Dissipative Non-Linear Sigma Models

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We present a study of dissipative $O(N)$ Non-Linear Sigma models in 0+1 dimensions, starting with classical motivations[1][1] and moving forward to calculations of the quantum mechanical propagators in the $O(2)$ and $O(3)$ NLSMs. Finally we employ large N techniques to study the finite temperature mass gap in the dissipative $O(N)$ model.

I. THE CALDEIRA-LEGGETT MODEL: POLYNOMIAL SPECTRAL FUNCTIONS

It is challenging to unitarily incorporate dissipation in quantum mechanics. The Caldeira-Legget Model is a prime QM Model that does the same in a very natural manner[1][1]. The physical system is coupled to a bath linearly which induces friction on the system. This model is ubiquitous, and has made its appearance in multiple areas of physics, particularly in the context of open quantum systems and open string theory[2][2] to name a few. More recently, the analysis of graviton effects in LIGO[4][4] can also be viewed as a generalisation of the CL model. Thus naturally the CL model warrants further studies. Furthermore this study in interesting and nontrivial in strongly interacting systems. In the limit of large degrees of freedom, one may even hope for analytical solutions.

The Caldeira-Leggett Model consists of 3 aspects, namely, a single oscillator, denoted Q, which is placed in a bath of oscillators, denoted q_i where $i = 1, 2, ...N$ for some large N, and the interaction between the two.The Lagrangian for this system can be written as -

$$L = \frac{1}{2}[\dot{Q}^2 - (\Omega^2 - \Delta\Omega^2)Q^2] - Q\sum_{i=1}^{N} f_i q_i + \frac{1}{2}\sum_{i=1}^{N}(\dot{q_i}^2 - \omega_i^2 q_i^2) \tag{1}$$

This Lagrangian describes the macroscopic oscillator $Q(t)$, linearly coupled to a "bath" of oscillators $q_i(t)$ that represent the environment, differing only by a weight f_i. The first term represents the lagrangian of the singular oscillator Q, which oscillates with the natural frequencyΩ, the third term represents the Lagrangian of the q_i oscillators when they are free to oscillate independently with natural frequency ω_i. The middle term describes the interaction between them and each individual interaction is weighted by f_i. Here, $\Delta\Omega^2 = -\sum_i \frac{f_i^2}{\omega_i^2}$. This term has been used to set the potential energy of the bath to zero. Solving the Euler-Lagrange equations for Q after integrating out the bath degrees of freedom q_i we obtain:

$$\ddot{Q} + (\Omega^2 - \Delta\Omega^2)Q + \int_{-\infty}^{t} F(t-\tau)Q(\tau)d\tau = 0 \tag{2}$$

Note that we have introduced the spectral function:

$$J(\omega) = \frac{\pi}{2}\sum_i \frac{f_i^2}{w_i}\delta(\omega - \omega_i) \tag{3}$$

and defined:

$$F(t) = -\frac{2}{\pi}\int_0^{\infty} J(\omega)\sin(\omega t)d\omega \tag{4}$$

This spectral function will enable us to specify the environment in which the Q oscillator evolves in time. We choose:

$$J(\omega) = \eta\Lambda^2\frac{\omega^n}{\omega^{n+1} + \Lambda^{n+1}} \tag{5}$$

with $n \in N$. Λ is used here to ensure that the friction diminishes while ω is large. This will be a model for friction when $\Lambda \gg \omega$, acting as a simple polynomial of degree n, and it goes to 0 for $\Lambda \ll \omega$. The spectral function above is a generalisation of the spectral function used in . Eq.(2) is exactly solvable for odd values of n and in the large Λ limit. The equation of motion is then:

$$\ddot{Q} - \frac{2\eta}{n+1}\sum_{k=0}^{\frac{n+1}{2}-1}\cos 2\theta\,\dot{Q} + \Omega^2 Q = 0 \tag{6}$$

with:

$$\theta = \frac{2k+1}{n+1}\pi \tag{7}$$

For $n = 1$, the equation becomes:

$$\ddot{Q} + \eta\dot{Q} + \Omega^2 Q = 0 \tag{8}$$

which just gives us the EOM of a damped harmonic oscillator. For higher odd values of n, the EOM becomes:

$$\ddot{Q} + \Omega^2 Q = 0 \tag{9}$$

We therefore see that for higher odd values of n, the Q oscillator completely decouples from the oscillator bath and acts as a free oscillator. Therefore, the main contribution to dissipation comes from the $n = 1$ spectral function. The even n case on evaluation gives divergences in intermediate steps, and is not exactly solvable like the odd n counterpart.

II. THE $O(2)$ NON-LINEAR SIGMA MODEL

The $O(2)$ NLSM describes 2 free particles($\Omega_i = 0$), Q_1 and Q_2 constrained to move on a circle(we choose the radius of the circle to be a). To talk about dissipation in such a model, we couple Q_1 and Q_2 to oscillator baths independently, and as a result of this, the net Lagrangian of the system becomes:

$$\mathcal{L}_{eff} = \mathcal{L}_1 + \mathcal{L}_2 + \lambda(Q_1^2 + Q_2^2 - a^2) \tag{10}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ refer to (1) with $Q \leftarrow Q_1$ and $Q \rightarrow Q_2$ respectively. We have introduced the constraint using Lagrange's method of undetermined multipliers (λ here being the Lagrange Multiplier). Taking our spectral function to be the $n = 1$ case of (5), and solving the Euler Lagrange equations in the large Λ limit, we obtain:

$$Q_1 = a\cos\left(Ae^{-\eta t} + B\right) \tag{11}$$

$$Q_2 = a\sin\left(Ae^{-\eta t} + B\right) \tag{12}$$

If we want to study the quantum mechanics of such a system, we consider the quantum mechanical propagator using the path integral approach. We approximate the path integral with a saddle point approximation about the classical path[3][3]. The constraint effectively enables us to remove Q_1 from (10), and the problem reduces effectively to a 1D problem. For the results below, we have assumed that the coupling to the bath is weak. Considering the boundary conditions:

$$Q_2(t_i) = Q_2^i \tag{13}$$

$$Q_2(t_f) = Q_2^f \tag{14}$$

The classical action turns out to be:

$$S_{cl} = \frac{a^2\left[\arcsin\left(\frac{Q_2^f}{a}\right) - \arcsin\left(\frac{Q_2^i}{a}\right)\right]^2}{2\left(t_f - t_i\right)}\left[1 - \eta(t_f + t_i)\right] \tag{15}$$

Now we can write down the expression for the corresponding quantum mechanical propagator:

$$\langle Q_2^f, t_f; Q_2^i, t_i\rangle = \mathcal{N}\exp\left[\frac{i}{\hbar}S_{cl}(Q_2^f, t_f; Q_2^i, t_i)\right] \tag{16}$$

where:

$$\mathcal{N} = \left[-\frac{1}{2\pi i\hbar}\frac{\partial^2 S_{cl}}{\partial Q_2^f \partial Q_2^i}\right]^{\frac{1}{2}} \tag{17}$$

III. THE $O(3)$ NON-LINEAR SIGMA MODEL

Carrying out a similar calculation as in the preceding section for a 3 particle system constrained to a 2-sphere with the effective Lagrangian being:

$$\mathcal{L}_{eff} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \lambda(Q_1^2 + Q_2^2 + Q_2^3 - a^2) \tag{18}$$

We obtain:

$$Q_1 = a\sin(Ae^{-\eta t} + B)\sin(Ct + D) \tag{19}$$

$$Q_2 = a\cos(Ae^{-\eta t} + B) \tag{20}$$

$$Q_3 = a\sin(Ae^{-\eta t} + B)\cos(Ct + D) \tag{21}$$

As we have done before, we can remove Q_3 from the Lagrangian using the sphere constraint. Under similar boundary conditions as the previous section, on calculating the classical action, we obtain:

$$S_{cl} = \frac{a^2\left[\arccos\left(\frac{Q_2^f}{a}\right) - \arccos\left(\frac{Q_2^i}{a}\right)\right]^2}{2(t_f - t_i)} + \frac{a^2\left[\arcsin\left(\frac{Q_1^f}{a\sqrt{a^2 - Q_2^{f\,2}}}\right) - \arcsin\left(\frac{Q_2^i}{a\sqrt{a^2 - Q_2^{i\,2}}}\right)\right]^2}{4(t_f - t_i)} - \frac{\left[\arcsin\left(\frac{Q_1^f}{a\sqrt{a^2 - Q_2^{f\,2}}}\right) - \arcsin\left(\frac{Q_2^i}{a\sqrt{a^2 - Q_2^{i\,2}}}\right)\right]^2 *}{4(t_f - t_i)} \left[\arccos\left(\frac{Q_2^f}{a}\right) - \arccos\left(\frac{Q_2^i}{a}\right)\right] * \left[Q_2^f\sqrt{a^2 - Q_2^{f\,2}} - Q_2^i\sqrt{a^2 - Q_2^{i\,2}}\right]$$

We can use the saddle point approximation[3][3] to evaluate the propagator from the above action.

IV. THE $O(N)$ NON-LINEAR SIGMA MODEL: ANALYSIS AS A LARGE N THEORY

Now we generalise our theory to n oscillators. A good amount of the dissipationless $O(N)$ NLSM(the quantum mechanical case) has been presented in [5][5], where we have made modifications by adding a term that corresponding to the dissipation. We now consider a N component vector field $\vec{n}(t)$, with each component coupled to its own oscillator bath. Since we are now working with N degrees of freedom, we will work with the density matrix to encode the information about the n components. Moving from the propagator to the density matrix simply requires us to make a Euclidean time transformation i.e: $t \rightarrow -i\tau$, where $\tau = \beta = \frac{1}{T}$(here we have set $\hbar$ to 1). When we make such a transformation, the boundary conditions of the field components become periodic i.e $\vec{n}(t) = \vec{n}(t+\beta)$. In particular, we will be interested in the partition function corresponding to the density matrix:

$$Z = Tr(\rho) = \int \mathcal{D}\vec{n}e^{-\int_0^\beta dt S_E} \tag{22}$$

Before we specify the explicit form of S_E, we need to introduce the $O(N)$ constraint. In particular, we know that the target manifold will be an n-sphere, and we will conveniently choose the radius of the same to be 1. Now, in such a case:

$$Z = \int \mathcal{D}\vec{n}e^{-\int_0^\beta dt S_E}\delta(\vec{n}^2 - 1) \tag{23}$$

The delta function enforces the constraint upon the system. We can rewrite:

$$\delta(\vec{n}^2 - 1) = \int_{-i\infty}^{i\infty}\mathcal{D}\lambda e^{-\lambda(\vec{n}^2 - 1)} \tag{24}$$

Here, we have taken a transformation from real λ to imaginary λ. Putting all of these together, we will define:

$$Z = \int_{-i\infty}^{i\infty} \mathcal{D}\lambda e^{-W[\lambda]} \tag{25}$$

where:

$$e^{-W[\lambda]} = \int \mathcal{D}\vec{n} e^{-\int_0^\beta dt[S_E + \lambda(\vec{n}^2-1)]} \tag{26}$$

The above quantity will be of interest to us when we calculate the mass gap, the equation for which being:

$$\frac{\partial W}{\partial \lambda} = 0 \tag{27}$$

We now construct the Euclidean Action(S_E). As a direct generalisation of (1), we can write:

$$S_E = \frac{1}{2\alpha_0}[(\partial_t \vec{n})^2 + \int_{-\infty}^{\infty} dt' \alpha(t-t')(\vec{n}(t)-\vec{n}(t'))^2] \tag{28}$$

where:

$$\alpha(t-t') = \frac{1}{2\pi}\int_0^\infty d\omega J(\omega) e^{-\omega|t-t'|} \tag{29}$$

The second term in the Hamiltonian comes on integrating the bath oscillators. This is analogous to the term used in [1][1]. To continue our calculations, we require to quadratize (28), as we desire the integral over n to be a simple gaussian integral. The first term can be quadratised as $\vec{n}(-\partial_t^2)\vec{n}$. On quadratising the second term we obtain:

$$(\vec{n}(t) - \vec{n}(t'))^2 = -4\vec{n}(t)\left(\sinh^2 \frac{(t-t')\partial_t}{2}\right)\vec{n}(t) \tag{30}$$

Making the shift $(t-t') \to x$, the second term becomes:

$$-4\vec{n}(t)\left(\int_{-\infty}^{\infty} dx \alpha(|x|) \sinh^2 \frac{x\partial_t}{2}\right)\vec{n}(t) \tag{31}$$

We can now evaluate the integral over n, which yields:

$$W[\lambda] = \frac{N}{2}\text{tr}\log\left(-\partial_t^2 + \lambda + 4\int_{-\infty}^{\infty} dx\alpha(|x|)\sinh^2\frac{x\partial_t}{2}\right) - \frac{1}{2\alpha_0}\int_0^\beta dt\lambda$$

When we take the large N limit, the fluctuations of λ become negligible with respect to t becomes negligible, and hence can be assumed to be constant. To evaluate the first term of $W[\lambda]$, we need to evaluate the trace, i.e:

$$\int_0^\beta dt \left(-\partial_t^2 + \lambda + 4\int_{-\infty}^{\infty} dx\alpha(|x|)\sinh^2\frac{x\partial_t}{2}\right)\delta(t-t')\Bigg|_{t-t'} \tag{32}$$

Now we write $\delta(t-t')$ in the following basis:

$$\delta(t-t') = \frac{1}{\beta}\sum_{n=-\infty}^{\infty} e^{-i\frac{2\pi n}{\beta}(t=t')} \tag{33}$$

Plugging this into (32), using the n=1 case of (5) and using the large Λ limit we obtain:

$$\frac{2}{N}W[m] = \sum_{n=-\infty}^{\infty}\log\left[\frac{4\pi^2n^2}{\beta^2} + m^2 - \frac{2\pi\eta}{\beta}n\right] - \frac{m^2\beta}{g} \tag{34}$$

Here we have defined $g = N\alpha_0$, and $\lambda = m^2$. Here, λ has the units of $[M]^2$, and can be thought of as the mass gap. Feeding (34) into (27) and evaluating the sum, we obtain the mass gap equation:

$$\frac{2}{g} = \frac{1}{\sqrt{\eta^2-4m^2}}\cot\frac{\beta}{4}\left(\eta - \sqrt{\eta^2-4m^2}\right) - \frac{1}{\sqrt{\eta^2-4m^2}}\cot\frac{\beta}{4}\left(\eta + \sqrt{\eta^2-4m^2}\right) \tag{35}$$

If we put $\eta = 0$ into the above equation we obtain:

$$m = \frac{g}{2}\coth\frac{\beta m}{2} \tag{36}$$

which matches with the mass gap equation from [5][5].

IV.A. Plots and Qualitative Analysis

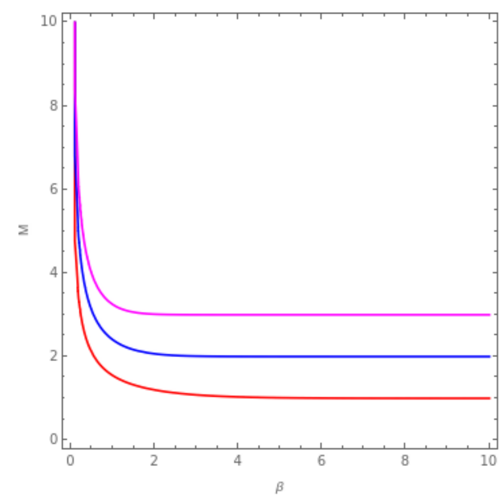

FIG. 1 Plot of eqn(36); mass gap(M) vs β. (Red, Blue, Magenta) correspond to g=(1,2,3) respectively.

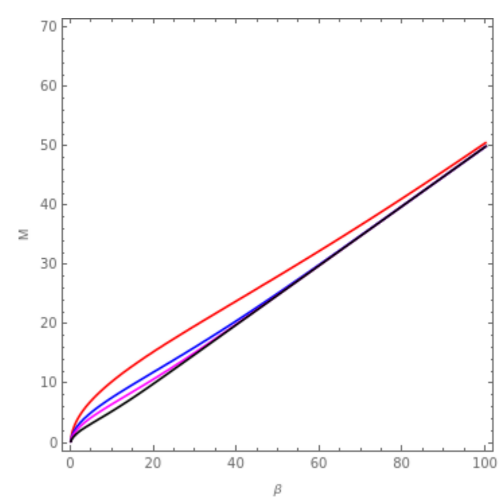

FIG. 2 Plot of eqn(36); mass gap(M) vs g. (Red, Blue, Magenta, Black) correspond to β=(0.1,0.2,0.3,0.6) respectively.

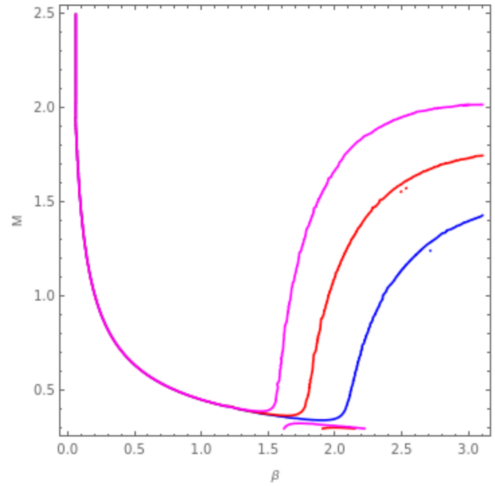

FIG. 3 Plot of eqn(35); mass gap(M) vs β corresponding to fixed $g = 0.2$ and (Blue, Red, Magenta) correspond to η=(3,3.5,4) respectively.

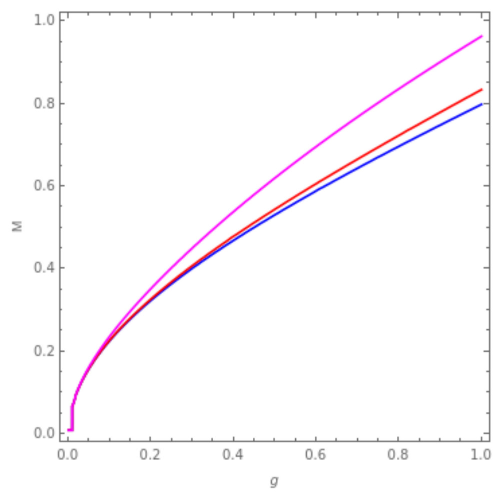

FIG. 4 Plot of eqn(35); mass gap(M) vs g corresponding to fixed $\beta = 2$ and $\eta = 6$ and (Blue, Red, Magenta) correspond to η=(2,2.5,3) respectively.

FIG. 1 and FIG. 2 are plots of dissipationless mass gap equation, and serve as a comparison to the case with dissipation.
FIG. 3 shows the dependence of the mass gap in eqn(35) on β for fixed $g = 0.2$. For high temperatures($\beta \to 0$), the behaviour is similar to that of the dissipationlesss case(FIG. 1), but as the temperature is decreased, there is a steep rise in the mass gap of the system, which is indicated in the plot in the region between $\beta = 1.5$ and $\beta = 2.2$. This is indicative of crossover behaviour in the system. Qualitatively it can be seen that the deviation from the dissipationless behaviour is dependent on the dissipation constant η. As η increases, the β at which the steep rise occurs decreases. Another point to note from the graph is that as the temperature decreases(β increases), the mass saturates. This saturation value is dependent on the dissipation constant η. It can be seen from the graph that for higher values of η, the saturation value of M also increases. The physical interpretation for this is that at low temperatures and and high dissipation, the system resists dynamics, which is what we would expect. From this it also becomes apparent that as $\eta \to 0$, the mass gap will show no steep rise, and will show behaviour similar to the dissipationless case.

FIG. 4 shows the dependence of the mass gap in eq(35) on g, keeping β fixed. Firstly, we see that as we increase g, the mass gap increases. This behaviour is common to both the dissipation and dissipationless cases. We interpret g as the 'coupling' constant, and it defines the extent to which the N oscillators are coupled to each other. The rate at which this graph increases depends on the dissipation constant η, which is apparent from the graph. As η increases, the rate at which the mass gap increases also increases. Thus for scenarios where dissipation is high, and coupling between the oscillators becomes high, for a fixed temperature, the mass gap will also be very high.

V. FUTURE WORK

So far in our analysis of the dissipative $O(N)$ NLSM , we have assumed that our vector field $\vec{n}(t)$ has only time degree of freedom in $\vec{n}(t)$. We are currently carrying out calculations by introducing space degrees of freedom. In this case we would need to rederive eqn(28) by appropriately generalising the lagrangian in eqn(1). This work will be a generalisation of the calculation carried out in [6][6], in which the calculation has been carried out for the dissipationless case. Once we build up the complete theoretical framework of the model, we intend to look for applications of the same, possibly in theories where the system has the ability to exchange energy with it's surroundings. Our analysis has been done for the $O(N)$ group, and our calculations might be useful in theories with spherical symmetries.

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