PHY 422A : Mathematical Methods II

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1 Group Theory Basics

Introduction

Symmetries play a very important role in dictating physics. They give strong constraints on the observables of our theory (starting from the point of deciding what are the good observables? *For instance in a gauge theory meaningful quantities are gauge invariant objects*. Symmetries can be analyzed nicely in the path integral formalism and give rise to *Ward identities* which are constraints respected by the observables. However sometimes symmetries go beyond the Lagrangian framework and can allow computation of observables. This is the case for instance in a conformal field theory.
Symmetries have a *group* structure. Once we understand symmetries via the theory of groups, we can compute the physics unavailable to us via perturbation. The group transformations can be represented as linear operations in vector spaces, and this leads naturally to finding *representations* of the group. Much of group theory, concerns classifying all possible representations..

Axioms

A group G, is a set with binary operation that assigns to every ordered pair, (g_1, g_2) of its elements, a third element g_3 usually written as $g_3 = g_1 g_2$. This binary operation of *product* follow the following rules,

1. Associativity : $g_1(g_2 g_3) = (g_1 g_2) g_3$.
2. Existence of identity, $e \in G$ s.t, $eg = g \ \forall g \in G$.
3. Existence of inverse, g^{-1} s.t., $g^{-1} g = e$.

There are many corollaries that follow from the above three axioms,

1. $gg^{-1} = e$, prove starting from $g^{-1}g = e$ and use associativity.
2. $ge = g$, use $eg = g$, the above, and associativity.
3. e is unique.
4. Inverse is unique.

Two elements are said to *commute* if, $g_1 g_2 = g_2 g_1$. If this is true $\forall g_i \in G$ then the group is *abelian*, else it is $non-abelian$ [1]. If G contains finite number of elements, *order* of G, $|G|$, then it is called a $finite$ group.

Some examples

1. The integers $\mathbb{Z}$ under addition. – Infinite What is the identity?

2. The set of functions,

$$\begin{aligned} f_1(z) &= z \qquad f_2(z) = \frac{1}{1-z} \qquad f_3(z) = \frac{z-1}{z}, \\ f_4(z) &= \frac{1}{z} \qquad f_5(z) = 1-z \qquad f_6(z) = \frac{z}{z-1}. \end{aligned}$$

   The product rule is,

$$(f_i, f_j) \mapsto f_i \circ f_j.$$

3. Integers modulo n under addition. – Finite.

4. Set of rotations in three dimensions, or set of 3-by-3 real matrices, O with, $OO^T = I$ and $\det O = 1$; this is $SO(3)$. This is an example of a *Lie* group.

5. Groups can be specified by list of *generators* and *relations*. *Cyclic* group of order n, C_n is given by generator a and relation, $a^n = e$.

6. The group $\mathbb{Z}_n$ is a finite group of order n. This is a *cyclic* group and the elements can be represented as roots of unity. e.g., $\mathbb{Z}_3 = \{1, \omega, \omega^2\}$.

7. Lorentz group generated by the transformation :

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}.$$

   Here ϕ is the boost angle. One can verify that the composition of two transformations adds up the boost angles.

[End of Lecture 1]

[1] There may be composition rules which are commutative but not associative. e.g., the arithmetic mean (example suggested by Radhika Prasad)

Some basic properties

1. *Subgroups* : A subset of a group (lets call it H) that forms a group. $\{e\}$ and G itself are *trivial* subgroups, others are *proper* subgroups. For a finite group, $|G|$ is divisible by $|H|$. This is $Lagrange's\ theorem$.

   To prove Langrange's theorem, one starts by listing out $A_1 = H = \{h_1, \dots h_m\}$. Then one picks $g_1 \in G$ and $g_1 \notin H$ and constructs the set, $A_2 = \{h_1g_1, h_2g_1, \dots, h_mg_1\}$. If this exhausts G then one stops, else one picks $g_2 \in G$ and $g_2 \notin H$ and $g_2 \notin A_2$ and constructs, $A_3 = \{h_1g_2, h_2g_2, \dots h_mg_2\}$. One repeats this until all elements of G are exhausted. Let the last list be, $A_k = \{h_1g_k, h_2g_k, \dots, h_mg_k\}$. Then one can show that all such lists are distinct. This can be proven by contradiction; let $h_ag_j = h_bg_l$, then $g_j = h_a^{-1}h_bg_l$. But this implies that $g_j \in A_l$. However, this is not the case by construction. Thus all are distinct. Now since k is an integer, we immediately have,

$$n = mk.\ (proved)$$

2. *Direct Product :* Given two groups F and G, we may arrive at a group, $F \otimes G$. The way to list elements of this is (f, g), where $f \in F$ and $g \in G$. The composition is via, $(f, g) \cdot (f', g') = (ff', gg')$. The identity element is (e_f, e_g).

   An example is $\mathbb{Z}_2 \otimes \mathbb{Z}_2 = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$. Note that this is different from the group, $\mathbb{Z}_4$.

3. *Multiplication Table :* An useful way to list down the group properties is by filling up the multiplication table :

	e	g_1	$\dots$	g_i	$\dots$	g_n
e	e	g_1	$\dots$	g_i	$\dots$	g_n
g_1	g_1	g_1^2	$\dots$	g_1g_i	$\dots$	g_1g_n
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
g_j	g_j	$\dots$	$\dots$	g_jg_i	$\dots$	g_jg_n
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
g_n	g_n	$\dots$	$\dots$	g_ng_i	$\dots$	g_n^2

The multiplication table follows the *once and only once* theorem. Every group element appears only once if we trace a particular row or a coloumn. This can be proven once again by contradiction. If we assume $g_j g_k = g_j g_l$, then this gives the contradiction $g_k = g_l$. This is a simple yet a very useful theorem. This rule immediately tells us,

$$\sum_{g} f(g) = \sum_{g'} f(gg') = \sum_{g'} f(g'g).$$

This is known as the *rearrangement lemma*.

4. *Homomorphism* A map $f : G \to G'$ which preserves the multiplicative structure of the group. i.e., acting on the group elements, g, g' we have, $f(gg') = f(g)f(g')$.

   A homomorphism becomes an isomorphism if the map is one-to-one and onto. [End of Lecture 2]

5. *Normal/Invariant subgroup*: H is a normal subgroup, if $g^{-1}Hg = H$. There is *scrambling* going on.

6. *Conjugacy :* Two group elements g_1, g_2 are said to be conjugate to each other if, there is an element $g \in G$ s.t., $g_2 = g^{-1}g_1 g$. We write this as, $g_1 \sim g_2$. It is an equivalence relation.

Example : In the rotation group SO(3), the conjugacy classes are the sets of rotations by same angle, but about different axis.

Permutation groups

The action of this group is to permute its elements. Since the once and only once rule need to be satisfied, one can show easily that every finite group of order n is some subgroup of the permutation group S_n.

The permutation group of n objets, S_n has order $n!$. If we order $G = \{e, g_1, g_2, \dots\}$ and then multiply from left by $g \in G$ then the ordered list, $\{g, gg_1, gg_2, \dots\}$ is a permutation of G. Thus any group G is a subgroup of the permutation group $S_{|G|}$. This is *Cayley's theorem*.

Permutation group is denoted using *cycles*. For example an element, π_1 in S_8 be denoted by,

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 & 5 & 4 & 7 & 6 & 8 \end{pmatrix} = (123)(45)(67)(8).$$

The rhs defines the cycle notation. Any permutation with the pattern, $(***)(**)(**)(*)$ is in the same conjugacy class as π_1.

Problem

If $\varphi : F \to G$ is a group homomorphism, and if we define $\mathrm{Ker}(\varphi)$ as the set of elements in F that map to e_G, then show that $\mathrm{Ker}(\varphi)$ is a normal subgroup of F.

Since it is a homomorphism, we also have, $\varphi(e_F) = e_G$. Now let the $\mathrm{Ker}(\varphi) = H = \{e_F, h_1, \dots, h_n\}$. Consider, $g \cdot h_i \cdot g^{-1}$, where $h_i \in H$ and $g \in F$, and lets act on this by the mapping, φ. We have:

$$\varphi(gh_ig^{-1}) = \varphi(g)\varphi(h_i)\varphi(g^{-1}) = \varphi(g)e_G\varphi(g^{-1}) = \varphi(gg^{-1}) = \varphi(e_F) = e_G.$$

By definition,

$$e_G = \varphi(h_l).$$

$$\therefore h_l = gh_ig^{-1} \ (proved).$$

[End of Lecture 3]

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7. *Cosets* :Given, $H \subseteq G$ with elements $H = \{h_1, h_2, \dots\}$, and $g \in G$, the left coset is $gH = \{gh_1, gh_2, \dots\}$. If two cosets intersect, then they coincide. A group can be written as,

$$G = g_1 H + g_2 H + \cdots$$

8. *Quotient group :* Given a normal subgroup H, take the set of cosets $G/H \equiv \{g_1 H, g_2 H, \dots\}$. Define product $(g_i H)(g_j H)$ by the coset where this lies, i.e., identify : $(g_i H)(g_j H) = g_k H$. As a result we now have a group. This is called the quotient group. The identity element is given by $e_G H$. The inverse of gH is $g^{-1}H$. The order of the quotient group is $|G|/|H|$.

Coxeter groups

This group is defined by the relation, $g \in G$, $g^2 = 1$ and also $(g_i g_j)^{n_{ij}} = 1$ for $n_{ij} \geq 2$. One can show that $n_{ij} = n_{ji}$. One starts with

$$\begin{aligned}
g_j^2 &= 1, \\
g_j (g_i g_j)^{n_{ij}} g_j &= 1, \\
g_j \underbrace{(g_i g_j)(g_i g_j) \dots (g_i g_j)}_{n_{ij} \text{ number of terms}} g_j &= 1, \\
\underbrace{(g_j g_i)(g_j g_i) \dots (g_j g_i)}_{n_{ij} \text{ number of terms}} g_j^2 &= 1, \\
(g_j g_i)^{n_{ij}} &= 1.
\end{aligned}$$

But by definition, $(g_j g_i)^{n_{ji}} = 1$. Therefore one has, $n_{ij} = n_{ji}$.

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9. *Simple group :* A group with no normal subgroups. Example : Cyclic group, 16 families of Lie groups, 26 sporadic groups including the *Monster* group.

2 Prelude to Lie groups

In contrast to finite groups, one can have infinite dimensional continuous groups. Here we focus on the rotation group where the elements denote rotations in space by angles. In 2 dimensions, the group is SO(2) and acts via,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.1}$$

The matrix is denoted by $R(\phi)$. In addition to length of a vector, rotation also leaves angles between two vectors invariant. Let us denote the vectors by, $\vec{u}, \vec{v}$. Then under the rotation : $\vec{u} \to \vec{u}' = R(\phi)\vec{u}$ and similarly $\vec{v} \to \vec{v}' = R(\phi)\vec{v}$. Since, $|\vec{u}|$, $|\vec{v}|$, as well as the angle between $\vec{u}$ and $\vec{v}$ stays invariant; this implies: $\vec{u} \cdot \vec{v} = \vec{u}' \cdot \vec{v}' = (R(\phi)\vec{u})^T (R(\phi)\vec{v}) = \vec{u}^T R(\phi)^T R(\phi)\vec{v}$. Thus the condition on the transformation is,

$$R(\phi)^T R(\phi) = 1. \tag{2.2}$$

This condition is not only satisfied by rotations, but also for instance by reflections. Real matrices satisfying this relation are called orthogonal matrices. We can check this explicitly for the matrices (2.1) by matrix multiplication :

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Next, using $\det AB = \det A \det B$, and $\det A = \det A^T$ we can conclude that,

$$\det R^T R = (\det R)^2 = \det I = 1.$$

Thus the determinant of the matrix is ± 1. Orthogonal matrices also contain reflections. For instance reflecting the x axis corresponds to the orthogonal matrix :

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

which has determinant -1. We can select out rotations from the set of orthogonal matrices by demanding determinant to be $+1$. Matrices with positive unity determinant are called *special*.

Rotating infinitesimally

A key insight of Lie was that the aspects of rotation group can be captured by focussing on infinitesimal rotations. Thus we write the small rotation through the small angle, θ as an expansion:

$$R(\theta) = I + A + \mathcal{O}(\theta^2).$$

Here A is order θ. The equation (2.2) implies that,

$$A = -A^T.$$

This means that A is antisymmetric. In two dimensions then the only possibility is,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Thus near identity rotations look like:

$$\begin{aligned} R(\theta) &= I + \theta J + \mathcal{O}(\theta^2) \\ &\simeq \begin{pmatrix} 1 & \theta \\ -\theta & 0 \end{pmatrix}. \end{aligned} \tag{2.3}$$

To get to the finite transformation, we may apply the above matrix operation many many times. Let the small angle $\theta = \frac{\phi}{N}$ where $N \to \infty$. Therefore we want to compute repeated action of $R(\theta)$ and see if after $N \to \infty$ number of times if we recover $R(\phi)$ from (2.1). We have :

$$\begin{aligned} \lim_{N\to\infty} \left(R(\phi/N)\right)^N &= \lim_{N\to\infty} \left(1 + \frac{\phi J}{N}\right)^N \\ &= e^{\phi J} = \sum_{n=0}^{\infty} \frac{\phi^n J^n}{n!} \\ &= \left(\sum_{m=0}^{\infty} (-1)^m \frac{\phi^{2m}}{(2m)!}\right) I + \left(\sum_{m=0}^{\infty} (-1)^m \frac{\phi^{2m+1}}{(2m+1)!}\right) J \\ &= \cos\phi I + \sin\phi J \\ &= \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = R(\phi). \end{aligned} \tag{2.4}$$

In the third equality above we used, $J^2 = -1$, therefore all even ($= 2m$) powers are just $(-1)^m$ while odd ($= 2m+1$) powers are $(-1)^m J$.

[End of Lecture 4]

SO(3)

For $SO(3)$, the rotations are in 3 dimensions. For 3×3, one may choose the following antisymmetric matrices,

$$J_1 = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_2 = -i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.5}$$

$$J_3 = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.6}$$

Thus analogous to $R = e^{\phi J}$ for $SO(2)$ in this case one can write a group element as,

$$R(\vec{\theta}) = e^{i \sum_{i=1}^{3} \theta_i J_i}. \tag{2.7}$$

Note that here we have chosen the generators to be Hermitian which is why there is the factor of i in the exponent. The algebra of these generators is :

$$[J_a, J_b] = -i\epsilon_{abc} J_c. \tag{2.8}$$

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3 Bits of Representation theory

The groups that we study here are conveniently represented using matrices. For every group element $g \in G$, we associate a matrix $D(g)$. In order to be a valid representation, this mapping of group element into matrices must be via a *homomorphism*, i.e.,

$$D(g_1)D(g_2) = D(g_1 g_2). \tag{3.9}$$

The identity element is the identity matrix. Inverse are simply matrix inverses. We already encountered some matrix representations with these property in (2.4) and (2.7). For finite groups, the multiplication table provides automatically a representation called the **regular** representation. In order to write down the matrices from the multiplication table in this representation, first order the group elements as g_i for $i = 1, 2, \ldots, N(G)$.[2] and represent every g_i as a column vector with all zeros except a 1 at the ith entry. Therefore the matrix, $D(g_k)$ in regular representation is just read from the character table, as $D(g_k)_{ij} = 1$ if the g_k-th row and g_i-th column in the multiplication table gives g_j else it is zero. Thus the regular representation is a $N(G)$ dimensional square matrix. The trivial representation is furnished by just the number 1, i.e., $D(g) = 1 \forall g \in G$.

A given group may have many representations. However only some of them are *irreducible*. An irreducible representation (written as *irrep* for short) is a representation which does not have any invariant subspace.

A reducible representation may be obtained by stacking up irreps ($D^i(g)$) in block diagonal form,

$$D(g) = \begin{pmatrix} D^1(g) & \ldots & \ldots \\ \ldots & D^2(g) & \ldots \\ \ldots & \ldots & D^3(g) \end{pmatrix}.$$

In this way, one can easily construct higher dimensional representations of a group. The above example shows the following irrep decomposition :

$$D(g) = \oplus_{i=1}^{3} D^i(g). \tag{3.10}$$

Character

Given a matrix representation, $D(g)$, the trace is an important quantity. In particular it depends on the representation used. The trace defines character of the representation :

$$\chi(g) = \text{Tr } D(g). \tag{3.11}$$

A part of the utility of the character is inherited from the cyclicity property of the trace. Suppose we transform our basis, then

$$D(g) \to D'(g) = SD(g)S^{-1}.$$

[2] It is customary to choose $g_1 = e$.

Under this similarity transformation, non-zero entries of the $D(g)$ matrix may become zero and vice-versa. These two representations should still be equivalent to each other, since we just changed the basis. The trace is preserved under similarity transformations, since

$$\text{Tr } D'(g) = \text{Tr } SD(g)S^{-1} = \text{Tr } S^{-1}SD(g) = \text{Tr } D(g) = \chi(g).$$

Thus given two non-equivalent representations the traces will turn out to be different. Another property of the character is that for the group elements belonging to the same conjugacy class, it is the same. This is because, if g and g' are in the same conjugacy class, then $\exists g'''$, such that, $g' = g'''gg'''^{-1}$. Thus,

$$\begin{aligned} \text{Tr } D(g') &= \text{Tr } D(g'''gg'''^{-1}) = \text{Tr } D(g''')D(g)D(g'''^{-1}) \\ &= \text{Tr } D(g''')D(g)D(g''')^{-1} = \text{Tr } D(g''')^{-1}D(g''')D(g) = \text{Tr } D(g) = \chi(g). \end{aligned} \tag{3.12}$$

Thus if $g, g' \in c$ where c is a conjugacy class then,

$$\chi(c) = \chi(g) = \chi(g'). \tag{3.13}$$

Characters depend on representations and are functions of the conjugacy class. Note:

$$\chi(e) = \text{dimension of the matrix representation.}$$

[End of Lecture 5]

3.1 Some *Great* Theorems

Unitarity theorem : For finite groups we can always choose the matrix representations to be unitary.
Schur's Lemma : If $D(g)A = AD(g)$, then $A = \lambda I$.
Great Orthogonality Theorem:

$$\sum_g D^{\dagger r}(g)^i_j D^s(g)^k_l = \frac{N(G)}{d_r}\delta^{rs}\delta^i_l\delta^k_j. \tag{3.14}$$

[End of Lecture 6]

3.1.1 Consequences of the orthogonality theorem

We start with equation :

$$\sum_g D^{\dagger r}(g)^i_j D^s(g)^k_l = \frac{N(G)}{d_r}\delta^{rs}\delta^i_l\delta^k_j. \tag{3.15}$$

Setting $j = i$ and $k = l$ and summing, we obtain

$$\sum_g \chi^{*r}(g)\chi^s(g) = N(G)\delta^{rs}. \tag{3.16}$$

since character is a function just of the conjugacy class, the above can be simplified further to,

$$\boxed{\sum_c n(c)\chi^{*r}(c)\chi^s(c) = N(G)\delta^{rs}.} \tag{3.17}$$

In the above the sum is over all conjugacy classes c, $n(c)$ is the number of elements in the conjugacy class c.
Viewing the characters as vectorial functions in the space of conjugacy classes, we have an orthogonality of vectors indexed by the irrep indices r, s. This is possible if the space of irreps form a subset of the space of classes.

$$N(R) \leq N(C). \tag{3.18}$$

One can also prove :

$$\boxed{\sum_r \chi^{*r}(c)\chi^r(c') = \frac{N(G)}{n(c)}\delta^{cc'}.} \tag{3.19}$$

Following a similar logic and this time viewing the characters as vectorial functions in the space of irreducible representations, we have an orthogonality indexed by conjugacy classes. This implies,

$$N(R) \geq N(C). \tag{3.20}$$

The only way the two inequalities, (3.18), (3.20) can be satisfied is when

$$\boxed{N(C) = N(R).} \tag{3.21}$$

Thus the number of conjugacy classes also give us the number of irreducible representations.
Now suppose we are given the character of some matrix representation D which in principle can be reducible. Let us assume that D contains n_r number of times the irreducible matrix representation D^r. In this case, we can write,

$$\chi(c) = \sum_r n_r \chi^r(c). \tag{3.22}$$

Therefore plugging this into, the l.h.s of the equation below we get,

$$\begin{aligned} \sum_c n(c)\chi^{*r}(c)\chi(c) = \sum_c n(c)\chi^{*r}(c)\sum_s n_s\chi^s(c) &= N(G)\sum_{c,s} n_s\delta^{rs}, \\ &= N(G)n_r. \end{aligned} \tag{3.23}$$

Thus we see that the operator,

$$\frac{n(c)}{N(G)}\chi^{*r}(c).$$

projects any reducible character into the irrep r subspace and gives the times the irrep r is contained upon summing over the class.
If we plug (3.22) into $\sum_c n(c)\chi(c)^*\chi(c)$ we get,

$$\begin{aligned} \sum_c n(c)\chi(c)^*\chi(c) &= \sum_{c,r,s} n_r n_s \chi^* r(c)\chi^s(c) = N(G)\sum_{r,s}\delta^{rs}n_r n_s \\ &= N(G)\sum_r n_r^2. \end{aligned} \tag{3.24}$$

For **Regular representations** the only non-zero character is for the identity conjugacy class, and this just give the dimension of the representation, which for this case is just the dimension of the group $N(G)$[3]. Hence from (3.23) we get,

$$\chi^{*r}(e)N(G) = N(G)n_r.$$

But $\chi^r(e) = d_r$, the dimension of irrep r. Therefore we obtain that in the regular representation,

$$d_r = n_r. \tag{3.25}$$

[3] i.e, $\chi^{\text{regular}}(e) = N(G)$

It is highly reducible, moreover the irrep r is contained d_r number of times! If we look at the regular representation for (3.24) we get:

$$N(G)^2 = N(G) \sum_r n_r^2, \;\; or, \;\; \sum_r n_r^2 = N(G). \tag{3.26}$$

Plugging in (3.25) into (3.26) we obtain,

$$\boxed{\sum_r d_r^2 = N(G).} \tag{3.27}$$

Character table : §II.3 of *Group Theory in a Nutshell for Physicists, A. Zee, Princeton University Press, 2016*

[End of Lecture 7]

4 Young tableaux for S_n

The irreps of S_n are in one to one correspondence with Young tableaux. The corresponding ones for the case of S_5 are given by, $\ydiagram{5}$, $\ydiagram{4,1}$, $\ydiagram{3,2}$, $\ydiagram{3,1,1}$, $\ydiagram{2,2,1}$, $\ydiagram{2,1,1,1}$, $\ydiagram{1,1,1,1,1}$. These also correspond to the cycle structure and thus the partition of integers. For instance, we can write $5 = 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1,$ and, $1+1+1+1+1$. The cycle structures respectively are, $(*****), (****)(*), (***)(**), (***)(*)(*), (**)(**)(*), (**)(*)(*)(*)$, and, $(*)(*)(*)(*)(*)$. The first row of the Young tableaux represents the largest cycle structure, second row the second largest and so on. The trivial representation is the last one which corresponds to :

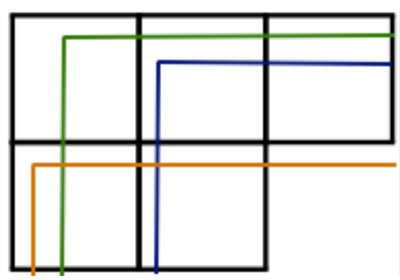
$$(1)(2)(3)(4)(5) = \ydiagram{1,1,1,1,1}$$

The number 5 has been partitioned into integers in 7 different ways [4], which corresponds to $N_c = 7$ conjugacy classes, which is also the possible cycle structures[5]. Since we have, $N_c = N_R$, these are also in correspondence with the number of irreducible representations. The Young diagrams can be used to calculate the dimensions of the irreducible representations using *Hook's law*.

[4] In Problem Set 2 you have seen $p(5) = 7$.

[5] The cycle structure is invariant under conjugation – see *here*

- Starting from the first row, pass a line through the boxes, from the right to the left.
- Next take a left-turn of 90^o in order to exit the figure. This turn makes a shape of hook.
- Count the number of boxes the hook passed through. Let the product of all possible hook lengths be called H.
- The dimension of the corresponding irreducible representation of S_n is given via
$$d_r = \frac{n!}{H}.$$
- Finally an example for the $(***)(**)$ cycle structure of S_5 :



The coloured lines in the above diagram show the non-trivial hooks, we calculate
$$d = \frac{5!}{4 \times 3 \times 2} = 5.$$

Calculating similarly we obtain: $\ydiagram{5}, d_1 = 1$, $\ydiagram{4,1}, d_2 = 4$, $\ydiagram{3,2}, d_3 = 5$, $\ydiagram{3,1,1}, d_4 = 6$, $\ydiagram{2,2,1}, d_5 = 5$, $\ydiagram{2,1,1,1}, d_6 = 4$, $\ydiagram{1,1,1,1,1}, d_7 = 1$. We can immediately verify (3.27), that,
$$n! = \sum_r d_r^2 = 5! = 120 = 1^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 1^2.$$

For the use of Young tableaux for $SU(N)$ see §8. [End of Lecture 8]

5 Group theory in Quantum Mechanics

Follow the discussion in §III.1 of *Group Theory in a Nutshell for Physicists, A. Zee, Princeton University Press, 2016* [End of Lecture 9]

6 Lie Groups

Continuous groups have infinite group elements. However to parametrize them one can often use finite number of real quantities. A convenient way to describe the transformations enacted by the group is via matrix representations where the finite number of real quantities enter as variables. Here we shall look at some important examples and count the number of the variables.

Since we are concerned with *Lie* groups, we will be looking close to the identity, thus the matrix group is expanded around the identity matrix,

$$D(g) = 1 + \epsilon X.$$

As we shall see in the next lecture, the matrices X are the generators, closing under Lie algebra, and can be exponentially parametrized to generate finite group transformations in the neighbourhood of identity.
The number of generators is same as the number of real variables describing the group. Each variable can be thought of as the strength of the transformation away from identity, in the direction of the corresponding generator element.

- the group $GL(n, \mathbb{C})$ is the group of $n \times n$ complex matrices, which are invertible. Since every entry of the matrix is an independent complex number, one has a total of $2n^2$ number of variables.

- the group $SL(n, \mathcal{C})$ is the above with the restriction that the determinant is 1.

Note: S in front of a group acronym, stands usually for special. This implies that the determinant is set to 1. This is a convenient way to normalize the group volume as we shall see later. This constraint implies tracelessness of the generators. This can be seen by using the expansion,

$$\det(1 + \epsilon A) = 1 + \epsilon \mathrm{tr} A + \mathcal{O}(\epsilon^2),$$

which can be derived, using

$$\det(1 + \epsilon A) = \prod_i (1 + \epsilon\lambda_i) = 1 + \epsilon \sum_i \lambda_i + \mathcal{O}(\epsilon^2) = 1 + \epsilon \mathrm{tr} A + \mathcal{O}(\epsilon^2).$$

where, λ_i's are the eigenvalues of the generator matrix X. Therefore, demanding $\det(1+\epsilon A) = 1$, sets $\mathrm{tr} X = 0$.

- *U(n)* The Unitary group satisfies, $UU^\dagger = 1$. These are complex matrices. For the generators the defining relationship implies,

  $$(1+\epsilon X)((1+\epsilon X)^\dagger = 1, \ \textit{or}, \ X = -X^\dagger.$$

  Thus X is a $n \times n$ complex matrix whose entries are related as, $X_{ij} = -X_{ji}^*$. The diagonals can only be of the form, $i \times$ something real. Thus there are n of them. The rest, come in pairs, that is, X_{ij} determines, X_{ji}. Thus there are $n(n-1)/2$ of these entries. Since these are complex numbers the number of reals is, $n(n-1)$. Hence total number of generators is $= n + n(n-1) = n^2$. From the condition that

  $$U_{ij} U_{jk}^\dagger = 1, \ \text{or}, \ U_{ij} U_{kj}^* = 1.$$

  we can conclude that $|U_{ij}| \leq 1$. This boundedness of the norm of U implies that U is a *compact* group.

- $SU(n)$ In this case the determinant condition gives one constraint among the entries. Thus the number of generators is $n^2 - 1$.

- *O(n) :* The orthonormal matrix group is a real matrix of size $n \times n$ satisfying, $OO^T = 1$. This implies that the generators satisfy,

  $$X_{ij} = -X_{ji}.$$

  Thus the generators are real antisymmetric matrices, and hence there are $n(n-1)/2$ of them. Since $\det AB = \det A \det B$ and $\det A^T = \det A$, we have, $\det OO^T = (\det O)^2 = 1$. Thus

  $$\det O = \pm 1.$$

- $SO(n)$: As the name suggests these special orthogonal groups are matrix groups which are real and orthogonal and also have determinant 1. However since the orthogonal group generators have all zeros in the diagonal, the tracelessness condition is automatically satisfied. Thus $SO(n)$ also has, $n(n-1)/2$ independent generators. This is the symmetry group of rotations in the Euclidean space.

- $Sp(2n, \mathbb{R})$: The real symplectic group of rank $2n$ is a $2n \times 2n$ matrix S which satisfies :
$$S^T \omega S = \omega. \tag{6.28}$$
where
$$\omega = \begin{pmatrix} 0_{n\times n} & -\mathbb{I}_{n\times n} \\ \mathbb{I}_{n\times n} & 0_{n\times n} \end{pmatrix}.$$
Note that even for $Sp(2n, \mathbb{C})$, when the entries are complex, the defining relation is (6.28), with a transpose. Thus the generators satisfy :
$$X^T \omega = -\omega X. \tag{6.29}$$
Parametrizing the generator with the following $n \times n$ matrices :
$$X = \begin{pmatrix} a_{n\times n} & b_{n\times n} \\ c_{n\times n} & d_{n\times n} \end{pmatrix}.$$
Now (6.29) gives the following constraints :
$$d = -a^T \;\; b = b^T, \quad c = c^T.$$
Thus we have one arbitrary real matrix, and two symmetric real matrices. Therefore the total number of generators is, $n^2 + 2 \times n(n+1)/2 = n(2n+1)$.

6.1 Geometry of $SU(2)$

The matrix group $A = SU(2)$ is a 2×2 complex matrix with unit determinant, and with $2^2 - 1 = 3$ independent real parameters. Additionally, there is the defining property that $A^\dagger = A^{-1}$. A convenient way to parametrize A is by,
$$A = \begin{pmatrix} x^0 + ix^3 & ix^1 + x^2 \\ ix^1 - x^2 & x^0 - ix^3 \end{pmatrix}. \tag{6.30}$$
The unit determinant condition is :
$$\sum_{i=0}^{3} (x^i)^2 = 1. \tag{6.31}$$
The inverse is,
$$A^{-1} = A^\dagger = \begin{pmatrix} x^0 - ix^3 & -ix^1 - x^2 \\ -ix^1 + x^2 & x^0 + ix^3 \end{pmatrix}. \tag{6.32}$$

Equation (6.31) describes a three dimensional sphere, S^3 with unit radius. The S^3 coordinates are x^1, x^2, x^3, and x^0 is determined via (6.31) as,

$$x^0 = \sqrt{1 - \sum_{j=1}^{3} (x^j)^2}. \tag{6.33}$$

Using the Pauli matrices[6],

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{6.34}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.35}$$

we can rewrite,

$$A = x^0 \mathbb{I}_{2\times 2} + ix^1\sigma^1 + ix^2\sigma^2 + ix^3\sigma^3.$$

Away from identity, there are 3 independent directions along the 3 generators. These near identity elements along the three directions are :

$$1 + i\epsilon\sigma^i, \text{ for } i \in \{1, 2, 3\}.$$

Let us find how the $1 + i\epsilon\sigma^3$ acts on the group element A,

$$A \to A \cdot (1 + i\epsilon\sigma^3) = (x^0 - \epsilon x^3) + i\sigma^1(x^1 - \epsilon x^2) + i\sigma^2(x^2 + \epsilon x^1) + i\sigma^3(x^3 + \epsilon x^0).$$

Thus the change in the parameters x^i ($i \in \{0, 1, 2, 3\}$), under the action of $(1 + i\epsilon\sigma^3)$ can be written as,

$$\delta \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \epsilon \begin{pmatrix} -x^3 \\ -x^2 \\ x^1 \\ x^0 \end{pmatrix} \tag{6.36}$$

Thus we can read-off the generator of this deformation in $i\sigma^3$ direction as,

$$L_3 = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} + x^0 \frac{\partial}{\partial x^3} \tag{6.37}$$

[6] Reminder : $\{\sigma^i, \sigma^j\} = 2\delta^{ij}\mathbb{I}_{2\times 2}$ and $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$

Similarly,

$$\begin{aligned} L_1 &= x^0 \frac{\partial}{\partial x^1} - x^3 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^3} \\ L_2 &= x^3 \frac{\partial}{\partial x^1} + x^0 \frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial x^3} \end{aligned} \tag{6.38}$$

Now, using (6.33),

$$\frac{\partial x^0}{\partial x^i} = -\frac{x^i}{x^0}, \text{ for, } i = 1, 2, 3.$$

With the above, we can explicitly check that,

$$[L_i, L_j] = -2\epsilon_{ijk} L_k. \tag{6.39}$$

This coincides with the albegra of $i\sigma^i$'s as it should, since both are representations of the generators of $SU(2)$.

[End of Lecture 10]

6.2 Lie Algebra

Let the number of independent real quantities needed to describe the continuous group G be N, and we call them as, α_a with $a = 1, 2, \ldots, N$. The group elements are functions (here we consider *smooth* ones) of α_a, $g \in G$, $g = g(\alpha_a)$. We choose, $\alpha_{a=0} = e$. Let the matrix representations of the group elements be $D(g(\alpha)) = D(\alpha)$. Thus with our choice about identity, we have,

$$D(0) = 1.$$

(note we interchangeably use 1 for the identity matrix $\mathbb{I}$). Expanding infinitesimally around $\alpha = 0$,

$$D(d\alpha) = 1 + i d\alpha_a X_a.$$

Here, $d\alpha_a$ is a small variation of α_a, and X_a's are the generators. We also have the implicit summation over the a index. From the above equation it is also clear that,

$$X_a = -i \frac{\partial}{\partial \alpha_a} D(\alpha) \bigg|_{\alpha_a = 0}.$$

We shall mostly be focussed on groups which have unitary representations. In that case, X_a are hermitian. Now following *Lie*, to generate a finite group transformation, we can keep acting by the infinitesimal transformation many number of times;

$$D(\alpha) = \lim_{k\to\infty}\left(1 + i\frac{\alpha_a X_a}{k}\right)^k = e^{i\alpha_a X_a}.$$

Note, that we have also used, $d\alpha_a = \alpha_a/k$. This is called the exponential mapping, how from a generator, we can get to finite group transformations (at least those connected to the identity element). This implies, that we can focus our study to the generators, which is nice since the generators form a vector space.

Now we want to see how the closure property of the group G, enforces the Lie algebra of the generators X. Closure implies,

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\delta_c X_x}.$$

This is same as,

$$i\delta_a X_a = \log\left(1 + e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1\right) = \log(1+K) \simeq K - \frac{1}{2}K^2.$$

We have just added and subtracted 1 within the parentheses in the r.h.s. Next we expand the exponentials in $K = e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1$, and keep till quadratic order in α_a, β_a's. Finally we get,

$$[\alpha_a X_a, \beta_b X_b] = 2i(\alpha_c + \beta_c - \delta_c)X_c = i\gamma_c X_c.$$

Next with the choice, $\gamma_c = \alpha_a \beta_b f_{abc}$, we can rewrite the above as,

$$[X_a, X_b] = i f_{abc} X_c.$$

Clearly the l.h.s is antisymmetric in a, b, thus,

$$f_{abc} = -f_{bac}.$$

From the hermiticity of X (which is true given a unitary representation),

$$[X_a, X_b]^\dagger = -i f^*_{abc} X_c = [X_b, X_a] = i f_{bac} X_c = -i f_{abc} X_c.$$

This shows that f_{abc} are real quantities.

Also once can go on to show,

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0.$$

This is called the Jacobi identity. [End of Lecture 11]

Adjoint Representation

It turns out that,

$$(X_a)_{bc} = -if_{abc}, \tag{6.40}$$

forms a representation of the generator. From the Jacobi identity one has,

$$f_{abd}f_{dcg} + f_{bcd}f_{dag} + f_{cad}f_{dbg} = 0.$$

Using the above, one can show that with the adjoint representation (6.40) one has group closure property.

6.3 Certain aspects of $SO(3)$

Refer to §$SO(3)$: The representation (2.5) is the adjoint representation. This is because one can check,

$$(J_a)_{bc} = -i\epsilon_{abc}.$$

Now to construct other representations of $SO(3)$ we focus on the algebra generated by,

$$J_\pm = J_x \pm iJ_y, \text{ and } J_z.$$

The relevant brackets give,

$$[J_\pm, J_z] = \mp J_\pm \text{ and } [J_+, J_-] = 2J_z \tag{6.41}$$

Since the generators do not commute we cannot simultaneously diagonalize all three of them. Let us choose to work with the J_z basis :

$$J_z|m\rangle = m|m\rangle. \tag{6.42}$$

Using the algebra (6.41) we have :

$$\begin{aligned} J_zJ_+|m\rangle &= (m+1)J_+|m\rangle. \\ J_zJ_-|m\rangle &= (m-1)J_-|m\rangle. \end{aligned} \tag{6.43}$$

Thus we can rename:

$$J_+|m\rangle = c_{m+1}|m+1\rangle,$$

and,

$$J_-|m\rangle = b_{m-1}|m-1\rangle.$$

Here, c_m, b_m are unfixed normalizations. The generators $J_\pm$ act as raising(lowering) operators. Since the generators are hermitian, we have $(J_\pm)^\dagger = J_\mp$.

$$\text{Since, } \langle m+1|J_+|m\rangle = c_{m+1}, \quad \text{and} \quad \langle m|J_-|m+1\rangle = b_m,$$

the hermiticity implies,

$$b_{m-1} = c_m^*.$$

Thereby,

$$J_-|m\rangle = c_m^*|m-1\rangle.$$

Thus we also have,

$$\begin{aligned} J_+J_-|m\rangle &= |c_m|^2|m\rangle \\ J_-J_+|m\rangle &= |c_{m+1}|^2|m\rangle. \end{aligned} \tag{6.44}$$

Since we are after finite irreps of $SO(3)$, we would need the states to terminate somewhere, so that it cannot be raised further. This defines the highest weight state. Let it have a J_z eigenvalue j. Then $J_z|j\rangle = j|j\rangle$. Now since this is the highest weight,

$$\langle j|J_-J_+|j\rangle = 0.$$

However, using (6.41) this is also same as,

$$\langle j|J_+J_- - 2J_z|j\rangle = |c_j|^2 - 2j.$$

where we also use (6.44). Thus we immediately have,

$$|c_j|^2 = 2j. \tag{6.45}$$

To figure out the rest of the normalizations, we use again the commutator, $[J_+, J_-]$ (6.41) along with (6.44),

$$\langle m|[J_+, J_-]|m\rangle = \langle m|J_+J_- - J_-J_+|m\rangle = |c_m|^2 - |c_{m+1}|^2 = \langle m|2J_z|m\rangle = 2m.$$

$$\therefore |c_n|^2 = |c_{n+1}|^2 + 2n. \tag{6.46}$$

Now we can use the above recursion relation multiple times to determine, $|c_{j-1}|^2, |c_{j-2}|^2, \ldots, |c_{j-s}|^2$ using the highest weight normalization value (6.45).

$$\begin{aligned} |c_{j-1}|^2 &= |c_j|^2 + 2(j-1) = 2(2j-1) \\ |c_{j-2}|^2 &= |c_{j-1}|^2 + 2(j-2) = 2(3j-(1+2)) \end{aligned} \tag{6.47}$$

Identifying the pattern, in the general case we have,

$$|c_{j-s}|^2 = 2((s+1)j - \sum_{k=1}^{s} k) = (s+1)(2j-s). \tag{6.48}$$

This tells us now, that we hit zero after going down $s = 2j$ steps, so, $|c_{-j}|^2 = 0$. Thus we have $2j+1$ states in total,

$$|j\rangle, |j-1\rangle, \ldots, |j-s\rangle, \ldots, |-j+1\rangle, |-j\rangle.$$

Thus this representation is $2j+1$ dimensional, i.e., the action of the $SO(3)$ group is described by a $2j+1 \times 2j+1$ matrix in this particular representation. Since $s = 2j$ needs to be an integer (number of times one lowers), $\implies j \in \frac{1}{2}\mathbb{Z}$. Thus j can be half-integers as well. Additionally we can also determine the normalization constant $|c_m|^2$ from (6.48) by setting $s = j - m$. Since we cannot determine the phase, we choose the c_m's to be reals, then we have,

$$\begin{aligned} J_+|m\rangle &= \sqrt{(j+1+m)(j-m)}|m+1\rangle \\ J_-|m\rangle &= \sqrt{(j+1-m)(j+m)}|m-1\rangle. \end{aligned} \tag{6.49}$$

A product representations of two such j and j' representations in general will be composed of other irreps of $SO(3)$. Let us see how this works out. Given the product representation, $j \otimes j'$, we can write a generic state in this representation as,

$$|j, m; j'm'\rangle.$$

[End of Lecture 12]

The way a group element $g \in G$ acts on the product states is the following:

$$D^{(r)}(g) \otimes D^{(s)}(g) \left(|j^{(r)}\rangle \otimes |j^{(s)}\rangle\right) = \left(D^{(r)}(g)|j^{(r)}\rangle\right) \otimes \left(D^{(s)}(g)|j^{(s)}\rangle\right).$$

In terms of generators, $D^{(r)} = 1 + i\theta_a X_a^{(r)}$, thus plugging this in the generator acts as,

$$X_a\left(|j\rangle \otimes |j'\rangle\right) = X_a|j\rangle \otimes |j'\rangle \oplus |j\rangle X_a|j'\rangle. \tag{6.50}$$

Thus the J_z eigenvalue of the state $|j, m; j', m'\rangle$ is $m + m'$ which at the most can be $j + j'$. Next, J_- can act on this state to produce, $|j, m-1; j', m'\rangle$ and $|j, m; j', m'-1\rangle$, both of whose J_z eigenvalue is $m + m' - 1$, which can at the most be $j+j'-1$. Thus this is part of the $j+j'-1$ highest weight representation. Inductively therefore,

$$j \otimes j' = (j+j') \oplus (j+j'-1) \oplus \cdots \oplus |j-j'|. \tag{6.51}$$

One can check that all the states are accounted for. In both sides there are a total of $(2j+1)(2j'+1)$ states. In the following we look at an example with $j = j' = \frac{1}{2}$. The highest weight state in the spectrum is, $|1/2, 1/2; 1/2, 1/2\rangle$. This has J_z value 1, and since this is highest weight, it is $= |1,1\rangle$. Acting by J_z on this gives, using (6.49), $J_-|1,1\rangle = \sqrt{2}|1,0\rangle$. However this should be same as $J_-|1/2,1/2;1/2,1/2\rangle = |1/2,-1/2;1/2,1/2\rangle + |1/2,1/2;1/2,-1/2\rangle$. Thus we identify,

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|1/2,-1/2;1/2,1/2\rangle + |1/2,1/2;1/2,-1/2\rangle\right). \tag{6.52}$$

Applying J_- once again to both hand sides of the above equation gives :

$$|1,-1\rangle = |\frac{1}{2},-\frac{1}{2};\frac{1}{2},-\frac{1}{2}\rangle. \tag{6.53}$$

This is the lowest weight state. The only other missing state in the decomposition is the singlet $|0\rangle$. One can easily convince that this is the state :

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|1/2,-1/2;1/2,1/2\rangle - |1/2,1/2;1/2,-1/2\rangle\right). \tag{6.54}$$

Note, that the sign is undetermined. The coefficients relating one basis of irreps to another are known as the **Clebsch-Gordon** coefficients,

$$|J,M\rangle = \sum_{m=-j}^{j} \sum_{m'=-j'}^{j'} |j,m;j'm'\rangle\langle j,m;j',m'|J,M\rangle. \tag{6.55}$$

Here in the r.h.s one has inserted identity by using completeness and the Clebsch-Gordon coefficients are the overlaps,

$$\langle j,m;j',m'|J,M\rangle.$$

[End of Lecture 13]

6.4 $SU(2)$ and $SO(3)$

Reminder: $SO(3)$ generates rotations of 3 dimensional vectors (§$SO(3)$):

$$x_i \to x'_i = R_{ij} x_j. \tag{6.56}$$

Here the matrix R, is made up of real entries, with the property that $RR^T = 1$, and $\det R = 1$. Whereas, an element of $SU(2)$ can be represented by the 2×2 complex matrix :

$$u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}. \tag{6.57}$$

This has 4 real quantities (a, b are complex numbers). Also the property, $uu^\dagger = 1$ needs to be satisfied. This holds provided,

$$|a|^2 + |b|^2 = 1. \tag{6.58}$$

This is also the necessary condition to have, $\det u = 1$. Now consider the matrix :

$$h = \sigma_i x_i. \tag{6.59}$$

Explicitly,

$$h = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}. \tag{6.60}$$

And enact a $SU(2)$ transformation on it,

$$\begin{aligned} h \to h' &= uhu^\dagger \qquad (6.61) \\ &= \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \\ &= \begin{pmatrix} x'_3 & x'_1 - ix'_2 \\ x'_1 + ix'_2 & -x'_3 \end{pmatrix} \qquad (6.62) \end{aligned}$$

From explicit matrix multiplication one can read off the transformed x'_i's :

$$\begin{aligned} x'_1 &= \left(\frac{1}{2}(a^2 - b^2) + c.c.\right) x_1 + \left(\frac{i}{2}((a^*)^2 + (b^*)^2) + c.c.\right) x_2 + (-ab + c.c.)\, x_3, \\ x'_2 &= \left(\frac{i}{2}((b^*)^2 - (a^*)^2) + c.c.\right) x_1 + \left(\frac{1}{2}(a^2 + b^2) + c.c.\right) x_2 + \left(\frac{i}{2} a^* b^* + c.c.\right) x_3, \\ x'_3 &= (a^* b + c.c.)\, x_1 + (ia^* b + c.c.)\, x_2 + \left(|a|^2 - |b|^2\right) x_3. \qquad (6.63) \end{aligned}$$

One can check that this transformation has the properties of an $SO(3)$ transformation. Constructing the R matrix from above :

$$R = \begin{pmatrix} \left(\frac{1}{2}(a^2 - b^2) + c.c.\right) & \left(\frac{i}{2}((a^*)^2 + (b^*)^2) + c.c.\right) & (-ab + c.c.) \\ \left(\frac{i}{2}((b^*)^2 - (a^*)^2) + c.c.\right) & \left(\frac{1}{2}(a^2 + b^2) + c.c.\right) & \left(\frac{i}{2}a^*b^* + c.c.\right) \\ (a^*b + c.c.) & (ia^*b + c.c.) & (|a|^2 - |b|^2) \end{pmatrix}. \tag{6.64}$$

For the above matrix all the entries are real. Furthermore,

$$RR^T = \begin{pmatrix} |a|^2 + |b|^2 & 0 & 0 \\ 0 & |a|^2 + |b|^2 & 0 \\ 0 & 0 & |a|^2 + |b|^2 \end{pmatrix}. \tag{6.65}$$

But this is just the identity matrix, due to (6.58). Similarly,

$$\det R = (|a|^2 + |b|^2)^3 = 1.$$

Thus R as defined in (6.64) represents $SO(3)$. Hence from the representation of $SU(2)$ we can write down the R matrix of $SO(3)$. Going back to (6.59) and (6.61) and writing in terms of components (with summation implicit),

$$h' = \sigma_j x'_j = u\sigma_i x_i u^\dagger. \tag{6.66}$$

Now using the property of Pauli matrices :

$$\text{Tr}\ (\sigma_j \sigma_k) = 2\delta_{jk}$$

we can solve for $\vec{x'}$ as follows. First multiply b.h.s of (6.66) by σ_k and then take trace:

$$\begin{aligned} \text{Tr}\ \sigma_k h' = \text{Tr}\ (\sigma_k \sigma_j)\, x'_j &= \text{Tr}\ \left(\sigma_k u \sigma_i u^\dagger\right) x_i \\ \text{or}\ \ 2x'_j \delta_{jk} &= \text{Tr}\ \left(\sigma_k u \sigma_i u^\dagger\right) x_i \\ \therefore x'_j &= \frac{1}{2}\text{Tr}\ \left(\sigma_k u \sigma_i u^\dagger\right) x_i. \end{aligned} \tag{6.67}$$

Thus

$$R_{ji} = \frac{1}{2}\text{Tr}\ \left(\sigma_j u \sigma_i u^\dagger\right).$$

Note from the above that both u and $-u$ give the same matrix element R_{ij}. Thus $SU(2)$ is not just $SO(3)$, rather, it covers $SO(3)$ twice. This choice of sign is expressed in the following isomorphism relation:

$$SO(3) \simeq SU(2)/\mathbb{Z}_2.$$

The upshot of this discussion is that the irreps of $SO(3)$ are also irreps of $SU(2)$. The generators of $SU(2)$ also satisfy the same Lie algebra as (2.8). However we will follow standard $SU(2)$ conventions and define the raising and lowering operators are bit differently :

$$J_{\pm} = \frac{J_1 \pm J_2}{\sqrt{2}}. \tag{6.68}$$

In comparison with (6.41) there is an extra prefactor of $1/\sqrt{2}$, due to which (6.49) is also modified to :

$$\begin{aligned} J_+|m\rangle &= \frac{\sqrt{(j+1+m)(j-m)}}{\sqrt{2}}|m+1\rangle \\ J_-|m\rangle &= \frac{\sqrt{(j+1-m)(j+m)}}{\sqrt{2}}|m-1\rangle. \end{aligned} \tag{6.69}$$

And, (6.41) is modified to,

$$[J_z, J_{\pm}] = \pm J_{\pm} \text{ and } [J_+, J_-] = J_z \tag{6.70}$$

7 Weights and Roots

The largest subset of commuting Hermitian generators are called the **Cartan subalgebra** : H_i, where $i = 1, 2, \ldots, m$ (m is called the rank of the Lie algebra). They thus satisfy,

$$H_i = H_i^{\dagger}, \quad \text{and} \ , \ [H_i, H_j] = 0.$$

Since these are simultaneously diagonalizable, we can choose a basis such that,

$$\text{Tr } [H_i H_j] = k_D \delta_{ij}. \tag{7.71}$$

Here D subscript denotes representation. In the case of $SU(2)$ the Cartan subalgebra, just consisted of J_3. (we could have also chosen any one of the J_a's). Thus as in $SU(2)$ we work with the states, that are eigenstates of H_i,

$$H_i|\mu, x, D\rangle = \mu_i|\mu, x, D\rangle. \tag{7.72}$$

Here the eigenvalues, μ_i are called the **weights**, and in general the vector, $(\mu_1, \mu_2, \ldots, \mu_m)$ is called the weight vector.

Adjoint Representations

Recall the definition of Adjoint representation from §$Adjoint_{Rep}$. In this case the states themselves can be labelled by the generators, X_a as, $|X_a\rangle$. One can also have a linear combination of generators as a state : $|\alpha X_a + \beta X_b\rangle$. The inner product can be defined as,

$$\langle X_a|X_b\rangle = \frac{1}{\lambda}\text{Tr}\ \left(X_a^\dagger X_b\right).$$

In the above, λ is just k_D as in (7.71) for the adjoint representation. Let us next see, how generators act on these states:

$$X_a|X_b\rangle = |X_c\rangle\langle X_c|X_a|X_b\rangle = |X_c\rangle\,[T_a]_{cb} = if_{abc}|X_c\rangle = |[X_a, X_b]\rangle. \tag{7.73}$$

Thus the generators in the adjoint representation act as commutators.
[End of Lecture 14]
The weights in the adjoint representation are called roots. Let E_α be a generator, then in adjoint $|E_\alpha\rangle$ is a state with weight : $H_i|E_\alpha\rangle = \alpha_i|E_\alpha\rangle$. Note that this implies :

$$[H_i, E_\alpha] = \alpha_i E_\alpha. \tag{7.74}$$

Next consider the state, $|E_\alpha^\dagger\rangle$. Let us calculate its root. Conjugating (7.74),

$$\begin{aligned} [E_\alpha^\dagger, H_i^\dagger] &= \alpha_i^* E_\alpha^\dagger, \\ \text{or, } [H_i, E_\alpha^\dagger] &= -\alpha_i E_\alpha^\dagger. \end{aligned} \tag{7.75}$$

Thus from the above it is clear that we can rewrite,

$$E_\alpha^\dagger = E_{-\alpha}. \tag{7.76}$$

Though we do not prove it here, it can be shown that the label α is unique for a generator, i.e., there cannot be a $\beta \neq \alpha$ such that, E_α and E_β are the same generator. This is called the *uniqueness theorem for the roots.* Since the states, $|E_\alpha\rangle$ are eigenstates of the Hermitian operators, H_i, we can and will choose a basis such that these are normalized, i.e.,

$$\langle E_\beta|E_\alpha\rangle = \frac{1}{\lambda}\text{Tr}\ (E_{-\beta}E_\alpha) = \delta_{\alpha\beta}. \tag{7.77}$$

Note that we also have,

$$\langle H_i|H_j\rangle = \delta_{ij}.$$

Next we show that (7.74) implies that the generators $E_{\pm\alpha}$ act as raising and lowering operators (similar to $J_\pm$ for $SU(2)$),

$$\begin{aligned} H_i E_{\pm\alpha}|\mu, D\rangle &= E_{\pm\alpha} H_i|\mu, D\rangle \pm \alpha_i E_{\pm\alpha}|\mu, D\rangle \\ &= (\mu_i \pm \alpha_i)\, E_{\pm\alpha}|\mu, D\rangle. \end{aligned} \tag{7.78}$$

In the above first equality we used the algebra, (7.74). This implies that the state, $E_\alpha|E_{-\alpha}\rangle$ has weight zero. Thus it must be linear combination of the Cartan generators. Thus,

$$E_\alpha|E_{-\alpha}\rangle = \beta_i|H_i\rangle. \tag{7.79}$$

But since $|E_{-\alpha}\rangle$ is in the adjoint representation, we also know :

$$E_\alpha|E_{-\alpha}\rangle = |[E_\alpha, E_{-\alpha}]\rangle. \tag{7.80}$$

Thus we conclude that,

$$|[E_\alpha, E_{-\alpha}]\rangle = \beta_i|H_i\rangle. \tag{7.81}$$

From the above using orthogonality relations we can write,

$$\begin{aligned} \beta_i &= \langle H_i|E_\alpha|E_{-\alpha}\rangle \\ &= \frac{1}{\lambda}\mathrm{Tr}\ (H_i[E_\alpha, E_{-\alpha}]) \\ &= \frac{1}{\lambda}\mathrm{Tr}\ ([H_i, E_\alpha]E_{-\alpha}) \\ &= \frac{\alpha_i}{\lambda}\mathrm{Tr}\ \left(E_\alpha E_{-\alpha}^\dagger\right) \\ &= \frac{\alpha_i}{\lambda}\mathrm{Tr}\ \left(E_{-\alpha}^\dagger E_\alpha\right) \\ &= \alpha_i\langle E_\alpha|E_\alpha\rangle = \alpha_i. \end{aligned} \tag{7.82}$$

Thus we have shown that $\beta_i = \alpha_i$. In the third and the fifth equalities we used the cyclicity property of the trace. Next, we can define :

$$E^\pm = \frac{1}{\alpha}E_{\pm\alpha}, \quad E_3 = \frac{\alpha\cdot H}{\alpha^2}. \tag{7.83}$$

Then using (7.74), one can derive:

$$[E_3, E^\pm] = \pm E^\pm, \quad \text{and } [E^+, E^-] = E_3. \tag{7.84}$$

The above is exactly the algebra of $SU(2)$ (6.70), thus we see that given the Cartan subalgebra and given a particular generator E_α we can construct an $SU(2)$ subalgebra for the group G. We shall call this algebra, $SU(2)_\alpha$.
[End of Lecture 15] Acting on a state in representation D,

$$E_3|\mu, x, D\rangle = \frac{\alpha \cdot \mu}{\alpha^2}|\mu, x, D\rangle. \tag{7.85}$$

Since this is an $SU(2)$ eigenvalue this must be either and integer or a half-integer; therefore,

$$\frac{2\alpha \cdot \mu}{\alpha^2} \in \mathbb{Z}. \tag{7.86}$$

Now let $(E^+)^p|\mu, x, D\rangle$ be a highest weight state, which means that there is some integer p for which,

$$(E^+)^{p+1}|\mu, x, D\rangle = 0.$$

Now the weight of this highest weight state can be directly evaluated :

$$E_3(E^+)^p|\mu, x, D\rangle \quad = \quad \left(\frac{\alpha \cdot \mu}{\alpha^2} + p\right)|\mu, x, D\rangle. \tag{7.87}$$

$[E_3, (E^+)^p]$*:* For the above computation we needed this commutator. The strategy is to work step by step :

$$\begin{aligned} E_3(E^+)^p &= E_3E^+(E^+)^{p-1} = E^+E_3(E^+)^{p-1} + (E^+)^p \\ &= (E^+)^2E_3(E^+)^{p-2} + 2(E^+)^p = \cdots = (E^+)^pE_3 + p(E^+)^p \end{aligned} \tag{7.88}$$

Thus,

$$[E_3, (E^+)^p] = p(E^+)^p.$$

Thus we have :

$$\frac{\alpha \cdot \mu}{\alpha^2} + p = j. \tag{7.89}$$

Similarly we construct a lowest weight state with,

$$(E^-)^q|\mu, x, D\rangle,$$

whose E_3 eigenvalue gives :

$$\frac{\alpha\cdot\mu}{\alpha^2} - q = -j. \tag{7.90}$$

We can now add (7.89) and (7.90) and obtain:

$$\frac{\alpha\cdot\mu}{\alpha^2} = -\frac{1}{2}\left(p-q\right). \tag{7.91}$$

Now, consider two generators, E_α and E_β. Both has their own $SU(2)$s. Consider first, $SU(2)_\alpha$. In this case (7.91) for the case when the state $|\mu, x, D\rangle = |E_\beta\rangle$ we have,

$$\frac{\alpha\cdot\beta}{\alpha^2} = -\frac{1}{2}(p-q). \tag{7.92}$$

Similarly for $SU(2)_\beta$ algebra with state, $|E_\alpha\rangle$ we have,

$$\frac{\beta\cdot\alpha}{\beta^2} = -\frac{1}{2}(p'-q'). \tag{7.93}$$

Next multiplying (7.92) and (7.93),

$$\frac{(\beta\cdot\alpha)^2}{\beta^2\alpha^2} = \frac{1}{4}(p-q)(p'-q') = \cos^2\theta_{\alpha\beta}. \tag{7.94}$$

In the above we see that the l.h.s is the angle between the root vectors $\theta_{\alpha\beta}$ whose cosine squared appears. Thus the product, $(p-q)(p'-q')$ which appears in the middle equality should be positive (since cosine squared is positive) and also should be an integer (since each of p, q, p', q' are integers). Thus there are only 4 cases :

$(p-q)(p'-q')$	$\theta_{\alpha\beta}$
0	90^o
1	60^o or 120^o
2	45^o or 135^o
3	30^o or 150^o
4	0^o or 180^o

In the above, the last case is again trivial. The zero degree case is ruled out due to the uniqueness theorem of the roots, while the 180^o case is the generator $E_{-\alpha}$.

7.1 SU(3)

Let us look at the group $SU(3)$ and how the above features fit in here. For $SU(3)$ there are 8 generators, which are traceless, hermitian matrices of rank 3. The standard basis for them are generalizations of Pauli matrices and are called *Gellmann* matrices in the literature. This also furnishes us with the fundamental representation of $SU(3)$.

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \ \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \ \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{7.95}$$

Conventionally, the generators of $SU(3)$ are defined as,

$$T_a = \frac{1}{2}\lambda_a. \tag{7.96}$$

They satisfy,

$$\text{Tr } (T_a T_b) = \frac{1}{2}\delta_{ab}. \tag{7.97}$$

One can check that the structure constants:

$$[T_a, T_b] = i f_{abc} T_c, \tag{7.98}$$

take the explicit values :

$$\begin{aligned}
f_{123} &= 1, \\
f_{147} &= -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2} \\
f_{458} &= f_{678} = \frac{\sqrt{3}}{2}.
\end{aligned} \tag{7.99}$$

The Cartan subalgebra in this case is generated by $T_3 = H_1$ and $T_8 = H_2$. The eigenvalues are simply,

$$(h_1, h_2) = (\pm\frac{1}{2}, \frac{\sqrt{3}}{6}), \ \ (0, -\frac{\sqrt{3}}{3}). \tag{7.100}$$

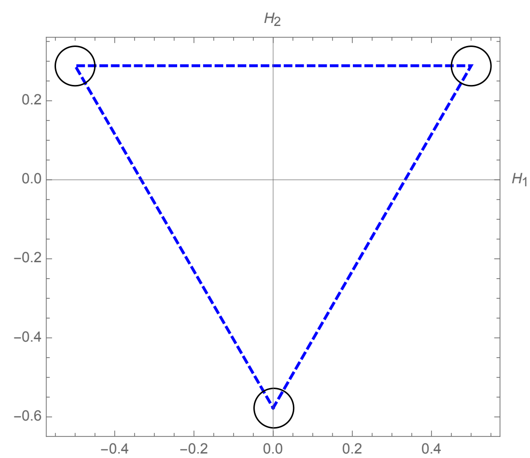


Figure 1: Weight diagram for $SU(3)$ fundamental, also called **3** due to its dimensionality.

These form an inverted equilateral triangle in the $H_1 - H_2$ plane. This is called the weight diagram.

The figure 1 is the weight diagram for $SU(3)$ in the fundamental representation. The other 6 generators generate movements between the points in this weight lattice. [End of Lecture 16]

One can thus determine the 6 roots :

$$
\begin{aligned}
E_{\pm 1,0} &= \frac{1}{\sqrt{2}}\left(T_1 \pm iT_2\right), \quad E_{\pm\frac{1}{2},\pm\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{2}}\left(T_4 \pm iT_5\right), \qquad (7.101)\\
E_{\mp\frac{1}{2},\pm\frac{\sqrt{3}}{2}} &= \frac{1}{\sqrt{2}}\left(T_6 \pm iT_7\right).
\end{aligned}
$$

To find the combinations of the generators appearing above we can act by the Cartans and fix the coefficients. For instance, to find $E_{\pm 1,0}$, by definition we need to have :

$$
H_1|E_{\pm 1,0}\rangle = \pm 1|E_{\pm 1,0}\rangle, \quad \text{and}, \quad H_2|E_{\pm 1,0}\rangle = 0|E_{\pm 1,0}\rangle. \qquad (7.102)
$$

This can be used to fix $E_{\pm 1,0}$ in terms of T_a upto a overall normalization. The latter is fixed by demanding,

$$
\langle E_{\pm 1,0}|E_{\pm 1,0}\rangle = 1. \qquad (7.103)
$$

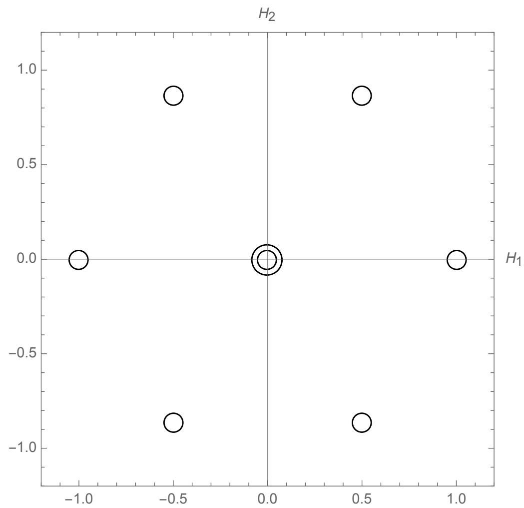


Figure 2: Weight diagram for $SU(3)$ adjoint, also called **8**.

Let us see how (7.101) satisfy the above requirements. We have already identified, $H_1 = T_3$ and $H_2 = T_8$. Thus,

$$\begin{aligned}
H_1|E_{\pm1,0}\rangle &= |[H_1, E_{\pm1,0}]\rangle = |[T_3, \frac{1}{\sqrt{2}}(T_1 \pm iT_2)]\rangle \\
&= \frac{1}{\sqrt{2}}|[T_3, T_1]\rangle \pm \frac{i}{\sqrt{2}}|[T_3, T_2]\rangle \\
&= \frac{1}{\sqrt{2}}|if_{31j}T_j\rangle \pm \frac{i}{\sqrt{2}}|if_{32j}T_j\rangle \\
&= \frac{1}{\sqrt{2}}|if_{312}T_2\rangle \pm \frac{i}{\sqrt{2}}|if_{321}T_1\rangle \\
&= \frac{i}{\sqrt{2}}|T_2\rangle \pm \frac{1}{\sqrt{2}}|T_1\rangle = \pm\frac{1}{\sqrt{2}}(|T_1\rangle \pm i|T_2\rangle) \\
&= \pm1|E_{\pm1,0}\rangle. \qquad (7.104)
\end{aligned}$$

Clearly since the individual states, $|T_a\rangle$ are ortho-normalized the overall normalization is $\frac{1}{\sqrt{2}}$. The weight diagram for the adjoint representation (which is 8 dimensional and also called **8**) has the shape of a hexagon along with two weights in the origin corresponding to the Cartans. See Fig.2.

Another important irrep of $SU(3)$ is the anti-fundamental representation

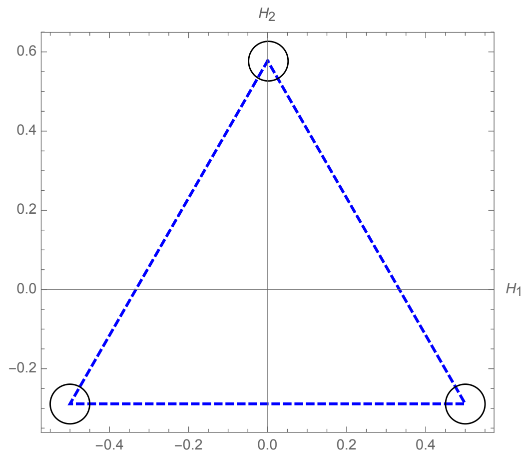


Figure 3: Weight diagram for $SU(3)$ anti-fundamental, also called $\bar{\mathbf{3}}$ due to its dimensionality and conjugation.

which also has dimensionality 3 and is obtained by conjugating the algebra.

$$(h_1, h_2) = (\mp\frac{1}{2}, -\frac{\sqrt{3}}{6}), \quad (0, \frac{\sqrt{3}}{3}). \tag{7.105}$$

Product representations

The weight diagrams can also be drawn for product representations. And then one can use it to decompose it into the contained irreducible representations. To see how this is done, we first look at $SU(2)$ where we know the decomposition rule (6.51). Let us consider the example

$$\frac{1}{2} \otimes \frac{1}{2}.$$

In the case of $SU(2)$ there is only one axis for the Cartan generator which we choose to be J_3. The first step is to list all the possible weights contained in the product representation. For the above example the weights are the J_3 eigenvalues of the four states $|\frac{1}{2}, \pm\frac{1}{2}; \frac{1}{2}, \pm\frac{1}{2}\rangle$. The J_3 values simply add up and gives us three possible values $-1, 0, 1$. However there are two zeros, corresponding to addition of $\frac{1}{2}$ and $-\frac{1}{2}$ which happens twice. Thus we have

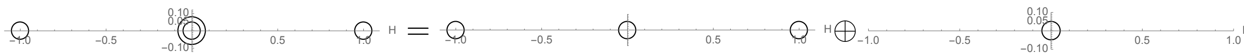


Figure 4: The weight diagram indicates $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$.

the following weight diagram as given in l.h.s of Fig. 4. The next step is to realize that in the weight diagrams of the irreducible representations of $SU(2)$ none of the weights are degenerate. For instance in the spin-j irrep of $SU(2)$, the possible J_3 eigenvalues, $-j, -j+1, \ldots, j-1, j$, occur only once. Thus the decomposition of the diagram is into the two diagrams as shown in the r.h.s of Fig. 4. These are nothing but the weight diagrams of $j = 1$ and $j - 0$. Thus we have derived,

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0.$$

For $SU(3)$ it is a bit tricky since in the irreps of this group, weights can have degenerate values. Clearly the adjoint weight diagram (Fig. 2) has double degeneracy at the origin reflecting the rank 2 feature of $SU(3)$. Let us look at the following example for $SU(3)$ for the case,

$$\mathbf{3} \otimes \bar{\mathbf{3}}.$$

In this case, following the first step let us list down all the possible weights in this product representation. We simply need to do all possible additions of the weights which are given in (7.100) to the weights in (7.105). We obtain the set,

$$(0,0),\ (-1,0),\ (1,0),\ (-\frac{1}{2}, \frac{\sqrt{3}}{2}),\ (-\frac{1}{2}, -\frac{\sqrt{3}}{2}),\ (-\frac{1}{2}, -\frac{\sqrt{3}}{2}),\ (\frac{1}{2}, -\frac{\sqrt{3}}{2}).$$

In the above $(0,0)$ appears thrice, all others appear once. This gives the weight diagram, l.h.s of Fig. 5 from which we can immediately see that it is built out of Fig. 2 and the singlet, which is the trivial 1 dimensional representation (just one weight in the origin). Thus we conclude:

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}. \tag{7.106}$$

[End of Lecture 17]

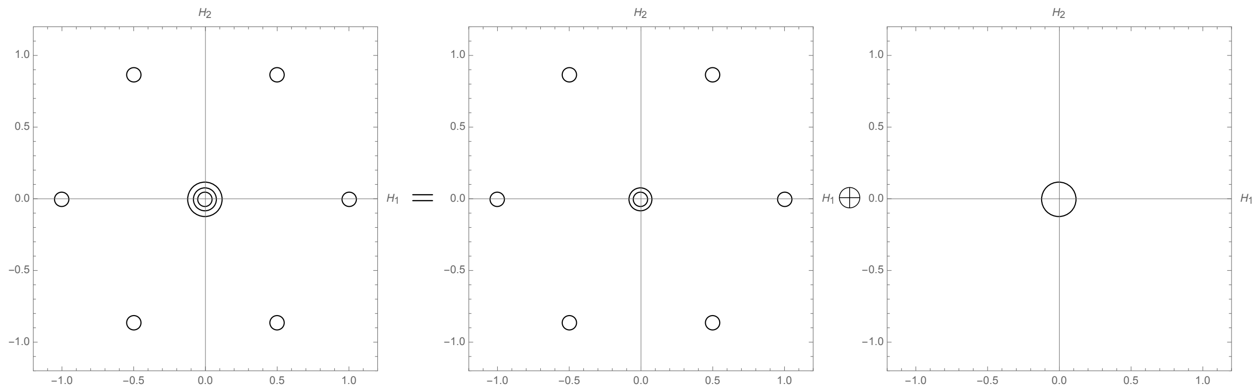
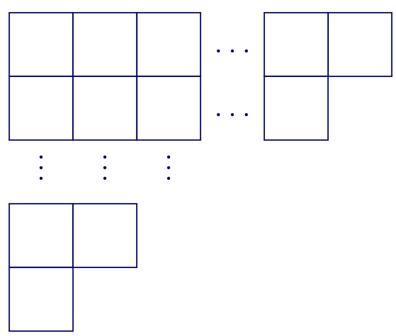


Figure 5: The weight diagram indicates $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$.

8 Young Tableaus for $SU(N)$

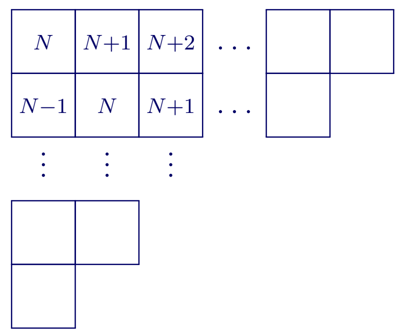
A standard Young diagram has the following form:



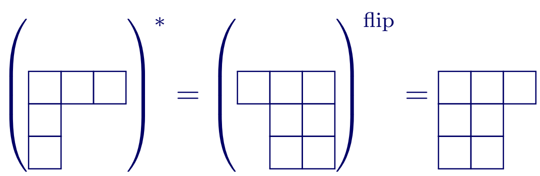
- Each tableaux corresponds to a irreducible representation.
- The trivial irrep (singlet) consists of N vertical boxes.
- There cannot be more boxes in a row below and on a column to the right.
- The boxes can be associated with numbers that can be used to calculate dimension of the irrep. The way numbers are assigned are as follows :
  - The top left box is given a number N for $SU(N)$.

– Numbers increase by one in a given row. The boxes in the top row next to the left most will have numbers: $N+1, N+2, \ldots$.

– Numbers decrease by one in a given column. In the left most column, the boxes below the N box have numbers, $N-1, N-2, \ldots$.



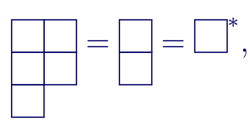
- The conjugate of an irrep via Young diagram is found by i) first replacing the j boxes in a column by $N-j$ boxes, and ii) then flipping the diagram w.r.t the left most vertical axis. For example in $SU(4)$,



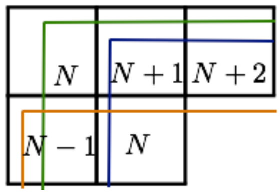
- The dimension of a Young diagram for $SU(N)$ can be calculated using,

$$d_r = \frac{\text{Factors}}{\text{Hooks}}.$$

Here the numerator is the product of all the entries in the boxes of the diagram, and the denominator is the product of all the hook numbers. For example we find for the irrep given by, , For $N = 3$ i.e., $SU(3)$ we obtain $d_r = 3$, this is the Young diagram,



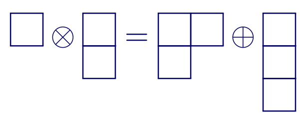
which is the anti-fundamental representation $\bar{\mathbf{3}}$ that as we know is three dimensional.



$$: \quad d_r = \frac{N^2(N+1)(N-1)(N-2)}{4 \times 3 \times 2}.$$

Figure 6: Non-trivial hooks are drawn in the given irrep of $SU(N)$.

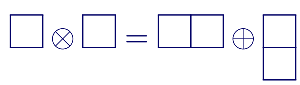
- The decomposition of product representations into irreps are obtained by all legal joinings of the second irrep in the tensor product with the first irrep. Ordering has to be maintained, for this see §12.2 of *Lie Algebras in Particle Physics* by *H. Georgi*. For instance in the case of $\mathbf{3} \otimes \bar{\mathbf{3}}$ we obtain :

  Computing the dimensions show that this is,

  $$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}.$$

  Another example can be for $SU(3)$, $\mathbf{3} \otimes \mathbf{3}$,

  Computing the dimensions show that this is,

  $$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}.$$

[End of Lecture 18]