Two dimensional Quantum Gravity

Abstract

Notes from reading course on Liouville conformal field theory held at IIT-Kanpur, in Spring 2020. The participants included Arjun Bagchi, Pinaki Banerjee, Ritankar Chatterjee, Diptarka Das, Bidyut Dey, Prateksh Dhivakar, Sudipta Dutta, Minhajul Islam, Kedar Kolekar, Mangesh Mandlik and, Punit Sharma.

Contents

1 References **2**

2 Introduction **2**

3 Liouville stress tensor **4**
- 3.1 Ward identity 5

4 Gauge-fixing to conformal gauge **7**

5 Weyl Invariance and Critical Dimensions **8**

6 Classical Limit, Equations of Motion and Solution **11**
- 6.1 Global classical solutions 15
- 6.2 Local classical solutions 17

A Appendix : General questions **22**

1 References

The following references were suggested:

1. Lectures on Liouville Theory and Matrix Models by the brothers Zamolodchikov.
2. Lectures on 2D Gravity and 2D String Theory by Ginsparg and Moore, hep-th/9304011
3. Notes on Quantum Liouville Theory and Quantum Gravity by Seiberg.
4. 2D Gravity and Random Matrices by Di Francesco, Ginsparg and Zinn-Justin, hep-th/9306153
5. Non-perturbative methods in 2d QFT by Abdalla, Abdalla and Rothe.
6. Theory of strings with boundaries by Alvarez, Nucl. Phys. B **216**, 125 (1983).
7. A review on Liouville field theory by Yu Nakayama, hep-th/0402009.
8. https://av.tib.eu/media/46725

2 Introduction

Scribe : Diptarka

- In a theory of gravity observables must be coordinate independent, one way out is to consider a topological theory of gravity. However we want a theory that is local. This issue is common also to 2D, however a better analytical control is possible.
- The theory of 2D quantum gravity is also the theory of gravity on the string worldsheet. Thus it is important to understand 2D gravity well to understand non-perturbative features of string theory.
- Some features of 2D gravity: all dynamics is captured by the Ricci scalar : $R(x)$.
- The metric has only 3 degrees of freedom, hence can be described by a function of two parameters, $\sigma(x)$. We will work with Euclidean signature and take the metric to be of the form: $g_{ab}(x) = e^{\sigma(x)}\delta_{ab}$.
- One can show,

$$R(x) = -e^{-\sigma(x)}\partial_a\partial^a\sigma(x). \tag{2.1}$$

- The most general local action that we can write down is therefore:

$$A = \mu\int\sqrt{g}d^2x + k\int R\sqrt{g}d^2x + \ldots. \tag{2.2}$$

  The $\ldots$ are terms which are irrelevant in the IR, while the only dynamical local term integrates to a topological quantity using the Gauss-Bonnet theorem.
- What we need to look at is gravity as an effective theory arising out of fluctuating matter living on a surface. However such a theory at a first go turns out to be non-local if the matter is massless, (and we integrate it out) or local only upto the correlation length (set by inverse mass scale) of the matter.
- This is not what we want. We want a local theory, thus we should not really integrate out the matter. There is a special kind of matter theory which actually allows us to describe the geometry dynamics (dynamics of the function $\sigma(x)$) with the help of a local action! This happens when the matter is conformal.

- Some reminders of a CFT : specified by $\{\Delta_i, \Phi_i\}$ and the central charge c. Under a change (infinitesimal) in $\sigma(x)$ we have, $\delta\Delta_i = -\Delta_i\Phi_i\delta\sigma(x)$ and $\delta g_{ab} = g_{ab}\delta\sigma(x)$. Also the Weyl anomaly is given via, $\theta = T_{ab}g^{ab} = -\frac{c}{12}R(x) + \mu$.
- The Liouville action arises as a reaction of conformal matter to Weyl variations. This is a local action and at least classically is itself also conformal.
- In the path integral for quantum gravity we need to integrate over :
  1. moduli
  2. σ
- One way to proceed is to fix the metric to some $\hat{g}$. If this is done then it turns out that, $A_{eff}[g] = A_{eff}[\hat{g}] + A_L[\sigma, \hat{g}]$, where,

$$A_L = -\frac{c}{96\pi}\int d^2x\ \sqrt{\hat{g}}\left(\hat{g}^{ab}\nabla_a\sigma\nabla_b\sigma - 2\sigma\hat{R}\right) + \mu\int d^2x\ e^\sigma\sqrt{\hat{g}}.$$

- Note that "naively" σ enters in a local way (which is what we wanted). However this is essentially because we have not integrated out the matter degrees of freedom, but simply related the theory of the matter degrees of freedom of one surface to that of another. The result of this translation between two geometries is a gift, that of a local action of σ !
- If we choose $\hat{g}_{ab} = \delta_{ab}$, then the action simplifies to,

$$A_L = -\frac{c}{96\pi}\int d^2x\ (\partial_a\sigma)^2 + \mu\int d^2x\ e^\sigma.$$

  The equation of motion of this action is,

$$\frac{c}{96\pi}\Delta\sigma = \mu e^\sigma. \tag{2.3}$$

  If we use this solution to compute $R(x)$ using (2.1) that gives, $R =$ constant.
- To quantize the theory we follow path integral formalism, we choose a particular gauge $\hat{g}_{ab} = \delta_{ab}$, where we have seen that A_L simplifies considerably. This is called the conformal gauge. Gauge fixing has to be upto diffeomorphisms. One can also find the Fadeev-Popov determinant. It turns out that this determinant is exactly of the form of the Liouville action itself with $c = -26$ (see §4 for a derivation).

$$\therefore A_{eff} = \frac{26-c}{96\pi}\int d^2x\ \ (\partial_a\sigma)^2 + \ldots$$

  And we are just left with,

$$Z_g = \int[\mathcal{D}\sigma]e^{-A_{eff}}.$$

  Of course we need to sum over the finite set of moduli.
- The unusual issue now is the fact that the UV cut-off itself of this theory is dependent on e^σ. The work of David-Distler-Kawai from 1988 shows that one can work with a σ independent cut-off provided one does a finite regularization of other parameters of the action. This is a way to understand the origin on μ in the Liouville action.
- The support for the DDK argument comes from matching calculations with,
  1. Matrix Models,
  2. Different gauge (Polyakov gauge).

3 Liouville stress tensor

Scribe : Sudipta

- Liouville action basically captures the response of the partition function with respect to the change of the weyl factor of the background metric.As discussed in the previous section the effective action can be decomposed as $A_{eff}[g] = A_{eff}[\hat{g}] + A_L[\sigma, \hat{g}]$
- Conservation of Stress tensor is

$$\nabla_\mu T^{\mu\nu} = 0 \tag{3.1}$$

we know for 2D manifolds any metric is conformally flat,hence can always be brought into the follwing form

$$g_{ab} = e^\sigma dz d\bar{z} \tag{3.2}$$

z and $\bar{z}$ are lightcone coordinates. Covariant derivative in conservation equation is with respect to the background metric,given by (2.2)

- In terms of components

$$g_{z\bar{z}} = \frac{1}{2}e^\sigma, g^{z\bar{z}} = 2e^{-\sigma} \tag{3.3}$$

And the levi-civita connections are given by

$$\Gamma^z_{zz} = \partial_z \sigma, \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \partial_{\bar{z}}\sigma \tag{3.4}$$

- Conservation equation

$$\nabla_{\bar{z}} T^{\bar{z}\bar{z}} + \nabla_z T^{z\bar{z}} = 0 \tag{3.5}$$

using the expressions for the levi civita connections we arrive at

$$\partial_{\bar{z}} T_{zz} + e^\sigma \partial_z (e^{-\sigma} T_{z\bar{z}}) = 0 \tag{3.6}$$

This implies that T_{zz} is **not Holomorphic** .

- Trace anomaly of CFTs coupled to any non trivial background is given by

$$< T^\mu_\mu >= 4e^{-\sigma} T_{z\bar{z}} = \alpha R \tag{3.7}$$

Hence

$$T_{z\bar{z}} = \frac{1}{4} e^\sigma \alpha R = -\alpha \partial_z \partial_{\bar{z}} \sigma \tag{3.8}$$

Using this in equation (2.6) we have

$$\partial_{\bar{z}} T_{zz} - e^\sigma \partial_z \Big(e^{-\sigma} \alpha \partial_z \partial_{\bar{z}} \sigma \Big) = 0 \tag{3.9}$$

$$\partial_{\bar{z}} T_{zz} = -\alpha (\partial_z \sigma)(\partial_z \partial_{\bar{z}} \sigma) + \alpha {\partial_z}^2 \partial_{\bar{z}} \sigma$$

$$= -\alpha \partial_{\bar{z}} \Big(\frac{\partial_z \sigma^2}{2} - \partial_z^2 \sigma \Big)$$

This implies

$$\partial_{\bar{z}} \Big[T_{zz} + \alpha (\frac{\partial_z \sigma^2}{2} - \partial_z^2 \sigma) \Big] = 0 \tag{3.10}$$

hence holomorphicity requires this extra term with T_{zz}.we can define t_{zz} such that $T = T_{zz} - \frac{\alpha}{2} t_{zz}$ is holomorphic.where $t_{zz} = -(\partial_z^2 \sigma - 2\partial_z^2 \sigma)$

- Under conformal transformations z $\rightarrow w(z,\bar{z})$, the weyl factor $\sigma(z,\bar{z})$ will transform as

$$\sigma(w,\bar{w}) = \sigma(z,\bar{z}) - log\Big(\frac{dw}{dz}\frac{d\bar{w}}{d\bar{z}}\Big) \tag{3.11}$$

Hence t_{zz} would transform as

$$t_{zz} = (\partial_z w)^2 t_{ww} - 2\{w,z\} \tag{3.12}$$

- T (Holomorphic component) should tarnsform as true tensor **(?)**.Then in order to compensate the schwarzian of t_{zz},T_{zz} should aslo transform with a schwarzian under conformal transformations. Hence

$$T(z) = (\partial_z w)^2 T_{ww} + \alpha\{w,z\} \tag{3.13}$$

3.1 Ward identity

- We can specially varry the metric without affecting the geometry, representing the changes of coordinate system. An infinitesimal coordinate transformation given by

$$x^\mu \rightarrow x^\mu + \epsilon^\mu \tag{3.14}$$

induces the variation

$$\delta g_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \tag{3.15}$$

- Now, we use the covariant action $\mathcal{A}[\phi, g]$ in the following functional integral

$$< O >= Z^{-1} \int O e^{-\mathcal{A}[\phi,g] D[\phi]} \tag{3.16}$$

where

$$O = O_1(x_1)O_2(x_2)....O_n(x_n) \tag{3.17}$$

- While doing functional integral over ϕ we note that by doing the change of coordinates we just relabel the integration variables. Hence the value of the integral remains same.

$$0 = - < \delta_\epsilon \mathcal{A} O > + < \delta_\epsilon O > \tag{3.18}$$

Here $\delta_\epsilon \mathcal{A}$ is the variation corresponding to the metric variation given before and is given by

$$\delta\mathcal{A} = \frac{1}{4\pi}\int \sqrt{g}\delta g_{\mu\nu} T^{\mu\nu} d^2x \tag{3.19}$$

$\delta_\epsilon O$ is given by

$$\delta_\epsilon O = \sum_k O_1(x_1)...\delta_\epsilon O_k(x_k)...O_n(x_n) \tag{3.20}$$

- Now, let us focus on the variation in the case of the Euclidean transformations

$$\epsilon^\mu(x) = \delta a^\mu + \delta\theta \epsilon_{\mu\nu}(x - x_0)^\nu \tag{3.21}$$

Physically this means rigid translations and rotations around some point x_0. $\epsilon_{\mu\nu}$ is antisymmetric tensor. For the above transformation O_k) transforms as

$$O_k(x_0) \rightarrow e^{is_k\delta\theta} O_k(x_0 + \delta a) \tag{3.22}$$

Here s_k is the spin of the field O_k. Therefore the euclidean variation gives us

$$\delta_\epsilon O_k(x_0) = \delta a^\mu \partial_\mu O_k(x_0) + is_k\delta\theta O_k(x_0 \tag{3.23}$$

- For more general transformation given by $x^\mu \to x^\mu + \epsilon^\mu$ we have to keep in mind that the variations $\delta_\epsilon O_k(x)$ are themselves local fields. That means they are linear combinations of the basic fields O_k with $\epsilon(x)$ dependent coefficients. The $\epsilon(x)$ must be linear and local. Here local dependence on $\epsilon(x)$ means that the coefficients of the variation $\delta_\epsilon O_k(x)$ may involve only ϵ and its derivatives taken at the same point x, i.e. $\epsilon^\mu(x)$, $\partial_\mu \epsilon^\nu(x)$, etc.
- Therefore, ultimately we end up with
$$\delta_\epsilon O_j(x) = \epsilon^\mu(x)\partial_\mu O_j(x) + \text{terms with derivatives of } \epsilon(x) \tag{3.24}$$
Here the first term ensures that for constant ϵ^μ we can recover equation (10).
- Now we can rewrite equation (5) as
$$\frac{1}{2\pi}\int \partial_\mu \epsilon_\nu(x) < T^{\mu\nu} O_1(x_1 ... O_N(x_N) > d^2x = \sum_k < O_1(x_1)...\delta_\epsilon O_k(x_k)...O_N(x_N) > \tag{3.25}$$
- The integration in the l.h.s. is over $\mathbb{R}^2$. Let us split it into two parts
$$\int_{\mathbb{R}^2} = \sum_k \int_{\mathbb{D}_\neg} + \int_{\bar{\mathbb{D}}} \tag{3.26}$$
where $\mathbb{D}_\sqsupset$ are small domains containing x_i and no other insertion point. In short $x_k \in \mathbb{D}_\neg$ and for $i \neq k$, $x_i \notin \mathbb{D}_\neg$. $\bar{\mathbb{D}}$ is the remaining part of $\mathbb{R}^2$, that is
$$\bar{\mathbb{D}} = \mathbb{R}^2 - \cup_i \mathbb{D}_\sqsupset \tag{3.27}$$
.
- Now let us transform the $\bar{\mathbb{D}}$ integral in equation (13) by parts which gives
$$-\frac{1}{2\pi}\int_{\bar{\mathbb{D}}} \epsilon_\nu(x) < \partial_\mu T^{\mu\nu}(x) O_1(x_1)...O_n(x_n) > + \text{boundary terms} \tag{3.28}$$
- This is the only term in equation no (12) which depends on $\epsilon(x)$ for $x \notin \mathbb{D}_\sqsupset$. Since r.h.s. of the equation does not depend on $\epsilon(x)$ for $x \notin \mathbb{D}_\sqsupset$, the above integrand must vanish for $x \in \bar{\mathbb{D}}$. Therefore the correlation function in the integrand. Making the domains $\mathbb{D}_\sqsupset$ arbitrarily small around x_i we conclude that it vanishes everywhere except when x coincides with one of the insertion points $x_1, x_2, ..., x_n$.
$$< \partial_\mu T^{\mu\nu}(x) O_1(x_1)...O_n(x_n) >= 0 \;\text{ if } x \neq x_1...x_n \tag{3.29}$$
Now, with the bulk term of equation (15) vanished, using Stokes formula, l.h.s of equation (12) can be re-written as
$$\sum_{i=1}^N \int_{\mathbb{D}_\sqsupset} \frac{\partial_\mu \epsilon_\nu(x)}{2\pi} < T^{\mu\nu} O > d^2x - \sum_{i=1}^N \oint_{\partial \mathbb{D}_\sqsupset} \frac{dx^\mu}{2\pi} \epsilon_\nu(x) < \epsilon_{\mu\lambda} T^{\lambda\nu}(x) O > \tag{3.30}$$
here $\partial\mathbb{D}_\sqsupset$ is the boundary of $\mathbb{D}_\sqsupset$s and the circular integration is done in counterclockwise direction.
- We can vary $\epsilon^\mu(x)$ independently in each domain $\mathbb{D}_\sqsupset$, and hence, equation (12) with l.h.s. in the form (17) must hold term by term. Therefore, we end up with
$$\delta_\epsilon O(x) = \int_{\mathbb{D}_\frown} \frac{d^2y}{2\pi} \partial_\mu \epsilon_\nu T^{\mu\nu}(y) O(x) - \oint_{\partial\mathbb{D}_\frown} \frac{dy^\mu}{2\pi} \epsilon_\nu(y) \epsilon_{\mu\lambda} T^{\lambda\nu}(y) O(x) \tag{3.31}$$

Isometries play an important role here. Isometries are pecial coordinate transformations leaving the form of the metric tensor unchanged (i.e. for which $\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = 0$). For such transformation the first part of (2.31) vanishes and the contour integral alone determines the variation.

4 Gauge-fixing to conformal gauge

Scribe : Kedar

- The path integral for 2D quantum gravity is

$$Z = \int \mathcal{D}g\mathcal{D}X e^{-S_X} \ , \tag{4.1}$$

where S_X is a matter action. We define the path integral measures $\mathcal{D}g$ and $\mathcal{D}X$ using Polyakov's approach. We turn the deformations δX^M into a Hilbert space by defining the inner product on the space of deformations, and imposing normalization as

$$||\delta X||^2 = \int d^2x\sqrt{g(x)}\delta X^M(x)\delta X_M(x) \ , \qquad \int \mathcal{D}(\delta X)e^{-||\delta X||^2} = 1 \ . \tag{4.2}$$

The most general local metric on the space of deformations δg_{ab} is

$$||\delta g||^2 = \int d^2x\sqrt{g}(G^{abcd} + ug^{ab}g^{cd})\delta g_{ab}\delta g_{cd} \ , \tag{4.3}$$

where $u > 0$ is an arbitrary fixed real number and $G^{abcd} = \frac{1}{2}(g^{ac}g^{bd} + g^{ad}g^{bc} - g^{ab}g^{cd})$ is the identity operator on the space of symmetric traceless tensors. Then decomposing the deformations δg_{ab} orthogonally

$$\delta g_{ab} = \delta h_{ab} + (2\delta\sigma)g_{ab} \ ; \qquad g^{ab}\delta h_{ab} = 0 \tag{4.4}$$

gives

$$||\delta g||^2 = \int d^2x\sqrt{g}G^{abcd}\delta h_{ab}\delta h_{cd} + 16u\int d^2\xi\sqrt{g}(\delta\sigma)^2 \ . \tag{4.5}$$

Imposing the normalization $\mathcal{D}(\delta g)e^{-\frac{1}{2}||\delta g||^2} = 1$ then gives the integration measure as $\mathcal{D}g \equiv \mathcal{D}h\mathcal{D}\sigma$.

- Gauge-fixing: For simplicity, we restrict to metrics of fixed topology of a sphere (*i.e.,* genus 0 and no boundaries). We want to gauge-fix to conformal gauge $\tilde{g}_{ab} = e^{\phi}\delta_{ab}$. Consider transformation of the metric under a diffeomorphism $x \to f(x)$,

$$g_{ab}(x) = \frac{\partial f^c}{\partial x^a}\frac{\partial f^d}{\partial x^b}\tilde{g}_{cd}(f(x)) = e^{\phi(f(x))}\frac{\partial f^c}{\partial x^a}\frac{\partial f_c}{\partial x^b} \ . \tag{4.6}$$

For the gauge-fixing to conformal gauge to be possible, the above transformation of g_{ab} should be non-singular *i.e.,* the Jacobian in passing to the variables ϕ, f^a should be non-zero. To see what we mean by the non-singularity, consider a further infinitesimal diffeomorphism $f^a \to f^a + \xi^a(f(x))$, under which

$$\delta g_{ab}(x) = \frac{\partial f^c}{\partial x^a}\frac{\partial f^d}{\partial x^b}\Big(\delta\tilde{g}_{cd} + \tilde{\nabla}_c\xi_d + \tilde{\nabla}_d\xi_c\Big)_{f(x)} \ . \tag{4.7}$$

- The non-singularity of the transformation (4.6) means that for any δg_{ab}, we can find ξ^a and ϕ such that (4.7) holds *i.e.,* $(\delta\phi e^{\phi}\delta_{cd} + \tilde{\nabla}_c\xi_d + \tilde{\nabla}_d\xi_c) = \gamma_{ab}$ has a non-trivial solution. Using an operator L which takes vectors to symmetric traceless 2nd rank tensors *i.e.,*

$$(L\xi)_{ab} = \tilde{\nabla}_a\xi_b + \tilde{\nabla}_b\xi_a - e^{\phi}\delta_{ab}\tilde{\nabla}^c\xi_c = \gamma_{ab} - \frac{1}{2}\delta_{ab}\gamma^c_c \equiv \psi_{ab} \ , \tag{4.8}$$

the non-singularity above means that $(L\xi)_{ab}$ has non-zero modes ψ_{ab}. The adjoint $L^\dagger$, defined through the inner product $(L^\dagger\psi, \xi) = (\psi, L\xi)$ takes symmetric traceless 2nd rank tensors to vectors and is given by

$$(L^\dagger\psi)_b = -\tilde{\nabla}^a\psi_{ab} \ . \tag{4.9}$$

- Comparing δg_{ab} in (4.7) with the orthogonal decomposition (4.4), we get $\delta h_{ab} = (L\xi)_{ab}$ and $2(\delta\sigma) = 2(\delta\phi) + \nabla^c\xi_c$. Changing variables from (h_{ab}, σ) to (ξ^a, ϕ) introduces a Jacobian,

$$\mathcal{D}h\mathcal{D}\sigma = \mathcal{D}\xi\mathcal{D}\phi \left|\frac{\partial(\sigma, h)}{\partial(\phi, \xi)}\right| = \mathcal{D}\xi\mathcal{D}\phi \ \det \begin{bmatrix} 1 & \# \\ 0 & L \end{bmatrix} = \mathcal{D}\xi\mathcal{D}\phi \ \det L = \mathcal{D}\xi\mathcal{D}\phi \, (\det L^\dagger L)^{\frac{1}{2}} \ . \tag{4.10}$$

- The matter action S_X and Faddeev-Popov determinant $(\det L^\dagger L)$ is invariant under diffeomorphisms, and the measure $\mathcal{D}\xi$ is independent of the conformal factor, so we can factor out the volume of the gauge group (diffemorphims), $\Omega_{diff} = \int \mathcal{D}\xi$ and write the path integral as

$$Z = \Omega_{diff} \int \mathcal{D}\phi\mathcal{D}X (\det L^\dagger L)^{\frac{1}{2}} e^{-S_X} \ . \tag{4.11}$$

- From L and $L^\dagger$ above, we get

$$\begin{aligned} (L^\dagger L\xi)_a &= -\nabla^b(\nabla_a\xi_b + \nabla_b\xi_a - g_{ab}\nabla^c\xi_c) \\ &= \nabla^2\xi_a - [\nabla^b, \nabla_a]\xi_b \\ &= -\Big(\nabla^2 + \frac{\mathcal{R}}{2}\Big)\xi_a \ , \end{aligned} \tag{4.12}$$

where we have used $[\nabla_a, \nabla_b]\xi_c = \mathcal{R}_{abc}{}^d\xi_d$ and $\mathcal{R}_{abcd} = \dfrac{\mathcal{R}}{2}(g_{ac}g_{bd} - g_{ad}g_{bc})$.

- Using the heat kernel method, the functional determinant above can be computed as

$$\log(\det L^\dagger L)^{\frac{1}{2}} = \log\Big[\det\Big(-\nabla^2 - \frac{\mathcal{R}}{2}\Big)\Big]^{\frac{1}{2}} = \frac{-26}{48\pi}\int d^2x \Big[\frac{\delta^{ab}}{2}\partial_a\phi\partial_b\phi + \mu e^\phi\Big] \ . \tag{4.13}$$

Thus we see that the Faddeev-Popov determinant for gauge-fixing to conformal gauge is the Liouville action with $c = -26$.

- Questions/Exercises :

1. Clarify the Polyakov's approach of defining path integral measure using metric on space of deformations of the fields and the normalization *i.e.*, (4.2) and (4.3).
2. Verify that $L^\dagger$ in (4.9) satisfies $(\psi, L\xi) = (L^\dagger\psi, \xi)$.
3. Work out the heat kernel and derive the functional determinant in (4.13).

5 Weyl Invariance and Critical Dimensions

Scribe : `Kedar`

Referrences:

1 Mostly "Ginsparg and Moore" [hep-th: 9304011] → Section: 2

2 Zamalodchikov and Zamalodchikov → Section: 21 and "Yu Nakayama" [hep-th/ 0402009] → 2.1

- We start with

$$Z[g] = \int \mathcal{D}_g X\mathcal{D}_g\phi e^{-S_X} \tag{5.1}$$

- Gauge fixing to $g_\mu = e^\phi \hat{g}_{\mu\nu}$, $\hat{g}_{\mu\nu} \rightarrow$ some fixed reference metric(commonly called "Fiducial Metric" in literature), we get (after modding out Ω_{diff}),

$$Z[g] = \int \mathcal{D}_g X \mathcal{D}_g \phi \big(det(L^\dagger L)\big)^{\frac{1}{2}} e^{-S_X} \tag{5.2}$$

We know, from string theory courses(i.e. Faddeev-Popov gauge fixing),

$$\Big(detL^\dagger L\Big)^{\frac{1}{2}} = \int \mathcal{D}(gh) e^{-S_{gh}(b,c,\bar{b},\bar{c})} \tag{5.3}$$

with $C_{gh} = -26$. Then

$$Z[g] = \int \mathcal{D}_g X \mathcal{D}_g \phi \big(det(L^\dagger L)\big)^{\frac{1}{2}} e^{-S_X} \tag{5.4}$$

$$S_X = \frac{1}{8\pi} \int d^2\xi \sqrt{g} g^{\mu\nu} \partial_\mu X^M \partial_\nu X^N \eta_{MN} \tag{5.5}$$

$$S_{gh} = \int d^2\xi (b\bar{\partial}c + \bar{b}\partial\bar{c}) \rightarrow \text{in complex coordinates} \tag{5.6}$$

$$S_L[\phi; g] = \int D^2\xi \sqrt{g} \Big[\frac{1}{2} f^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \mathcal{R}\phi + \mu e^\phi\Big] \tag{5.7}$$

$$\tag{5.8}$$

- Lets do a Weyl transformation from $g \rightarrow \hat{g}$. Under this,

$$\int \mathcal{D}_{e^\phi \hat{g}}(X) e^{-S_X[g]} = e^{\frac{C_m}{48\pi} S_L[g]} \int \mathcal{D}_{\hat{g}}(X) e^{-S_X[\hat{g}]}, \tag{5.9}$$

$$\int \mathcal{D}_{e^\phi \hat{g}}(gh) e^{-S_{gh}[g]} = e^{\frac{C_m}{48\pi} S_L[g]} \int \mathcal{D}_{\hat{g}}(gh) e^{-S_{gh}[\hat{g}]}. \tag{5.10}$$

- However, the measure for the Liouville field, ϕ implicitly depends on ϕ and so we cannot use the above technique to compute the transformation: $\int \mathcal{D}_{e^\phi \hat{g}}(\phi) \rightarrow \int \mathcal{D}_{\hat{g}}(\phi)$
- The measure $\mathcal{D}_{\hat{g}}(\phi)$ satisfies(since $Z[e^\phi \hat{g}]$ is constructed diffeomorphism invariantly),

$$||^2 \delta\phi||_g^2 = \int d^2\xi \sqrt{g} (\delta\phi)^2 = \int d^2 \sqrt{\hat{g}} e^\phi (\delta\phi)^2 \tag{5.11}$$

It is to be noted that "e^ϕ" in NOT Gaussian!

- We want this to transform it to the standard gaussian measure, i.e.

$$||\delta\phi||_{\hat{g}}^2 = \int d^2\xi \sqrt{\hat{g}} (\delta\phi)^2, \tag{5.12}$$

- We need to obtain the Jacobian for this transformation and include it into the Liouville action. However, it is difficult to obtain this Jacobian from first principles or above technique for X and gh fields. So following DDK. we will assume/guess the form of the "renormalized Liouville action" by assuming its "locality","diffeomorphism invariance" and "conformal invariance". Let us know if somebody understand what the previous highted text means.
- (See the footnote on page-9 of Nakayama, where, Nakayama says, the derivation of this is discussed)!

- Here, we will proceed with the guesswork, following DDK, and write

$$Z[g] = \int \mathcal{D}_{e^{\phi}\hat{g}}(X)\mathcal{D}_{e^{\phi}\hat{g}}(gh)e^{-S_X - S_{gh}} \tag{5.13}$$

$$= \int \mathcal{D}_{e^{\phi}\hat{g}}(X)\mathcal{D}_{e^{\phi}\hat{g}}(gh)J(\phi, \hat{g}) \tag{5.14}$$

$$\tag{5.15}$$

Where,

$$J(\phi, \hat{g}) = exp\Big[- \int d^2\xi \sqrt{\hat{g}}\big(\tilde{a}\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \tilde{b}\hat{\mathcal{R}}\phi + \mu e^{\tilde{c}\phi}\big)\Big] \tag{5.16}$$

- Z is defined to be reparameterization invariant and should depend only on $g = e^{\phi}\hat{g}$(upto diffeomorphism) and not on the fiducial metric $\hat{g}$. So, Z should be invariant under change in $\hat{g}$ with g fixed, i.e.

$$g = e^{\phi}\hat{g} = e^{\phi}e^{\epsilon}\tilde{g} = e^{\tilde{\phi}}\tilde{g} \tag{5.17}$$

Where, $\phi = \tilde{\phi} - \epsilon$
Infinitesimally it looks like, $\delta\hat{g} = \epsilon(\xi)\hat{g}$ and $\delta\phi = -\epsilon(\xi)$.

- Under this infinitesimal variation, the path integral becomes

$$Z = \int \mathcal{D}_{\tilde{g}}(X)\mathcal{D}_{\tilde{g}}(gh)\mathcal{D}_{\tilde{g}}(\tilde{\phi})e^{-S_X[\tilde{g}]-S_{gh}[\tilde{g}]}J(\tilde{\phi}, \tilde{g})e^{\Delta S} \tag{5.18}$$

Invariance of Z under $\delta\hat{g}$ and $\delta\phi$ above implies, $\Delta S = 0$. Where

$$\Delta S = \left(\frac{c_M + c_{gh} + 1}{48\pi}\right)S_L[\epsilon; \tilde{g}]$$
$$- \int d^2\xi \Big[\sqrt{\tilde{g}}e^{\epsilon}\tilde{a}e^{-\epsilon}\tilde{g}^{\mu\nu}(-2\partial_\mu\tilde{\phi}\partial_\nu\epsilon) + \tilde{b}\sqrt{\tilde{g}}(-\tilde{\mathcal{R}}\epsilon - \tilde{\phi}\tilde{\nabla}^2\epsilon) + \mu e^{(1-\tilde{c})\epsilon}\big)\Big] \tag{5.19}$$

(1) in the above equation comes from transformation of the measure. Following two items are to be clarified!

1. $\mathcal{D}_{\hat{g}}\phi$, Which we have defined to be Gaussian
2. $||\delta\phi||^2_{\hat{g}} = \int d^2\xi\sqrt{g}(\delta\phi)^2 \rightarrow$ a single free boson with central charge=1. The variantion terms in $\Delta S = 0$ give

$$\left(\frac{c_M + c_{gh} + 1}{48\pi} + \tilde{b}\right)\int d^2\xi\sqrt{\tilde{g}}\tilde{R}\epsilon = 0 \Rightarrow \tilde{b} = -\left(\frac{c_M + c_{gh} + 1}{48\pi}\right) \tag{5.20}$$

$$\left(2\tilde{a} - \tilde{b}\right)\int d^2\xi\sqrt{\tilde{g}}\epsilon\Box\tilde{\phi} = 0 \Rightarrow \tilde{a} = \frac{\tilde{b}}{2} = -\left(\frac{c_M + c_{gh} + 1}{48\pi}\right) \tag{5.21}$$

For $c_M = D$ and $c_{gh} = -26$, $\tilde{a} = \frac{25-D}{96\pi}$, $\tilde{b} = \frac{25-D}{48\pi}$, we get

$$J[\phi, \tilde{g}] = exp\left[-\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\Big[\Big(\frac{25-D}{12}\Big)\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \Big(\frac{25-D}{6}\Big)\hat{\mathcal{R}}\phi + 8\pi\mu e^{\tilde{c}\phi}\Big]\right] \tag{5.22}$$

Rescale, $\phi = \sqrt{\frac{12}{25-D}}\varphi \equiv \frac{2}{Q}\varphi$, where, $Q \equiv \sqrt{\frac{25-D}{3}}$

$$J = exp\Big[- \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\Big[\hat{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + Q\hat{\mathcal{R}}\varphi + 8\pi\mu e^{2\frac{\tilde{c}}{Q}\varphi}\Big] \tag{5.23}$$

- The total path integral for the matter-ghost CFT coupled to the Liouville field φ (i.e. ϕ) is

$$Z = \int \mathcal{D}_{\hat{g}}(X)\mathcal{D}_{\hat{g}}(gh)\mathcal{D}_{\hat{g}}(\tilde{\varphi})e^{-S_X - S_{gh}}e^{-S_0[\varphi;\hat{g}]}, \tag{5.24}$$

Where,

$$S_0 = \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\Big[\hat{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + Q\hat{\mathcal{R}}\varphi + 8\pi\mu e^{2\frac{\tilde{c}}{Q}\varphi}\Big] \tag{5.25}$$

- Conformal invariance of matter-ghost-Liouville theory: We see that with $c_M = D$, $c_{gh} = -26$, and $c_0 = 1 + 3Q^2 = 26 - D$, the total central charge of the full matter-ghost-Liouville theory is,

$$C_{\text{total}} = c_M + c_{gh} + c_0 = D - 26 + 26 - D = 0. \tag{5.26}$$

- **Questions/Exercises :**

  [1.] Apriori do we know/expect S_0 is a CFT?

  [2.] Show that S_0 is a CFT by showing that the algebra of L_m modes of $T_{\mu\nu} \sim \frac{\delta S_0}{delta\hat{g}^{\mu\nu}}$ is the Virasoro Algebra. Also find that $c_0 = 1 + 3Q^2$.

6 Classical Limit, Equations of Motion and Solution

Scribe : Bidyut

Referrences:

1 Notes on 2d quantum gravity and Liouville theory by Harold Erbin

2 Zamalodchikov and Zamalodchikov Section 15

- Crtical Exponent from LCFT action

$$\mathcal{Z}[e^{\sigma\hat{g}}] = e^{\frac{c}{48\pi}\int\sqrt{\hat{g}}[R\sigma + \frac{1}{2}\hat{g}(\partial\sigma)^2]}\mathcal{Z}[\hat{g}] \tag{6.1}$$

Now we have $g_{\mu\nu} \to L^2 g_{\mu\nu}$. Then our action will be

$$\begin{aligned}\mathcal{Z}[L^2\hat{g}] =& \quad e^{\frac{c}{48\pi}\log L^2\int\hat{R}\sqrt{\hat{g}}d^2x}\mathcal{Z}[\hat{g}]\\ =& \quad e^{\frac{c}{6}(1-\gamma)\log L^2}\mathcal{Z}[\hat{g}]\\ =& \quad A^\kappa\mathcal{Z}[\hat{g}]\end{aligned} \tag{6.2}$$

where $\kappa = \frac{c}{6}(1-\gamma)$ How does it come?

$$\sqrt{\hat{g}}\hat{R} = 8\pi\eta\,\delta(x - x_0) \tag{6.3}$$

$$\begin{aligned}ds^2 &= \frac{dzd\bar{z}}{(z\bar{z})^2\eta}\\ &= e^\sigma dzd\bar{z}\end{aligned} \tag{6.4}$$

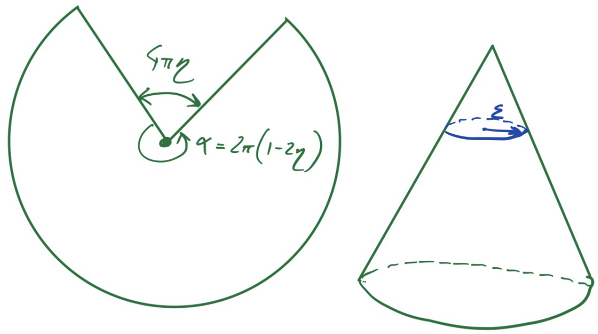

Figure 1: The conical singularity and its resolution

So

$$\sigma = -2\eta \log(z\bar{z})$$

$$\text{for, } |Z| > \epsilon \quad e^{\sigma} = \frac{1}{(z\bar{z})^{2\eta}}$$

$$\text{and for, } |Z| < \epsilon \quad e^{\sigma} = \frac{1}{(\epsilon^2)^{2\eta}}$$

- <u>Classical Limit:</u>
  The action

$$A[\sigma, \hat{g}] = \frac{1}{\hbar} S_L[\sigma, \hat{g}]$$

Here

$$S_L = \frac{1}{2\pi} \int \sqrt{\hat{g}}[\hat{R}\sigma + \frac{1}{2}\hat{g}(\partial\sigma)^2 + \Lambda e^{\sigma}]$$

$$\frac{1}{\hbar} = \frac{26 - c}{24}$$

Here for classical limit we have to take $\hbar \to 0$ and then corresponding limit for central charge $c = -\infty$. Now we will focus on the solutions of classical equation of motion for above action. Equation of motion

$$-\Delta_{\hat{g}}\sigma + \hat{R} + \Lambda e^{\sigma} = 0 \tag{6.5}$$

Here $\Delta_{\hat{g}}$ is Laplacian. We know that for the transformation

$$g_{\mu\nu} = e^{\sigma}\hat{g} \tag{6.6}$$

The scalar curvature associated with the metric $\hat{g}$ has the form

$$\sqrt{\hat{g}}(\hat{R} - \Delta_{\hat{g}}\sigma) = \sqrt{g}R \tag{6.7}$$

Then one can write equation of motion as

$$R + \Lambda = 0 \tag{6.8}$$

Since Λ describe the geometries classically so both the positive and negative conditions are important here for different situation. Now let us take some particular domain of manifold with $\hat{g}$ to be a flat Euclidean metric then $\hat{R} = 0$. In complex coordinates $(z, \bar{z})$

$$ds^2 = e^{\sigma(z,\bar{z})} dz d\bar{z} \tag{6.9}$$

Then equation of motion becomes

$$4\partial_z \partial_{\bar{z}} \sigma = \Lambda e^{\sigma} \tag{6.10}$$

This is the Liouville equation.
Now we will try to understand some properties of the above equation. We define two quantities

$$t = -\partial_z \sigma \partial_z \sigma + 2\partial_z^2 \sigma \qquad \bar{t} = -\partial_{\bar{z}} \sigma \partial_{\bar{z}} \sigma + 2\partial_{\bar{z}}^2 \sigma \tag{6.11}$$

By taking derivative with respect to $\bar{z}$ and z one can see that these are holomorphic and anti-holomorphic function respectively. We can also see that if one take variation of the action with respect to $\hat{g}_{\mu\nu}$ that given t and $\bar{t}$ are components of classical energy momentum tensor. Varying action

$$\delta S[\sigma, \hat{g}] = -\frac{1}{4\pi} \int \sqrt{\hat{g}} \delta \hat{g}^{\mu\nu} \mathcal{T}_{\mu\nu} d^2x \tag{6.12}$$

where

$$\mathcal{T}_{\mu\nu} = -\partial_\mu \sigma \partial_\nu \sigma + \hat{g}_{\mu\nu}(\frac{1}{2}(\partial\sigma)^2 + \Lambda e^{\sigma}) + 2(\partial_\mu \partial_\nu \sigma - \hat{g}_{\mu\nu} \partial^2 \sigma) \tag{6.13}$$

Contracting with $\hat{g}_{\mu\nu}$ we have

$$\mathcal{T}^\mu_\mu = 2(\Lambda e^{\sigma} - \partial^2 \sigma) \tag{6.14}$$

If we consider the equation of motion along with $\hat{R} = 0$ then the trace will vanish. One can easily identify that $\mathcal{T}_{zz} = t(z)$ and $\mathcal{T}_{\bar{z}\bar{z}} = \bar{t}(\bar{z})$.

By taking the field as $e^{-\frac{\sigma}{2}}$ and doing second derivative with respect to z we can see that

$$4\partial_z^2 e^{-\frac{\sigma}{2}} = t e^{-\frac{\sigma}{2}} \tag{6.15}$$

and

$$4\partial_{\bar{z}}^2 e^{-\frac{\sigma}{2}} = \bar{t} e^{-\frac{\sigma}{2}} \tag{6.16}$$

These looks like eigenvalue equation

$$-4\partial_z^2 \psi(z) = t\psi(z) \tag{6.17}$$
$$-4\partial_{\bar{z}}^2 \psi(\bar{z}) = \bar{t}\psi(\bar{z}) \tag{6.18}$$

Now Let's discuss the transformation properties of these differential equations. If we take a transformation $z \to \omega(z)$ and $\bar{z} \to \bar{\omega}(\bar{z})$ which leads to

$$\sigma \to \sigma(\omega, \bar{\omega}) = \sigma(z, \bar{z}) - \log(\frac{\partial \omega}{\partial z} \frac{\partial \bar{\omega}}{\bar{z}}) \tag{6.19}$$

then metric will be

$$ds^2 = e^{\sigma} d\omega d\bar{\omega} \tag{6.20}$$

and operator will be

$$\partial_z^2 = \frac{\partial^2 \omega}{\partial z^2}\partial_\omega + (\frac{\partial \omega}{\partial z})^2 \partial_\omega^2 \tag{6.21}$$

One can write equation (6.17) as

$$4\partial_\omega^2 \psi(\omega) = t(\omega)\psi(\omega) \tag{6.22}$$

where

$$\psi(\omega) = (\partial_\omega z)^{-\frac{1}{2}} \psi\left(z(\omega)\right)$$

While the function $t(\omega)$ transform as

$$t(\omega) = (\partial_\omega z)^2 \, t\left(z(\omega)\right) + 2\{z, \omega\}$$

- Solution:-
  Above mentioned eigenvalue equation are very important for Liouville field theory. If we know the solutions of these eigenvalue equations then we can calculate σ:
  Let we choose
  $$\psi(z) = \begin{pmatrix} \psi_1(z) \\ \psi_2(z) \end{pmatrix}$$
  Here ψ_1 and ψ_2 are two linearly independent solutions of the holomorphic equation. So the Wronskian will be some constant. We choose Wronskian as
  $$W = \psi_1 \psi_2' - \psi_2 \psi_1' = 1$$
  For anti-holomorphic equation we also have two independent solutions and we can choose them such a way so that they also satisfy same Wronskian condition. Now take this as a solution and we will verify this combination
  $$\sigma(z, \bar{z}) = -2\log\left[\bar{\psi}(\bar{z})\, \Lambda\, \psi(z)\right] + \log 8 \tag{6.23}$$
  where, Λ is a 2×2 constant matrix whose determinant is
  $$\det \Lambda = -\Lambda$$
  Let,
  $$\bar{\chi}(\bar{z}) = \bar{\psi}(\bar{z})\Lambda.$$
  So we can write (6.23) as
  $$\sigma(z, \bar{z}) = -2\log\left[\bar{\chi}(\bar{z})\psi(z)\right] + \log 8$$

  Then,
  $$\begin{aligned} -4\partial_z\partial_{\bar{z}}\,\sigma =& -4\partial_z\partial_{\bar{z}}\left[-2\log\left[\bar{\chi}(\bar{z})\psi(z)\right] + \log 8\right] \\ =& 8\partial_z\left[\frac{(\partial_{\bar{z}}\bar{\chi})\psi}{\bar{\chi}\psi}\right] \\ =& \frac{8\left\{(\partial_{\bar{z}}\,\bar{\chi}\,\partial_z\psi)(\chi\psi) - (\partial_{\bar{z}}\,\bar{\chi}\,\psi)(\bar{\chi}\partial_z\psi)\right\}}{(\bar{\chi}\psi)^2} \\ =& \frac{8}{(\bar{\chi}\psi)^2}\left[(\bar{\chi}_1\partial_{\bar{z}}\,\bar{\chi}_2 - \bar{\chi}_2\partial_{\bar{z}}\,\bar{\chi}_1)(\psi_1\partial_z\,\psi_2 - \psi_2\partial_z\,\psi_1)\right] \end{aligned}$$

Now if we use back

$$\bar{\chi}(\bar{z}) = \bar{\psi}(\bar{z})\mathsf{\Lambda}\,,$$

then after some straightforward algebra we get

$$-4\partial_z\partial_{\bar{z}}\,\sigma = \frac{8}{(\bar{\chi}\psi)^2}\left[(\bar{\psi}_1\partial_{\bar{z}}\,\bar{\psi}_2 - \bar{\psi}_2\partial_{\bar{z}}\,\bar{\psi}_1)(\Lambda_{11}\Lambda_{22} - \Lambda_{21}\Lambda_{12})\right]$$

Since Wronskian is 1 so one can get,

$$\begin{aligned} -4\partial_z\partial_{\bar{z}}\,\sigma =& \frac{8\det\mathsf{\Lambda}}{(\bar{\chi}\psi)^2} \\ =& -\frac{8\Lambda}{(\bar{\chi}\psi)^2} \\ =& -\Lambda e^{\sigma} \end{aligned}$$

where $e^{\sigma} = \frac{8}{(\bar{\chi}\psi)^2}$ or $\sigma(z,\bar{z}) = -2\log\left[\bar{\chi}(\bar{z})\psi(z)\right] + \log 8$.
So finally

$$\sigma(z,\bar{z}) = -2\log\left[\bar{\psi}(\bar{z})\,\mathsf{\Lambda}\,\psi(z)\right] + \log 8$$

is a solution of

$$-4\partial_z\partial_{\bar{z}}\,\sigma = -\Lambda e^{\sigma}$$

where $\psi(z)$ and $\bar{\psi}(\bar{z})$ can be get from eigenvalue equations(6.17) and (6.18).
Here we have problem with the above solution since the solutions are not real and not single valued. For reality problem one can just take matrix $\mathsf{\Lambda}$ Hermitian. For the single-valued problem we have to analyze monodromy properties of the differential equation which will be in 6th lecture.

6.1 Global classical solutions

The monodromy is defined via taking a solution basis around a complex curve, which we denote by C.

$$\psi(C * z) = M(C)\psi(z)$$

Here C is a monodromy and $M(c)$ is some matrix representing the monodromy where $M(C)\,\epsilon\,SL(2,C)$. Where M is satisfies the condition

$$M^{\dagger}\mathsf{\Lambda}M = \mathsf{\Lambda}$$

For this M the solution σ are invariant under change of ψ
Now for Positive Curvature:
Since we know

$$R + \Lambda = 0$$

so for +ve curvature $\Lambda < 0$ and $\mathsf{\Lambda}$ is defined as

$$\mathsf{\Lambda} = \sqrt{-\Lambda}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Similarly for negative Curvature:
$\Lambda > 0$ and $\mathsf{\Lambda}$ is defined as

$$\mathsf{\Lambda} = \sqrt{\Lambda}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

. <u>Special Case:</u>
Let's discuss a simple solution with positive curvature. The simplest solution then is sphere where

$$\psi_1(z) = 1$$
$$\psi_2(z) = z$$

so that Wronskian $W = \psi_1\psi_2' - \psi_2\psi_1' = 1$. Now pluging this ψ_1 and ψ_2 in the solution of σ (6.23) one can get

$$\begin{aligned}\sigma =& -2\log\left[\begin{pmatrix}1 & \bar{z}\end{pmatrix}\sqrt{-\Lambda}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ z\end{pmatrix}\right] + \log 8\\ =& \log\left[-\frac{8}{\Lambda(1+z\bar{z})^2}\right]\end{aligned}$$

So we have,

$$\begin{aligned}ds^2 =& e^{\sigma}dzd\bar{z}\\ =& -\frac{8}{\Lambda}\frac{1}{(1+z\bar{z})^2}dzd\bar{z}\end{aligned} \tag{6.24}$$

$$\mathcal{Z}[e^{\sigma_{cl}}\delta_{ab}\,M_{CFT}] = e^{-S_L[\sigma_{cl}]}\mathcal{Z}[\delta_{ab}M_{CFT}] \tag{6.25}$$

$$\mathcal{Z}_{sphere} = e^{-S_L}\mathcal{Z}_{plane}$$

In flat background(i.e Plane) $\hat{R} = 0$ and $\hat{g} = 1$

$$S = \frac{1}{2\pi}\int[2(\partial_z\sigma)(\partial_{\bar{z}}\sigma) + \Lambda e^{\sigma}]d^2z$$

then also $e^{\sigma} = -\frac{8}{\Lambda}\frac{1}{(1+z\bar{z})^2}$.

Now,

$$8\int_{|z|<L} dzd\bar{z}\frac{z\bar{z}}{(1+z\bar{z})^2} \to 8\,\pi(\log L^2 - 1)$$

Here we take L very high and consider only the disk $|z| < L$. So here we have a divergence so we have to add a $S_{cl} = -4\log L^2$ counter term
Another thing is that if we choose

$$\psi_1(z) = 1$$
$$\psi_2(z) = z$$

so that Wronskian is 1. Now from (6.17)

$$-4\partial_z^2\psi = t\psi$$

one can say that t should be 0.

$$\sqrt{\hat{g}}\hat{R} = 8\pi\eta\delta(x - x_0)$$

again to remove singularity, see Fig. 1.

$$ds^2 = \frac{dzd\bar{z}}{(z\bar{z})^{-2\eta}} \qquad |z| > \epsilon$$
$$ds^2 = \frac{dzd\bar{z}}{(\epsilon^2)^{-2\eta}} \qquad |z| < \epsilon$$

So partition function

$$\mathcal{Z} \sim \exp\left[-\frac{1}{\hbar}(-4\eta^2 \log \epsilon^2)\right]$$

Here ϵ is very small.

6.2 Local classical solutions

Scribe : `Ritankar`

Reference :

Zamolodchikov brothers, pages 100-104.

We shall now focus on local solutions that come about from singularities in the classical stress tensor of the Liouville theory.

- Specifically we study the role of isolated singularities of the function $t(z)$ in the equation
$$-4\partial_z^2\psi(z) = t(z)\psi(z) \tag{6.26}$$
.
- Here we focus on the regular singularities. We do not have to bother about first order poles. For first order poles, the solutions $\psi(z)$ are regular at the first order poles. The second order poles of $t(z)$ significant role.
- Let $t(z)$ behaves as $t(z) \to \frac{r}{z^2}$ as $z \to 0$ (r is real). Using Frobenius method we find that there are two solutions of equation (6.26) in this case, which behave like $\psi_1(z) \to z^\eta$ and $\psi_2(z) \to z^{1-\eta}$ as z approaches 0 where η staisfies the indicial equation
$$r = 4\eta(1 - \eta) \tag{6.27}$$
- Apart from η we also occasionally use another parameter λ to parametrize r. λ is given by
$$\eta = \frac{1}{2} - \frac{\lambda}{2}, \qquad 1 - \eta = \frac{1}{2} + \frac{\lambda}{2} \tag{6.28}$$
which implies $r = 1 - \lambda^2$.

– Depending on value of r, we can consider three different types of monodromy.

– For $r < 1$ the monodromy is called Elliptic monodromy. In this case λ (and hence, η) is real. Here we assume that $\lambda > 0$. We also choose the basis (ψ_1, ψ_2) in such a way that at $z \to 0$ we can have

$$\psi(z) = \begin{bmatrix} \psi_1(z) \\ \psi_2(z) \end{bmatrix} \to \frac{1}{\sqrt{\lambda}} \begin{bmatrix} z^{\frac{1}{2}-\frac{\lambda}{2}} \\ z^{\frac{1}{2}+\frac{\lambda}{2}} \end{bmatrix} \tag{6.29}$$

– If we take the curve C_0 to be going around the point $z = 0$ and sufficiently close to 0, then the monodromy matrix associated with it becomes

$$M(C_0) = \begin{bmatrix} -e^{-i\pi\lambda} & 0 \\ 0 & e^{i\pi\lambda} \end{bmatrix}, \qquad trM(C_0) = -2cos(\pi\lambda) \in [-2:2] \tag{6.30}$$

– Note that this matrix belongs to both $SU(2)$ and $SU(1,1)$. That means both positive and negative curvature are consistent with this kind of singularity. Therefore the expression of $\sigma(z,\bar{z})$, in terms of $\psi(z)$ and $\bar{\psi}(\bar{z})$ takes the following form

$$\sigma(z,\bar{z}) = -2\log(\psi_1(z)\bar{\psi}_1(\bar{z}) \pm \psi_2(z)\bar{\psi}_2(\bar{z})) + \log(\pm(-8\lambda^2/\Lambda)) \tag{6.31}$$

Here the plus and minus sign corresponds to the cases of positive and negative curvatures respectively.

– Since the second term in the argument of the logarithm is z^λ times the first term with $\lambda > 0$, at $z \to 0$, the first term necessarily dominates over the second term. Therefore at $z \to 0$

$$\sigma(z,\bar{z}) \to -2\eta\log(z\bar{z}) \tag{6.32}$$

– The singularity of above kind y corresponds to conical point at $z = 0$

$$\sqrt{g}R(z,\bar{z}) = -4\partial_z\partial_{\bar{z}}\sigma(z,\bar{z}) = 8\pi\delta^{(2)}(z) + reg \tag{6.33}$$

Equation (6.31) global solution of the constant curvature problem, whose global character depends on the sign of the curvature.

– Now we shall discuss the case of positive curvature. In this case the metric corresponding to equation (6.31) is

$$ds^2 = e^\sigma dzd\bar{z} = -\Big(\frac{8\lambda^2}{\Lambda}\Big)\frac{dzd\bar{z}}{(z\bar{z})^{1-\lambda}(1+(z\bar{z})^\lambda)^2} \tag{6.34}$$

– While at all finite nonzero z this metric is smooth, it is invariant under the transformation $z \to w(z) = \frac{1}{z}$ and hence, has another conical singularity at $z = \infty$

$$\sqrt{g}R(w,\bar{w}) = -4\partial_w\partial_{\bar{w}}\sigma(w,\bar{w}) = 8\pi\delta^{(2)}(w) + reg \tag{6.35}$$

Therefore the Liouville action diverges at $|z| \to 0$ and $|z| \to \infty$.

– Now we discuss the negative curvature for which $\sigma(z,\bar{z})$ is equation (**??**) with -ve sign. Therefore the metric in this case, is

$$ds^2 = -\Big(\frac{8\lambda^2}{\Lambda}\Big)\frac{dzd\bar{z}}{(z\bar{z})^{1-\lambda}(1-(z\bar{z})^\lambda)^2} \tag{6.36}$$

If we take the domain $|z| < 1$ then we find the geometry of the Poicare e disk with the conical point somewhere inside.

– Hyperbolic monodormy occurs when $r > 1$. In this case $\lambda = \sqrt{1-r}$ is purely imaginary

$$\lambda = ip \tag{6.37}$$

Taking $p > 0$ and ensuring that the Wronskian is unity we introduce the following basis

$$\psi(z) = \begin{bmatrix} \psi_1(z) \\ \psi_2(z) \end{bmatrix} \to \frac{1}{\sqrt{p}} \begin{bmatrix} z^{\frac{1}{2}-\frac{ip}{2}} \\ -iz^{\frac{1}{2}+\frac{ip}{2}} \end{bmatrix} \tag{6.38}$$

where $z \to 0$.

– The monodromy matrix has the following form

$$M(C_0) = \begin{bmatrix} -e^{\pi p} & 0 \\ 0 & -e^{-\pi p} \end{bmatrix} \in SL(2,R), \qquad trM(C_0) = -2\cosh(\pi p) < 2 \tag{6.39}$$

– We see that the matrix belongs to $SL(2,R)$. That means only -ve curvature is consistent with this type of singularity. Therefore, the matrix Λ takes the following form in this basis

$$\mathbf{\Lambda} = i\sqrt{\Lambda} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{6.40}$$

– Therefore $\sigma(z,\bar{z})$ becomes

$$\sigma(z,\bar{z}) = -2\log[(z\bar{z})^{\frac{1}{2}+i\frac{p}{2}} + (z\bar{z})^{\frac{1}{2}-i\frac{p}{2}}] + \log\left(\frac{8p^2}{\Lambda}\right) \tag{6.41}$$

$$= -2\log[|z|\cos(p\log(z))] + +\log\left(\frac{2p^2}{\Lambda}\right) \tag{6.42}$$

In polar coordinates τ and θ, ($z = e^{\tau+i\theta}$, and $\bar{z} = e^{\tau+i\theta}$), the metric becomes

$$ds^2 = \frac{2p^2}{\Lambda}\frac{d\tau^2 + d\theta^2}{\cos^2(p\tau)} \tag{6.43}$$

– This metric tensor has singularities at $\tau_n = p^{-1}(\frac{\pi}{2}+n\pi)$. They are at the concentric circles in z-plane, given by $|z| = e^{\tau_n}$. Since the singularities are like this and no singularity at $z = 0$, we cannot expect it to be a $z \to 0$ asymptotic of something else. Yet this metric has some interesting features which we are going to discuss below.

– We would focus on one of the regular regions

$$-\frac{\pi}{2} < p\tau < \frac{\pi}{2} \tag{6.44}$$

In z-plane this region is annulus between two circles given by $|z| = e^{-\frac{\pi}{2}p}$ and $|z| = e^{\frac{\pi}{2}p}$. The structure of the circles are similar to the absolute of Poincare disk. This is geometry of hyperboloid on one sheet and occasionally referred to as ADS_2.

– Taking $\tau = it$ in the metric given in equation (6.43) we find the Minkowski space-time pseudo-metric.

$$ds^2 = \frac{2p^2}{\Lambda}\frac{-dt^2 + d\theta^2}{\cosh^2(pt)} \tag{6.45}$$

This can be interpreted as a 2d universe evolving from small circle of zero size at $t \to -\infty$, expands to a finite size of the order $\sqrt{p/\Lambda}$ and then at $t \to \infty$, again shrinks to zero size. See Fig. 2.

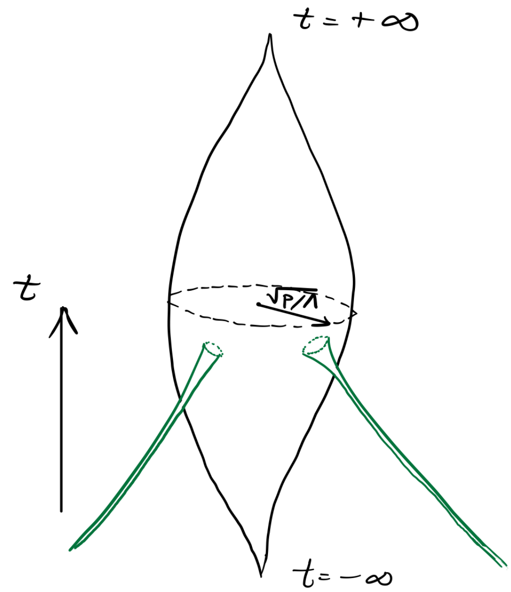

Figure 2: A visualization of metric (6.45). The green long *legs* represent the local parabolic solutions.

- Let us recall that the expression of $t(z) = \frac{r}{z^2} = \frac{1+p^2}{z^2}$ was for $z \to 0$, i.e. it is just the asymptotic form and it may have some less singular correction term. Therefore, it is perfectly possible that the resulting geometry outside of some circle becomes smooth. In that case this hyperbolic monodromy describes an opening to a smooth geometry. The boundary circle outside which the resulting geometry is smooth has a structure of the absolute.
- The parabolic monodromy corresponds to $r = 1$ and in that case the differential equation becomes

$$4\partial_z^2\psi(z) + \frac{1}{z^2}\psi(z) = 0 \tag{6.46}$$

- It has two linearly independent solutions $\sqrt{z}$ and $\sqrt{z}\log z$. We take the following basis for $z \to 0$

$$\psi(z) = \begin{bmatrix}\psi_1(z)\\ \psi_2(z)\end{bmatrix} \to \begin{bmatrix}\sqrt{z}\log z\\ i\sqrt{z}\end{bmatrix} \tag{6.47}$$

- The monodromy matrix $M(C_0)$ the takes the following form

$$M(C_0) = \begin{bmatrix}-1 & -2\pi\\ 0 & -1\end{bmatrix} \in SL(2,R) \tag{6.48}$$

- Since the monodromy matrix belongs to $SL(2,R)$, the matrix $\mathbf{\Lambda}$ again takes the form of equation (6.40). Then again using the expression of $\sigma(z,\bar{z})$ in terms of ψ and $\bar{\psi}$, we arrive at

$$ds^2 = \frac{8}{\lambda}\frac{dzd\bar{z}}{(z\bar{z})\log^2(z\bar{z})} \tag{6.49}$$

- The curvature in this case becomes

$$\sqrt{g}R(z,\bar{z}) = 4\pi\delta^2(z) + reg \tag{6.50}$$

An interesting feature of the geometry is this: geodesic distance from $z = 0$ to any other point is infinite. This can be visualised as an infinitely long leg growing from the surface.

A Appendix : General questions

- Can the procedure of defining path integral measure for the metric in Lec-4 be generalized to higher dimensions, generically? Or the generalization in higher dimensions is admissible only for "conformal gravity"?