Problem Set - 07

10/05/2020

1. **sine-Gordon Solitons** A very important non-linear PDE in addition to KdV is the sine-Gordon equation:

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \sin\phi = 0.$$

Here the sine-Gordon field $\phi = \phi(x,t)$. Find the solitary wave solution of the above equation assuming that the solution as well as its derivatives decay at large distances and times. What is the main qualitative difference of this soliton from the KdV soliton encountered in class?

Hint : Write $\phi(x,t) = f(x - vt) = f(\eta)$ *where* v *is the velocity of the soliton. As* $\eta \to \infty$, $f, \partial_\eta f, \partial_\eta^2 f \to 0$.

2. **QM at constant Electric field :** The time-independent Schrödinger's equation for a one-dimensional particle in a constant electric field, E, is of the form,

$$\left(-\partial_x^2 + Ex + \lambda\right)\psi = 0.$$

For the above differential equation find the diagonal part of the Greens function $G(x,x) = D(x)$.

(a) Use the Gelfand-Dikii method to write down the expansion of $D(x)$. Work out explicitly till order 3 at least.
Hint : The GD solution is $D(x) = \frac{1}{2\sqrt{\lambda}}\left(1 - \frac{b_1(x)}{2\lambda} + \dots\right)$ *where,* $\left(q\partial_x + \partial_x q - \frac{1}{2}\partial_x^3\right) b_n = \partial_x b_{n+1}$.

(b) Notice the pattern in the terms, use this to resum the series.
Hint : The following identity may be useful :

$$\sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} \frac{x^n}{4^n} = \frac{1}{\sqrt{1+x}}.$$

(c) Why would you expect that the answer from (b) is the right one?
Hint : $Ai(x)Bi(x) \approx 1/\sqrt{x}$, *where* Ai *and* Bi *are the Airy functions.*

3. **Non-linear Schrödinger equation** Show that with,

$$L = \begin{pmatrix} i\partial_x & \chi^* \\ \chi & i\partial_x \end{pmatrix},$$

and

$$P = \begin{pmatrix} i|\chi|^2 & \chi'^* \\ -\chi' & -i|\chi|^2 \end{pmatrix},$$

the equation,

$$L_t = [L, P].$$

corresponds to the NLSE : $i\dot{\chi} = \chi'' - 2|\chi|^2\chi$..