On the Distribution of Various Voronoi Association Cells in Cellular Networks

Abhishek Gupta

***Abstract*—In this paper, we compute the distribution of various Voronoi association cells in a cellular networks. We consider a cellular network with BSs located as a Poisson point process (PPP) in 1D and 2D. We compute the empirical distribution of Voronoi cells of the kth closest BSs for $k = 1 : 4$ and compare them with the distribution of typical cell. We also investigate the second order Voronoi cells of the typical, closest and second closest BSs.**

I. Introduction

In a cellular network, users are associated to BSs according to some association law. Therefore the area is divided among BS with each BS having one region (termed association cell) where all users are connected to the corresponding BS. For example, under the average received power based association (closest distance based), the association cells are given by Voronoi cells. Although analytical expressions for distribution of these cells are not available, their empirical distribution are available in the past literature [1].

There may be many applications where distribution of various other association cells may be needed. For example consider a case where each user wish to connect k closest BSs to utilize the macro diversity. In that case, it is important to characterize the area and load on the association cells of these BSs. Hence, compute the empirical distribution of Voronoi cells of the kth closest BSs for $k = 1 : 4$ and compare them with the distribution of typical cell. We also compute the additional load on the BSs serving these users which are not in their Voronoi cell, but for which the BS is the second closest BS.

II. System model

We consider a cellular network with BSs located as a Poisson point process (PPP) $\Phi = \{\mathbf{X}_i\}$ in 1D and 2D. In particular, We consider a user at the origin. which is associated to the closest BS.

The tagged BS's cell's area can be approximated as a gamma random variable with distribution

$$f_X(x) = \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-\frac{x}{\theta}} \quad \text{for } x > 0 \text{ and } k, \theta > 0 \quad (1)$$

with $k = (3n + 3)/2$ and $\theta = 2/(\lambda(3n + 1))$. Here n is the dimension of the space.

We let $\lambda = 1$ without loss of generality.

III. 1D Cellular Network

We first consider 1D deployment of BSs. See Fig. 1.

A. Gupta is with the Department of Electrical Engineering at IIT Kanpur.

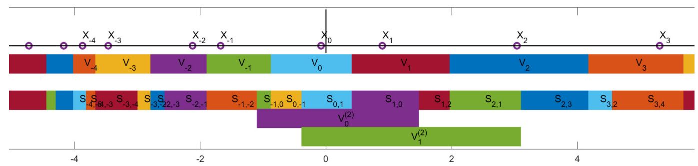

Fig. 1. 1D BS Deployment. X_i are ordered on the line. Their Voronoi cells V_i are shown. The one BS containing the origin in their Voronoi cell is numbered as 0. The second order Voronoi cells S are also marked. In particular S_{ij} denotes the region where i is the closest and j is the second closest BS. The region $V_i^{(2)}$ denotes the region where i is one of the two closest BSs.

A. kth Closest BS's Cell

In 1D BS deployment, X_i are ordered on the line. The one BS containing the origin in their Voronoi cell is numbered as 0. For ith BS, the Voronoi cell is indicated as V_i.

Lemma 1. *For 1D case, the tagged BS's cell's area is a gamma random variable with distribution*

$$f_X(x) = \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-\frac{x}{\theta}} \quad \textit{for } x > 0 \quad (2)$$

with $k = 3$ and $\theta = 1/2$.

Proof. See Appendix A. □

The PDF of the length of different Voronoi cells can be approximated using Gamma distribution

$$f_X(x) = \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-\frac{x}{\theta}} \quad \text{for } x > 0 \text{ and } k, \theta > 0 \quad (3)$$

with parameters given in the Table in Fig. 2.

Cell	Mean	k	θ
Typical Cell	1.0148	1.9801	0.5125
V_0 1st Closest Point Cell	1.4996	2.9922	0.5012
2nd Closest Point Cell	0.9974	1.9988	0.4990
3rd Closest Point Cell	0.9984	1.9971	0.4999
4th Closest Point Cell	0.9909	2.0505	0.4833

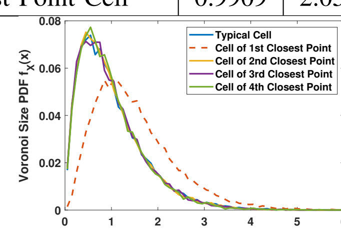

Fig. 2. Empirical Distribution of Different Voronoi Cells in 1D

Hypothesis: The first closest cell area is gamma distributed with $\Gamma(3,.5)$ with mean area 1.5. For $i>1$, ith closest cell area is gamma distributed with $\Gamma(2,.5)$ with mean area 1. The typical cell area is gamma distributed with $\Gamma(2,.5)$ with mean area 1.

B. Second Order Voronoi and Association Cell

The second order Voronoi tessellation is obtained such that each cell a particular pair (i,j) as the first two closest BSs. In particular S_{ij} denotes the region where i is the closest and j is the second closest BS. See Fig. 1.

The region $V_i^{(2)}$ denotes the region where i is one of the two closest BSs which is given as

$$V_i^{(2)} = S_{i-1,i} \cup S_{i,i-1} \cup S_{i,i+1} \cup S_{i+1,i}.$$

Let us call them M_{LL}, M_L, M_R and M_{RR} cells for the BS.

The PDF of the length of $V_i^{(2)}$ can be approximated using Gamma distribution with parameters given in the following Table:

Cell	Mean	k	θ
Typical Cell	2.03	4.00	0.51
$V_0^{(2)}$ 1st Closest Point Cell	2.49	4.91	0.51
2nd Closest Point Cell	2.49	4.94	0.51
3rd Closest Point Cell	1.99	3.97	0.50
4th Closest Point Cell	2.01	3.97	0.51

It has also been observed that 0th BSs' M_L and M_R cells $S_{0,-1}$, $S_{0,1}$ are distributed as $\Gamma(1.17,.64)$ with mean 0.75 while M_{LL} and M_{RR} cells $S_{-1,0}$, $S_{1,0}$ distributed as $\Gamma(1,.5)$ with mean 0.5 .

On the other hand, the second closest BSs' M_L and M_R cells are distributed as $\Gamma(1,.5)$ with mean 0.5 while M_{LL} and M_{RR} cells distributed as $\Gamma(1.17,.64)$ with mean 0.75 .

For rest BSs' all M cells are distributed as $\Gamma(1,.5)$ with mean 0.5. The same behavior is observed by the typical cell.

Hypothesis: The first and second $V^{(2)}$ cells are gamma distributed with $\Gamma(5,.5)$ with mean area 2.5. The rest $V^{(2)}$ cells and typical cell area is gamma distributed with $\Gamma(4,.5)$ with mean area 1.

IV. 2D Distribution

We now consider 2D deployment. See Fig. 3.

A. kth Closest BS's Cell

X_i are ordered according to their distance from the origin. Their Voronoi cells V_i are shown. The one BS closest to the origin is numbered as 0 which contain the origin in its Voronoi cell. For ith BS, the Voronoi cell is indicated as V_i. The PDF of the length of different Voronoi cells can be approximated using Gamma distribution

$$f_X(x) = \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-\frac{x}{\theta}} \quad \text{for } x>0 \text{ and } k,\theta>0 \tag{4}$$

with parameters given in the Table in Fig 4.

Observation: The first closest cell area is gamma distributed with $\Gamma(4.5,1/3.5)$ with mean area 1.28. For $i>2$,

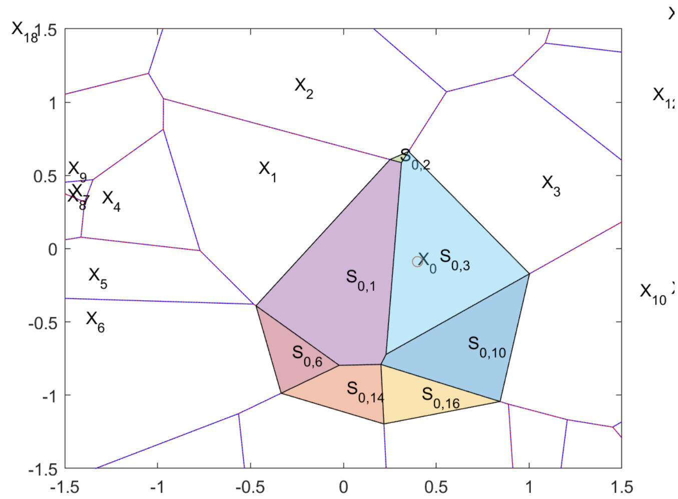

Fig. 3. 2D BS Deployment. X_i are ordered according to their distance from the origin. Their Voronoi cells V_i are shown. The one BS closest to the origin is numbered as 0 which contain the origin in its Voronoi cell. The second order Voronoi cells S are also marked. In particular S_{ij} denotes the region where i is the closest and j is the second closest BS. The region $V_i^{(2)}$ denotes the region where i is one of the two closest BSs.

Cell	Mean	k	θ
Typical Cell	1.0110	3.5282	0.2865
V_0 1st Closest Point Cell	1.2811	4.5442	0.2819
V_1 2nd Closest Point Cell	1.0546	3.6757	0.2869
V_2 3rd Closest Point Cell	1.0215	3.5254	0.2898
V_3 4th Closest Point Cell	1.0075	3.5044	0.2875

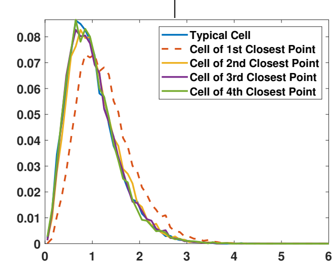

Fig. 4. Empirical Distribution of Different Voronoi Cells in 2D

ith closest cell area is gamma distributed with $\Gamma(3.5,1/3.5)$ with mean area 1. The typical cell area is gamma distributed with $\Gamma(3.5,1/3.5)$ with mean area 1.

B. Second order Voronoi

The second order Voronoi tessellation is obtained such that each cell a particular pair (i,j) as the first two closest BSs. In particular S_{ij} denotes the region where i is the closest and j is the second closest BS. See Fig. 5.

The region $V_i^{(2)}$ denotes the region where i is one of the two closest BSs which is given as

$$V_i^{(2)} = \cup_{j\neq i} S_{j,i} \cup S_{i,j}.$$

The PDF of the length of $V_i^{(2)}$ can be approximated using Gamma distribution with parameters given in the following

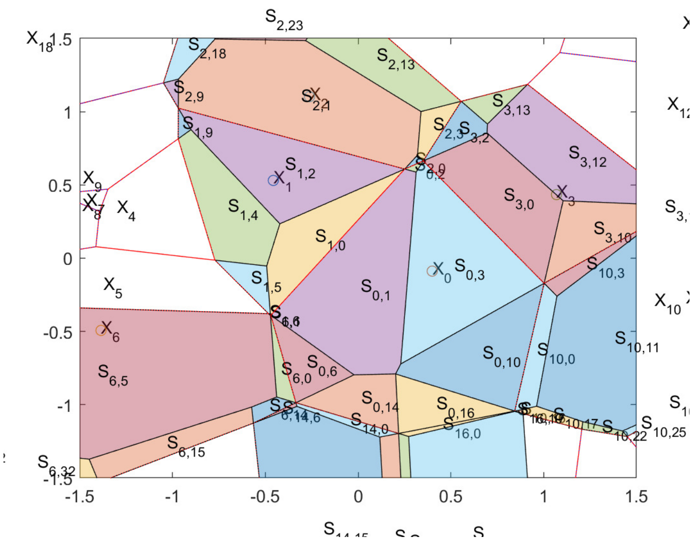

Fig. 5. 2D BS Deployment. X_i are ordered according to their distance from the origin. The region $V_0^{(2)}$ denotes the region where i is one of the two closest BSs which consists of all S_{ij} where either i or j is 0.

Table:

Cell	Mean	k	θ
$V^{(2)}$ Typical Cell	2.03	6.9	0.29
$V_0^{(2)}$ 1st Closest Point Cell	2.34	8.22	0.28
$V_1^{(2)}$ 2nd Closest Point Cell	2.23	7.98	0.28

Observation: The first $V^{(2)}$ cells are gamma distributed with $\Gamma(8.2, 1/3.5)$ with mean area 2.34. The second $V^{(2)}$ cell area is gamma distributed with $\Gamma(8, 0.38)$ with mean area 2.23. The typical $V^{(2)}$ cells are gamma distributed with $\Gamma(7, 1/3.5)$ with mean area 2.

V. UE Distribution in Gamma Cells

In this section, we will compute the distribution of number of UEs in any cell which is gamma distributed with parameter (k, θ)*i.e.*

$$f_X(x) = \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-\frac{x}{\theta}} \quad \text{for } x > 0 \text{ and } k, \theta > 0 \qquad (5)$$

We will assume the UEs are distributed as PPP with density μ which is equal to the ratio of UE density and BS density.

Since UEs are PPP, the number of UEs in an area a is Poisson distributed *i.e.*

$$\mathbb{P}[N = n] = \exp(-\mu a) \frac{1}{n} (\mu a)^n.$$

Since a is random here, hence

$$\begin{aligned}
\mathbb{P}[N = n] &= \mathbb{E}\left[\exp(-\mu a) \frac{1}{n!} (\mu a)^n\right] \\
&= \int_0^\infty \exp(-\mu a) \frac{1}{n!} (\mu a)^n \frac{1}{\theta^k \Gamma(k)} a^{k-1} e^{-\frac{a}{\theta}} \mathrm{d}a \\
&= \frac{1}{n!} \frac{1}{\theta^k \Gamma(k)} \mu^n \int_0^\infty \exp\left(-\mu a - \frac{a}{\theta}\right) a^{n+k-1} \mathrm{d}a \\
&= \frac{1}{n!} \frac{1}{\theta^k \Gamma(k)} \mu^n \frac{1}{(\mu + \frac{1}{\theta})^{n+k}} \Gamma(n+k) \\
&= \frac{\Gamma(n+k)}{n! \Gamma(k)} \left(\frac{1}{\theta}\right)^k \mu^n \frac{1}{(\mu + \frac{1}{\theta})^{n+k}}.
\end{aligned}$$

The probability that there are no users in the cell is

$$\mathbb{P}[N = 0] = \theta^{-k} \left(\mu + \frac{1}{\theta}\right)^{-k}.$$

VI. Conclusions

It can be concluded that Voronoi cells of kth closest BS for $k > 2$ is the same as the typical cell.

Appendix A
Proof of Lemma 1

Let BSs at the positive axis be denoted as $X_1, X_2 \cdots$ in the order of their distance from origin. Similarly BSs at negative side are denoted as $X_{-1}, X_{-2} \cdots$. Also denote $Y_i = \|X_{-i}\| = -X_{-i}$ for $i = 1, 2, \cdots$.

Let us first assume that $X_1 < Y_1$. Then the origin will be in the Voronoi cell of X_1. The tagged Voronoi cell V will be the Voronoi cell of the point at X_1.

As clear from the illustration, this is $\left[\frac{X_{-1}+X_1}{2}, \frac{X_1+X_2}{2}\right]$. The length of the tagged Voronoi cell is

$$\begin{aligned}
|V| &= \frac{X_1 + X_2}{2} - \frac{X_{-1} + X_1}{2} \\
&= \frac{1}{2}(X_2 - X_1) + \frac{1}{2}(X_1 - X_{-1}) \\
&= \frac{1}{2}(X_2 - X_1) + \frac{1}{2} X_1 + \frac{1}{2} Y_1
\end{aligned}$$

We know that X_1, Y_1 and $X_2 - X_1$ have an exponential distribution with parameter λ and are independent to each other.

Hence,

$$X_1 + X_2 - X_1 + Y_1 = \Gamma(3, 1/\lambda)$$

Therefore, $|V|$ has gamma distribution $\Gamma(3, 1/(2\lambda))$

References

[1] J.-S. Ferenc and Z. Néda, "On the size distribution of poisson voronoi cells," *Physica A: Statistical Mechanics and its Applications*, vol. 385, no. 2, pp. 518–526, 2007. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0378437107007546