Functional Integral Representation for the S-matrix in Integrable 2d Conformal Field Theories

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Abstract

We study semiclassical scattering in Liouville theory, in SL(2,R)/U(1) WZW theory and in SL(3,R) Toda theory, using the relations between the in and out fields defined by the chiral currents of these theories. By these relations we investigate the generating functional F that corresponds to the semiclassical S-matrix. In particular, we construct the Legendre transform of F in a closed form. The construction scheme is similar for these three theories and the obtained functionals have a compact form. This provides a basis for the functional integral representation of the S-matrix in these theories, which earlier was known only for Liouville theory.

Keywords: S-matrix, Integrability, Conformal symmetry, Liouville theory, Coset WZW theory.

1 Introduction

In this paper we investigate the S-matrix of integrable 2d CFTs semiclassically. In particular, we generalize the semiclassical results, obtained in [1, 2] for the S-matrix of Liouville theory [3–5], to SL(2,ℝ)/U(1) WZW theory [6, 7] and to SL(3,ℝ) Toda theory [8–10]. These theories are completely integrable classically in the sense that the general solution of the equations of motion is represented as a function of the chiral and the anti-chiral fields [8, 11, 12]. By the form of the general solution one can analyze the behavior of the interacting fields at the time asymptotics, which shows that the interacting fields of these theories asymptotically become free massless fields. These free fields itself have the chiral structure, which allows to express the parameterizing chiral fields in terms of the asymptotic (*in* or *out*) fields in a compact form. As a result one obtains the interacting fields in terms of the asymptotic fields and one also finds explicit relations between the *in* and *out* fields. These relations define a non-trivial canonical transformation and the corresponding generating functional F provides the semiclassical S-matrix.

Integrability of Liouville theory is based on the 2d conformal symmetry generated by the stress tensor components (T and $\bar{T}$). Two other theories have additional chiral and anti-chiral currents (W and $\bar{W}$), which define higher symmetries. The chiral currents have a free-field form in terms of the asymptotic fields and due to the chiral structure their form in terms of the *in* and *out* fields is the same. This leads to an infinite set of equations for the generating functional F, which describe the symmetries of the theories.

In quantum theory these equations are generalized for the S-matrix, using the free-field representation of the quantum currents, which are known in a closed form. The subject of S-matrix of scattering states in Liouville was initiated in [13]. In this approach one easily finds low level transition amplitudes, which appear rational functions of $\hbar$ and the momentum zero modes. Due to a rather simple form of these amplitudes one can try to generalize the answer by recursive relations [1]. However, these relations even at the classical level are non-trivial and it is hard to get the answer for the general transitions in a compact form.

An alternative scheme based on a new type functional integral representation of the S-matrix was proposed in [1] and [2] for Liouville theory. In special cases this functional integral is exactly calculable and reproduces the low level amplitudes precisely. The main goal of the present paper is a generalization of the scheme for other integrable theories.

2 Results and Discussion

2.1 Generating function in Liouville theory

In this section we first briefly summarize some results of [1, 2] which are relevant for the semiclassical analysis of the S-matrix in Liouville theory. Then we propose a new scheme for the construction of the generating functional of the canonical transformation between the *in* and *out* fields.

We study Liouville theory on a cylindrical spacetime with spatial coordinate σ given on the unit circle. The Liouville field Φ satisfies the equation of motion

$$\partial_{\bar{x}}\partial_x\Phi + \mu^2\, e^{2\Phi} = 0\,, \tag{2.1}$$

where $x = \tau + \sigma$ and $\bar{x} = \tau - \sigma$ are the chiral and the anti-chiral coordinates, respectively. At the time asymptotics, $\tau \to \mp\infty$, the Liouville field turns into the massless free fields which split into the sums of the chiral and anti-chiral parts

$$\Phi_{\rm in} = \phi(x) + \bar{\phi}(\bar{x})\,, \qquad \Phi_{\rm out} = \tilde{\phi}(x) + \tilde{\bar{\phi}}(\bar{x})\,. \tag{2.2}$$

The chiral fields obey the mode expansion

$$\phi(x) = q + \tfrac{1}{2}px + i\sum_{n\neq 0}\frac{a_n}{n}\, e^{-inx}\,, \qquad \tilde{\phi}(x) = \tilde{q} + \tfrac{1}{2}\tilde{p}x + i\sum_{n\neq 0}\frac{b_n}{n}\, e^{-inx}\,, \tag{2.3}$$

with $a_{-n} = a_n^*$, $b_{-n} = b_n^*$, and a similar structure has the anti-chiral part.

The scattering process does not mix the chiralities. This means that the chiral *out*-field $\tilde{\phi}(x)$ depends only on the chiral *in*-field $\phi(x)$. Therefore, it is sufficient to investigate this dependance, since the anti-chiral part is similar.

To analyze the relation between ϕ and $\tilde{\phi}$, we use the chiral stress tensor of Liouville theory, $T = (\partial_x\Phi)^2 - \partial_x^2\Phi$, which in terms of the asymptotic fields becomes

$$T(x) = \phi'^{\,2}(x) - \phi''(x) = \tilde{\phi}'^{\,2}(x) - \tilde{\phi}''(x)\,. \tag{2.4}$$

The Fourier modes in (2.3) satisfy the canonical Poisson brackets relations

$$\{p, q\} = \{\tilde{p}, \tilde{q}\} = 1\,, \qquad \{a_m, a_n\} = \{b_m, b_n\} = \frac{im}{2}\,\delta_{m+n,0}\,, \tag{2.5}$$

defined by the symplectic form

$$\omega = \delta p\wedge\delta q + 2i\sum_{m>0}\frac{\delta a_m^*\wedge\delta a_m}{m} = \delta\tilde{p}\wedge\delta\tilde{q} + 2i\sum_{m>0}\frac{\delta b_m^*\wedge\delta b_m}{m}\;. \tag{2.6}$$

It is important to note that the zero modes of the *in*-field are given on the half-plane $p > 0$ and one has $\tilde{p} = -p$, similarly to the scattering of a particle in the exponential potential. The reflection of the momentum zero mode, together with $T_{\rm in}(x) = T_{\rm out}(x)$ given by (2.4), provides rather complicated relations between the other modes of the *in* and *out* chiral fields, which lead to a non-trivial S-matrix.

Since the asymptotic fields are related to each other by a canonical transformation, the difference between the pre-symplectic forms of the *out* and *in* fields is an exact 1-form. Using also the relation $\tilde{p} = -p$, from (2.6) we obtain

$$(\tilde{q} + q)\delta p - 2i\sum_{m>0}\frac{b_m}{m}\,\delta b_m^* - 2i\sum_{m>0}\frac{a_m^*}{m}\,\delta a_m = \delta F[p, b^*, a]\,, \tag{2.7}$$

where $F[p, b^*, a]$ is the generating functional of the canonical transformation and we have

$$\frac{\partial F}{\partial p} = \tilde{q} + q\,, \qquad \frac{\partial F}{\partial a_m} = -\frac{2i}{m}\,a_m^*\,, \qquad \frac{\partial F}{\partial b_m^*} = -\frac{2i}{m}\,b_m\,. \tag{2.8}$$

According to the Dirac's correspondence between the generating function of a canonical transformation and the related quantum unitary map, the exponent $e^{\frac{i}{\hbar}F}$ is associated with the semiclassical S-matrix, and our aim is to investigate the functional F.

To find F from the equations (2.8), first we have to express the variables $(\tilde{q}+q, b, a^*)$ in terms of (p, b^*, a) from the relations between the *in* and *out* fields and then integrate (2.8). However, even the first stage of this scheme appears to be a nontrivial task.

Possible solutions of this problem were described in [1,2]. Below we present a modified scheme which we later generalize in the next two sections for two other integrable theories.

To start, we introduce the positive and negative frequency parts of the chiral fields $\phi(x)$ and $\tilde{\phi}(x)$

$$\begin{aligned} a(x) &= i\sum_{m>0}\frac{a_m}{m}\,e^{-imx}\,, \qquad & a^*(x) &= -i\sum_{m>0}\frac{a_m^*}{m}\,e^{imx}\,, \\ b(x) &= i\sum_{m>0}\frac{b_m}{m}\,e^{-imx}\,, \qquad & b^*(x) &= -i\sum_{m>0}\frac{b_m^*}{m}\,e^{imx}\,, \end{aligned} \tag{2.9}$$

and rewrite the relation (2.4) between the asymptotic fields,

$$\phi'' - \tilde{\phi}'' = \left(\phi' + \tilde{\phi}'\right)\left(\phi' - \tilde{\phi}'\right)\,, \tag{2.10}$$

in terms of two other chiral fields $\varphi(x)$ and $j(x)$ given by

$$\varphi(x) = \phi(x) + \tilde{\phi}(x) = \tilde{q} + q + a(x) + b(x) + a^*(x) + b^*(x) = \sum_{n\in\mathbb{Z}}\varphi_n\,e^{-inx}\,, \tag{2.11}$$

$$j(x) = p + 2a'(x) - 2b^{*\,\prime}(x) = \sum_{n\in\mathbb{Z}} j_n\,e^{-inx}\,. \tag{2.12}$$

From (2.3) and (2.9) follows

$$\phi'(x) - \tilde{\phi}'(x) = j(x) - \varphi_+'(x) + \varphi_-'(x)\,, \tag{2.13}$$

and (2.10) can be written as

$$\big(j(x) - \varphi_+'(x) + \varphi_-'(x)\big)' = \varphi'(x)\,\big(j(x) - \varphi_+'(x) + \varphi_-'(x)\big)\,, \tag{2.14}$$

where $\varphi_+(x)$ and $\varphi_-(x)$ are the positive and the negative frequency parts of $\varphi(x)$, respectively,

$$\varphi_+(x) = a(x) + b(x) = \sum_{m>0}\varphi_m\,e^{-imx}\,, \qquad \varphi_-(x) = a^*(x) + b^*(x) = \sum_{m>0}\varphi_{-m}\,e^{imx}\,. \tag{2.15}$$

Integration of (2.14) yields

$$j(x) = \varphi_+'(x) - \varphi_-'(x) + \mu_0\,e^{\varphi(x)}\,, \tag{2.16}$$

where μ_0 is an integration constant.

A more detailed analysis of the relation between the chiral asymptotic fields shows (see eq. (4.18) in [1]) that (2.16) holds for $\mu_0 = \mu$, where μ is the 'cosmological' constant in the Liouville equation (2.1). In this case (2.16) can be written as

$$j(x) = 2\pi\,\frac{\delta A[\varphi]}{\delta\varphi(x)}\,, \tag{2.17}$$

where $A[\varphi]$ is the functional

$$A[\varphi] = -\frac{i}{2} \sum_{n \in \mathbb{Z}} |n| \varphi_{-n} \varphi_n + \mu \int_0^{2\pi} \frac{dx}{2\pi}\, e^{\varphi(x)} \,. \tag{2.18}$$

Based on (2.17), we introduce the Legendre transformation $A[\varphi] \mapsto \tilde{F}[j]$:

$$\tilde{F}[j] = \sum_{n \in \mathbb{Z}} j_{-n} \varphi_n \; - A[\varphi] \,. \tag{2.19}$$

From this equation follows

$$\frac{\partial \tilde{F}}{\partial j_n} = \varphi_{-n} \,, \tag{2.20}$$

and comparing it to (2.11)-(2.12), we find that the functional F defined by

$$F = \tilde{F} + 2i \sum_{m>0} \frac{b^*_m a_m}{m} \,, \tag{2.21}$$

is a solution of (2.8). Thus, the generating functional F is defined by $\tilde{F}[j]$, which is the Legendre transform of the functional (2.18).

As our main interest lies in the quantum S-matrix, for which (2.10) along with its quantum deformations provide sufficient structure, we can formulate the problem without any reference to the integration constant μ_0 that appears in (2.16). Indeed, since (2.10) does not involve zero modes, we can work with an action that avoids introducing μ_0 altogether. Therefore, from (2.16) we solve for μ_0:

$$p = \mu_0 e^{\tilde{q}+q} \int_0^{2\pi} \frac{dy}{2\pi}\, e^{\check{\varphi}(y)} \,, \tag{2.22}$$

with $\check{\varphi} = \varphi_+ + \varphi_-$.

Then, by (2.16) and (2.22), the non-zero modes j_n are represented as

$$j_n = -i|n|\varphi_n + \frac{p}{\int_0^{2\pi} \frac{dy}{2\pi}\, e^{\check{\varphi}(y)}} \int_0^{2\pi} \frac{dx}{2\pi}\, e^{\check{\varphi}(x)}\, e^{inx} = \frac{\partial \check{A}[\check{\varphi}]}{\partial \varphi_{-n}} \,, \tag{2.23}$$

where the new functional is

$$\check{A}[\check{\varphi}] = -\frac{i}{2} \sum_{n \neq 0} |n| \varphi_{-n} \varphi_n + p \log\left(\int_0^{2\pi} \frac{dx}{2\pi}\, e^{\check{\varphi}(x)} \right) \,. \tag{2.24}$$

As above, we introduce the Legendre transformation $\check{A}[\check{\varphi}] \mapsto \check{F}[\check{j}]$, where now the zero modes are not involved

$$\check{F}[\check{j}] = \langle \check{j} | \check{\varphi} \rangle - \check{A}[\check{\varphi}] \,, \quad \text{where} \quad \langle \check{j} | \check{\varphi} \rangle = \sum_{n \neq 0} j_n \varphi_{-n} \,, \tag{2.25}$$

i.e. p here has no dual variable and in (2.24) it plays the role of a parameter.

By the same arguments, leading to (2.21), we now find that for $n \neq 0$

$$\frac{\partial F}{\partial j_n} = \frac{\partial}{\partial \check{j}_n}\left(\check{F} + 2i\sum_{m>0}\frac{b_m^* a_m}{m}\right). \tag{2.26}$$

In quantum theory, the condition $T_{\rm in} = T_{\rm out}$ provides a system of linear differential equations for the functional $\tilde{\mathcal{S}}[\check{j}]$, which is a quantum counterpart of the functional $e^{\frac{i}{\hbar}\check{F}[\check{j}]}$, and it describes the S-matrix up to a p-dependent factor $R(p)$, which is the reflection amplitude given by the 2-point function of [13, 14]. Therefore, one can normalize this functional by the condition $\tilde{\mathcal{S}}[0] = 1$ and represent it as $\tilde{\mathcal{S}}[\check{j}] = Z[\check{j}]/Z[0]$.

A possible candidate for $Z[\check{j}]$ is the functional integral

$$Z[\check{j}] = \int \prod_{n\neq 0} d\varphi_n \, e^{\frac{i}{\hbar}\left(\langle \check{j}|\check{\varphi}\rangle - \check{A}[\check{\varphi}]\right)}, \tag{2.27}$$

where $\check{A}[\check{\varphi}]$ is given by (2.24). The validity of this representation is based on the fact that the saddle point calculation of this integral gives the correct semiclassical result.

Note that the functional $\check{A}[\check{\varphi}]$ contains the zero mode p as a parameter, which plays the role of a coupling constant. It was shown in [2] that at $p = i\hbar$ and $p = 2i\hbar$ the functional integral (2.27) can be calculated in a closed form. The obtained answer coincides with low level amplitudes (at $p = i\hbar$ and $p = 2i\hbar$), which were calculated in [1] exactly in the operator formalism. This gives strong argument in favor of (2.27).

By the same arguments as in (2.27) one can also consider the functional integral

$$\tilde{Z}[\check{j}] = \int d\varphi_0 \prod_{n\neq 0} d\varphi_n \, e^{\frac{i}{\hbar}\left(p\varphi_0 + \langle \check{j}|\check{\varphi}\rangle - A[\varphi]\right)}, \tag{2.28}$$

where $A[\varphi]$ is given by (2.18). Integration of φ_0 here, relates the integral (2.28) to (2.27) by $\tilde{Z}[\check{j}] = \Gamma\left(\frac{ip}{\hbar}\right)\left(\frac{i\mu}{\hbar}\right)^{-\frac{ip}{\hbar}} Z[\check{j}]$, which means that these functional integral representations are equivalent.

Note that a shift of the zero mode of φ by $\log(p/\mu)$ has the effect of replacing the prefactor in the potential term of A (cf. (2.18)) by p and to cancel the first term in the exponent of (2.28). This should not affect the complete amplitudes but is advantageous in the diagrammatic evaluation of $Z[\check{j}]$, cf. ref. [1].

2.2 Generating function in SL(2,ℝ)/U(1) WZW theory

In this section we consider the scattering problem for SL(2,ℝ)/U(1) WZW theory using the scheme described in the previous section for Liouville theory leading to the functional integral representation of the S-matrix. In particular, we will construct the functional A, which is related to the generating functional of the canonical transformation between the asymptotic fields by a Legandre transformation.

We describe the scattering problem in $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ WZW theory by the action

$$S = \frac{1}{2}\int d\tau \int_0^{2\pi}\frac{d\sigma}{2\pi}\,\frac{(\partial_\tau\Phi_1)^2 + (\partial_\tau\Phi_2)^2 - (\partial_\sigma\Phi_1)^2 - (\partial_\sigma\Phi_2)^2}{1 + e^{2\Phi_1}}, \tag{2.29}$$

where (σ, τ) are the standard spacetime coordinates on the cylinder as in Liouville theory and (Φ_1, Φ_2) are scalar fields. The Lagrangian in (2.29) corresponds to a sigma-model and it is again convenient to use the chiral coordinates $(x = \tau + \sigma, \bar{x} = \tau - \sigma)$. The equation of motion for the complex field $\Phi := \Phi_1 + i\Phi_2$ then reads

$$\partial_{\bar{x}}\partial_x\Phi = \frac{e^{\Phi^*+\Phi}}{1+e^{\Phi^*+\Phi}}\,\partial_{\bar{x}}\Phi\,\partial_x\Phi\,. \tag{2.30}$$

This equation provides the chirality conditions

$$\partial_{\bar{x}}T = 0\,, \qquad \partial_x\bar{T} = 0\,, \qquad \partial_{\bar{x}}U = 0\,, \qquad \partial_x\bar{U} = 0\,, \tag{2.31}$$

where $(T, \bar{T})$ are the standard components of the stress tensor in this sigma-model

$$T = \frac{\partial_x\Phi^*\,\partial_x\Phi}{1+e^{\Phi^*+\Phi}}\;, \qquad \bar{T} = \frac{\partial_{\bar{x}}\Phi^*\,\partial_{\bar{x}}\Phi}{1+e^{\Phi^*+\Phi}}\;, \tag{2.32}$$

and $(U, \bar{U})$ have the following form

$$U = \frac{\partial_x^2\Phi}{\partial_x\Phi} - \partial_x\Phi + \frac{\partial_x\Phi^*}{1+e^{\Phi^*+\Phi}}\;, \qquad \bar{U} = \frac{\partial_{\bar{x}}^2\Phi}{\partial_{\bar{x}}\Phi} - \partial_{\bar{x}}\Phi + \frac{\partial_{\bar{x}}\Phi^*}{1+e^{\Phi^*+\Phi}}\,. \tag{2.33}$$

From (2.31)-(2.33) follows that the field $V = e^{-\Phi}$ satisfies the linear equations [12]

$$\partial_x^2V = U(x)\partial_xV + T(x)\,V\,, \qquad \partial_{\bar{x}}^2V = \bar{U}(\bar{x})\partial_{\bar{x}}V + \bar{T}(\bar{x})\,V\,. \tag{2.34}$$

Note that the Lagrangian in (2.29) takes a more familiar form [6] in terms of the field V

$$\mathcal{L} = \frac{1}{4\pi}\,\frac{\partial_{\bar{x}}V^*\partial_xV + \partial_xV^*\partial_{\bar{x}}V}{1+V^*V}\,. \tag{2.35}$$

Due to (2.34), the field $V(x, \bar{x})$ can be written as

$$V(x,\bar{x}) = \psi(x)\bar{\psi}(\bar{x}) + \tilde{\psi}(x)\bar{\tilde{\psi}}(\bar{x})\,, \tag{2.36}$$

where $\big(\psi(x), \tilde{\psi}(x)\big)$ and $\big(\bar{\psi}(\bar{x}), \bar{\tilde{\psi}}(\bar{x})\big)$ are linearly independent solutions of the first and the second equation in (2.34), respectively, i.e.

$$\Psi''(x) = U(x)\,\Psi'(x) + T(x)\,\Psi(x)\,, \qquad \tilde{\Psi}''(x) = U(x)\,\tilde{\Psi}'(x) + T(x)\,\tilde{\Psi}(x)\,. \tag{2.37}$$

As in Liouville theory, (2.36) provides the general solution of the equations of motion of the model [12], which splits in different monodromy classes. Scatterings are described by the hyperbolic monodromy, where the parameterizing chiral fields are the free-field exponentials $\psi(x) = e^{-\phi_1(x)-i\phi_2(x)}$, $\tilde{\psi}(x) = e^{-\tilde{\phi}_1(x)-i\tilde{\phi}_2(x)}$, and one can show that they are related by

$$\psi^*(x)\,\tilde{\psi}(x) - \tilde{\psi}^*(x)\psi(x) = i\,. \tag{2.38}$$

The free fields obey the standard mode expansions

$$\phi_1(x) = q_1 + \tfrac{1}{2}p_1x + i\sum_{n\neq0}\frac{a_n}{n}\,e^{-inx}\,, \qquad \tilde{\phi}_1(x) = \tilde{q}_1 + \tfrac{1}{2}\tilde{p}_1x + i\sum_{n\neq0}\frac{b_n}{n}\,e^{-inx}\,, \tag{2.39}$$

$$\phi_2(x) = q_2 + \tfrac{1}{2}p_2x + i\sum_{n\neq0}\frac{c_n}{n}\,e^{-inx}\,, \qquad \tilde{\phi}_2(x) = \tilde{q}_2 + \tfrac{1}{2}\tilde{p}_2x + i\sum_{n\neq0}\frac{d_n}{n}\,e^{-inx}\,, \tag{2.40}$$

where

$$\tilde{p}_1 = -p_1 < 0\,, \qquad \tilde{p}_2 = p_2 + 2N\,, \tag{2.41}$$

with integer N, which describes a winding number. We consider the case $N = 0$.

In this case, similar to Liouville theory, the field $e^{2\Phi_1}$ vanishes at the time asymptotics. Therefore, at $\tau \to \mp\infty$, the interacting fields Φ_1 and Φ_2 become free massless fields and the form of the chiral fields $T(x)$ and $U(x)$ is the same in terms of the *in* and the *out* fields. To simplify the analysis of the chiral fields in terms of the asymptotic fields it is convenient to introduce the W-current

$$W(x) = \frac{i}{4}\,\left(U(x) - U^*(x)\right) T(x)\,. \tag{2.42}$$

Then, the conditions $T_{\rm in}(x) = T_{\rm out}(x)$ and $W_{\rm in}(x) = W_{\rm out}(x)$ provide the following relations between the free chiral asymptotic fields

$$\phi_1'^2 + \phi_2'^2 = \tilde{\phi}_1'^2 + \tilde{\phi}_2'^2\,, \tag{2.43}$$

$$\phi_1'^2\phi_2' + \phi_2'^3 + \tfrac{1}{2}\phi_1''\phi_2' - \tfrac{1}{2}\phi_1'\phi_2'' = \tilde{\phi}_1'^2\tilde{\phi}_2' + \tilde{\phi}_2'^3 + \tfrac{1}{2}\tilde{\phi}_1''\tilde{\phi}_2' - \tfrac{1}{2}\tilde{\phi}_1'\tilde{\phi}_2''\,. \tag{2.44}$$

Below we will follow the steps described in the previous section for Liouville theory.

The Poisson brackets of the Fourier modes in (2.39) are defined by the symplectic form of the chiral fields, which are similar for all four fields and, for example, for ϕ_1 it reads

$$\omega = \delta p_1 \wedge \delta q_1 + 2i \sum_{m>0} \frac{\delta a_m^* \wedge \delta a_m}{m}\,. \tag{2.45}$$

Since the asymptotic fields are related to each other by a canonical transformation, we can introduce the generating functional of this transformation defined by

$$\delta F = (\tilde{q}_1 + q_1)\delta p_1 - (\tilde{q}_2 - q_2)\delta p_2 - 2i \sum_{m>0} \frac{1}{m}\left(b_m\,\delta b_m^* + d_m\,\delta d_m^* + a_m^*\,\delta a_m + c_m^*\,\delta c_m\right), \tag{2.46}$$

where the 1-form on the right hand side is the difference between the presymplectic forms of the *out* and the *in* fields and we have used the relations (2.41), for $N = 0$.

Thus, F is treated as a function of the modes $(p_1, p_2, b_m^*, d_m^*, a_m, c_m)$, with $m > 0$, and its partial derivatives satisfy the relations

$$\begin{aligned}
&\frac{\partial F}{\partial p_1} = \tilde{q}_1 + q_1\,, \qquad \frac{\partial F}{\partial p_2} = q_2 - \tilde{a}_2\,,\\
&\frac{\partial F}{\partial b_m^*} = -\frac{2i}{m}\,b_m, \qquad \frac{\partial F}{\partial d_m^*} = -\frac{2i}{m}\,d_m\,, \qquad \frac{\partial F}{\partial a_m} = -\frac{2i}{m}\,a_m^*, \qquad \frac{\partial F}{\partial c_m} = -\frac{2i}{m}\,c_m^*\,.
\end{aligned} \tag{2.47}$$

Our goal again is to describe the functional F and, like in Liouville theory, we introduce the new chiral fields

$$\varphi(x) = \phi_1(x) + \tilde{\phi}_1(x) = \sum_{n\in\mathbb{Z}} \varphi_n\, e^{-inx}, \quad \chi(x) = \phi_2(x) + \tilde{\phi}_2(x) = p_2 x + \sum_{n\in\mathbb{Z}} \chi_n\, e^{-inx}, \tag{2.48}$$

$$j(x) = p_1 + 2a'(x) - 2b^{*\,\prime}(x) = \sum_{n\in\mathbb{Z}} j_n\, e^{-inx}, \quad k(x) = 2c'(x) - 2d^{*\,\prime}(x) = \sum_{n\neq 0} k_n\, e^{-inx}. \tag{2.49}$$

Here $a(x)$ is the positive frequency part of $\phi_1(x)$, $b^*(x)$ is the negative frequency part of $\tilde{\phi}_1(x)$, *etc.*, as in (2.9). Note that $\chi(x)$ is not periodic and $k(x)$ has no zero mode.

Now we use the equations (2.43)-(2.44) to relate the fields in (2.48) and (2.49). From (2.43) follows

$$\frac{\phi_1' - \tilde{\phi}_1'}{\phi_2' + \tilde{\phi}_2'} = \frac{\tilde{\phi}_2' - \phi_2'}{\phi_1' + \tilde{\phi}_1'}\,. \tag{2.50}$$

We denote the ratio here by $R(x)$ and by (2.48) obtain

$$\phi_1' = \frac{1}{2}(\varphi' + R\chi')\,, \quad \tilde{\phi}_1' = \frac{1}{2}(\varphi' - R\chi')\,, \quad \phi_2' = \frac{1}{2}(\chi' - R\varphi')\,, \quad \tilde{\phi}_2' = \frac{1}{2}(\chi' + R\varphi')\,. \tag{2.51}$$

Inserting this expressions in (2.44), we find the equation for $R(x)$

$$R' - (R + R^3)\varphi' = 0\,. \tag{2.52}$$

Its integration yields

$$R(x) = \frac{c\, e^{\varphi(x)}}{\sqrt{1 - c^2 e^{2\varphi(x)}}}\,, \tag{2.53}$$

where c is an integration constant.

Similarly to (2.13), from (2.48)-(2.49) one has

$$\phi_1' - \tilde{\phi}_1' = j - \varphi_+' + \varphi_-'\,, \qquad \phi_2' - \tilde{\phi}_2' = k - \chi_+' + \chi_-'\,, \tag{2.54}$$

where $\varphi_\pm$ and $\chi_\pm$ are defined as in (2.15). Then, by (2.51) and (2.53) we find

$$j = \varphi_+' - \varphi_-' + \frac{c\, e^{\varphi(x)}\, \chi'}{\sqrt{1 - c^2 e^{2\varphi(x)}}}\,, \qquad k = \chi_+' - \chi_-' - \frac{c\, e^{\varphi(x)}\, \varphi'}{\sqrt{1 - c^2 e^{2\varphi(x)}}}\,. \tag{2.55}$$

Situation here is similar to (2.16), where the integration constant μ_0 was fixed from more specific relation between the asymptotic fields than the identity of the chiral currents. In this case the integration constant c can be specified from (2.38), which is equivalent to $2\sin\left(\phi_2 - \tilde{\phi}_2\right) = e^{\phi_1 + \tilde{\phi}_1}$, and it provides $|c| = 1/2$.

However, our final goal is the construction of the functional integral where the coordinate zero mode and the integration constant are excluded. Therefore, we keep the integration constant in (2.55) and rewrite these equations as

$$j(x) = 2\pi \frac{\delta A[\varphi, \chi]}{\delta \varphi(x)}\,, \qquad k_n = \frac{\partial A[\varphi, \chi]}{\partial \chi_{-n}}\,, \tag{2.56}$$

where $A[\varphi, \check{\chi}]$ is the functional

$$A[\varphi, \check{\chi}] = -\frac{i}{2}\sum_{n\in\mathbb{Z}} |n|\varphi_{-n}\varphi_n - \frac{i}{2}\sum_{n\in\mathbb{Z}} |n|\chi_{-n}\chi_n + \int_0^{2\pi} \frac{dx}{2\pi}\, \arcsin[c\, e^{\varphi(x)}]\, \chi'(x)\,. \tag{2.57}$$

Based on (2.56), we introduce the Legendre transformation $A[\varphi, \check{\chi}] \mapsto \tilde{F}[j, k]$:

$$\tilde{F}[j, k] = \sum_{n\in\mathbb{Z}} j_{-n}\, \varphi_n + \sum_{n\neq 0} k_{-n}\, \chi_n - A[\varphi, \check{\chi}]\,. \tag{2.58}$$

This provides

$$\frac{\delta\tilde{F}[j,k]}{\delta j(x)} = \frac{1}{2\pi}\,\varphi(x)\,, \qquad \frac{\partial\tilde{F}[j,k]}{\partial k_n} = \chi_{-n}\,, \tag{2.59}$$

and according to (2.47)-(2.48) the functional

$$\tilde{F} + 2i\sum_{m>0}\frac{b_m^*\,a_m}{m} + 2i\sum_{m>0}\frac{d_m^*\,c_m}{m} \tag{2.60}$$

reproduces F up a term which is only p_2 dependent.

Since $\tilde{F}[j,k]$ and $A[\varphi,\check{\chi}]$ are related by the Legendre transformation (2.58), the saddle point calculation of the integral

$$\int d\varphi_0\prod_{n\neq 0} d\varphi_n\,d\chi_n\,\exp\frac{i}{\hbar}\left(p_1\varphi_0 + \sum_{n\neq 0}\left(j_{-n}\,\varphi_n + k_{-n}\,\chi_n\right) - A[\varphi,\check{\chi}]\right) \tag{2.61}$$

yields $e^{\frac{i}{\hbar}\tilde{F}}$, and it describes non-zero mode transition of the S-matrix at list semiclassically.

If we integrate in (2.61) only the zero mode φ_0 by the saddle point method, we get

$$\int\prod_{n\neq 0} d\varphi_n\,d\chi_n\,\exp\frac{i}{\hbar}\left(\sum_{n\neq 0}\left(j_{-n}\varphi_n + k_{-n}\chi_n\right) - \check{A}[\check{\varphi},\check{\chi}]\right)\,, \tag{2.62}$$

where

$$\check{A}[\check{\varphi},\check{\chi}] = A[\varphi,\check{\chi}] - p_1\varphi_0\,, \tag{2.63}$$

and the zero mode φ_0 is obtained as a function of $(p_1, p_2, \varphi_n, \chi_n)$ from the equation

$$p_1 = \int_0^{2\pi}\frac{dx}{2\pi}\,\frac{c\,e^{\varphi(x)}\,\chi'(x)}{\sqrt{1-c^2e^{2\varphi(x)}}}\,. \tag{2.64}$$

One can look for the solution of φ_0 in power series of the non-zero modes $(\varphi_n,\,\chi_n)$ and use it in (2.62).

2.3 Generating function in SL(3,$\mathbb{R}$) Toda theory

Here we apply the scheme described in the previous sections to SL(3,$\mathbb{R}$) Toda theory, which we describe by the action [15]

$$S = \int dx\,d\bar{x}\left[\partial_{\bar{x}}\Phi_1\,\partial_x\Phi_1 + \partial_{\bar{x}}\Phi_2\,\partial_x\Phi_2 - 2\mu^2\,e^{\Phi_1}\cosh\left(\sqrt{3}\Phi_2\right)\right]. \tag{2.65}$$

The spacetime is assumed again cylindrical and we use the same chiral coordinates.

The equations of motion obtained from this action

$$\partial_{\bar{x}}\partial_x\Phi_1 + \mu^2 e^{\Phi_1}\cosh\left(\sqrt{3}\Phi_2\right) = 0\ , \qquad \partial_{\bar{x}}\partial_x\Phi_2 + \sqrt{3}\,\mu^2\,e^{\Phi_1}\sinh\left(\sqrt{3}\Phi_2\right) = 0\ , \tag{2.66}$$

provide the chirality conditions $\partial_{\bar{x}}T = 0$, $\partial_{\bar{x}}W = 0$, where T and W currents are

$$T = (\partial_x\Phi_1)^2 + (\partial_x\Phi_2)^2 - 2\partial_x^2\Phi_1\ , \tag{2.67}$$

$$W = (\partial_x\Phi_2)^3 - 3(\partial_x\Phi_1)^2\,\partial_x\Phi_2 + \tfrac{3}{2}\partial_x^2\Phi_1\,\partial_x\Phi_2 + \tfrac{9}{2}\partial_x^2\Phi_2\,\partial_x\Phi_1 - \tfrac{3}{2}\partial_x^3\Phi_2\ . \tag{2.68}$$

The potential term in the action (2.65) vanishes at the time asymptotics, and we have

$$\begin{aligned} \Phi_\alpha(\tau,\sigma) &\longrightarrow \phi_\alpha(x)+\bar{\phi}_\alpha(\bar{x}), \quad \text{at } \tau\to-\infty, \\ \Phi_\alpha(\tau,\sigma) &\longrightarrow \tilde{\phi}_\alpha(x)+\tilde{\bar{\phi}}_\alpha(\bar{x}), \quad \text{at } \tau\to+\infty, \end{aligned} \tag{2.69}$$

for $\alpha=(1,2)$. This defines the stress tensor (2.67) and the W-current (2.68) in terms of the asymptotic fields

$$T=(\phi_1')^2+(\phi_2')^2-2\phi_1''=(\tilde{\phi}_1')^2+(\tilde{\phi}_2')^2-2\tilde{\phi}_1''\,, \tag{2.70}$$

$$\begin{aligned} W&=(\phi_2')^3-3(\phi_1')^2\,\phi_2'+\tfrac{3}{2}\phi_1''\,\phi_2'+\tfrac{9}{2}\phi_2''\,\phi_1'-\tfrac{3}{2}\phi_2''' \\ &=(\tilde{\phi}_2')^3-3(\tilde{\phi}_1')^2\,\tilde{\phi}_2'+\tfrac{3}{2}\tilde{\phi}_1''\,\tilde{\phi}_2'+\tfrac{9}{2}\tilde{\phi}_2''\,\tilde{\phi}_1'-\tfrac{3}{2}\tilde{\phi}_2'''\,. \end{aligned} \tag{2.71}$$

Thus, the scattering problem for SL(3,$\mathbb{R}$) Toda theory is described by two chiral *in*-fields (ϕ_1,ϕ_2) and two chiral *out*-fields $(\tilde{\phi}_1,\tilde{\phi}_2)$, as in SL(2,$\mathbb{R}$)/U(1) WZW theory.

The expansion of these fields in Fourier modes is again (2.39) and the relation between the momentum zero modes is also similar $\tilde{p}_1=-p_1<0$, $\tilde{p}_2=p_2$.

On this basis, we introduce four new chiral fields

$$\varphi=\phi_1+\tilde{\phi}_1\,,\quad \chi=\phi_2-\tilde{\phi}_2\,,\quad j=p_1+2a'-2b'^*\,,\quad k=p_2+2c'+2d'^*\,, \tag{2.72}$$

where the positive and negative frequency parts are defined as before

$$\begin{aligned} \phi_1'&=\tfrac{1}{2}\,p_1+a'+a'^*\,, &\qquad \tilde{\phi}_1'&=-\tfrac{1}{2}\,p_1+b'+b'^*\,, \\ \phi_2'&=\tfrac{1}{2}\,p_2+c'+c'^*\,, &\qquad \tilde{\phi}_2'&=\tfrac{1}{2}\,p_2+d'+d'^*\,. \end{aligned} \tag{2.73}$$

The difference here is in the choice of the field χ and the current k. The field χ now is periodic and k has nonzero momentum mode. This choice is more natural than the one given in (2.48)-(2.49). Note that the same choice fails for SL(2,$\mathbb{R}$)/U(1) theory due to (2.38), which forbids to treat $\phi_1+\tilde{\phi}_1$ and $\phi_2-\tilde{\phi}_2$ as independent fields.

From (2.72) we have

$$\phi_1'=\varphi_-'+\tfrac{1}{2}\,j\,,\quad \tilde{\phi}_1'=\varphi_+'-\tfrac{1}{2}\,j\,,\quad \phi_2'=\tfrac{1}{2}\,k+\chi_-'\,,\quad \tilde{\phi}_2'=\tfrac{1}{2}\,k-\chi_+'\,. \tag{2.74}$$

Hence, (2.70)-(2.71) relate the fields (φ,χ) to the currents (j,k). By (2.70) one can solve k algebraically and (2.71) then reduces to a second order differential equation for j. Its integration, after a suitable choice of the integration constants, leads to the solution

$$j=\varphi_+'-\varphi_-'+\sqrt{\tilde{\chi}'^{\,2}+2e^{\varphi}\cosh\tilde{\chi}}\,, \tag{2.75}$$

$$k=\chi_+'-\chi_-'+\frac{\frac{2}{3}\,\chi''-\frac{1}{3}\,\varphi'\chi'+\frac{2}{\sqrt{3}}\,e^{\varphi}\sinh\tilde{\chi}}{\sqrt{\tilde{\chi}'^{\,2}+2e^{\varphi}\cosh\tilde{\chi}}}, \tag{2.76}$$

where $\tilde{\chi}=\frac{1}{\sqrt{3}}\,\chi$.

Since the transformation from the *in*-fields to the *out* fields is canonical, the currents in (2.75) and (2.76) have to be represented as the variational derivatives

$$j(x)=2\pi\,\frac{\delta A[\varphi,\chi]}{\delta\varphi(x)}\,,\qquad k(x)=2\pi\,\frac{\delta A[\varphi,\chi]}{\delta\chi(x)}\,, \tag{2.77}$$

and we find the functional

$$A = \int_0^{2\pi} \frac{dx}{2\pi} \left(\varphi'_+ \varphi_- + \chi'_+ \chi_- + 2\sqrt{\tilde{\chi}'^{\,2} + 2e^{\varphi} \cosh \tilde{\chi}} + \tilde{\chi}' \log \frac{\sqrt{\tilde{\chi}'^{\,2} + 2e^{\varphi} \cosh \tilde{\chi}} - \tilde{\chi}'}{\sqrt{\tilde{\chi}'^{\,2} + 2e^{\varphi} \cosh \tilde{\chi}} + \tilde{\chi}'} \right) . \tag{2.78}$$

Its Legendre transformation $A \mapsto \tilde{F}$, defined by (2.78) and

$$\tilde{F}(j,k) = \langle j|\varphi\rangle + \langle k|\chi\rangle - A[\varphi,\chi] \,, \tag{2.79}$$

provides

$$\frac{\delta F[j,k]}{\delta j(x)} = \frac{1}{2\pi}\, \varphi(x) \,, \qquad \frac{\delta A[j,k]}{\delta k(x)} = \frac{1}{2\pi}\, \chi(x) \,. \tag{2.80}$$

The choice of the fields φ and χ defines the relation between the functionals $\tilde{F}$ and F. In this case it is given by

$$F = \tilde{F} + 2i \sum_{m>0} \frac{b^*_m\, a_m}{m} - 2i \sum_{m>0} \frac{d^*_m\, c_m}{m} \,. \tag{2.81}$$

3 Conclusion

In the conclusion we present a schematic overview of the key methodological elements developed throughout this paper.

We denote by a and b the set of the positive frequency modes of the *in* and *out* fields respectively, and by a^* and b^* the corresponding negative frequency modes.

The map from the *in*-field to the *out*-field is canonical, which is expressed by

$$\omega = i\, \delta b^* \wedge \delta b = i\, \delta a^* \wedge \delta a \,, \tag{3.1}$$

where ω is the free-field symplectic form.

We denote by $F(b^*, a)$ the generating functional of the canonical transformation between the asymptotic fields, which depends on the negative frequency modes of the *out*-field and the positive frequency modes of the *in*-field.

Thus, the differential of F is

$$\delta F(b^*, a) = -i\, b\, \delta b^* - i\, a^* \delta a \,, \tag{3.2}$$

and its derivatives are

$$\frac{\partial F(b^*, a)}{\partial a} = -i\, a^* \,, \qquad \frac{\partial F(b^*, a)}{\partial b^*} = -i\, b \,. \tag{3.3}$$

To get equations for F in a closed form, one has to express the modes (a^*, b) in terms of (b^*, a), using the relations between the asymptotic fields, which we formally write as

$$T(b^*, b) = T(a^*, a) \;. \tag{3.4}$$

These relations define infinite set of nonlinear equations for (a^*, b) and its solution is a hard problem even for Liouville theory. A similar problem is to solve (b^*, a) from (3.4) in terms of (a^*, b).

However, one can introduce the new fields

$$\varphi^* = -i(a^* + \epsilon b^*)\,, \qquad \varphi = i(b + \epsilon a)\,. \tag{3.5}$$

which allow to solve (3.4) with respect to (b^*, a) in terms of (φ^*, φ), where $\epsilon = \pm 1$.

Then, due to (3.3), the solution of (3.4) is represented as

$$b^* = -\frac{\partial A}{\partial \varphi}\,, \qquad a = \frac{\partial A}{\partial \varphi^*}\,, \tag{3.6}$$

where the functional $A[\varphi^*, \varphi]$ is obtained from the explicit form of the solution.

By (3.6) we introduce the Legendre transformation $A \mapsto \tilde{F}$:

$$\tilde{F}[b^*, a] = a\varphi^* - b^*\varphi - A[\varphi^*, \varphi]\,, \tag{3.7}$$

which defines F as

$$F[b^*, a] = \tilde{F}[b^*, a] + i\,\epsilon\, b^*\, a\,. \tag{3.8}$$

The quantum version of the Legendre transformation (3.7) can be treated as

$$e^{\frac{i}{\hbar}\,\tilde{F}[b^*,a]} = \int d\varphi^*\, d\varphi\, e^{\frac{i}{\hbar}(a\varphi^* - b^*\varphi - A[\varphi^*,\varphi])}\,, \tag{3.9}$$

since the saddle point calculation of the integral reproduces the classical relation (3.8).

As it was mentioned in Section 2, in Liouville theory this relation was checked in [1] and [2] by various quantum calculations.

Low level transition amplitudes are known both in SL(2,$\mathbb{R}$)/U(1) WZW theory and in SL(3,$\mathbb{R}$) Toda theory in a closed form [16]. This defines a part of the left hand side of (3.9), and it is natural to calculate the corresponding right hand side of (3.9), investigating the functional integral in (3.9) for the constructed functionals (2.57) and (2.78).

We are also looking for the generalization of the method to other integrable models, where the first natural candidates are supersymmetric extensions of the considered theories, or the same theories with conformally invariant boundary conditions.

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