Coverage in Asymmetric HetNets with User-Access Restrictions

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I. System Model and Metrics

Let us assume a k tier network with each tier's BS deployed according to a PPP. BSs of ith tier are located according to a homogeneous PPP Φ_i of intensity λ_i in the Euclidean plane. For simplicity, we will take identical tiers, however each having different BS density λ_i. Mobile users are independently located according to some other stationary point process $\Phi_{i,\mathrm{u}}$.

The BSs transmit constantly with fixed power p_t to a single desired mobile user on any particular time-frequency resource, i.e. orthogonal multiple access within a cell. Therefore, the mobile user sees interference from all other BSs of all tiers in the network but not from its own desired BS. Let us consider single slope path-loss propagation with α path-loss exponent. Signals attenuate with distance according to the standard power-law path loss propagation model with path loss exponent $\alpha > 2$. Specifically, we assume that the average received power at distance r is $p_{\mathrm{rx}}(r) = p_\mathrm{t} c r^{-\alpha}$ where c is the near field gain (*i.e.*path-loss experienced at $r = 1$). Let $p = p_\mathrm{t} c$, then $p_{\mathrm{rx}}(r) = pr^{-\alpha}$.

The $\mathtt{SNR} = \frac{p}{\sigma^2}$ is defined to be the average received $\mathtt{SNR}$ at a distance of $r = 1$.

Random channel effects are incorporated by a multiplicative random value G_0 for the desired signal and G_i for ith interferer. For simplicity we assume these all correspond to Rayleigh fading with mean 1, so G_0 and $\{G_i\}$ all are iid and follow an exponential distribution with mean 1.

Consider a UE of the first network.

Denote the set of tiers that allow connection to this UE by $\mathcal{I}$.

Suppose the combined density of tiers it can connect to is $\mu_1 = \sum_{\mathcal{I}} \lambda_i$ while the combined density of tiers closed to this UE is $\mu_2 = \sum_{[1:k]\setminus\mathcal{I}} \lambda_i$. Let us denote the accessible BS PP as

$$\Psi_1 = \bigcup_{\mathbb{1}\in\mathcal{I}} \Phi_i.$$

Similarly the PP denoting the non-accessible BSs is

$$\Psi_2 = \bigcup_{\mathbb{1}\in[1:k]\setminus\mathcal{I}} \Phi_i.$$

Both Ψ_1 and Ψ_2 are homogeneous PPPs with density μ_1 and μ_2.

Assume that the user is at the origin. This user connects to the BS that provides the highest average $\mathtt{SNR}$ among the accessible BSs.

Let us assume the user is connected to the BS $\mathbf{B_o}$. Recall $\mathbf{B_o} \in \Psi_1$. Due to association criterion, this BS is the closest BS among all BSs in Ψ_1. In other words, no other BS in Ψ_1 can be closer than R. Therefore, using the void probability of Ψ_1,

$$\mathbb{P}\left[R > r\right] = \mathbb{P}\left[\text{No BS closer than } r\right] = e^{-\mu_1 \pi r^2}. \qquad (1)$$

The pdf of R is

$$f_R(r) = \frac{\mathrm{d}F_R(r)}{\mathrm{d}r} = 2\pi\mu_1 r e^{-\mu_1\pi r^2} \mathbf{1}\left(r \geq 0\right). \qquad (2)$$

Now, the desired signal power is given as

$$S = G_0 p R^{-\alpha}. \qquad (3)$$

The interference is given as

$$I = \sum_{\mathbf{X}_i\in\Psi_1\setminus\{\mathbf{B_o}\}} G_i p \|\mathbf{X}_i\|^{-\alpha} + \sum_{\mathbf{X}_i\in\Psi_2} G_i p \|\mathbf{X}_i\|^{-\alpha} \qquad (4)$$

$$I = I_1 + I_2 \qquad (5)$$

where I_1 and I_2 denotes the interference from BSs in Ψ_1 and Ψ_2.

The SINR of the mobile user at a random distance R from its associated BS can be expressed as

$$\mathtt{SINR} = \frac{G_0 p R^{-\alpha}}{\sigma^2 + I}. \qquad (6)$$

II. Coverage Probability

We are interested in deriving the probability of coverage in a downlink cellular network. The coverage probability is defined as

$$\mathrm{p_c}(\tau, \alpha) \triangleq \mathbb{P}[\mathtt{SINR} > \tau]. \qquad (7)$$

Conditioning on the serving BS being at a distance $R = r$ from the typical user, the probability of coverage relative to an SINR threshold τ can be written as

$$\mathrm{p_c}(\tau, \alpha) = \mathbb{E}_R\left[\mathbb{P}[\mathtt{SINR} > \tau \mid R = r]\right]$$
$$= \int_{r>0} \mathbb{P}[\mathtt{SINR} > \tau \mid r] f_R(r)\mathrm{d}r$$

Using the distribution $f_R(r)$, we get

$$\mathrm{p_c}(\tau, \alpha) = \int \mathbb{P}\left[\frac{G_0 p R^{-\alpha}}{\sigma^2 + I} > \tau \;\middle|\; R = r\right] e^{-\pi\mu_1 r^2} 2\pi\mu_1 r \mathrm{d}r$$
$$= 2\pi\mu_1 \int e^{-\pi\mu_1 r^2} \mathbb{P}[G_0 > \tau p^{-1} R^\alpha(\sigma^2 + I) \mid R = r] r\mathrm{d}r.$$

Using the fact that $G_0 \sim \exp(1)$, the inner probability term can be further simplified as

$$\mathbb{P}[G_0 > \tau p^{-1} R^\alpha(\sigma^2 + I) \mid R = r]$$

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$$
\begin{aligned}
&= \mathbb{E}_I \left[\mathbb{P}[G_0 > \tau p^{-1} R^{\alpha}(\sigma^2 + I) \mid R = r, I] \right] \\
&= \mathbb{E}_I \left[\exp(-\tau p^{-1} r^{\alpha}(\sigma^2 + I)) \right] \\
&= e^{-p^{-1}\tau r^{\alpha}\sigma^2} \mathcal{L}_I(\tau p^{-1} r^{\alpha}), \qquad (8)
\end{aligned}
$$

where $\mathcal{L}_I(s)$ is the interference Laplace transform. Therefore

$$
\mathrm{p_c}(\tau, \alpha) = 2\pi\mu_1 \int e^{-\pi\mu_1 r^2} e^{-\tau p^{-1} r^{\alpha}\sigma^2} \mathcal{L}_I(\tau p^{-1} r^{\alpha}) r \mathrm{d}r. \quad (9)
$$

To proceed further, we need the Laplace transform of random variable I at s conditioned on the distance $R = r$ of $\mathbf{B_o}$ from the origin, which we denote as $\mathcal{L}_I(s)$. The Laplace transform I is given as

$$
\mathcal{L}_I(s) = \mathcal{L}_{I_1}(s)\mathcal{L}_{I_2}(s).
$$

I_1 is caused by interfering BSs in Ψ_1. Due to $\mathbf{B_o}$, being the closest one among all BSs in Ψ_1, all the interfering BSs are located outside the ball $\mathcal{B}(0, R)$. Therefore,

$$
\begin{aligned}
\mathcal{L}_{I_1}(s) &= \mathbb{E}\left[e^{-sI}\right] \\
&= \mathbb{E}_{\Psi_1, \{G_i\}} \left[\exp\left(-s \sum_{\mathbf{X}_i \in \Psi_1 \setminus \{\mathbf{B_o}\}} G_i p \|\mathbf{X}_i\|^{-\alpha} \right) \right] \\
&= \mathbb{E}_{\Psi_1, \{G_i\}} \left[\prod_{\mathbf{X}_i \in \Psi_1 \setminus \{\mathbf{B_o}\}} \exp(-sG_i p \|\mathbf{X}_i\|^{-\alpha}) \right]. \qquad (10)
\end{aligned}
$$

Using the PGFL of marked PPP with respect to the function $f(\mathbf{x}, G) = \exp(-sGp\|\mathbf{x}\|^{-\alpha})$and then employing a transformation to polar coordinates $\mathbf{x} = (x, \theta)$, we get

$$
\begin{aligned}
&\mathcal{L}_{I_1}(s) = \\
&\exp\left(-2\pi\lambda \int_r^{\infty} \left(1 - \mathbb{E}_G[\exp(-sGpx^{-\alpha})]\right) x \mathrm{d}x \right). \quad (11)
\end{aligned}
$$

The integration range excludes a ball centered at 0 and radius r since the closest interferer has to be farther than the desired BS, which is at distance r. Since $G_i \sim \exp(1)$, the moment generating function of an exponential random variable gives

$$
\begin{aligned}
\mathcal{L}_{I_1}(s) &= \exp\left(-2\pi\lambda \int_r^{\infty} \left(1 - \frac{1}{1 + spx^{-\alpha}}\right) x \mathrm{d}x \right) \\
&= \exp\left(-2\pi\lambda \int_r^{\infty} \left(\frac{1}{1 + (sp)^{-1}x^{\alpha}}\right) x \mathrm{d}x \right). \quad (12)
\end{aligned}
$$

Similarly, I_2 is caused by interfering BSs in Ψ_2 which can be located anywhere in $\mathbb{R}^2$. Therefore,

$$
\begin{aligned}
\mathcal{L}_{I_2}(s) &= \mathbb{E}\left[e^{-sI}\right] \\
&= \mathbb{E}_{\Psi_2, \{G_i\}} \left[\exp\left(-s \sum_{\mathbf{X}_i \in \Psi_2} G_i p \|\mathbf{X}_i\|^{-\alpha} \right) \right] \\
&= \mathbb{E}_{\Psi_2, \{G_i\}} \left[\prod_{\mathbf{X}_i \in \Psi_2} \exp(-sG_i p \|\mathbf{X}_i\|^{-\alpha}) \right]. \qquad (13)
\end{aligned}
$$

Solving in the similar fashion, we get

$$
\begin{aligned}
\mathcal{L}_{I_2}(s) &= \exp\left(-2\pi\mu_2 \int_0^{\infty} \left(1 - \frac{1}{1 + spx^{-\alpha}}\right) x \mathrm{d}x \right) \\
&= \exp\left(-2\pi\mu_2 \int_0^{\infty} \left(\frac{1}{1 + (sp)^{-1}x^{\alpha}}\right) x \mathrm{d}x \right). \quad (14)
\end{aligned}
$$

Armed now with an expression for the Laplace transform of the interference, we proceed to the main result. From (12) and (14) we have

$$
\begin{aligned}
\mathcal{L}_I(\tau p^{-1} r^{\alpha}) = &\exp\left(-2\pi\mu_1 \int_r^{\infty} \frac{\tau}{\tau + (x/r)^{\alpha}} x \mathrm{d}x \right), \\
&\times \exp\left(-2\pi\mu_2 \int_0^{\infty} \frac{\tau}{\tau + (x/r)^{\alpha}} x \mathrm{d}x \right),
\end{aligned}
$$

and employing a change of variables $u = (x/r)^2 \tau^{-\frac{2}{\alpha}}$ results in the expression

$$
\mathcal{L}_I(\tau p^{-1} r^{\alpha}) = \exp\left(-\pi r^2 \tau^{2/\alpha} \left(\mu_1 \rho(\tau, \alpha) + \mu_2 \beta(\alpha)\right) \right), \quad (15)
$$

where

$$
\rho(\tau, \alpha) = \int_{\tau^{-2/\alpha}}^{\infty} \frac{1}{1 + u^{\alpha/2}} \mathrm{d}u, \quad \beta(\alpha) = \int_0^{\infty} \frac{1}{1 + u^{\alpha/2}} \mathrm{d}u. \quad (16)
$$

The following theorem provides the final expression for the coverage probability, which is found by plugging (15) into (9) and simplifying, with a final substitution of $v = r^2$.

Theorem 1. *The probability of coverage of a typical randomly located mobile user is*

$$
\begin{aligned}
&\mathrm{p_c}(\tau, \alpha) = \\
&\pi\mu_1 \int_0^{\infty} e^{-\pi\mu_1 v(1 + \tau^{2/\alpha}\rho(\tau,\alpha)) - \pi\mu_2 v \tau^{2/\alpha}\beta(\alpha) - \tau v^{\alpha/2}\sigma^2/p} \mathrm{d}v, \quad (17)
\end{aligned}
$$

This fairly simple integral expression already hints at some of the key dependencies on the SINR distribution in terms of the network parameters. However, it can be further simplified in three practical special cases that we now explore.

III. Special Cases

We now consider three special cases where the expression in Theorem 1 can be further simplified. These correspond to exploring the high SNR regime $\mathtt{SNR} \to 0$ – equivalently referred to as the "no noise" or "interference-limited" case – and to the case where the path loss exponent is constrained to be $\alpha = 4$, which is a fairly typical value for terrestrial propagation at moderate to large distances. There are three such combinations of these simplifications that we consider.

A. Noise still present, $\alpha = 4$

In this case, the probability of coverage can be written as

$$
\begin{aligned}
&\mathrm{p_c}(\tau, \alpha) = \\
&\pi\mu_1 \int_0^{\infty} e^{-\pi v(\mu_1(1 + \sqrt{\tau}\rho(\tau,4)) + \mu_2\sqrt{\tau}\pi/2) - \tau p^{-1}\sigma^2 v^2} \mathrm{d}v,
\end{aligned}
$$

where $\rho(\tau, 4)$ can be computed as

$$
\rho(\tau, 4) = \int_{\sqrt{\tau}}^{\infty} \frac{1}{1 + u^2} \mathrm{d}u = \arctan\sqrt{\tau}. \quad (18)
$$

B. Interference-limited, any path loss exponent

The coverage probability for the noiseless case can be easily obtained from Theorem 1 by substituting $\sigma^2 = 0$ and evaluating the now trivial $e^{ax}dx$ integral. The result is given by the following simple expression

$$\mathrm{p_c}(\tau, \lambda, \alpha) = \frac{1}{1 + \sqrt{\tau}\rho(\tau, \alpha) + (\mu_2/\mu_1)\sqrt{\tau}\beta(\alpha)}. \tag{19}$$

C. Interference-limited, $\alpha = 4$

When the path loss exponent $\alpha = 4$, the no noise coverage probability can be further simplified to

$$\mathrm{p_c}(\tau, \lambda, 4) = \frac{1}{1 + \sqrt{\tau}\arctan\sqrt{\tau} + (\mu_2/\mu_1)\sqrt{\tau}\pi/2}. \tag{20}$$