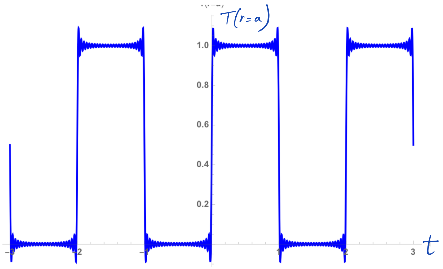
5. Note, that the boundary profile of $T(r=a)$ can be expressed as a Fourier series :-

$$\sum_{n=-\infty}^{\infty} \frac{T_0}{n\pi} \sin\left(\frac{n\pi}{2}\right) e^{\frac{in\pi}{\tau}(t-\tau/2)} = T(r=a,t) \quad - (1)$$

This can be derived easily by using orthogonality property of $e^{in\theta}$ (when integrated over a period).

Below we plot this by truncating the sum from -50 to 50, for $T_0 = \tau = 1$



Now we make the ansatz $T(r,t) = \sum_{n=-\infty}^{\infty} a_n(r)\, e^{\frac{in\pi}{\tau}(t-\frac{\tau}{2})}$; plugging into the heat equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = \frac{1}{\kappa}\frac{\partial T}{\partial t}$$

(note that from the 3D Laplacian due to spherical symmetry we have focussed on the $l = m = 0$ angular contribution.)

Hence we get an equation for $a_n(r)$:

$$r^2 a_n'' + 2r a_n' - \frac{r^2}{\kappa} \frac{in\pi}{\tau} a_n = 0.$$

This can be solved in terms of j_{-1} and y_{-1} Bessels with imaginary components. Spherical Bessel J with imaginary argument, diverges at the origin $r=0$; therefore we reject it.

Hence we have $a_n(r) = c_n\, y_{-1}\left(\frac{1+i}{\sqrt{2}}\sqrt{\frac{n\pi}{\kappa\tau}}\, r\right) = c_n\, \mathrm{sinc}\left((-1)^{3/4}\sqrt{\frac{n\pi}{\kappa\tau}}\, r\right)$

$\mathrm{sinc}(x) = \frac{\sin x}{x}$.

Now; c_n can be found using $a_n(r=a) = \frac{T_0}{n\pi}\sin\left(\frac{n\pi}{2}\right)$.

Thus:

$$c_n = \frac{T_0}{n\pi} \frac{\sin\frac{n\pi}{2}}{\mathrm{sinc}\left((-1)^{3/4}\sqrt{\frac{n\pi}{\kappa\tau}}\, a\right)}$$

Hence;

$$T(r,t) = \sum_{n=-\infty}^{\infty} \frac{T_0}{n\pi} \frac{\sin\frac{n\pi}{2}\ \mathrm{sinc}\left((-1)^{3/4}\sqrt{\frac{n\pi}{\kappa\tau}}\, r\right)}{\mathrm{sinc}\left((-1)^{3/4}\sqrt{\frac{n\pi}{\kappa\tau}}\, a\right)} e^{\frac{in\pi}{\tau}(t-\frac{\tau}{2})}$$

This is a plot of $T(r=0)$ as a function of t; with the sum restricted to -50 to 50.