PHY-422A : Midsemester Exam

Comments: 5 questions, 40 marks. Please put a box around your final answers.

2020/02/22

1. Find the *irrep* decomposition from the product representation : (6,6) of $SU(4)$. That is from :

$$\begin{smallmatrix}\square\\\square\end{smallmatrix} \otimes \begin{smallmatrix}\square\\\square\end{smallmatrix} = ?$$

   (a) List all the legal Young tableaus resulting in the decomposition. **[5 pts]**

   (b) Use hook's law to calculate the resulting dimensionalities of the *irreps*. **[5 pts]**

2. $SU(2)$:

   (a) Draw the weight diagram corresponding to the product representation $j \otimes j' = \frac{5}{2} \otimes \frac{3}{2}$. Thereby, using weight diagrams, show that you get the expected decomposition. **[4 pts]**

   (b) Identify the highest weight states $|4,4\rangle$ and $|3,3\rangle$ in terms of the states $|j = \frac{5}{2}, m; j' = \frac{3}{2}, m'\rangle$. **[2 pts]**

   (c) Express $|j = 3, m = 2\rangle$ in terms of the states $|j = \frac{5}{2}, m; j' = \frac{3}{2}, m'\rangle$. **[2 pts]**

3. **S$_4$:**

   (a) List the conjugacy classes of S_4. Hence find the number of irreps. **[2 pts]**

   (b) Use Young tableaux and hook's law to determine the dimension of the irreps. Verify that $N(G) = \sum_r d_r^2$. **[3 pts]**

   (c) *Character Table :* Fill in the missing entries below. χ_1 is the trivial irrep. The left column has the information on n_c. **[3 pts]**

$(*)_{n_c}$	χ_1	χ_2	χ_3	χ_4	χ_5
$(1)_1$					
$(12)_6$		-1	1	-1	0
$(12)(34)_3$		1	-1		
$(123)_8$			0		-1
$(1234)_6$		-1		1	0

4. $SO(3)$ **from** $SU(2)$

   (a) Verify that

   $$u = \begin{pmatrix} \frac{i}{2} & \frac{i}{\sqrt{2}} + \frac{1}{2} \\ \frac{i}{\sqrt{2}} - \frac{1}{2} & -\frac{i}{2} \end{pmatrix}$$

   is an element of $SU(2)$. **[2 pts]**

   (b) Find the rotation matrix $R(\phi)$ corresponding to u, using the relationship between $SU(2)$ group element and $SO(3)$ group element. **[4 pts]**

5. Consider the following mapping of a real number x to another real number x',

   $$x \to x' = \frac{ax+b}{cx+d}, \quad \text{where,} \quad a,b,c,d \in \mathbb{R} \text{ and,} \quad ad - bc \neq 0.$$

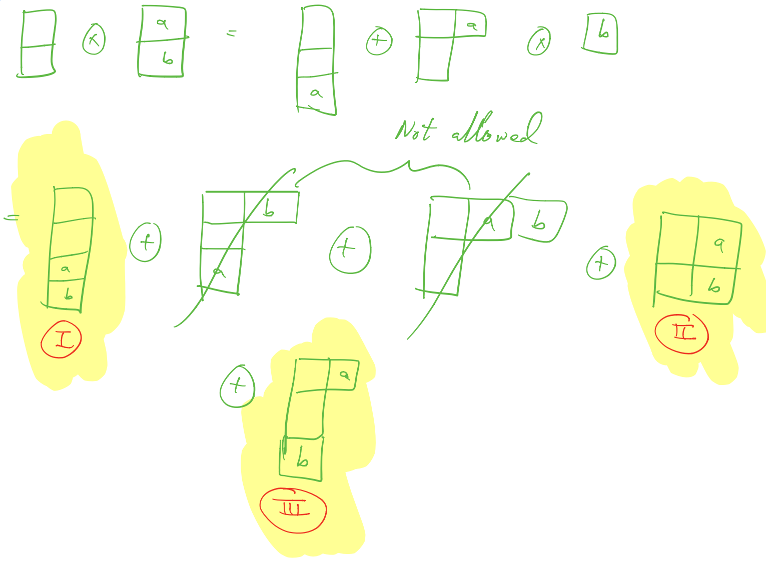
   (a) Show that these transformations form a group. **[4 pts]**

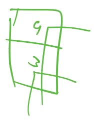
   (b) Show that the cross ratio of four points,

   $$\frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_3 - x_2)}$$

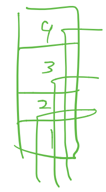
   is invariant under the transformation. [**4 pts**]

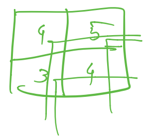
1(a) (5 pts)

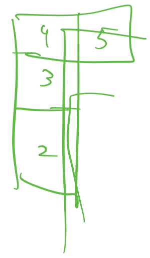


 $= \frac{12}{2} = 6$

1(b) (5 pts)

I  : $\frac{4\times3\times2\times1}{4\times3\times2} = 1$

II  : $\frac{4\times5\times\cancel{4}\times\cancel{3}}{\cancel{3}\times\cancel{2}\times\cancel{2}} = 20$

III  : $\frac{\cancel{4}\times5\times3\times\cancel{2}}{\cancel{4}\times\cancel{2}} = 15$

$6\times6 = 36 = 1 + 20 + 15$.

2(a) (4 pts)

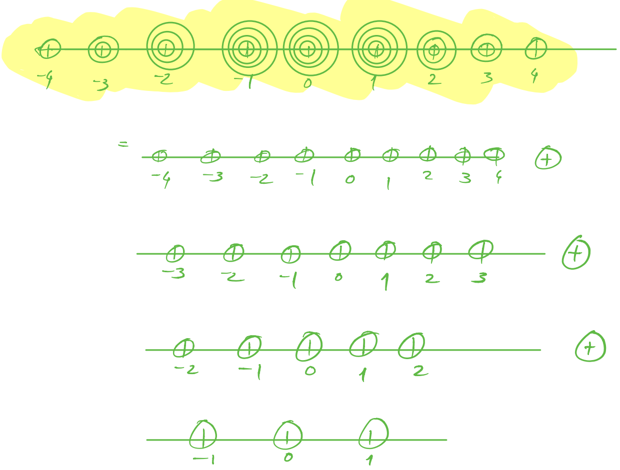
$j \otimes j = \frac{5}{2} \otimes \frac{3}{2}$

$\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}$

$\oplus$

$\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$

$4, 3, 2, 1, 3, 2, 1, 0, 2, 1, 0, -1,$
$1, 0, -1, -2, 0, -1, -2, -3, -1, -2,$
$-3, -4$



$= 4 \oplus 3 \oplus 2 \oplus 1$

$24 = 9 + 7 + 5 + 3$

2(b) (2 pts)

• $|4,4\rangle = \left|\frac{5}{2},\frac{5}{2};\frac{3}{2},\frac{3}{2}\right\rangle$

• $|3,3\rangle = \alpha\left|\frac{5}{2},\frac{5}{2};\frac{3}{2},\frac{1}{2}\right\rangle + \beta\left|\frac{5}{2},\frac{3}{2};\frac{3}{2},\frac{3}{2}\right\rangle$

We know $J_+|3,3\rangle = 0$ and $J_+\left(\alpha\left|\frac{5}{2},\frac{5}{2};\frac{3}{2},\frac{1}{2}\right\rangle + \beta\left|\frac{5}{2},\frac{3}{2};\frac{3}{2},\frac{3}{2}\right\rangle\right)$

$= \alpha\sqrt{3}\left|\frac{5}{2},\frac{5}{2};\frac{3}{2},\frac{3}{2}\right\rangle + \beta\sqrt{5}\left|\frac{5}{2},\frac{5}{2};\frac{3}{2},\frac{3}{2}\right\rangle$

$= (\alpha\sqrt{3} + \beta\sqrt{5})\left|\frac{5}{2},\frac{5}{2};\frac{3}{2},\frac{3}{2}\right\rangle$ thus we can chose $\alpha = 1$; $\beta = -\frac{\sqrt{3}}{\sqrt{5}}$

So; $|3,3\rangle = \left|\frac{5}{2},\frac{5}{2};\frac{3}{2},\frac{1}{2}\right\rangle - \sqrt{\frac{3}{5}}\left|\frac{5}{2},\frac{3}{2};\frac{3}{2},\frac{3}{2}\right\rangle$

Normalizing: $|3,3\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{5}\left|\frac{5}{2},\frac{5}{2};\frac{3}{2},\frac{1}{2}\right\rangle - \sqrt{3}\left|\frac{5}{2},\frac{3}{2};\frac{3}{2},\frac{3}{2}\right\rangle\right)$

2(c) (2 pts)

$J_-|3,3\rangle = \sqrt{6}|3,2\rangle = \frac{1}{2\sqrt{2}}\left(5\left|\frac{5}{2},\frac{3}{2};\frac{3}{2},\frac{1}{2}\right\rangle + 2\sqrt{5}\left|\frac{5}{2},\frac{5}{2};\frac{3}{2},-\frac{1}{2}\right\rangle - 2\sqrt{6}\left|\frac{5}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2}\right\rangle - 3\left|\frac{5}{2},\frac{3}{2};\frac{3}{2},\frac{1}{2}\right\rangle\right)$

thus; $|3,2\rangle = \frac{1}{2\sqrt{2}}\left(\frac{5}{\sqrt{6}}\left|\frac{5}{2},\frac{3}{2};\frac{3}{2},\frac{1}{2}\right\rangle + 2\sqrt{\frac{5}{6}}\left|\frac{5}{2},\frac{5}{2};\frac{3}{2},-\frac{1}{2}\right\rangle - 2\left|\frac{5}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2}\right\rangle - \frac{3}{\sqrt{6}}\left|\frac{5}{2},\frac{3}{2};\frac{3}{2},\frac{1}{2}\right\rangle\right)$

Normalizing:
$|3,2\rangle = \frac{1}{\sqrt{13}}\left(\frac{5}{\sqrt{6}}\left|\frac{5}{2},\frac{3}{2};\frac{3}{2},\frac{1}{2}\right\rangle + 2\sqrt{\frac{5}{6}}\left|\frac{5}{2},\frac{5}{2};\frac{3}{2},-\frac{1}{2}\right\rangle - 2\left|\frac{5}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2}\right\rangle - \frac{3}{\sqrt{6}}\left|\frac{5}{2},\frac{3}{2};\frac{3}{2},\frac{1}{2}\right\rangle\right)$

3(a) (2 pts)

Partitions of 4:

(*)(*)(*)(*) $1+1+1+1$

(**)(*)(*) $2+1+1$

(**)(**) $2+2$

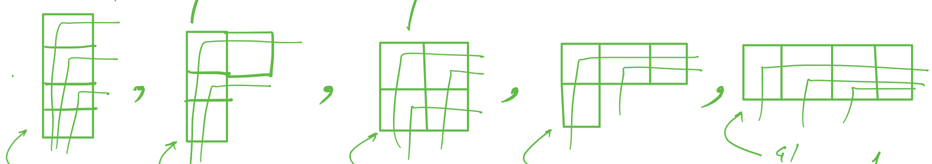
(***)(*) $3+1$

(****) 4

$\therefore$ There are 5 irreps and hence there are 5 conjugacy classes.

3(b) (3 pts)

The S_4 irreps can be represented as:



$\frac{4!}{4\times3\times2}=1$, $\frac{4!}{4\times2}=3$, $\frac{4!}{3\times2\times2}=2$, $\frac{4!}{4\times2}=3$, $\frac{4!}{4\times3\times2}=1$.

$$N(G) = 4! = 24 = 1^2+3^2+2^2+3^2+1^2 = \sum_r d_r^2$$

3(c) (3 pts)

$(*)_{n_c}$	χ_1	χ_2	χ_3	χ_4	χ_5
$(1)_1$	1	1	3	3	2
$(12)_6$	1	-1	1	-1	0
$(12)(34)_3$	1	1	-1	-1	2
$(123)_8$	1	1	0	0	-1
$(1234)_6$	1	-1	-1	1	0

4(a) (2 pts)

$$\mathcal{U} = \begin{pmatrix} \frac{i}{2} & \frac{i}{\sqrt2}+\frac12 \\ \frac{i}{\sqrt2}-\frac12 & -\frac{i}{2} \end{pmatrix}$$

$$\mathcal{U}^\dagger = \begin{pmatrix} -\frac{i}{2} & -\frac{i}{\sqrt2}-\frac12 \\ -\frac{i}{\sqrt2}+\frac12 & \frac{i}{2} \end{pmatrix}$$

$$\mathcal{U}\mathcal{U}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}_{2\times2}$$

$\det \mathcal{U} = 1$

$\therefore \mathcal{U} \in SU(2)$.

4(b) (4 pts)

$$\mathcal{U} = \sigma_3 \frac{i}{2} + \sigma_1 \frac{i}{\sqrt2} + \sigma_2 \frac{i}{2} \Rightarrow \mathcal{U}^\dagger = -\frac{i}{2}\sigma_3 - \sigma_1\frac{i}{\sqrt2} - \sigma_2 \frac{i}{2}$$

$$R_{ji} = \frac12 \mathrm{Tr}\left(\sigma_j \mathcal{U} \sigma_i \mathcal{U}^\dagger\right)$$

• $\mathcal{U}\sigma_1 = \left(\frac{i}{2}\sigma_3 + \frac{i}{\sqrt2}\sigma_1 + \frac{i}{2}\sigma_2\right)\sigma_1 = -\sigma_1\left(\frac{i}{2}\sigma_3 + \frac{i}{\sqrt2}\sigma_1 + \frac{i}{2}\sigma_2\right) + 2\sigma_1\frac{i}{\sqrt2}\sigma_1 = -\sigma_1\mathcal{U} + i\sqrt2$

• $\mathcal{U}\sigma_2 = -\sigma_2\mathcal{U} + i$

• $\mathcal{U}\sigma_3 = -\sigma_3\mathcal{U} + i$

Now: • $R_{11} = \frac12 \mathrm{Tr}\left(\sigma_1\mathcal{U}\sigma_1\mathcal{U}^\dagger\right) = \frac12\mathrm{Tr}\left[\sigma_1\left(-\sigma_1\mathcal{U} + i\sqrt2\right)\mathcal{U}^\dagger\right]$
$= \frac12 \mathrm{Tr}\left[-\mathbb{1}\right] + \frac{i}{\sqrt2}\mathrm{Tr}\left(\sigma_1\mathcal{U}^\dagger\right) = \frac12\mathrm{Tr}(-\mathbb{1}) + \frac{i}{\sqrt2}\left(\frac{-i}{\sqrt2}\right)\mathrm{Tr}(\mathbb{1})$
$= \frac12\mathrm{Tr}(-\mathbb{1}) + \frac12\mathrm{Tr}(\mathbb{1}) = 0.$

Thus also useful: • $\mathrm{Tr}\left(\sigma_2\mathcal{U}^\dagger\right) = -i$ and • $\mathrm{Tr}\left(\sigma_3\mathcal{U}^\dagger\right) = -i$
• $\mathrm{Tr}\left(\sigma_1\mathcal{U}^\dagger\right) = -\sqrt2\, i$

• $R_{12} = \frac12\mathrm{Tr}\left(\sigma_1\mathcal{U}\sigma_2\mathcal{U}^\dagger\right) = \frac12\mathrm{Tr}\left(\sigma_1\left(-\sigma_2\mathcal{U}+i\right)\mathcal{U}^\dagger\right) = \frac{i}{2}\mathrm{Tr}\left(\sigma_1\mathcal{U}^\dagger\right) = \frac{1}{\sqrt2}.$

• $R_{13} = \frac12\mathrm{Tr}\left(\sigma_1\mathcal{U}\sigma_3\mathcal{U}^\dagger\right) = \frac12\mathrm{Tr}\left(\sigma_1\left(-\sigma_3\mathcal{U}+i\right)\mathcal{U}^\dagger\right) = \frac{i}{2}\mathrm{Tr}\left(\sigma_1\mathcal{U}^\dagger\right) = \frac{1}{\sqrt2}.$

• $R_{21} = \frac12\mathrm{Tr}\left(\sigma_2\mathcal{U}\sigma_1\mathcal{U}^\dagger\right) = \frac12\mathrm{Tr}\left(\sigma_2\left(-\sigma_1\mathcal{U}+i\sqrt2\right)\mathcal{U}^\dagger\right) = \frac{i}{\sqrt2}\mathrm{Tr}\left(\sigma_2\mathcal{U}^\dagger\right) = \frac{1}{\sqrt2}.$

• $R_{22} = \frac12\mathrm{Tr}\left(\sigma_2\mathcal{U}\sigma_2\mathcal{U}^\dagger\right) = \frac12\mathrm{Tr}\left(\sigma_2\left(-\sigma_2\mathcal{U}+i\right)\mathcal{U}^\dagger\right) = \frac12\mathrm{Tr}(-\mathbb{1}) + \frac{i}{2}\mathrm{Tr}\left(\sigma_2\mathcal{U}^\dagger\right) = -1 + \frac12 = -\frac12$

• $R_{23} = \frac12\mathrm{Tr}\left(\sigma_2\mathcal{U}\sigma_3\mathcal{U}^\dagger\right) = \frac12\mathrm{Tr}\left(\sigma_2\left(-\sigma_3\mathcal{U}+i\right)\mathcal{U}^\dagger\right) = \frac{i}{2}\mathrm{Tr}\left(\sigma_2\mathcal{U}^\dagger\right) = \frac12$

• $R_{31} = \frac12\mathrm{Tr}\left(\sigma_3\mathcal{U}\sigma_1\mathcal{U}^\dagger\right) = \frac12\mathrm{Tr}\left(\sigma_3\left(-\sigma_1\mathcal{U}+i\sqrt2\right)\mathcal{U}^\dagger\right) = \frac{i}{\sqrt2}\mathrm{Tr}\left(\sigma_3\mathcal{U}^\dagger\right) = \frac{1}{\sqrt2}.$

• $R_{32} = \frac12\mathrm{Tr}\left(\sigma_3\mathcal{U}\sigma_2\mathcal{U}^\dagger\right) = \frac12\mathrm{Tr}\left(\sigma_3\left(-\sigma_2\mathcal{U}+i\right)\mathcal{U}^\dagger\right)$
$= \frac{i}{2}\mathrm{Tr}\left(\sigma_3\mathcal{U}^\dagger\right) = \frac12.$

• $R_{33} = \frac12\mathrm{Tr}\left(\sigma_3\mathcal{U}\sigma_3\mathcal{U}^\dagger\right) = \frac12\mathrm{Tr}\left(\sigma_3\left(-\sigma_3\mathcal{U}+i\right)\mathcal{U}^\dagger\right)$
$= -\frac12\mathrm{Tr}(\mathbb{1}) + \frac{i}{2}\mathrm{Tr}\left(\sigma_3\mathcal{U}^\dagger\right)$
$= -1 + \frac12 = -\frac12.$

$$R \equiv \begin{pmatrix} 0 & 1/\sqrt2 & 1/\sqrt2 \\ 1/\sqrt2 & -1/2 & 1/2 \\ 1/\sqrt2 & 1/2 & -1/2 \end{pmatrix}$$

$RR^T = \mathbb{1}_{3\times3}$ (check).

5(a) (4 pts)

- Show closure :—

Let g be the action
$x \longrightarrow x' = \frac{ax+b}{cx+d} = g\circ x$

Let g' be the action
$x \longrightarrow x'' = \frac{a'x+b'}{c'x+d'} \equiv g'\circ x$

The combined action:

$$g'\circ g\circ x = g'\circ \frac{ax+b}{cx+d} = \frac{a\frac{a'x+b'}{c'x+d'}+b}{c\frac{a'x+b'}{c'x+d'}+d}$$

$$= \frac{(aa'+bc')x+(ab'+bd')}{(ca'+c'd)x+(cb'+dd')} = \frac{a''x+b''}{c''x+d''} \quad —(1)$$

Thus the combined action is also of the same form, in particular if $\{a,b,c,d\}$ and $\{a',b',c',d'\} \in \mathbb{R}$ then from (1) $\{a'',b'',c'',d''\}$ also $\in \mathbb{R}$.
This establishes closure.

- The identity is $e\circ x = x$ this is achieved by $a=1,\ b=0,\ c=0,\ d=1$ —(2)
note $a=-1,\ b=0,\ c=0,\ d=-1$ also does the job

5(b) (4 pts)

The cross-ratio is $\frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_4)(x_3-x_2)} = z$

This goes to:

$$z \to g\circ z = \frac{\left(\frac{ax_1+b}{cx_1+d} - \frac{ax_2+b}{cx_2+d}\right)\left(\frac{ax_3+b}{cx_3+d} - \frac{ax_4+b}{cx_4+d}\right)}{\left(\frac{ax_1+b}{cx_1+d} - \frac{ax_4+b}{cx_4+d}\right)\left(\frac{ax_3+b}{cx_3+d} - \frac{ax_2+b}{cx_2+d}\right)}$$

$$= \frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_4)(x_3-x_2)} = z \text{ ; thus invariant!}$$

- The inverse should satisfy:

$$g\circ g^{-1}\circ x = e\circ x = x$$

thus given $\{a,b,c,d\}$ of g; we need to determine $\{a',b',c',d'\}$ of g'; such that their composition is identity.

From (1) we know the general composition rule; therefore, we demand (using (2))

$$\left.\begin{aligned} aa'+bc' &= 1\\ ab'+bd' &= 0\\ ca'+c'd &= 0\\ cb'+dd' &= 1\end{aligned}\right\} (3)$$

Solving for the unknowns $\{a',b',c',d'\}$ in term of the knowns $\{a,b,c,d\}$ we obtain:

$$a' = \frac{\pm d}{ad-bc} \qquad c' = \frac{\mp c}{ad-bc}$$

$$b' = \frac{\mp b}{ad-bc} \qquad d' = \frac{\pm a}{ad-bc}$$

Thus for existence of inverse we also need
$ad-bc \neq 0.$