CFT Readings

Abstract

Miscellaneous notes from reading course on CFT (mainly 2D) held in Monsoon 2020. The participants include Athira P. V. , Diptarka Das, Bidyut Dey, Suchetan Das and Anurag Sarkar.

1 Derivation of descendant Ward identity

We start with the following expression,

$$< \phi_n^{-k}(z)\,\phi_1(z_1)\,....\,\phi_N(z_N) > . \tag{1.1}$$

where, $\phi_i's$ with $i = 1, ..., N$ are the primary fields and ϕ_n^{-k} is a secondary field of the conformal family $[\phi_n]$.
We know from equation 3.12 of [1],

$$T(\zeta)\,\phi_n(z) = \sum_{k=0}^{\infty} (\zeta - z)^{-2+k}\,\phi_n^{-k}(z) \tag{1.2}$$

then,

$$\phi_n^{-k}(z) = \oint \frac{dw}{2\pi i}\,(w-z)^{1-k}\,T(w)\,\phi_n(z). \tag{1.3}$$

From equation (1.3), it follows that

$$\begin{aligned} &< \phi_n^{-k}(z)\,\phi_1(z_1)\,....\,\phi_N(z_N) > \\ &\qquad = \oint_z \frac{dw}{2\pi i}\,(w-z)^{1-k} < (T(w)\,\phi_n(z))\,\phi_1(z_1)\,....\,\phi_N(z_N) > \\ &\qquad = -\sum_{i=1}^{N} \oint_{z_i} \frac{dw}{2\pi i}\,(w-z)^{1-k} < \phi_n(z)\phi_1(z_1)\,..\,T(w)\phi_i(z_i)\,..\,\phi_N(z_N) > \end{aligned} \tag{1.4}$$

the integration contour is deformed to get the third line from second line.
By using the OPE of primary fields we will get,

$$\begin{aligned} &< \phi_n^{-k}(z)\,\phi_1(z_1)\,....\,\phi_N(z_N) > = \\ &-\oint_{z_i} \frac{dw}{2\pi i}\,(w-z)^{1-k} \Big(\sum_{i=1}^{N} \Big[\frac{h_i}{(w-z_i)^2} + \frac{1}{w-z_i}\frac{\partial}{\partial z_i} \Big] \Big) < \phi_n(z)\,\phi_1(z_1)\,....\,\phi_N(z_N) > \end{aligned}$$

where, $\Delta_1, \Delta_2, ..., \Delta_N$ are dimensions of the fields $\phi_1, \phi_2, ..., \phi_N$ respectively. We can do the contour integral,

$$\oint dw\,\frac{1}{(w-z)^{k-1}}\,\frac{1}{(w-z_i)^2} = 2\pi i\,\frac{1-k}{z_i - z} \tag{1.5}$$

and,

$$\oint dw \, \frac{1}{(w-z)^{k-1}} \, \frac{1}{w-z_i} \, \frac{\partial}{\partial z_i} \; = \; 2\pi i \, \frac{1}{(z_i-z)^{k-1}} \, \frac{\partial}{\partial z_i} \tag{1.6}$$

By using these expressions we will get,

$$\begin{aligned} &< \phi_n^{-k}(z) \, \phi_1(z_1) \, ... \, \phi_N(z_N) \; > = \\ &\Big(\sum_{i=1}^{N} \Big[h_i \frac{k-1}{z_i - z} \; - \; \frac{1}{(z_i-z)^{k-1}} \, \frac{\partial}{\partial z_i} \Big] \Big) < \; \phi_n(z) \, \phi_1(z_1) \, ... \, \phi_N(z_N) \; > \end{aligned} \tag{1.7}$$

2 Virasoro blocks using Ward identities

Given the conformal Ward identity here we prove equation (15.2.7) from [2],

$$c_{mn}^{i\{k,\bar{k}\}} = \sum_{\{k',\bar{k}'\}} \mathcal{M}^{-1}_{\{k\},\{k'\}} \mathcal{M}^{-1}_{\{\bar{k}\},\{\bar{k}'\}} \mathcal{L}_{-\{k\}} \bar{\mathcal{L}}_{-\{\bar{k}\}} \langle \mathcal{O}_m(\infty,\infty) \mathcal{O}_n(1,1) \mathcal{O}_i(z_1,\bar{z}_1) \rangle_{S^2} \Big|_{z_1=0} . \tag{2.1}$$

From the definition of OPE we have,

$$\mathcal{O}_m(z,\bar{z}) \mathcal{O}_n(0,0) = \sum_{i,\{k,\bar{k}\}} z^{-h_m - h_n + h_i + N} \bar{z}^{-\bar{h}_m - \bar{h}_n + \bar{h}_i + \bar{N}} c_{mn}^{i\{k,\bar{k}\}} L_{-\{k\}} \bar{L}_{-\{\bar{k}\}} \cdot \mathcal{O}_i(0,0). \tag{2.2}$$

In the above equation, $L_{-\{k\}} = \prod_i L_{-k_i}$ such that, $\sum_i k_i = N$. We also have the Ward identity of descendant with primaries :

$$\langle [L_{-k_1} \dots L_{-k_\ell} \bar{L}_{-l_1} \dots \bar{L}_{-l_m} \cdot \mathcal{O}_1(z_1)] \dots \mathcal{O}_n(z_n) \rangle_{S^2} = \mathcal{L}_{-k_\ell} \dots \mathcal{L}_{-k_1} \bar{\mathcal{L}}_{-l_m} \bar{\mathcal{L}}_{-l_1} \langle \mathcal{O}_1(z_1) \dots \mathcal{O}_n(z_n) \rangle_{S^2} . \tag{2.3}$$

In the above (see eq(1.7)),

$$\mathcal{L}_{-k} = \sum_{i=2}^{n} \left[\frac{h_i(k-1)}{(z_i - z)^k} - \frac{1}{(z_i - z_1)^{k-1}} \frac{\partial}{\partial z_i} \right] .$$

The equation (2.3) can be derived by using the OPEs between stress tensors and primaries. Using the state-operator correspondence, note that we can write eq(2.2) as,

$$\mathcal{O}_m(1,1) | \mathcal{O}_n \rangle = \sum_{i,\{k,\bar{k}\}} c_{mn}^{i\{k,\bar{k}\}} L_{-\{k\}} \bar{L}_{-\{\bar{k}\}} | \mathcal{O}_i \rangle . \tag{2.4}$$

Note $z, \bar{z}$ from eq(2.2) disappears since they are taken to 1. The conjugate of the above equation is,

$$\langle \mathcal{O}_n | \mathcal{O}_m(1,1) = \sum_{i,\{k,\bar{k}\}} c_{mn}^{i\{k,\bar{k}\}*} \langle [L_{-\{k\}} \bar{L}_{-\{\bar{k}\}}] \cdot \mathcal{O}_i | . \tag{2.5}$$

Next, note

$$\Big\langle [L_{-\{k\}} \bar{L}_{-\{\bar{k}\}}] \cdot \mathcal{O}_i \Big| [L_{-\{k'\}} \bar{L}_{-\{\bar{k}'\}}] \cdot \mathcal{O}_j \Big\rangle = \delta_{ij} \mathcal{M}_{\{k\},\{k'\}} \mathcal{M}_{\{\bar{k}\},\{\bar{k}'\}} . \tag{2.6}$$

Here, the matrices $\mathcal{M}_{k,k'}$ are the inner product matrices. See eq(15.1.6)-eq(15.1.11) of [2]. The inverse is by definition,

$$\sum_\beta \mathcal{M}_{\gamma\beta} \left(\mathcal{M}^{-1} \right)_{\beta\alpha} = \delta_{\gamma\alpha} .$$

Therefore we have,

$$\begin{aligned}
&\sum_{\{k',\bar{k}'\}} \left\langle [L_{-\{k\}}\bar{L}_{-\{\bar{k}\}}]\cdot\mathcal{O}_i \middle| [L_{-\{k'\}}\bar{L}_{-\{\bar{k}'\}}]\cdot\mathcal{O}_j \right\rangle \mathcal{M}^{-1}_{\{k'\},\{l\}}\mathcal{M}^{-1}_{\{\bar{k}'\},\{\bar{l}\}} \\
&\qquad = \delta_{ij} \sum_{\{k',\bar{k}'\}} \mathcal{M}_{\{k\},\{k'\}}\mathcal{M}^{-1}_{\{k'\},\{l\}}\mathcal{M}_{\{\bar{k}\},\{\bar{k}'\}}\mathcal{M}^{-1}_{\{\bar{k}'\},\{\bar{l}\}} \\
&\qquad = \delta_{ij}\delta_{\{k\}\{l\}}\delta_{\{\bar{k}\}\{\bar{l}\}}.
\end{aligned} \tag{2.7}$$

Putting this to use firstly,

$$\begin{aligned}
\langle\mathcal{O}_n|\mathcal{O}_m(1,1)|[L_{-\{k'\}}\bar{L}_{-\{\bar{k}'\}}]\cdot\mathcal{O}_j\rangle &= \sum_{i,\{k,\bar{k}\}} c_{mn}^{i\{k,\bar{k}\}*}\langle[L_{-\{k\}}\bar{L}_{-\{\bar{k}\}}]\cdot\mathcal{O}_i|[L_{-\{k'\}}\bar{L}_{-\{\bar{k}'\}}]\cdot\mathcal{O}_j\rangle, \\
&= \sum_{i,\{k,\bar{k}\}} c_{mn}^{i\{k,\bar{k}\}*}\delta_{ij}\mathcal{M}_{\{k\},\{k'\}}\mathcal{M}_{\{\bar{k}\},\{\bar{k}'\}}, \\
&= \sum_{\{k,\bar{k}\}} c_{mn}^{j\{k,\bar{k}\}*}\mathcal{M}_{\{k\},\{k'\}}\mathcal{M}_{\{\bar{k}\},\{\bar{k}'\}}
\end{aligned} \tag{2.8}$$

Hence, multiplying by Graam matrix inverses and contracting properly (note since matrix elements are written down order is unimportant),

$$\begin{aligned}
&\sum_{\{k',\bar{k}'\}} \langle\mathcal{O}_n|\mathcal{O}_m(1,1)|[L_{-\{k'\}}\bar{L}_{-\{\bar{k}'\}}]\cdot\mathcal{O}_j\rangle\mathcal{M}^{-1}_{\{k'\},\{l\}}\mathcal{M}^{-1}_{\{\bar{k}'\},\{\bar{l}\}} \\
&\qquad = \sum_{\{k,\bar{k}\}}\sum_{\{k',\bar{k}'\}} c_{mn}^{j\{k,\bar{k}\}*}\mathcal{M}_{\{k\},\{k'\}}\mathcal{M}^{-1}_{\{k'\},\{l\}}\mathcal{M}_{\{\bar{k}\},\{\bar{k}'\}}\mathcal{M}^{-1}_{\{\bar{k}'\},\{\bar{l}\}} \\
&\qquad = c_{mn}^{j\{l,\bar{l}\}*}
\end{aligned} \tag{2.9}$$

Next using the fact that $\mathcal{M}_{ij} = \mathcal{M}_{ji}$ hence also true for its inverse, and that $c^{j\{l,\bar{l}\}}$ is real, upon using the eq(2.3) and relabelling indices, we obtain equation(2.1).

References

[1] A. Belavin, A. M. Polyakov, and A. Zamolodchikov, "Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory," *Nucl. Phys. B* **241** (1984) 333–380.

[2] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12, 2007.