Aspects of Matrix Models

Anurag Sarkar (191021)
Physics Department, IIT Kanpur.

Under the supervision of Prof.Diptarka Das

In this report, we study various aspects of matrix models focusing in particular on the double scaling limit. After a review of both the large N saddle method and the method of orthogonal polynomials, we develop a short Python code to generate relevant recursion for orthogonal polynomials and then we focus on entanglement entropy. We then define a reduced density matrix by tracing over "complementary" eigenvalues. We derive an integral expression for the reduced density matrix using large N methods. We also report on the progress using the method of orthogonal polynomials.

I. INTRODUCTION

In matrix models, we have partition function of the form:

$$e^Z = \int dM e^{-\mathrm{tr}(V(M))} \tag{1}$$

where, M is an $N \times N$ matrix, and the integral is over all such matrices. For real physical observables, we can take M to be hermitian. Then, the measure dM is defined as $dM = \prod_i dM^i_i \prod_{i<j} d\mathrm{Re}M^i_j d\mathrm{Im}M^i_j$ with $i, j = 1, ..., N$. We can also define a correlator of the form:

$$\langle M...M \rangle = \int dM \, e^{-\mathrm{tr}(V(M))} \, M^{i_1}_{j_1}...M^{i_n}_{j_n} \tag{2}$$

A. Where does it arise?

a. 2D gravity path integral: In case of 2D gravity, we have partition function of the form:

$$Z = \sum_h \int \mathcal{D}g \, e^{-S(g)} \tag{3}$$

where, the sum over all the topologies is represented by h, the genus or the number of handles of the surface; and the Einstein action for pure gravity reads $S(g) = \int d^d x \sqrt{g}(KR + \Lambda)$ in which g_{ij} is the metric tensor, R is the curvature and K, Λ are two coupling constants.

The integral $\int \mathcal{D}g$ over the metric on the surface is generally difficult to calculate. To deal with this problem, if we can discretize the surface, it becomes much easier to calculate. As an example, we can consider a random triangulation of the surface, in which the surface is constructed from triangles as seen in the figure 1. The summation over all such random triangulations gives us the discrete analogue to the integral $\mathcal{D}g$ over all possible geometries:

$$\sum_h \int \mathcal{D}g \;\; \rightarrow \sum_{random\ triangulations} \tag{4}$$

Consequently, the partition function becomes:

$$Z = \sum_h \int \mathcal{D}g \, e^{-\beta A + \gamma\chi} \tag{5}$$

where, A is the area: $A = \int \sqrt{g}$ and χ is the Euler character: $\chi = (1/4\pi)\int \sqrt{g}R = 2 - 2h$. Now, to perform the integral over all possible geometries ie, in discretized form a sum over random triangulations, we use a certain matrix integral as a generating function for random triangulations. As we are considering triangles, we may consider the partition function to be:

$$e^Z = \int dM \, e^{-\frac{1}{2}\mathrm{tr}M^2 + \frac{g}{\sqrt{N}}\mathrm{tr}M^3} \tag{6}$$

where M is an $N \times N$ hermitian matrix, the integral is understood to be defined by analytic continuation in coupling g. Then after adjusting the variables properly, each diagram gives an overall factor:

$$N^{V-E+F} = N^{\chi} = N^{2-2h} \tag{7}$$

where, V, E, F are the total number of vertices, edges and faces (generated by the loops) in the diagram. Thus, we can expand the partition function in powers of N as:

$$Z(g, N) = \sum_h Z_h(g) \, N^{2-2h} \tag{8}$$

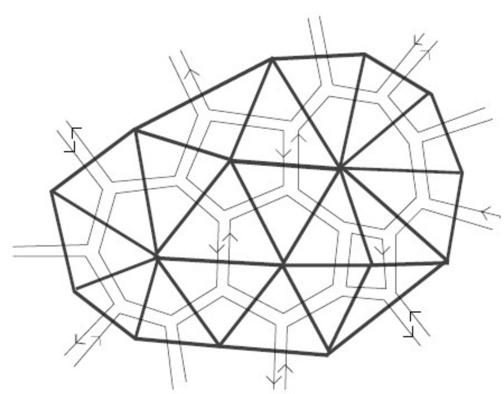

FIG. 1. A piece of a random triangulation of a surface *(source: P.Ginsparg and G. Moore, Lectures on 2D Gravity and 2D String Theory)*

B. Critical behaviour

From above, it follows:

$$Z(g) = N^2 Z_0(g) + Z_1(g) + N^{-2} Z_2(g) + .. = \sum_h N^{2-2h} Z_h(g) \tag{9}$$

where Z_h gives the contribution from the surfaces of genus h. In the large N limit, we take $N \to \infty$; then only the genus zero contribution survives, because all the other terms are of $O(N^0)$ or lower.

But, it turns out that the successive coefficients $Z_h(g)$ in (9) diverge at some critical value of the coupling $g = g_c$. For higher genus contributions, the following expression can be derived:

$$Z_h(g) \sim (g_c - g)^{(2-\gamma_{str})\chi/2} \tag{10}$$

where γ_{str} is some critical exponent. We see that the contribution from higher genus are enhanced for $g \to g_c$ as $\chi < 0$ ($h > 0$) for $\gamma_{str} < 2$. So, if we take the limits $N \to \infty$ and $g \to g_c$ in a correlated manner instead of taking them independently, we can compensate the large N suppression with the $g \to g_c$ enhancement. This would result in a non-trivial contribution from the higher genus. To introduce both of the limits simultaneously, we may consider double scaling. To do this, we define a new parameter κ such that:

$$\kappa^{-1} \equiv N(g - g_c)^{(2-\gamma_{str})\chi/2} \tag{11}$$

If we have:

$$Z_h(g) \sim f_h(g - g_c)^{(2-\gamma_{str})\chi/2} \tag{12}$$

Then, $N^{2-2h} Z_h(g)$ becomes:

$$N^{2-2h} Z_h(g) = \kappa^{2h-2} f_h \tag{13}$$

Thus, (9) can be written as:

$$Z = \kappa^{-2} f_0 + f_1 + \kappa^2 f_2 = \sum_h \kappa^{2h-2} f_h \tag{14}$$

The desired limits then can be taken simultaneously ie, taking $N \to \infty$ and $g \to g_c$ while keeping κ fixed. This is known as the double scaling limit.

C. Outline

In the second section, we discuss the *Method of Orthogonal Polynomials*. Here, we implement an algorithm to find the relevant recursions in case of an arbitrary even $V(M)$. To determine these coefficients, a python code has been written, which is attached in the appendix section.

In the third section, we discuss the process of finding a *reduced density matrix in eigenvalue space for the 0-dimensional Matrix Model*. This is done using saddle point approximation, and the approach using orthogonal polynomials is also briefly discussed.

In the fourth section, we discuss about some ongoing works and some future prospects of this model.

II. METHOD OF ORTHOGONAL POLYNOMIALS

We rewrite (1) in the form:

$$e^Z = e^{-\mathrm{tr}(V(M))} = \int \prod_{i=1}^{N} d\lambda_i \Delta^2(\lambda) e^{-\sum_i V(\lambda_i)} \tag{15}$$

where $V(M)$ is some general polynomial. Here we have diagonalized M, and λ_i's are the N eigenvalues of the matrix. The $\Delta(\lambda)$ is the Vandermonde determinant, defined as:

$$\Delta(\lambda) = \prod_{i<j} (\lambda_j - \lambda_i) \tag{16}$$

Due to the antisymmetry of Δ in interchanging any two eigenvalues, we can write $\Delta(\lambda) = \det(\lambda_i^{j-1})$. Now, to solve (15) we define an infinite set of polynomials $P_n(\lambda)$, which are mutually orthogonal with respect to the measure:

$$\int_{-\infty}^{\infty} d\lambda\, e^{-V(\lambda)} P_n(\lambda) P_n(\lambda) = h_n \delta_{nm} \tag{17}$$

The P_n's are called the orthogonal polynomials, and h_n is a constant. The polynomials have the form: $P_n(\lambda) = \lambda^n + \sum_{i=0}^{n-1} b_i \lambda^i$. Consequently, we can now write:

$$\Delta(\lambda) = \det \lambda_i^{j-1} = \det P_{j-1}(\lambda_i) \tag{18}$$

We can now substitute the above expression in place of $\Delta(\lambda)$ in (15), with $\det P_{j-1}(\lambda_i) = \sum (-1)^\pi \prod_k P_{i_k - 1}(\lambda_k)$. Due to the orthogonality condition, only the terms containing $P_j(\lambda_i)$ pairs with same i, j's will contribute, and there will be $N!$ such terms, thus (15) reduces to:

$$e^Z = N! \prod_{i=0}^{N-1} h_i = N!\, h_0^N \prod_{k=1}^{N-1} f_k^{N-k} \tag{19}$$

where, $f_k = (h_k / h_{k-1})$. Now, if we implement $N \to \infty$ limit, then k/N can be taken as a continuous variable ξ running from 0 to 1. Similarly, f_k/N also becomes a continuous function $f(\xi)$. Then for large N, we can write (19) as:

$$\frac{Z}{N^2} = \frac{1}{N^2}(\log N! + N \log h_0) + \frac{1}{N} \sum_k (1 - \frac{k}{N}) \log f_k$$

Thus, up to an additive constant:

$$\frac{Z}{N^2} \sim \int_0^1 d\xi (1 - \xi) \ln f(\xi) \tag{20}$$

For simplicity, we now assume $V(\lambda)$ is an even polynomial. As P_i's form a complete basis, we can express $\lambda P_n(\lambda)$ as a linear combination of the lower P_i's. Thus, $\lambda P_n(\lambda) = \sum_{i=0}^{n+1} a_i P_i(\lambda)$, where $a_i = h_i^{-1} \int e^{-V} \lambda P_n P_i$. Now, in the expansion, $\int e^{-V} \lambda P_n P_n$ is 0 for even V, and for the other coefficients, a_i will be 0 for $i < (n-1)$. Thus the only non zero terms will be a_{n+1} and a_{n-1} and then we can write:

$$\lambda P_n = P_{n+1} + r_n P_{n-1} \tag{21}$$

Using this recursion, we have:

$$\int e^{-V} \lambda P_n P_{n-1} = r_n h_{n-1} = h_n \tag{22}$$

This shows that $f_n = r_n$. Similarly, we also have:

$$n h_n = \int e^{-V} \lambda P_n' P_n = r_n \int e^{-V} V' P_n P_{n-1} \tag{23}$$

This equation will allow us to determine r_n for a particular $V(\lambda)$.
Consider a general even $V(\lambda)$ of the form λ^{2p}, then V' is of the form λ^{2p-1}. Thus we need to compute integrals of the form: $\int e^{-V} V' P_n P_{n-1} \sim \int e^{-V} \lambda^{2p-1} P_n P_{n-1}$. We have derived a Python code to obtain the result, which is attached in the appendix. Using this code, we state here the results for some p values:

$$\begin{aligned}
p = 2 &: r_n \\
p = 4 &: r_n(r_{n-1} + r_n + r_{n+1}) \\
p = 6 &: r_n(r_{n-1}r_{n+2} + r_{n+1}^2 + 2r_{n+1}r_n + r_{n-1}r_{n+1} \\
&\quad r_n^2 + 2r_n r_{n-1} + r_{n-1}^2 + r_{n-1}r_{n-2})
\end{aligned}$$

So, if we consider a $V(\lambda)$ of the form:

$$V(\lambda) = \frac{N}{2g}(\lambda^2 + \lambda^4 + b\lambda^6) \tag{24}$$

then, using (23) and substituting the values we derived above, we have:

$$\begin{aligned}
gn = r_n + \frac{2}{N} r_n(r_{n-1} + r_n + r_{n+1}) + \frac{3b}{N^2} r_n(r_{n-1}r_{n+2} + r_{n+1}^2 \\
+ 2r_{n+1}r_n + r_{n-1}r_{n+1} r_n^2 + 2r_n r_{n-1} + r_{n-1}^2 + r_{n-1}r_{n-2})
\end{aligned} \tag{25}$$

Now, taking the $N \to \infty$ limit, n becomes a continuous variable ξ and we have, $r_n \to r(\xi)$ and $r_{n\pm1} \to r(\xi \pm \epsilon)$. Then, to the leading order in $1/N$ we can write (25) as:

$$g\xi = r + 6r^2 + 30br^3 = W(r) \tag{26}$$

$$= g_c + \frac{1}{2} W''(r_c)(r - r_c)^2 \tag{27}$$

Then, using (20), we can ultimately derive:

$$\frac{Z}{N^2} \sim (g_c - g)^{-\gamma_{str}+2} \tag{28}$$

where $\gamma_{str} = -1/2$. This is of the same form of (10) for genus 0 ($h = 0$). Thus keeping only the leading order terms, we get the genus 0 partition function. We can also obtain the higher genus partition functions by keeping the higher order terms in (26) and taking the double scaling limit as discussed earlier. For example, keeping the terms up to first order, we can write (26) as:

$$g\xi = W(r) + 2r(\xi)(1 + 15br(\xi))(r(\xi+\epsilon) + r(\xi-\epsilon) - 2r(\xi)) \tag{29}$$

Applying proper change of variables and double scaling limit, we get an expression for Z, which matches (14), as discussed earlier. For the general m^{th} order critical point, we can take $W' = W'' = ... = W^{(m-1)} = 0$ at $r = r_c$, then $W(r) \sim g_c + (1/m!)W^{(m)}(r_c)$ with $\gamma_{str} = -1/m$.

III. REDUCED DENSITY MATRIX

A. By Saddle Point Approximation

Following the discussion in the previous section, we can write:

$$e^Z = \int dM e^{-(\frac{N}{g})\mathrm{tr}\, V(M)} = \int \prod_{i=1}^{N} d\lambda_i \Delta^2(\lambda)\, e^{-(\frac{N}{g})\sum_i V(\lambda_i)} \tag{30}$$

where g is the coupling constant. Since partition function $Z = \mathrm{tr}\,\rho$, we define:

$$\rho(\lambda) = \Delta^2(\lambda)\, e^{-(\frac{N}{g})\sum_i V(\lambda_i)} \tag{31}$$

Lets consider a subspace A of two eigenvalues λ_1, λ_2. Then a natural definition of a reduced density matrix:

$$\begin{aligned}
\rho_A(\lambda_1, \lambda_2) &= \int \prod_{i=3}^{N} d\lambda_i \prod_{j<k} (\lambda_j - \lambda_k)^2\, e^{-(\frac{N}{g})\sum_{j=1}^{N} V(\lambda_j)} \\
&= (\lambda_1 - \lambda_2)^2 e^{-\frac{N}{g}(V(\lambda_1)+V(\lambda_2))} \\
&\times \int \prod_{i=3}^{N} d\lambda_i \prod_i (\lambda_i - \lambda_1)^2(\lambda_i - \lambda_2)^2 \prod_{i<j}(\lambda_j - \lambda_i)^2 \\
&\qquad e^{-(\frac{N}{g})\sum_{i=3}^{N} V(\lambda_j)}
\end{aligned} \tag{32}$$

We can evaluate the Saddle Point Equation with respect to a single eignevalue λ_i. Equation of motion is given by:

$$\frac{1}{N}\Big[\frac{2}{\lambda_i - \lambda_1} + \frac{2}{\lambda_i - \lambda_2} + \sum_{j\neq i} \frac{2}{\lambda_j - \lambda_i}\Big] = \frac{1}{g} V'(\lambda_i) \tag{33}$$

To solve this we introduce:

$$\omega(z) = \frac{1}{N}\mathrm{tr}\, \frac{1}{M - z} = \frac{1}{N}\sum_{i=3}^{N} \frac{1}{\lambda_i - z} \tag{34}$$

Now, we take the distribution of eigenvalues to be $\rho(\lambda) = \frac{1}{N}\sum_i \delta(\lambda - \lambda_i)$. This becomes continuous in large N limit and we can write:

$$\omega(z) = \int \frac{\rho(\lambda) d\lambda}{\lambda - z} \tag{35}$$

with normalization condition $\int \rho(\lambda')d\lambda' = 1$. Then, using analytic continuation, we can write (33) as:

$$2P\int \frac{\rho(\lambda')\,d\lambda'}{\lambda-\lambda'} = \frac{V'(\lambda)}{g} - \frac{2}{N}\Big[\frac{1}{\lambda-\lambda_1} + \frac{1}{\lambda-\lambda_2}\Big] \quad (36)$$

which is equivalent to

$$\lim_{\epsilon\to 0}\big[\omega(z+i\epsilon)+\omega(z-i\epsilon)\big] = -\frac{V'(\lambda)}{g} + \frac{2}{N}\Big[\frac{1}{z-\lambda_1} + \frac{1}{z-\lambda_2}\Big] \quad (37)$$

And we can write the eigenvalue density as:

$$\rho(z) = \lim_{\epsilon\to 0}\frac{1}{2\,i\,\pi}(\omega(z+i\epsilon)-\omega(z-i\epsilon)) \quad (38)$$

Considering the potential $V(z)$ has only one minima, we can take a one cut solution with two branch points at a and b:

$$\tilde{\omega}(z) = \frac{\omega(z)}{\sqrt{(z-a)(z-b)}} \quad (39)$$

For an even potential of the form: $V(\lambda) = \frac{\lambda^2}{2} + \frac{\lambda^4}{4}$, the branch points are distributed about 0 symmetrically. Thus we take the branch points to be $\pm a$. After going through the computations, we obtain from (32):

$$\begin{aligned}&\log\rho_A(\lambda_1,\lambda_2)\\ &= 2\log|(\lambda_1-\lambda_2)| - \frac{N}{g}(V(\lambda_1)+V(\lambda_2))\\ &+ N^2\Big[\int d\lambda d\mu\rho(\lambda)\rho(\mu)\log|(\lambda-\mu)| - \frac{1}{g}\int d\lambda\rho(\lambda)V(\lambda)\Big]\\ &+ N\Big[\int d\lambda\rho(\lambda)\log|(\lambda-\lambda_1)| + \int d\lambda\rho(\lambda)\log|(\lambda-\lambda_1)|\Big]\end{aligned} \quad (40)$$

Integrating EOM (33) with respect to λ_i we have,

$$\begin{aligned}\frac{V(\lambda)}{2g} &= \frac{1}{N}\Big[\log(\frac{\lambda_1-\lambda}{\lambda_1}) + \log(\frac{\lambda_2-\lambda}{\lambda_2})\Big]\\ &+ \int d\mu\rho(\mu)\Big[\log|(\mu-\lambda)| - \log|\mu|\Big]\end{aligned} \quad (41)$$

Using this in (40) one can get,

$$\begin{aligned}&\log\rho_A(\lambda_1,\lambda_2)\\ &= 2\log|(\lambda_1-\lambda_2)| - \frac{N}{g}(V(\lambda_1)+V(\lambda_2))\\ &+ N\log(\lambda_1\lambda_2) + N^2\Big[\int d\lambda\rho(\lambda)(\log|\lambda| - \frac{V(\lambda)}{2g})\Big]\end{aligned} \quad (42)$$

where we used normalization condition $\int d\lambda\rho(\lambda) = 1$ and $\rho(\lambda)$ is given by:

$$\begin{aligned}\rho(\lambda) &= \frac{1}{2\pi g}(1+\lambda^2+\frac{a^2}{2})\sqrt{a^2-\lambda^2}\\ &+ \frac{1}{N\pi}\frac{\sqrt{a^2-\lambda^2}}{(\lambda-\lambda_1)\sqrt{\lambda_1^2-a^2}}\end{aligned} \quad (43)$$

From this, we can compute the entropy:

$$S = \int d\lambda_1 d\lambda_2 \rho_A(\lambda_1,\lambda_2)\log\rho_A(\lambda_1,\lambda_2) \quad (44)$$

This is currently being computed in [3].

B. By The Method of Orthogonal Polynomials

We start with a similar expression as (32):

$$\rho_A(\lambda_1,\lambda_2) = \int\prod_{i=3}^{N} d\lambda_i \Delta^2(\lambda)\, e^{-\sum_{j=1}^{N} V(\lambda_j)} \quad (45)$$

Where, $\Delta(\lambda)$ is defined in (16):

$$\Delta(\lambda) = \det P_{j-1}(\lambda_i) = \sum_{\sigma}(-1)^{\pi_\sigma}\prod_{l=1}^{N} P_{ml} \quad (46)$$

where we have used the notation $P_{m-1}(\lambda_l) \equiv P_{ml}$. Then, (45) can be written as:

$$\rho_A(\lambda_1,\lambda_2) = \int\prod_{i=3}^{N} d\lambda_i\, \sum_{\sigma}(-1)^{\pi_\sigma}\prod_{l=1}^{N}P_{ml}\,\sum_{\tau}(-1)^{\pi_\tau}\prod_{k=1}^{N}P_{nk} \quad (47)$$

Because of the orthogonality condition, all the cross terms obtained from multiplying two determinants will be zero, thus finally giving us:

$$\begin{aligned}\rho_A = \frac{(N-1)!}{2}&\prod_{j=0}^{N-1} h_j e^{-V(\lambda_1)-V(\lambda_2)}\\ &\times\sum_{m}\sum_{n\neq m}\frac{(P_{m1}P_{n2}-P_{m2}P_{n1})^2}{h_m h_n}\end{aligned} \quad (48)$$

Now, we consider $V(\lambda)$ to be quadratic in λ. Then the P_n is simply the Hermite polynomials H_n. As we need the leading order term in P_n to have unit coefficient, we define:

$$P_n(\lambda) = 2^{-n/2}\,H_n\left(\frac{\lambda}{\sqrt{2}}\right)$$

Taking $V(\lambda) = \frac{\lambda^2}{2}$, we have the orthogonality condition:

$$\int P_n(\lambda)P_m(\lambda)e^{-V(\lambda)} = \sqrt{2\pi}n!\,\delta_{mn} \quad (49)$$

Thus, we have $h_n = \sqrt{2\pi}n!$. After expanding $\log\rho_A$ in the expression (48) and keeping up to the first order term, we finally obtain:

$$
\begin{aligned}
&\int d\lambda_1 d\lambda_2\, \rho_A \log \rho_A \\
&= (Z + 2N + 3 - \log(2N^2))\frac{e^Z}{2} \\
&+ \frac{2e^Z}{N(N-1)} \sum_m \sum_{m \neq n} \sum_i \sum_{i \neq j} \frac{1}{h_m h_n h_i h_j} \int d\lambda_1 d\lambda_2 e^{-V_1 - V_2} \\
&\qquad \times (P_{m1}^2 P_{n2}^2 P_{i1}^2 P_{j2}^2 - 2P_{m1}P_{n1}P_{i1}^2 P_{m2}P_{n2}P_{j2}^2 \\
&\qquad + (P_{m1}P_{n1}P_{i1}P_{j1})(P_{m2}P_{n2}P_{i2}P_{j2})) \qquad (50)
\end{aligned}
$$

Here, $V_1 = V(\lambda_1)$ and $V_2 = V(\lambda_2)$. In deriving the above expression, we have used (19). This is currently in progress.

IV. CONCLUSION AND FUTURE DIRECTIONS

Though the concept of Matrix Models is relatively new, it already shows some quite promising results. For example, in section 2, the derived result indicates a critical exponent $\gamma_{str} = -1/2$, which corresponds to the case of pure gravity and matches order by order for all genus. This method is valid for higher order critical points also, where we have to consider subleading terms (as mentioned briefly at the end of that section).

As mentioned in the previous sections, the matrix integrals can be approached both by saddle point approximation and the method of orthogonal polynomials. Although the saddle point method is quite promising, the computation involved in the process gets more and more complicated for higher order potentials or potentials involving multiple minima; this involves overlapping branch cuts and essential singularities arising from the square root and logarithmic terms. Whereas, in case of orthogonal polynomials this problem does not arise. The final expressions (44) and (50) obtained at the end of section is currently being computed. Subsequently, the simplicity of matrix model results for correlation functions has initiated a rapid evolution of continuum Liouville technology so that as well many correlation functions can be computed in both approaches and are found to coincide.

Appendix

```
from math import *
import sympy as sy
n,r_n=sy.symbols("n,r_n")

'''p=power of lambda'''
p=int(input())

def step(p,n,r):
    a=[n]         #list of the polynomials
    b=[]
    g=[1]         #list of the coefficients
    h=[]
    for i in range(p):
        h=[]
        b=[]
        for j in range(len(a)):
            c=a[j]+1
            d=a[j]-1
            e=g[j]*1
            f=g[j]*(r+a[j]-n)
            b.append(c)
            b.append(d)
            h.append(e)
            h.append(f)
        a=b
        g=h
    q=[]
    for l in range(len(a)):
        if a[l]==(n-1):
            q.append(g[l])
    return q

k=step(p,n,r_n)

print(k)
print('number of terms:', len(k))
```


[1] P. Ginsparg and G. Moore, *Lectures on 2D Gravity and 2D String Theory*,June 11–19, 1992 at TASI summer school, Boulder, CO.

[2] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin, *2D Gravity and Random Matrices*, Physics Reports (1993).

[3] D. Das, S. Das, B. Dey, and A. Sarkar, *Entanglement Entropy in Matrix Models* (work in progress).