

$$\begin{aligned} x_1 = y \quad &\& \quad x_2 = y', \\ x_1' = y' = x_2 \quad &\& \quad x_2' = y''. \end{aligned} \tag{5.45}$$

Thus, we can write two coupled first order linear equation,

$$\begin{aligned} x_1' &= 0x_1 + 1x_2, \\ x_2' &= -a_0 x_1 - a_1 x_2. \end{aligned} \tag{5.46}$$

Thus, we have converted a 2^{nd} order equation to two coupled 1^{st} order equations, which we can solve following the method shown in the previous section. We can do it for even higher order equations, like a 3^{rd} order equation,

$$y''' + a_2 y'' + a_1 y' + a_0 y = 0 \Rightarrow y''' = -a_0 y - a_1 y' - a_2 y''. \tag{5.47}$$

Using the following substitution,

$$x_1 = y, x_2 = y', x_3 = y'', \tag{5.48}$$

the above equation can be converted to three 1^{st} order equations,

$$\begin{aligned} x_1' &= 0x_1 + 1x_2 + 0x_3, \\ x_2' &= 0x_1 + 0x_2 + 1x_3, \\ x_3' &= -a_0 x_1 - a_1 x_2 - a_2 x_3. \end{aligned} \tag{5.49}$$

Let us see an example, where we solve the following 2^{nd} order linear equation using this method,

$$y'' + 5y_1 - 6y = 0. \tag{5.50}$$

Using the substitution $x_1 = y, x_2 = y'$, we get two 1^{st} order equations to solve,

$$\begin{aligned} x_1' &= 0 + x_2, \\ x_2' &= 6x_1 - 5x_2. \end{aligned} \tag{5.51}$$

The eigenvalues and eigenvectors for matrix $A = \begin{pmatrix} 0 & 1 \\ 6 & -5 \end{pmatrix}$ are,

$$\lambda_1 = 1, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \& \quad \lambda_2 = -6, \begin{pmatrix} 1 \\ -6 \end{pmatrix}. \tag{5.52}$$

The D matrix is given by $D = \begin{pmatrix} 1 & 0 \\ 0 & -6 \end{pmatrix}$ and the B matrix is given by, $B = \begin{pmatrix} 1 & 1 \\ 1 & -6 \end{pmatrix}$. Using Eq. 5.38 we can write

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -6 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{5.53}$$

and the solutions are $z_1 = c_1 e^t$ and $z_2 = c_2 e^{-6t}$. Thus, the solution for x_1 and x_2 are,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -6 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{5.54}$$

Now, since $y = x_1$, we write the final solution as, $y = c_1 e^t + c_2 e^{-6t}$.

5.5.1 Exercise

1. $y'' + y' - 2y = 0$

   Answer: $y = c_1 e^t + c_2 e^{-2t}$