Vibrations in a granular sphere

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1 Introduction

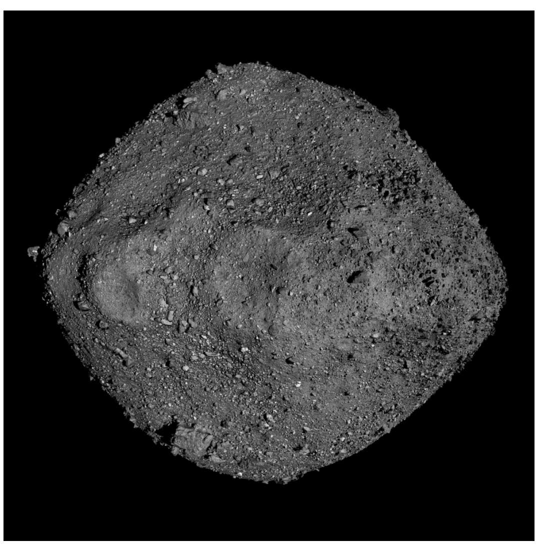
Asteroids that are smaller than a few kilometres in size are typically classified as rubble piles—collections of loose grains held together by self-gravity. The most common configuration for these rubble-pile asteroids is a top-like shape, as depicted in Fig. 1a. Due to their small size, these asteroids have extremely weak gravity and thus lack an atmosphere. Without an atmosphere, any space debris that enters the asteroid's vicinity directly impacts its surface, forming craters and causing seismic shaking. This seismic activity can, in turn, trigger further movement of the surface material.

Most of the impactor energy is spent in the cratering process. However a small percentage of impactor's kinetic energy do travel as seismic energy. The seismic efficiency,

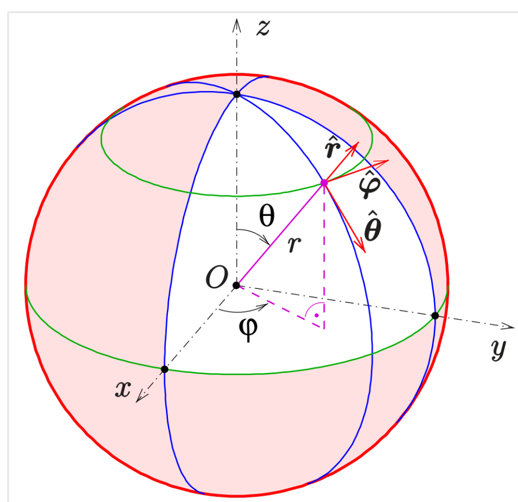
$$\eta = \frac{\text{Seismic energy}}{\text{Impactor's kinetic energy}}, \tag{1}$$

is generally of the order of $10^{-5} - 10^{-7}$. Since the size of the asteroid is very small, it is feasible that the seismic energy travels through out the body and we observe a seismic reverberation. An interesting consequence of this global reverberation is the motion of the surface materials due to seismic shaking induced fluidization. However, the height of the surface layer that becomes unstable during such seismic events remains uncertain.

To address these questions, it is essential to first understand the vibrational modes of a rotating, self-gravitating granular sphere. Therefore, in this manuscript, we will investigate the vibrations of a granular sphere using elastic theory.



(a)



(b)

Figure 1: (a) Asteroid 101955 Bennu, which was recently visited by the OSIRIS-REx spacecraft, which also returned with samples from its surface. (b) A schematic representation of spherical coordinates is shown, where θ represents the meridional angle, and ϕ represents the azimuthal angle. Source: Google

2 Stress inside a gravitating sphere

In the case of a linear elastic body, the stress state can be determined by employing equilibrium conditions and compatibility relations (or constitutive models when using displacement fields). However, for a granular material, such strain-stress relationships do not exist due to the presence of both elastic and plastic deformation components. Even if we consider only elastic deformation, the elastic moduli depend on pressure, making the stress-strain relations inherently non-linear.[Wal87, JL03, MGJS04, AFP13]. To find the stress state inside a gravitating sphere, We will use three different approaches:

1. Linear elastic model with effective moduli
2. Non-linear elastic model
3. Mohr-Coulomb criteria.

Motivated by the geometry of the problem, we will use spherical coordinates (r, θ, ϕ) as shown in Fig. 1b. The strain-displacement relations in spherical coordinates are given as follows,

$$\begin{aligned}
&\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \qquad \epsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \\
&\epsilon_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta\cos\theta}{r\sin\theta}, \quad \epsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \\
&\epsilon_{r\phi} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right), \quad \epsilon_{\theta\phi} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi\cos\theta}{r\sin\theta}\right),
\end{aligned} \tag{2}$$

where u_r, u_θ, and u_ϕ are the negative displacement components in the r, θ, and ϕ directions, respectively. Please note that as grains can not exert tension, we have taken compression as positive and extension as negative. Similarly, compressive forces will be taken positive and extensional forces as negative everywhere in the subsequent sections.

Since, the only force is gravitation which depends solely on r, we expect that the displacements u_i are independent of θ and ϕ. Using this, we obtain the strain field as

$$\begin{aligned}
&\epsilon_{rr} = \frac{du_r}{dr}, \quad \epsilon_{\theta\theta} = \frac{u_r}{r}, \quad \epsilon_{\phi\phi} = \frac{u_r}{r} + \cot\theta\frac{u_\phi}{r} \\
&\epsilon_{r\theta} = \frac{1}{2}\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right), \quad \epsilon_{r\phi} = \frac{1}{2}\left(\frac{du_\phi}{dr} - \frac{u_\phi}{r}\right), \quad \epsilon_{\theta\phi} = -\frac{u_\phi\cos\theta}{2r\sin\theta}.
\end{aligned} \tag{3}$$

Since, there is a point symmetry about the centre in the problem, we expect $u_\phi = u_\theta$ and $\epsilon_{\theta\theta} = \epsilon_{\phi\phi}$, which is only possible when $u_\phi = u_\theta = 0$. This further simplifies the strain-displacement relation to

$$\epsilon_{rr} = \frac{\mathrm{d}u_r}{\mathrm{d}r}, \qquad \epsilon_{\theta\theta} = \epsilon_{\phi\phi} = \frac{u_r}{r}, \qquad \epsilon_{r\theta} = \epsilon_{\theta\phi} = \epsilon_{r\phi} = 0. \tag{4}$$

Similarly, the equilibrium conditions in the spherical coordinates reduces to

$$\frac{\mathrm{d}\sigma_{rr}}{dr} + \frac{2}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) + \rho g(r) = 0. \tag{5}$$

Note that the stress state in a granular body is history dependent and since the formation history of these asteroids are not known precisely, any stress calculation is an approximation. Assuming accretion (symmetric deposition layer by layer) as the formation mechanism for these asteroids, [BG63] has derived a stress state using linear elastic model.

2.1 Linear model

We assume that the granular sphere can be modeled as a linear elastic body with effective Lamé parameters λ_e and μ_e.

Using the constitutive relations for linear elastic material and strain field given by (4), the stress components become

$$\sigma_{rr} = (\lambda_e + 2\mu_e)\frac{\mathrm{d}u_r}{\mathrm{d}r} + 2\lambda_e \frac{u_r}{r}, \qquad \sigma_{\theta\theta} = \sigma_{\phi\phi} = \lambda_e \frac{\mathrm{d}u_r}{\mathrm{d}r} + 2(\lambda_e + \mu_e)\frac{u_r}{r}, \qquad \sigma_{r\theta} = \sigma_{\theta\phi} = \sigma_{r\phi} = 0. \quad (6)$$

Substituting above in the equilibrium relation (5), we obtain

$$\frac{\mathrm{d}^2 u_r}{dr^2} + \frac{2}{r}\frac{\mathrm{d}u_r}{\mathrm{d}r} - \frac{2}{r^2}u_r = -\frac{\rho g(r)}{\lambda_e + 2\mu_e}. \quad (7)$$

The equation is non-homogeneous and hence contains a homogenous solution u_r^h and a particular solution u_r^p. The homogeneous part of the above equation is a *Cauchy-Euler* equation and hence can be solved by substituting $u_r = r^m$, which gives

$$u_r^h = \frac{C_1}{r^2} + C_2 r. \quad (8)$$

Since, we want the solution to remain finite at $r = 0$, we conclude $C_1 = 0$. For getting the particular solution of (7), we substitute $t = \ln r$ in it and simplify to get

$$\frac{\mathrm{d}^2 u_r}{dt^2} + \frac{\mathrm{d}u_r}{\mathrm{d}t} - 2u_r = -\frac{\rho r^2 g(r)}{\lambda_e + 2\mu_e} = -\frac{4\pi G\rho^2 e^{3t}}{3(\lambda_e + 2\mu_e)}, \quad (9)$$

where we have replaced $g(r)$ by the gravitational force inside a sphere. Since (9) is a second order constant coefficient ODE, we can find the particular solution using method of undetermined coefficients ([Kre07], pp. 79). Combining the particular and general solution of (7), we obtain

$$u_r = -\frac{2}{15(\lambda_e + 2\mu_e)}\pi G\rho^2 r^3 + C_1 r, \quad (10)$$

where constant C_1 can be found by the condition that the top surface is traction free $\sigma_{rr} = 0$. The displacement, strain, and stress components are given by:

$$u_r = \frac{2\pi G\rho^2 r}{15(\lambda_e + 2\mu_e)}\left[\frac{5\lambda_e + 6\mu_e}{3\lambda_e + 2\mu_e}R^2 - r^2\right], \quad (11)$$

$$e_{rr} = \frac{2\pi G\rho^2 r}{15(\lambda_e + 2\mu_e)}\left[\frac{5\lambda_e + 6\mu_e}{3\lambda_e + 2\mu_e}R^2 - 3r^2\right], \quad (12)$$

$$e_{\theta\theta} = e_{\phi\phi} = \frac{2\pi G\rho^2}{15(\lambda_e + 2\mu_e)}\left[\frac{5\lambda_e + 6\mu_e}{3\lambda_e + 2\mu_e}R^2 - r^2\right], \quad (13)$$

$$\sigma_{rr} = \frac{2\pi G\rho^2(5\lambda_e + 6\mu_e)}{15(\lambda_e + 2\mu_e)}\left(R^2 - r^2\right), \quad (14)$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{2\pi G\rho^2}{15(\lambda_e + 2\mu_e)}\left[(5\lambda_e + 6\mu_e)R^2 - (5\lambda_e + 2\mu_e)r^2\right], \quad (15)$$

where R is the radius of the sphere. The expression can transform to exactly match the solution in [BG63].

The deformation is not homogenous except near the centre. Hence, assuming a homogenous deformation for calculating the stress field is not correct as usually done for rubblepiles. A counter intuitive result from (14)-(15) is that the stress field is not hydrostatic ($\sigma_{rr} \neq \sigma_{\theta\theta}$) unless $\mu_e = 0$. Also, in soil mechanics, the *lithostatic* stress is given by

$$\sigma_{rr} = \rho g h = \frac{4\pi}{3}G\rho^2 R(R - r). \quad (16)$$

To make (14) and (15) consistent, we assume $\mu_e = 0$, which reduces (11)-(15) to

$$u_r = \frac{2\pi G\rho^2 r}{15\lambda_e}\left[\frac{5}{3}R^2 - r^2\right], \tag{17}$$

$$e_{rr} = \frac{2\pi G\rho^2 r}{15\lambda_e}\left[\frac{5}{3}R^2 - 3r^2\right], \tag{18}$$

$$e_{\theta\theta} = e_{\phi\phi} = \frac{2\pi G\rho^2}{15\lambda_e}\left[\frac{5}{3}R^2 - r^2\right], \tag{19}$$

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{2\pi G\rho^2}{3}\left(R^2 - r^2\right). \tag{20}$$

The solution for the stress matches with that given in [Hol01, KBW05] and is now lithostatic. However, since we have assumed that the effective shear modulus is zero, the rubblepile will not be able to transmit shear waves. This is in contradiction with the experiments where shear waves are observed [ZAN07].

2.2 Non-linear elastic model

Motivated by the free energy for the linear elastic material, [JL03] proposed a free energy for the granular elastic material,

$$w(\epsilon_{ij}) = \mathcal{B}\sqrt{\Delta}\left(\frac{2}{5}\Delta^2 + \frac{\epsilon_s^2}{\xi}\right), \tag{21}$$

where following definitions are used

$$\Delta \equiv \epsilon_{ll}, \quad \epsilon_s^2 \equiv \epsilon_{ij}^0\epsilon_{ij}^0, \quad \epsilon_{ij}^0 \equiv \epsilon_{ij} - \frac{1}{3}\Delta\delta_{ij}. \tag{22}$$

Here $\mathcal{B}$ and ξ are material parameters. Using the free energy w, we can write stress as,

$$\sigma_{ij} = \frac{\partial w}{\partial \epsilon_{ij}} = K\Delta\delta_{ij} + 2\mu\epsilon_{ij}^0, \tag{23}$$

where the bulk modulus K and shear modulus μ are given as

$$K = \mathcal{B}\sqrt{\Delta}\left(1 + \frac{1}{2}\frac{\epsilon_s^2}{\Delta^2\xi}\right), \quad \mu = \sqrt{\Delta}\frac{\mathcal{B}}{\xi}. \tag{24}$$

Note that the elastic moduli are now strain dependent and hence the constitutive relations are non-linear. ϵ_{ij} are only the elastic strains while the relations (4) are for total strains. We will assume that the equilibrium is attained very slowly such that the plastic strain always remained zero and the elastic strain is approximately equal to the total strain. With this assumption, we can express elastic moduli using (24) as

$$K = \mathcal{B}\sqrt{\frac{\mathrm{d}u_r}{\mathrm{d}r} + \frac{2u_r}{r}}\left(1 + \frac{1}{3}\frac{\left(\frac{\mathrm{d}u_r}{\mathrm{d}r} - \frac{u_r}{r}\right)^2}{\left(\frac{\mathrm{d}u_r}{\mathrm{d}r} + \frac{2u_r}{r}\right)^2\xi}\right) \tag{25}$$

$$\mu = \frac{\mathcal{B}}{\xi}\sqrt{\frac{\mathrm{d}u_r}{\mathrm{d}r} + \frac{2u_r}{r}}. \tag{26}$$

Substituting these elastic moduli in (23), we get

$$\sigma_{rr} = \frac{\mathcal{B}\left(r^2\left(\xi+\frac{5}{3}\right)\left(\frac{\mathrm{d}u_r}{\mathrm{d}r}\right)^2 + 4r\left(\xi+\frac{1}{6}\right)u_r\frac{\mathrm{d}u_r}{dr} + 4u_r^2\left(\xi-\frac{7}{12}\right)\right)}{r^{\frac{3}{2}}\sqrt{\frac{\mathrm{d}u_r}{\mathrm{d}r}r + 2u_r\,\xi}} \tag{27}$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{\mathcal{B}\left(r^2\left(\xi-\frac{1}{3}\right)\left(\frac{\mathrm{d}u_r}{\mathrm{d}r}\right)^2 + 4r\left(\xi-\frac{1}{3}\right)u_r\frac{\mathrm{d}u_r}{dr} + 4u_r^2\left(\xi+\frac{5}{12}\right)\right)}{r^{\frac{3}{2}}\sqrt{\frac{\mathrm{d}u_r}{\mathrm{d}r}r + 2u_r\,\xi}}. \tag{28}$$

Substituting above equations in the equilibrium equation (5), we obtain the non-linear system of ODE for the displacement field u_r and strain $y = \mathrm{d}u_r/\mathrm{d}r$ as,

$$\begin{aligned}\left[\left(4\xi+\frac{5}{3}\right)u_r^2 + 4\left(\xi+\frac{7}{6}\right)ryu_r + r^2y^2\left(\xi+\frac{5}{3}\right)\right]\frac{\mathrm{d}y}{\mathrm{d}r} + \frac{2\rho g}{3\mathcal{B}}\sqrt{r}\xi\left(2u_r+ry\right)^{\frac{3}{2}}&\\ +\frac{2}{r^2}\left(\left(4\xi+3\right)u_r^2 + 4ry(\xi+1)\,u_r + r^2y^2\left(\xi+1\right)\right)\left(yr-u_r\right) &= 0 \qquad (29)\\ \frac{\mathrm{d}u_r}{\mathrm{d}r} - y &= 0. \qquad (30)\end{aligned}$$

Also, the boundary conditions are

$$\begin{aligned} u_r &= 0 \qquad && r = 0\\ \sigma_{rr} &= 0 \qquad && r = R. \end{aligned} \tag{31}$$

Solving this system of ODE is not easy because (a) (29) is not well behaved near $r = 0$ and (b) the boundary conditions are known at two different boundaries and hence we need to use shooting method. As, we cannot use boundary conditions at $r = 0$, we need to obtain the solution for u_r when $r \to 0$. Near centre, the gravitational force is zero and using symmetry about $r = 0$, we obtain $\mathrm{d}\sigma_{ij}/\mathrm{d}r = 0$. Hence from the equilibrium equation (5), we can observe that $\sigma_{rr} = \sigma_{\theta\theta}$. Equating stresses from (27) and (28), we get

$$\frac{\mathrm{d}u_r}{\mathrm{d}r} - \frac{u_r}{r} = C_1 \implies \frac{u_r}{r} = C_2 + C_1 \ln r. \tag{32}$$

The second term goes to negative infinity as $r \to 0$ which implies $C_2 = 0$ and we get the displacement field same as (8).

The second boundary condition when used with (27), gives

$$y = \frac{\left(-6\xi - 1 + 3\sqrt{-3\xi+4}\right)u_r}{\left(3\xi+5\right)R}, -\frac{\left(6\xi + 1 + 3\sqrt{-3\xi+4}\right)u_r}{\left(3\xi+5\right)R}. \tag{33}$$

For a typical cohesionless material with the friction angle 30°, the material parameter $\xi = 5/3$. Using this, we find that $\sigma_{rr} = 0$ at the surface happens only when $y = u_r = 0$. For the case of high friction angles, ξ can be less than 5/3 and we may have multiple boundary conditions. Hence, at the surface $\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\phi\phi} = \Delta = 0$. The condition $\Delta = 0$ implies

$$y = -2\frac{u_r}{r}. \tag{34}$$

As $\Delta \to 0$ pressure becomes zero and there is possibility that the stress state reaches the Coulomb yield surface given by

$$\frac{\sigma_s}{P} = \sqrt{\frac{2}{\xi}}, \quad \text{where} \quad P = \frac{\sigma_{ll}}{3}, \quad \sigma_s^2 = \sigma_{ij}^0\sigma_{ij}^0, \quad \text{and} \quad \sigma_{ij}^0 = \sigma_{ij} - P\delta_{ij}. \tag{35}$$

Substituting (27) and (28) in the above equation, we obtain yield criteria as

$$y = -\frac{\left(6\sqrt{\frac{1}{\xi}}\,\xi - \sqrt{\frac{1}{\xi}} - \sqrt{3}\right)u_r}{\left(3\sqrt{\frac{1}{\xi}}\,\xi + \sqrt{\frac{1}{\xi}} - 2\sqrt{3}\right)r}, -\frac{\left(6\sqrt{\frac{1}{\xi}}\,\xi - \sqrt{\frac{1}{\xi}} + \sqrt{3}\right)u_r}{\left(3\sqrt{\frac{1}{\xi}}\,\xi + \sqrt{\frac{1}{\xi}} + 2\sqrt{3}\right)r}, \tag{36}$$

which for the case of $\xi = 5/3$ becomes

$$y = -1.073\frac{u_r}{r}, \; -4.427\frac{u_r}{r}. \tag{37}$$

This implies that before pressure going to zero the material will yield and hence the elastic theory described above will not work beyond this point. Therefore, the non-linear elastic theory is only valid till a radius after which the yielding happens. This radius at which the yield happens is not known and hence (37) can not be used as a boundary condition , a case similar to (32) where C_2 is unknown. Beyond this radius, we can estimate stress using lithostatic model

$$\sigma_{rr} = \frac{4\pi}{3}G\rho^2(R-r), \qquad \sigma_{\theta\theta} = \sigma_{\phi\phi} = k\sigma_{rr}, \tag{38}$$

where k is the Earth-pressure coefficient [Ned92]. Note that the second condition in (37) is irrelevant since material will yield due to first condition.

We can estimate the value of C_2 from the stress relation (20) at $r \to 0$, which gives

$$C_2 = \left(\frac{2\pi}{9\sqrt{3}\mathcal{B}}G\rho^2R^2\right)^{\frac{2}{3}}. \tag{39}$$

Using this as a reference value, we will apply the shooting method, accepting the solution where yielding occurs at a point beneath the surface such that σ_{rr}, as described by equations (27) and (38), are equal. This continuity of σ_{rr} gives the condition

$$\frac{18u_r^{\frac{3}{2}}\left(81\xi^3 - 90\xi^2 + 27\sqrt{3}\,\xi^{\frac{7}{2}} - 45\sqrt{3}\,\xi^{\frac{3}{2}} - 2\sqrt{3}\,\sqrt{\xi} - 27\xi\right)\mathcal{B}}{r^{\frac{3}{2}}\left(\sqrt{3}\,\sqrt{\xi}+1\right)^{\frac{3}{2}}\xi\left(3\xi + 2\sqrt{3}\,\sqrt{\xi}+1\right)\left(3\sqrt{\xi}+\sqrt{3}\right)^3} = \frac{4\pi}{3}G\rho^2(R-r) \tag{40}$$

Figure 2a shows the solution for strains u_r/r and y. The strains ϵ_{ij} decreases monotonously with the radius. At $r = 0.99$, $y \to 1.073$ and hence the material yields. After this point the elastic theory fails and we observe oscillation in the strains. Figure 2b compares the stress calculated using linear elastic and non-linear elastic model. At the surface σ_{rr} coincides with the hydrostatic elastic stress σ^h but departs from it inside the sphere. The error in the stress is maximum at the centre which is around 20%. Hence, we conclude that the hydrostatic stress is not a bad assumption for the granular spherical asteroid.

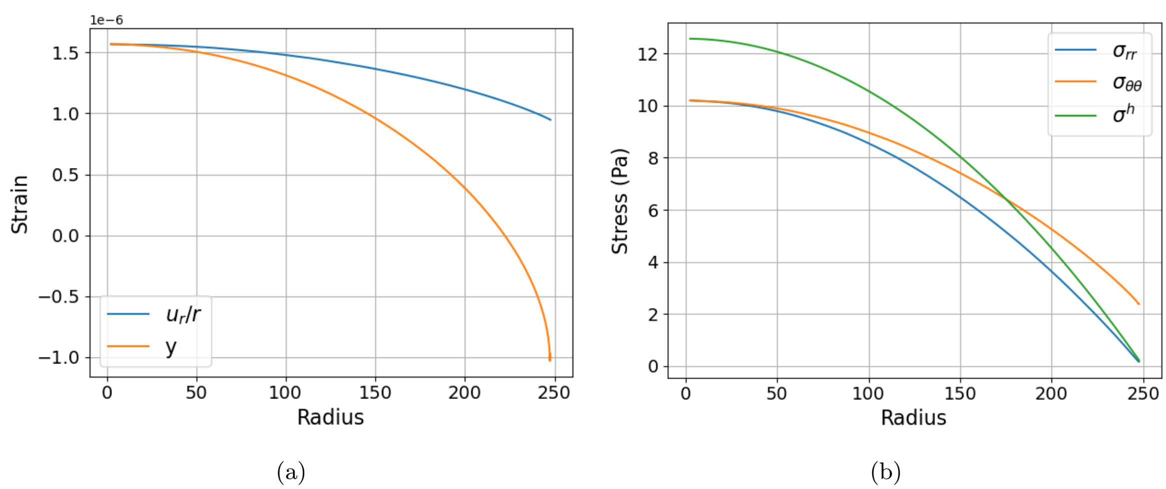


Figure 2: (a) Strains and (b) stresses calculated for a 500 m diameter granular asteroid. σ^h is the hydrostatic stress calculated using linear elastic theory and σ_{rr}, $\sigma_{\theta\theta}$ are from the nonlinear elastic model.

2.3 Mohr-Coulomb criteria

The stress field inside a rubblepile can also be calculated without resorting to elastic moduli if we assume grains to be cohesionless [Hol01]. The equilibrium equation (5), after substituting the gravity field becomes

$$\frac{\mathrm{d}\sigma_{rr}}{dr} + \frac{2}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) - \frac{4\pi}{3} G\rho^2 r = 0. \tag{41}$$

Since the gravity field is linear in r, we assume the stress field of the form

$$\sigma_{rr} = C_1 + C_2 r^2 \quad \text{and} \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = C_3 + C_4 r^2. \tag{42}$$

Substituting this guess in (41) and using second boundary condition of (31), we get the following form for the stresses

$$\sigma_{rr} = -C_2(-R^2 + r^2) \quad \text{and} \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = -C_2(-R^2 + 2r^2) - \frac{2\pi}{3} G\rho^2 r^2. \tag{43}$$

For a Mohr-Coulomb material the shear stress should always be less than or equal to the friction angle as shown in Fig. 3a. For a cohesionless material if $\sigma_{rr} = 0$, then other two principal stresses should also be zero otherwise any Mohr's circle that is drawn will intersect the failure line at more than one points (see Fig. 3b) which is not physically possible. Hence, for a cohesionless material $\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\phi\phi} = 0$ at the $r = R$. This gives the final condition using which we get

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{2\pi G\rho^2}{3}\left(R^2 - r^2\right), \tag{44}$$

which is exactly equal to equation (20). Even though the linear elastic solution and Mohr-Coulomb criterion gives the same stress field, the former approach does not need any assumption on shear modulus and can still support shear waves.

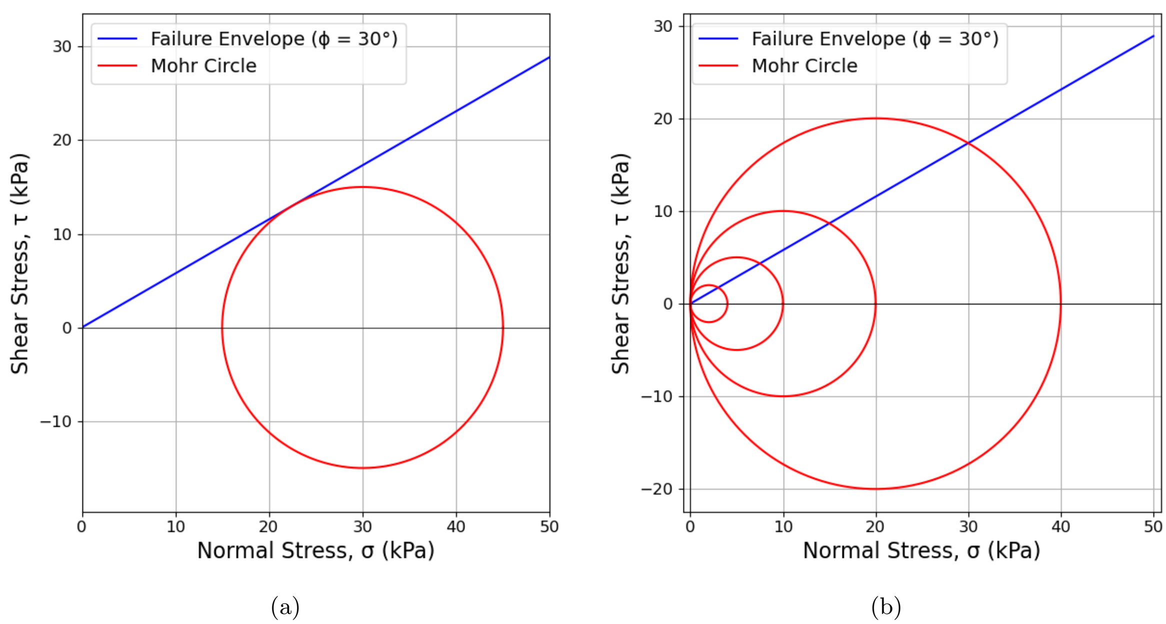


Figure 3: (a) Mohr-Coulomb yield criteria for cohesionless grains. (b) Violation of Mohr-Coulomb criteria when one the principal stresses is non-zero and other is zero.

3 Spherical waves

One of the simple wave that can exist in the spherical body is a symmetric P-wave such that the displacement is only a function of r. This wave can exist is there is a blast at the centre of the sphere. This will give us the first fundamental frequency of the sphere.

3.1 wave equation

To derive the wave equation, we will use the Hamilton's principle and will assume that the body behaves elastically. We start by considering the radial displacement field $u_r(r,t)$ which is now function of time also. The strain tensor

$$\epsilon = \frac{1}{2}\left[(\nabla \mathbf{u})^T + \nabla \mathbf{u} + (\nabla \mathbf{u})^T \cdot \nabla \mathbf{u}\right], \tag{45}$$

for the above displacement field becomes

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r} + \frac{1}{2}\left(\frac{\partial u_r}{\partial r}\right)^2, \quad \epsilon_{\theta\theta} = \epsilon_{\phi\phi} = \frac{u_r}{r} + \frac{1}{2}\left(\frac{u_r}{r}\right)^2. \tag{46}$$

We assume that the stresses generated by the strains are much smaller than the pre-existing stresses and hence the strain energy density becomes

$$W = \sigma_{ij}\epsilon_{ij}. \tag{47}$$

The total strain energy U for the spherical body is obtained by integrating the strain energy density over the volume

$$U = \int_V W\, dV = \int_0^R \int_0^\pi \int_0^{2\pi} (\sigma_{rr}\epsilon_{rr} + \sigma_{\theta\theta}\epsilon_{\theta\theta} + \sigma_{\phi\phi}\epsilon_{\phi\phi}) r^2 \sin\theta\, d\phi\, d\theta\, dr$$

The work done by the gravitational force of the sphere becomes

$$\mathcal{G} = \int_V \rho g u_r\, dV = T = \int_0^R \int_0^\pi \int_0^{2\pi} \rho g u_r r^2 \sin\theta\, d\phi\, d\theta\, dr. \tag{48}$$

Similarly, total kinetic energy T of the spherical body is given by:

$$T = \int_V \frac{1}{2}\rho\left(\frac{\partial u_r}{\partial t}\right)^2 dV = T = \int_0^R \int_0^\pi \int_0^{2\pi} \frac{1}{2}\rho\left(\frac{\partial u_r}{\partial t}\right)^2 r^2 \sin\theta\, d\phi\, d\theta\, dr \tag{49}$$

According to Hamilton's Principle, the integral of the Lagrangian ($L = T - U + \mathcal{G}$) over time must be stationary:

$$\delta \int_{t_1}^{t_2} (T - U + \mathcal{G})\, dt = 0 \tag{50}$$

Substituting the expressions for T, U and $\mathcal{G}$, we obtain

$$\delta \int_{t_1}^{t_2} \left[\int_0^R \int_0^\pi \int_0^{2\pi} \left(\frac{1}{2}\rho\left(\frac{\partial u_r}{\partial t}\right)^2 - (\sigma_{rr}\epsilon_{rr} + \sigma_{\theta\theta}\epsilon_{\theta\theta} + \sigma_{\phi\phi}\epsilon_{\phi\phi}) + \rho g u_r\right) r^2 \sin\theta\, d\phi\, d\theta\, dr\right] dt = 0$$

Above equation can be simplified using (45)-(50),

$$\int_{t_1}^{t_2}\int_0^R -\rho\frac{\partial^2 u_r}{\partial t^2}\delta u_r - \sigma_{rr}\left\{\delta\left(\frac{\partial u_r}{\partial r}\right) + \frac{\partial u_r}{\partial r}\delta\left(\frac{\partial u_r}{\partial r}\right)\right\} - \frac{2\sigma_{\theta\theta}}{r}\delta u_r - 2\frac{u_r}{r^2}\delta u_r + \rho g \delta u_r r^2 dr dt = 0$$

$$\int_{t_1}^{t_2}\int_0^R \left\{-\rho\frac{\partial^2 u_r}{\partial t^2}r^2 + \frac{d(\sigma_{rr}r^2)}{dr} + \frac{\partial}{\partial r}\left(\sigma_{rr}\frac{\partial u_r}{\partial r}r^2\right) - \frac{2\sigma_{\theta\theta}}{r}\left(1 + \frac{u_r}{r}\right)r^2 + \rho g r^2\right\}\delta u_r dr dt$$
$$- \left(\sigma_{rr} + \sigma_{rr}\frac{\partial u_r}{\partial r}\right) r^2 \delta u_r \Big|_0^R = 0 \tag{51}$$

The natural boundary condition is the usual one, $\sigma_{rr} = 0$. Using the equilibrium condition (5), we obtain the wave equation as

$$\rho\frac{\partial^2 u_r}{\partial t^2} = \sigma_{rr}\left\{\frac{\partial^2 u_r}{\partial r^2} + \frac{2}{r}\frac{\partial u_r}{\partial r}\right\} + \frac{\partial \sigma_{rr}}{\partial r}\frac{\partial u_r}{\partial r} - \sigma_{\theta\theta}\frac{2u_r}{r^2}. \tag{52}$$

The last term is the extra term that is not available in usual spherical waves. We will do the modal analysis of the above wave equation using Finite Difference Method. We discretize the spatial derivatives using central differences (except at the boundary) to get

$$\rho \frac{\mathrm{d}^2 u_i(t)}{\mathrm{d}t^2} = \sigma_{rr,i}\left(\frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{(\Delta r)^2} + \frac{1}{r_i}\frac{u_{i+1}(t) - u_{i-1}(t)}{\Delta r}\right) + \left(\frac{\sigma_{rr,i+1} - \sigma_{rr,i-1}}{2\Delta r}\right)\frac{u_{i+1}(t) - u_{i-1}(t)}{2\Delta r} \tag{53}$$

At $r = 0$, the particle does not move owing to the symmetry of the problem. At $r = R$, we assume the Neumann boundary condition $\mathrm{d}u_r/dr = 0$, which is generally true for the free surface. However, for the grains, we have seen in the previous section that the elasticity theory does not hold.

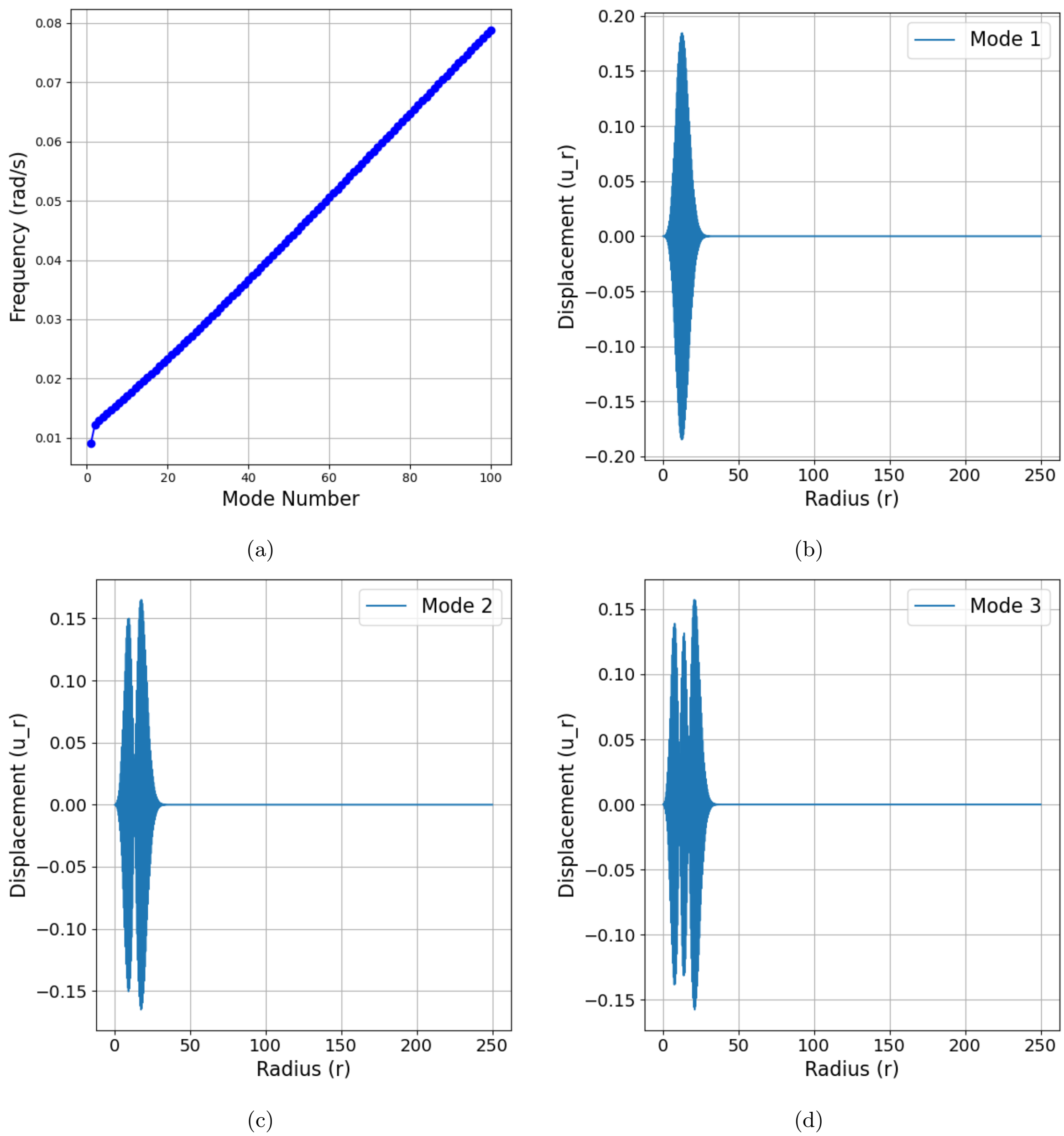


Figure 4: (a) Natural frequencies of the system. (b)-(d) Eigen functions of the system.

First hundred frequencies are shown in 4a and first three modes are shown in Fig. 4b-4d. From these figures, we can observe that the vibrations are localised near the centre only. This may happen due to geometric decay of the waves as it moves out from the centre.

4 Discussions and conclusion

We have derived the state stress in a rotating granular body using three different approaches. We found that the linear elastic and Mohr-Coulomb model gave the same result. The stress from non-linear elastic model matches with that of other approaches near the surface. We used this stress state to do the modal analysis of a spherical wave.

The analysis can be extended further for a rotating sphere with cohesive grains. This will find a direct application to the top-shaped asteroids. The wave speed and fundamental frequency calculated via this approach can be used to calculate energy dissipation during the transmission of the seismic waves [RJMGO05]. Furthermore, the stress induced due to wave transmission can be coupled with the Mohr-Coulomb criterion for estimating the depth of landslide triggered by the travelling waves. We hope to show these applications in future.

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