Proof that operator norm is a norm

January 7, 2013

Some definitions:

1. Any norm $\|.\|$ (vector or matrix) satisfies the following properties:

$$\begin{aligned}
&(\text{P1}) && \|\mathbf{x}\| \geq 0 && \forall \ \ \mathbf{x} && (1)\\
&(\text{P2}) && \|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = 0 && && (2)\\
&(\text{P3}) && \|t\mathbf{x}\| = |t| \, \|\mathbf{x}\| && \forall \ \ t \in \mathbb{R}, \mathbf{x} && (3)\\
&(\text{P4}) && \|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| && \forall \ \ \mathbf{x}, \mathbf{y} && (4)
\end{aligned}$$

2. The operator norm for a matrix $\mathbf{A}$ is defined as

$$\|\mathbf{A}\|_{a,b} := \sup_{\|\mathbf{x}\|_b \leq 1} \|\mathbf{A}\mathbf{x}\|_a \tag{5}$$

for any two vector norms $\|.\|_a$ and $\|.\|_b$ which satisfy P1-P4.

3. The $\sup$ notation is defined as (for any function f and constraint set $\mathcal{X}$)

$$\sup_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}) \geq f(\mathbf{v}) \ \ \forall \ \ \mathbf{v} \in \mathcal{X} \tag{6}$$

For example,

$$\sup_{\|\mathbf{x}\|_b \leq 1} \|\mathbf{A}\mathbf{x}\|_a \geq \|\mathbf{A}\mathbf{v}\|_a \ \ \forall \ \ \|\mathbf{v}\|_b \leq 1 \tag{7}$$

The operator norm satisfies all the four properties of a norm.
(P1) This one is trivial.

$$\begin{aligned}
\|\mathbf{A}\mathbf{x}\|_a &\geq 0 && \text{for all } \mathbf{A} \text{ and } \mathbf{x} && (8)\\
\Rightarrow \sup_{\mathbf{x}} \|\mathbf{A}\mathbf{x}\|_a &\geq 0 && \text{for all } \mathbf{A} && (9)\\
\Rightarrow \sup_{\|\mathbf{x}\|_b \leq 1} \|\mathbf{A}\mathbf{x}\|_a &\geq 0 && \text{for all } \mathbf{A} && (10)
\end{aligned}$$

(P2) First consider $\mathbf{A} = \mathbf{0}$, the all-zero matrix. Clearly,

$$\|\mathbf{0}\|_{a,b} = \sup_{\|\mathbf{x}\|_b \leq 1} \|\mathbf{0x}\|_a = 0 \tag{11}$$

Next suppose that $\|\mathbf{A}\|_{a,b} = 0$ for some $\mathbf{A}$. Then,

$$\begin{aligned}
\sup_{\|\mathbf{x}\|_b \leq 1} \|\mathbf{Ax}\|_a = 0 & & (12)\\
\Rightarrow \|\mathbf{Av}\|_a = 0 & \quad \text{for all } \|\mathbf{v}\|_b \leq 1\text{, by definition} & (13)\\
\Rightarrow \mathbf{Av} = 0 & \quad \text{from (P2), for all } \|\mathbf{v}\|_b \leq 1 & (14)\\
\Rightarrow \mathbf{A} = \mathbf{0} & & (15)
\end{aligned}$$

The last step follows since, if $\mathbf{A}$ has even a single non-zero entry, it will always be possible to find an $\mathbf{v} : \|\mathbf{v}\|_b \leq 1$, such that $\mathbf{Av} \neq 0$.

(P3) Trivial, since

$$\sup_{\|\mathbf{x}\|_b \leq 1} \|t\mathbf{Ax}\|_a = \sup_{\|\mathbf{x}\|_b \leq 1} |t| \, \|\mathbf{Ax}\|_a = |t| \sup_{\|\mathbf{x}\|_b \leq 1} \|\mathbf{Ax}\|_a \tag{16}$$

from (P3) for vector norm.

(P4) Consider the rhs

$$\begin{aligned}
\|\mathbf{A}\|_{a,b} + \|\mathbf{B}\|_{a,b} &= \sup_{\|\mathbf{x}\|_b \leq 1} \|\mathbf{Ax}\|_a + \sup_{\|\mathbf{y}\|_b \leq 1} \|\mathbf{By}\|_a & \text{by definition} \quad (17)\\
&\geq \|\mathbf{Av}\|_a + \|\mathbf{Bu}\|_a & \forall \;\; \|\mathbf{v}\|_b \leq 1, \|\mathbf{u}\|_b \leq 1 \quad (18)
\end{aligned}$$

This statement holds for all $\mathbf{u}$, $\mathbf{v}$ such that $\|\mathbf{v}\|_b \leq 1, \|\mathbf{u}\|_b \leq 1$. Therefore it also holds when $\mathbf{u} = \mathbf{v}$ with $\|\mathbf{u}\|_b \leq 1$, i.e.,

$$\begin{aligned}
\|\mathbf{A}\|_{a,b} + \|\mathbf{B}\|_{a,b} &\geq \|\mathbf{Au}\|_a + \|\mathbf{Bu}\|_a & \forall \;\; \|\mathbf{u}\|_b \leq 1 \quad (19)\\
&\geq \|\mathbf{Au} + \mathbf{Bu}\|_a & \forall \;\; \|\mathbf{u}\|_b \leq 1 \;\; \text{from (P4)} \quad (20)
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\mathbf{A}\|_{a,b} + \|\mathbf{B}\|_{a,b} &\geq \sup_{\|\mathbf{u}\|_b \leq 1} \|\mathbf{Au} + \mathbf{Bu}\|_a & (21)\\
&= \|\mathbf{A} + \mathbf{B}\|_{a,b} & (22)
\end{aligned}$$