On the Distribution of Various Voronoi Association Cells in Cellular Networks

Abhishek Gupta

Abstract—In this paper, we compute the distribution of various Voronoi association cells in a cellular networks. We consider a cellular network with BSs located as a Poisson point process (PPP) in 1D and 2D. We compute the empirical distribution of Voronoi cells of the kth closest BSs for $k = 1 : 4$ and compare them with the distribution of typical cell. We also investigate the second order Voronoi cells of the typical, closest and second closest BSs.

I. INTRODUCTION

In a cellular network, users are associated to BSs according to some association law. Therefore the area is divided among BS with each BS having one region (termed association cell) where all users are connected to the corresponding BS. For example, under the average received power based association (closest distance based), the association cells are given by Voronoi cells. Although analytical expressions for distribution of these cells are not available, their empirical distribution are available in the past literature [1].

There may be many applications where distribution of various other association cells may be needed. For example consider a case where each user wish to connect $k$ closest BSs to utilize the macro diversity. In that case, it is important to characterize the area and load on the association cells of these BSs. Hence, compute the empirical distribution of Voronoi cells of the $k$th closest BSs for $k = 1 : 4$ and compare them with the distribution of typical cell. We also compute the additional load on the BSs serving these users which are not in their Voronoi cell, but for which the BS is the second closest BS.

II. SYSTEM MODEL

We consider a cellular network with BSs located as a Poisson point process (PPP) $\Phi = \{X_i\}$ in 1D and 2D. In particular, We consider a user at the origin, which is associated to the closest BS.

The tagged BS’s cell’s area can be approximated as a gamma random variable with distribution

$$f_X(x) = \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-\frac{x}{\theta}} \quad \text{for } x > 0$$  \hspace{1cm} (2)

with $k = 3$ and $\theta = 1/2$.

Proof. See Appendix A.  \hfill $\square$

The PDF of the length of different Voronoi cells can be approximated using Gamma distribution

$$f_X(x) = \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-\frac{x}{\theta}} \quad \text{for } x > 0 \text{ and } k, \theta > 0$$  \hspace{1cm} (3)

with parameters given in the Table in Fig. 2.

<table>
<thead>
<tr>
<th>Cell</th>
<th>Mean</th>
<th>$k$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Typical Cell</td>
<td>1.0148</td>
<td>1.9801</td>
<td>0.5125</td>
</tr>
<tr>
<td>$V_0$ 1st Closest Point Cell</td>
<td>1.4996</td>
<td>2.9922</td>
<td>0.5012</td>
</tr>
<tr>
<td>2nd Closest Point Cell</td>
<td>0.9974</td>
<td>1.9988</td>
<td>0.4990</td>
</tr>
<tr>
<td>3rd Closest Point Cell</td>
<td>0.9984</td>
<td>1.9971</td>
<td>0.4999</td>
</tr>
<tr>
<td>4th Closest Point Cell</td>
<td>0.9999</td>
<td>2.0505</td>
<td>0.4833</td>
</tr>
</tbody>
</table>

III. 1D CELLULAR NETWORK

We first consider 1D deployment of BSs. See Fig. 1.

A. Gupta is with the Department of Electrical Engineering at IIT Kanpur.
Hypothesis: The first closest cell area is gamma distributed with \( \Gamma(3, 5) \) with mean area 1.5. For \( i > 1 \), \( i \)th closest cell area is gamma distributed with \( \Gamma(2, 5) \) with mean area 1. The typical cell area is gamma distributed with \( \Gamma(2, .5) \) with mean area 1.

B. Second Order Voronoi and Association Cell

The second order Voronoi tessellation is obtained such that each cell a particular pair \((i, j)\) as the first two closest BSs. In particular \( S_{ij} \) denotes the region where \( i \) is the closest and \( j \) is the second closest BS. See Fig. 1.

The region \( V_i^{(2)} \) denotes the region where \( i \) is one of the two closest BSs which is given as

\[
V_i^{(2)} = S_{i-1,i} \cup S_{i,i-1} \cup S_{i,i+1} \cup S_{i+1,i}.
\]

Let us call them \( M_{LL}, M_L, M_R \) and \( M_{RR} \) cells for the BS.

The PDF of the length of \( V_i^{(2)} \) can be approximated using Gamma distribution with parameters given in the following Table:

<table>
<thead>
<tr>
<th>Cell</th>
<th>Mean</th>
<th>( k )</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Typical Cell</td>
<td>2.03</td>
<td>4.00</td>
<td>0.51</td>
</tr>
<tr>
<td>( V_0^{(2)} ) 1st Closest Point Cell</td>
<td>2.49</td>
<td>4.91</td>
<td>0.51</td>
</tr>
<tr>
<td>2nd Closest Point Cell</td>
<td>2.49</td>
<td>4.94</td>
<td>0.51</td>
</tr>
<tr>
<td>3rd Closest Point Cell</td>
<td>1.99</td>
<td>3.97</td>
<td>0.50</td>
</tr>
<tr>
<td>4th Closest Point Cell</td>
<td>2.01</td>
<td>3.97</td>
<td>0.51</td>
</tr>
</tbody>
</table>

It has also been observed that 0th BSs’ \( M_L \) and \( M_R \) cells \( S_{0,-1}, S_{0,1} \) are distributed as \( \Gamma(1.17, .64) \) with mean 0.75 while \( M_{LL} \) and \( M_{RR} \) cells \( S_{-1,0}, S_{1,0} \) distributed as \( \Gamma(1, 5) \) with mean 0.5.

On the other hand, the second closest BSs’ \( M_L \) and \( M_R \) cells are distributed as \( \Gamma(1, 5) \) with mean 0.5 while \( M_{LL} \) and \( M_{RR} \) cells distributed as \( \Gamma(1.17, .64) \) with mean 0.75.

For rest BSs’ all \( M \) cells are distributed as \( \Gamma(1, 5) \) with mean 0.5. The same behavior is observed by the typical cell.

Hypothesis: The first and second \( V^{(2)} \) cells are gamma distributed with \( \Gamma(5, .5) \) with mean area 2.5. The rest \( V^{(2)} \) cells and typical cell area is gamma distributed with \( \Gamma(4, .5) \) with mean area 1.

IV. 2D DISTRIBUTION

We now consider 2D deployment. See Fig. 3.

A. \( k \)th Closest BS’s Cell

\( X_i \) are ordered according to their distance from the origin. Their Voronoi cells \( V_i \) are shown. The one BS closest to the origin is numbered as 0 which contain the origin in its Voronoi cell. For \( i \)th BS, the Voronoi cell is indicated as \( V_i \). The PDF of the length of different Voronoi cells can be approximated using Gamma distribution

\[
f_X(x) = \frac{1}{\theta \Gamma(k)} x^{k-1} e^{-\frac{x}{\theta}} \quad \text{for } x > 0 \text{ and } k, \theta > 0 \quad (4)
\]

with parameters given in the Table in Fig 4.

Observation: The first closest cell area is gamma distributed with \( \Gamma(4.5, 1/3.5) \) with mean area 1.28. For \( i > 2 \), \( i \)th closest cell area is gamma distributed with \( \Gamma(3.5, 1/3.5) \) with mean area 1. The typical cell area is gamma distributed with \( \Gamma(3.5, 1/3.5) \) with mean area 1.

B. Second order Voronoi

The second order Voronoi tessellation is obtained such that each cell a particular pair \((i, j)\) as the first two closest BSs. In particular \( S_{ij} \) denotes the region where \( i \) is the closest and \( j \) is the second closest BS. See Fig. 5.

The region \( V_i^{(2)} \) denotes the region where \( i \) is one of the two closest BSs which is given as

\[
V_i^{(2)} = \cup_{j \neq i} S_{j,i} \cup S_{i,j}.
\]

The PDF of the length of \( V_i^{(2)} \) can be approximated using Gamma distribution with parameters given in the following
The typical area $V$ which consists of all BSs within radius $R$ of the origin. The region $V_0^{(2)}$ which is equal to the ratio of UE density and BS density.

We will assume the UEs are distributed as PPP with density $\lambda$. Since UEs are PPP, the number of UEs in an area $a$ is Poisson distributed, i.e.

$$f_X(x) = \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-\frac{x}{\theta}} \quad \text{for } x > 0 \text{ and } k, \theta > 0 \quad (5)$$

We will assume the UEs are distributed as PPP with density $\mu$ which is equal to the ratio of UE density and BS density. Since UEs are PPP, the number of UEs in an area $a$ is Poisson distributed, i.e.

$$\mathbb{P}[N = n] = \exp(-\mu a) \frac{1}{n!} (\mu a)^n.$$

Since $a$ is random here, hence

$$\mathbb{P}[N = n] = \mathbb{E} \left[ \exp(-\mu a) \frac{1}{n!} (\mu a)^n \right]$$

$$= \int_0^\infty \exp(-\mu a) \frac{1}{n!} (\mu a)^n \frac{1}{\theta^k \Gamma(k)} a^{k-1} e^{-\frac{a}{\theta}} da$$

$$= \frac{1}{n!} \frac{1}{\theta^k \Gamma(k)} \frac{1}{\mu^k} \int_0^\infty \exp \left( -\mu a - \frac{a}{\mu} \right) a^{n+k-1} da$$

$$= \frac{1}{n!} \frac{1}{\theta^k \Gamma(k)} \frac{1}{\mu^k} \frac{1}{(\mu + 1/\theta)^{n+k}} \Gamma(n+k)$$

$$= \frac{1}{n! \theta^k \Gamma(k)} \frac{1}{(\mu + 1/\theta)^{n+k}} \Gamma(n+k)$$

The probability that there are no users in the cell is

$$\mathbb{P}[N = 0] = \theta^{-k} \left( \mu + \frac{1}{\theta} \right)^{-k}.$$

### VI. Conclusions

It can be concluded that Voronoi cells of $k$th closest BS for $k > 2$ is the same as the typical cell.

### Appendix A

**Proof of Lemma 1**

Let BSs at the positive axis be denoted as $X_1, X_2 \ldots$ in the order of their distance from origin. Similarly BSs at negative side are denoted as $X_{-1}, X_{-2} \ldots$. Also denote $Y_i = \|X_i\| = |X_{-i}|$ for $i = 1, 2, \ldots$.

Let us first assume that $X_1 < Y_1$. Then the origin will be in the Voronoi cell of $X_1$. The tagged Voronoi cell $V$ will be the Voronoi cell of the point at $X_1$.

As clear from the illustration, this is $\left[ \frac{X_{-1} + X_1}{2}, \frac{X_{-1} + X_2}{2} \right]$. The length of the tagged Voronoi cell is

$$|V| = \frac{X_1 + X_2}{2} - \frac{X_{-1} + X_1}{2}$$

$$= \frac{1}{2} (X_2 - X_1) + \frac{1}{2} (X_1 - X_{-1})$$

$$= \frac{1}{2} (X_2 - X_{-1}) + \frac{1}{2} X_1 + \frac{1}{2} Y_1$$

We know that $X_1, Y_1$ and $X_2 - X_1$ have an exponential distribution with parameter $\lambda$ and are independent to each other.

Hence,

$$X_1 + X_2 - X_1 + Y_1 = \Gamma(3, 1/\lambda)$$

Therefore, $|V|$ has gamma distribution $\Gamma(3, 1/(2\lambda))$

### References