

Research Article

Comparison of Different Confidence Intervals under Type-I Censoring Scheme

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In a life testing experiment, the successive failure times at putting n units under test are recorded under consideration that the advance fixed experiment is terminated at time T . This type of censoring scheme, called conventional Type-I censoring scheme, is used widely. In this paper, we assume that the items' failure times are independent and distributed with exponential lifetime distribution with parameter θ . The estimator with maximum likelihood method is obtained in an exact form and its distribution is also obtained with the unknown parameter. We proposed the exact confidence interval, for estimators $\hat{\theta}$ and $\ln \hat{\theta}$, asymptotic confidence intervals, confidence interval under likelihood ratio test, and finally, two bootstrap confidence intervals. Under the Bayesian approach, the unknown parameter is estimated and the corresponding credible interval is obtained considering the prior information formulated with the inverted gamma distribution. The Monte Carlo simulation study is used to compare different methods. Finally, for illustrative purposes, the real dataset is used and analyzed.

1. Introduction

Suppose that n identical items are placed under a life testing experiment. Then, independent and identically distributed (i.i.d.) lifetime of each item is distributed with exponential lifetime distribution with parameter θ . Hence, the lifetime random variable X has the probability density function (PDF) with parameter $\theta > 0$ presented by

$$f(x; \theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases} \quad (1)$$

The test continues to advance until fixed time T is reached to terminate the experiment. The experiment is done under consideration that the failed items through the

test are not replaced. This type of censoring scheme is known as conventional Type-I censoring scheme which is quite important in reliability acceptance test in MIL-STD-781C [1].

The Type-I censoring scheme was introduced by Epstein [2], and the estimator of two-sided confidence interval for the parameter θ was proposed by Epstein [3] without any formal proof. The proposition presented by Epstein was modified slightly by Fairbanks et al. [4] to present a simple new set of confidence intervals. The maximum likelihood estimator (MLE) of the parameter θ and the corresponding exact distribution of the estimator as well as one-sided confidence interval were obtained by Chen and Bhattacharya [5]. Under consideration that, Bayesian approach with inverted gamma prior distribution, the Bayesian point and corresponding two-sided credible interval of the mean lifetime estimator is considered by Draper and Guttman [6].

Recently, the maximum likelihood estimator under right-censored linear regression data is presented by Yu [7] and for mean ranked acceptance sampling plan under exponential distribution is presented by Hussain et al. [8]. The problem of building two-sided confidence interval with respect to the exact distribution of $\hat{\theta}$ was presented by Chen and Bhat-tacharya [5].

The problem of the exact distribution of $\hat{\theta}$ under Type-I censoring scheme is more serious to obtain. And, several books on reliability or survival analysis such as, Bain [9], Barlow and Proschan, [10] and Lawless [11] do not talk about this problem. It might be worth mentioning that, specially as an application in the analysis repairable system, the problem of random sampling from a truncated exponential distribution. This is discussed in detail given by Bain and Engelhardt [12] (Chapter 9) as well as distribution of the sum of the observations taken from a truncated exponential distribution. As the same conditional density of the sum of observed failure times given number of failures in a time-censored Weibull process, it turns out in the repairable systems framework. In natural setting this density is given of sampling from the truncated exponential distribution see, Bain, Engelhardt and Wright [13], see also Bain and Weeks [14] in this respect.

In statistics, we are using a sample of data to estimate an interval of interested parameter value. The interval estimation in the paper is discussed in different forms. Under frequentist method, different confidence intervals are proposed, but under Bayesian approach, the credible interval is proposed. Also, there is a common form of interval estimation, such as fiducial intervals, tolerance intervals, and prediction intervals. In the literature, there are different novel situations as a guide on how interval estimates are formulated. Both credible intervals and confidence intervals are different but have a similar standing. Confidence intervals can be applied in more situations in parametric and non-parametric models other than credible interval. The problem of testing the performance of interval estimation procedures involves approximations of various kinds, and there is a need to check that the actual performance of a procedure is close to what is claimed. The coverage probability and interval length are the key concepts associated with interval estimators. We know that a higher coverage is obtained under longer interval length and lower coverage is obtained under shorter interval length. In statistics, we face a number of interval estimators of the population parameters, and a decision has to be made on what the "best" method of estimation is. The key concepts are that the better coverage probability goes with smaller interval length and vice versa, and it is useful to have some practical way of combining these measures.

Under consideration the Type-I censoring scheme, the exact and asymptotic distributions of $\hat{\theta}$ are obtained. Also, based on the two values $\hat{\theta}$ and $\ln \hat{\theta}$, the exact and the corresponding asymptotic confidence intervals of $\hat{\theta}$ are derived. Also, based on the likelihood ratio test (LRT), the confidence interval, the Bayes credible interval for inverted gamma prior, and the exact expressions for $E(\hat{\theta})$ and $E(\theta')$ are derived from considering the unknown parameter. Finally, different methods are compared with Monte Carlo

simulations, and the real-life dataset is applied to all different methods.

This paper is organized as follows. The problem and the MLE of the unknown parameter with its distribution are formulated in Section 2. Confidence intervals as well as credible intervals are presented in Section 3. Different methods are compared through the numerical experiments in Section 4. The real dataset is analyzed in Section 5. Finally, in Section 6, we draw the conclusions from our work.

2. The MLE and Its Distribution

In this section, we constructed the exact conditional moment generating function that can be used to build exact conditional confidence intervals of the parameter θ under the assumption of exponential distribution.

2.1. MLE. Suppose that $X_1, X_2, \dots, X_n$ denote the lifetimes of the n identical items put under test, where X_i is distributed with exponential distribution with parameter θ and PDF given in (1). The ordered lifetimes of these items is denoted by $X_{1:n}, X_{2:n}, \dots, X_{n:n}$. Suppose that, the number of items that fails upto and including the prefixed time point T is denoted by D . So, the observations under conventional Type-I censoring scheme are defined by

$$\{X_{1:n}, X_{2:n}, \dots, X_{D:n}\}, \quad (2)$$

where

$$X_{D:n} < T < X_{D+1:n} \text{ and } 0 \leq D \leq n. \quad (3)$$

Let us define

$$S = \begin{cases} \sum_{i=1}^D X_{i:n} + (n-D)T & \text{if } D \geq 1 \\ nT & \text{if } D = 0, \end{cases} \quad (4)$$

where S defines the total time on test. Therefore, based on the exponential Type-I censored data, the likelihood function of θ can be written as

$$L(\theta) = \frac{n!}{(n-D)!} \theta^{-D} e^{-S/\theta}. \quad (5)$$

From the likelihood function (5) and for $D \geq 1$, the MLE of θ exists and is given by

$$\hat{\theta} = \frac{S}{D}, \quad (6)$$

and that of $\hat{\theta}$ does not exist for $D = 0$.

2.2. Distribution of the MLE. Here, the distribution of $\hat{\theta}$ show by the conditional distribution of $\hat{\theta}$ for $1 \leq D \leq \infty$ and in other words obtain the distribution of S given $D = d$, and $1 \leq d \leq \infty$, then unconditional distribution is obtained. It can be done, as given by Hoem [15] by using the convolution property of the uniform distribution or as suggested by Chen and Bhattacharya [5] by the conditional moment generating function (MGF) approach. Here we are using the MGF approach. The conditional MGF of $\hat{\theta}$ given $D \geq 1$ is given by

$$\begin{aligned}\Phi_{\hat{\theta}}(\omega) &= E\left(e^{\omega\hat{\theta}}|D \geq 1\right) = \sum_{d=1}^n E\left(e^{\omega\hat{\theta}}|D = d\right)P(D = d) \\ &= (1 - q^n)^{-1} \sum_{d=1}^n \binom{n}{d} [q^{1-\omega\theta/d}]^{n-d} [1 - q^{1-\omega\theta/d}]^{n-d} \left[1 - \frac{\omega\theta}{d}\right]^{-d},\end{aligned}\quad (7)$$

where $\mathbf{p} = 1 - \mathbf{e}^{-T/\theta}$ and $\mathbf{q} = 1 - \mathbf{p}$. The proof of (7) is given in Appendix A.

Note that, the conditional distribution of the sum of observations given $(D = d)$ is conditional distributed as Type-I censoring observation taken from one-parameter exponential distribution is the same as the sum of observations in a random sample of size d , taken from truncated exponential distribution Bain [9] or Chen and Bhattacharya [5]. Then, the MLE distributed as a function obtained by

taking the expectation relative to D (truncated above 0) of the conditional distribution of $\sum_{i=1}^d X_{i:n} + (n - D)T/d$ given $D = d$.

Now we are going to calculate the value of the mean and variance of $\hat{\theta}$ for Type-I censoring scheme. It is clear that the problem of obtaining the first two moments $\hat{\theta}$ may be complicated. However, from the conditional MGF, they can be obtained as follows.

$$\begin{aligned}E(\hat{\theta}) &= \Phi_{\hat{\theta}}(0) = (1 - q^n)^{-1} \left[\left(\theta - \frac{T}{p} \right) (1 - q^n) + T p^n \right] + (1 - q^n)^{-1} T n \sum_{d=1}^n \binom{n}{d} \frac{1}{d} p^d q^{n-d} \\ &= \theta + \text{Bias}(\theta),\end{aligned}\quad (8)$$

where

$$\text{Bias}(\theta) = (1 - q^n)^{-1} T n \sum_{d=1}^n \binom{n}{d} \frac{1}{d} p^d q^{n-d} - \frac{T}{p}. \quad (9)$$

Similarly,

$$E(\hat{\theta}^2) = \Phi''_{\hat{\theta}}(0) = (1 - q^n)^{-1} \sum_{d=1}^n \binom{n}{d} p^d q^{n-d} \left[\left(\theta - \frac{Tq}{p} + T \left(\frac{n}{d} - 1 \right) \right)^2 + \frac{\theta^2}{d} - \frac{T^2 q}{dp} \left(1 + \frac{q}{p} \right) \right]. \quad (10)$$

It is interesting to observe that the bias is positive unless, and as $T \rightarrow \infty$, i.e., when the Type-I censoring scheme becomes complete sample life testing experiment. In that case $E(\hat{\theta})^{\text{TM}} \theta$ which is nothing but an expected value of $\hat{\theta}$ complete life testing experiment. The second term on the right-hand side (8) can be written in recursive relation (see, for example, Govindrajulu [16]).

3. Different Confidence and Credible Intervals

In this section, the different confidence intervals are proposed as follows.

- (1) Based on, the exact distribution of $\hat{\theta}$.
- (2) Based on, the asymptotic distribution of $\hat{\theta}$ and $\ln \hat{\theta}$.
- (3) Based on, the likelihood ratio test.

- (4) Bootstrap-p and bootstrap-t confidence intervals.

We have also proposed one Bayes credible interval of θ based on the inverted gamma prior on θ .

3.1. Exact Confidence Interval. The exact distribution of $\hat{\theta}$ was used by Chen and Bhattacharya [5] to present the one-sided confidence interval of the parameter θ . Also, the two-sided confidence interval of θ can be obtained easily based on the exact distribution of $\hat{\theta}$, in the conventional Type-I censoring scheme under the consideration that $P(\hat{\theta} \geq b)$ is an increasing function of θ as it was originally proposed by Chen and Bhattacharya [14]. Using Chen and Bhattacharya [14], we observe that

$$P_{\theta}(\hat{\theta} \leq b) = (1 - q^n)^{-1} \sum_{d=1}^n \binom{n}{d} \sum_{k=0}^d \binom{d}{k} (-1)^k q^{(n-d+k)} \int_0^{A_{k,d}} f_{\chi^2_{2d}}(u) du, \quad (11)$$

where $f_{\chi^2/2, d}(u)$ present the chi-square PDF with $2d$ degrees of freedom, and

$$A_{kd} = \frac{2d}{\theta} < b - \frac{T}{d}(n - d + k); \text{ for } \langle a \rangle = \max\{0, a\}. \quad (12)$$

Under the consideration of Type-I censoring (11), we observe the cumulative distribution function (CDF) of $\hat{\theta}$. Hence, the symmetric $(1 - \alpha)100\%$ confidence interval (b_1, b_2) under Type-I censoring is formulated by choosing b_1 and b_2 such that

$$P_{\theta}(\hat{\theta} \leq b_1) = \frac{\alpha}{2} \text{ and } P_{\theta}(\hat{\theta} \leq b_1) = 1 - \frac{\alpha}{2}. \quad (13)$$

3.2. Asymptotic Confidence Interval. The limiting properties of the MLEs are used to obtain the asymptotic distribution of $\hat{\theta}$. Then, the asymptotic distribution of $\hat{\theta}$, is asymptotically normally distributed with mean θ and variance $I[(\theta)]^{-1}$ where

$$I(\theta) = E\left(-\frac{\partial^2}{\partial \theta^2} \ln l(\theta)\right) = E\left(\frac{2S}{\theta^3} - \frac{D}{\theta^2}\right). \quad (14)$$

The asymptotic confidence interval based on the asymptotic distribution of $\hat{\theta}$ can be easily calculated. It is observed that

$$E(S|D \geq 1) = -\frac{n(p\theta - TQ^n)}{1 - q^n}. \quad (15)$$

The proof of (15) is given in Appendix B and

$$E(D) = \sum_{d=1}^n d(1 - q^n)^{-1} \binom{n}{d} p^d q^{n-d}. \quad (16)$$

Hence, using (15) and (16), we can obtain (14). Therefore, for $D > 0$, the $100(1 - \alpha)\%$ asymptotic confidence interval for θ is

$$\hat{\theta} \pm Z_{\alpha/2} I[(\theta)]^{-1/2}. \quad (17)$$

As reported in Meeker et al. [17], the confidence interval based on the asymptotic theory of $\ln \hat{\theta}$ is superior than $\hat{\theta}$. Then, the approximate $100(1 - \alpha)\%$ confidence interval under $D > 0$ of $\ln \theta$ is given by

$$\ln \hat{\theta} \pm Z_{\alpha/2} \left(E\left(\frac{2S}{\theta}\right) - D \right). \quad (18)$$

Therefore, the $100(1 - \alpha)\%$ approximate confidence interval for $D > 0$ for θ becomes

$$e^{\ln \hat{\theta} \pm Z_{\alpha/2} (E(2S/\theta) - D)^{-1/2}} = \hat{\theta} e^{\pm Z_{\alpha/2} (E(2S/\theta) - D)^{-1/2}}, \quad (19)$$

which can be easily calculated by using (15) and (16).

3.3. Confidence Interval Based on the LRT. From Meeker and Escobar [11], we observe that the confidence interval based

on the LRT is often superior than the confidence interval based on the asymptotic distribution of the MLE. So, we propose, as given by Lawless [8], the confidence interval based on the LRT for constructing confidence intervals for the gamma parameters. But similar methods can be easily adopted here. Under testing of hypothesis problem, firstly consider

$$H_0: \theta = \theta_0 \text{ vs } H_1: \theta \neq \theta_0, \quad (20)$$

and the likelihood ratio statistic presented by

$$\Lambda = -2[\ln L(\theta_0) - \ln L(\hat{\theta})], \quad (21)$$

where $L(\cdot)$ is the likelihood function (see, for example, (5)). For a large sample size, the distribution of Λ is approximated under H_0 as χ^2 with one degree of freedom. Then, the exact distribution of Λ is independent of θ_0 for scale parameter θ . Therefore, we constructed the $100(1 - \alpha)\%$ confidence interval, $C(\theta)$ of θ , by

$$C(\theta) = \theta: -D \ln \theta - \frac{S}{\theta} \geq \ln L(\hat{\theta}) + \frac{1}{2} \chi^2_{1, 1-\alpha}, \quad (22)$$

where the upper $100(1 - \alpha)\%$ percentile point of the $\chi^2_{1, 1-\alpha}$ distributed with one degree of freedom. Since, by definition, $\theta > 0$, $-D \ln \theta - S/\theta$ is a unimodal function and therefore for a given α , $C(\theta)$ is a unique interval.

3.4. Bootstrap Confidence Intervals. In this section, based on the bootstrap technique, we propose two confidence intervals, namely, (a) the percentile bootstrap (Boot-p) confidence interval proposed by Efron [18] and (b) the bootstrap-t (Boot-t) confidence interval proposed by Hall [19]. Different authors have shown interest in bootstrap techniques in the problem of building the bootstrap confidence intervals (see, for example, Almarashi and Abd-Elmougod [20] and Abd-Elmougod and Mahmoud [21]). The following algorithms described the steps needed to construct the Boot-up and Boot-t confidence intervals of θ .

Boot-p method:

- (1) Obtain $\hat{\theta}$ and the MLE of θ as discussed before.
- (2) Generate a bootstrap sample $\{X^*_{1:n}, \dots, X^*_{D^*:n}\}$ using $\hat{\theta}$ and T . Obtain $\hat{\theta}^*$ and the MLE of θ based on the bootstrap sample.
- (3) Step 2 is repeated, NBOOT times.
- (4) Let $\text{CDF}(x) = P(\hat{\theta}^* \leq x)$ be the CDF of $\hat{\theta}^*$. Define $\hat{\theta}_{\text{Boot-p}}(x) = \text{CDF}^{-1}(x)$ for a given x . The approximate $100(1 - \alpha)\%$ confidence interval of θ is given by

$$\left(\hat{\theta}_{\text{Boot-p}}\left(\frac{\alpha}{2}\right), \hat{\theta}_{\text{Boot-p}}\left(1 - \frac{\alpha}{2}\right) \right). \quad (23)$$

The following method may be used to construct the Boot-t confidence interval of θ .

Boot-t method:

- (1) Obtain $\hat{\theta}$ and the MLE of θ as discussed before.
- (2) By using $\hat{\theta}$ and T , the bootstrap sample $\{X_{1:n}^*, \dots, X_{D^*:n}^*\}$ is generated. Obtain $\hat{\theta}^*$ and the MLE of θ based on the bootstrap sample. Also, compute $\hat{V}(\hat{\theta}^*)$ using (fish-info) for $D^* \geq 1$.
- (3) The statistic value T^* is determined by $T^* = \hat{\theta}^* - \hat{\theta}/\sqrt{\hat{V}(\hat{\theta}^*)}$.
- (4) Repeat Step 2 and Step 3, NBOOT times.
- (5) Let $\text{CDF}(x) = P(\hat{\theta}^* \leq x)$ be the CDF of $\hat{\theta}^*$. Define $\hat{\theta}_{\text{Boot}-t}(x) = \hat{\theta} + \sqrt{\hat{V}(\hat{\theta}^*)} \text{CDF}^{-1}(x)$ for a given x . So, the approximate 100 $(1 - \alpha)\%$ confidence interval of θ is given by

$$\left(\hat{\theta}_{\text{Boot}-t}\left(\frac{\alpha}{2}\right), \hat{\theta}_{\text{Boot}-t}\left(1 - \frac{\alpha}{2}\right) \right). \quad (24)$$

3.5. Bayesian Credible Interval. In this section, we provide the Bayesian analysis of the above-mentioned problem. In the context of exponential lifetimes, θ may be reasonably modelled by the inverted gamma prior. Following the approach of Draper and Guttman [6], it is assumed that θ has an inverted gamma prior with parameters $\alpha > 0$ and $r > 0$ with the following PDF:

$$\pi(\theta) = \begin{cases} \frac{a^r}{\Gamma(r)\theta^{r+1}} e^{-a/\theta} & \theta > 0, \\ 0 & \theta \leq 0. \end{cases} \quad (25)$$

When $\alpha = r = 0$, one obtains the non-informative prior on θ . The posterior PDF of θ based on the above inverted gamma prior is

$$\pi(\theta|\text{data}) \propto \theta^{-(D+r+1)} e^{-S+a/\theta}. \quad (26)$$

From post-theta, it is clear that the posterior distribution of θ is also an inverted gamma distribution, and we can easily obtain the Bayes estimate of θ under the squared error loss function as

$$\hat{\theta}_{\text{Bayes}} = \frac{S+a}{D+r}. \quad (27)$$

Interestingly, the Bayes estimate based on the non-informative prior coincides with the MLE. Hence, the posterior distribution of θ is used to obtain credible interval of θ . Note that the posterior distribution of $Z = 2(S+a)/\theta$ is a X^2 distribution having a positive integer $2(D+r)$ degrees of freedom. Hence, 100 $(1 - \alpha)\%$ credible interval of θ is presented by

$$\left(\frac{2(S+a)}{X_{2(D+r),\alpha/2}^2}, \frac{2(S+a)}{X_{2(D+r),1-\alpha/2}^2} \right), \quad (28)$$

for $(D+r) > 0$. For the non-integer value of $2(D+r)$, gamma distribution can be applied to formulate credible interval of θ . Then, the non-informative prior can be used if

no prior information is available to construct a credible interval for θ . If some prior information is available, then some positive values of a and r may be used to construct a credible interval for θ (Table 1).

4. Numerical Experiments

Since the performances of the different methods cannot be compared theoretically, we use Monte Carlo simulations to compare different methods for different sample sizes and for different censoring times (T). Pentium IV processor is used for all the computations, and we use the random number generator RAN2 of Press et al. [1].

We consider different n and T values, and in all the cases, we have considered $\theta = 10$. For each case, we have computed the 95% confidence intervals using all the five methods proposed here. For comparison purposes, we have also computed the 95% credible interval of θ based on the non-informative prior as discussed before. For each dataset, we have computed the length of the confidence interval of θ based on (a) the exact distribution of $\hat{\theta}$ (Table 1), (b) the asymptotic distribution of θ (Table 2), (c) the asymptotic distribution of $\ln\hat{\theta}$ (Table 3), (d) the likelihood ratio test (Table 4), (e) Boot-p procedure (Table 5), and (f) Boot-t procedure (Table 6) as discussed in Section 3. We have also computed 95% Bayes credible interval as discussed in Section 3 based on the non-informative prior (Table 7). For the asymptotic confidence interval, if the lower limit is negative, it is replaced by zero. We have repeated the experiment 1000 times and reported the average lengths of the confidence intervals and the coverage percentages. In each box, the first quantity represents the average length of the confidence/credible interval, the second quantity within brackets $\{(\cdot)\}$ represents the coverage percentages, and the third quantity within the brackets $\{[\cdot]\}$, if any, represents the non-existence percentage of the MLE.

Some of the points are quite clear from the above experiments. It is observed that for fixed T , as n increases, the length of the intervals based on all the above-mentioned procedures decreases. It indicates the consistency property of the MLE. Similarly, for fixed n , as T increases, the length of the intervals decreases. All the methods more or less maintain the coverage percentages at the nominal level (95% in this case) unless n and or T is very small.

It is also observed that the exact confidence interval does not work very well for small values of n and T . It is also observed that the Bayes credible interval works quite well for small values of n and T . By comparing Table 5 with Table 7, at $n = 50$, we observed that the coverage percentages of Boot-p serve well and are closer to 95% than Bayes credible intervals (4 best out of 7 choices). Also, the widths of Boot-p are almost all shorter than those of Bayes credible intervals. In fact, most of the times, it is better than the other methods considered here. The two terms of their average lengths and coverage percentages are used to compare different confidence intervals. It is observed that the confidence intervals based on the asymptotic distribution of $\hat{\theta}$ and $\ln\hat{\theta}$ and Boot-p confidence interval perform quite well, unless n and T are very small. Since the Boot-t method is involved numerically

TABLE 1: Exact confidence interval with coverage percentages.

T	$n = 10$	$n = 15$	$n = 25$	$n = 50$
0.50	3.192 (94.8) [0.6]	3.113 (95.3) [0.1]	2.185 (96.6)	1.102 (96.7)
0.75	3.053 (94.4)	2.425 (96.0)	1.509 (95.7)	0.870 (96.1)
1.00	2.671 (94.2)	1.915 (95.5)	1.215 (95.0)	0.760 (95.5)
1.25	2.333 (93.9)	1.622 (94.8)	1.073 (94.4)	0.699 (94.7)
1.50	2.029 (94.5)	1.434 (94.6)	0.993 (94.0)	0.659 (94.2)
1.75	1.810 (94.8)	1.314 (94.7)	0.940 (94.8)	0.634 (94.7)
2.00	1.638 (94.8)	1.234 (94.7)	0.903 (94.7)	0.616 (94.8)

TABLE 2: Confidence interval based on the asymptotic distribution of $\hat{\theta}$ with coverage percentages.

T	$n = 10$	$n = 15$	$n = 25$	$n = 50$
0.50	2.936 (90.1) [0.6]	2.255 (90.9) [0.1]	1.478 (92.1)	0.949 (94.9)
0.75	2.231 (91.3)	1.667 (92.4)	1.195 (93.6)	0.799 (95.1)
1.00	1.919 (92.0)	1.423 (93.2)	1.052 (92.8)	0.717 (94.5)
1.25	1.716 (94.4)	1.308 (93.5)	0.970 (92.5)	0.669 (93.8)
1.50	1.609 (96.3)	1.236 (94.4)	0.921 (93.0)	0.637 (93.5)
1.75	1.557 (98.1)	1.189 (96.0)	0.888 (94.2)	0.617 (93.1)
2.00	1.529 (97.4)	1.166 (96.9)	0.867 (95.1)	0.602 (92.8)

TABLE 3: Confidence interval based on the asymptotic distribution of $\hat{\theta}$ with coverage percentages.

T	$n = 10$	$n = 15$	$n = 25$	$n = 50$
0.50	6.730 (94.9) [0.6]	3.391 (95.8) [0.1]	1.607 (96.0)	0.984 (95.7)
0.75	3.354 (96.2)	1.885 (96.3)	1.262 (96.1)	0.819 (95.1)
1.00	2.517 (96.8)	1.536 (96.4)	1.099 (95.6)	0.732 (95.7)
1.25	1.934 (96.8)	1.397 (96.0)	1.007 (95.4)	0.681 (95.2)
1.50	1.774 (98.1)	1.310 (97.1)	0.953 (95.7)	0.648 (95.4)
1.75	1.694 (98.3)	1.255 (98.2)	0.916 (97.0)	0.626 (94.7)
2.00	1.656 (97.6)	1.228 (98.0)	0.894 (96.7)	0.611 (94.5)

and the confidence intervals based on the asymptotic distributions are larger than the Bayes credible interval, we recommend to use the Bayes credible interval in all the cases. It may be mentioned that the exact confidence intervals are difficult to obtain. One needs to solve two non-linear equations to compute the exact confidence intervals. Therefore, it may be avoided. We have further reported the bias values (Table 8) for different n and T ; as expected, the biases tend to zero, as n and T increase.

TABLE 4: Confidence interval based on the LRT with coverage percentages.

T	$n = 10$	$n = 15$	$n = 25$	$n = 50$
0.50	6.828 (93.1) [0.6]	3.777 (93.9) [0.1]	1.777 (95.6)	1.040 (95.9)
0.75	3.705 (94.4)	2.143 (95.9)	1.348 (95.6)	0.844 (95.6)
1.00	2.807 (95.7)	1.685 (95.3)	1.156 (95.8)	0.750 (95.6)
1.25	2.213 (96.7)	1.511 (95.6)	1.051 (95.9)	0.695 (95.1)
1.50	1.999 (97.9)	1.407 (97.7)	0.991 (95.8)	0.660 (95.6)
1.75	1.895 (98.0)	1.342 (97.8)	0.951 (96.7)	0.638 (95.0)
2.00	1.849 (97.2)	1.311 (97.2)	0.927 (96.9)	0.621 (93.8)

TABLE 5: Confidence interval based on Boot-p method with coverage percentages.

T	$n = 10$	$n = 15$	$n = 25$	$n = 50$
0.50	3.482 (91.2) [0.9]	2.933 (93.3)	1.922 (93.9)	1.079 (94.8)
0.75	3.209 (92.5) [0.1]	2.209 (92.8)	1.377 (93.7)	0.857 (94.7)
1.00	2.822 (93.2)	1.708 (91.8)	1.159 (94.1)	0.746 (94.6)
1.25	2.319 (95.3)	1.488 (93.0)	1.043 (92.5)	0.694 (94.3)
1.50	1.962 (96.1)	1.349 (93.6)	0.967 (92.2)	0.656 (94.4)
1.75	1.771 (96.9)	1.241 (94.9)	0.917 (92.1)	0.628 (94.7)
2.00	1.681 (97.5)	1.176 (96.2)	0.878 (93.1)	0.609 (94.4)

TABLE 6: Confidence interval based on Boot-t method with coverage percentages.

T	$n = 10$	$n = 15$	$n = 25$	$n = 50$
0.50	12.168 (62.7) [0.9]	11.263 (76.3)	4.391 (78.2)	1.466 (83.9)
0.75	11.620 (68.7) [0.1]	6.081 (80.0)	2.113 (80.0)	1.029 (87.1)
1.00	10.072 (74.9)	3.370 (82.7)	1.569 (83.4)	0.847 (89.7)
1.25	6.424 (76.4)	2.555 (83.7)	1.327 (86.5)	0.768 (90.3)
1.50	4.126 (79.2)	2.145 (85.0)	1.180 (84.8)	0.712 (90.1)
1.75	3.268 (77.5)	1.773 (85.5)	1.092 (86.7)	0.671 (90.9)
2.00	2.933 (77.2)	1.611 (84.7)	1.031 (86.7)	0.645 (91.8)

5. Data Analysis

In this section, we have analyzed one dataset taken from Bain [9] and applied different methods discussed so far. So, the following dataset is used (see Table 9).

Dataset. From exponential population, suppose 20 items are put on a life testing experiment and the experiment is continued for 150 hours, which is prefixed before the starting of the experiment. During that period, 13 items have failed at the following hours: 3, 19, 23, 26, 37, 38, 41, 45, 58, 84, 90, 109, and 138.

TABLE 7: Bayes credible interval with coverage percentages.

T	$n = 10$	$n = 15$	$n = 25$	$n = 50$
0.50	11.469 (93.6) [0.6]	5.160(94.1)[0.1]	1.906 (95.6)	1.076 (96.1)
0.75	5.040 (94.8)	2.379 (95.7)	1.413 (95.8)	0.877 (95.4)
1.00	3.539 (95.6)	1.796 (95.4)	1.211 (95.7)	0.744 (95.5)
1.25	2.450 (96.4)	1.594 (95.7)	1.106 (96.4)	0.715 (95.0)
1.50	2.172 (97.4)	1.475 (97.6)	1.042 (96.5)	0.676 (95.4)
1.75	2.036 (97.3)	1.401 (97.2)	0.998 (96.5)	0.653 (95.7)
2.00	1.979 (96.1)	1.366 (96.8)	0.971 (96.7)	0.635 (94.8)

TABLE 8: Biases of the MLE.

T	$n = 10$	$n = 15$	$n = 25$	$n = 50$
0.50	0.21376	0.23803	0.14083	0.04991
0.75	0.22445	0.15675	0.07483	0.02989
1.00	0.17015	0.10048	0.04816	0.02068
1.25	0.12980	0.07039	0.03471	0.01546
1.50	0.09469	0.05188	0.02655	0.01200
1.75	0.07228	0.03986	0.02092	0.00962
2.00	0.05643	0.03170	0.01690	0.00780

TABLE 9: Different 95% confidence and credible intervals of θ .

Method	Confidence/credible interval
Exact	(58.80143, 224.46170)
Based on $\hat{\theta}$	(62.90728, 208.01579)
Based on $\ln\hat{\theta}$	(79.28743, 231.43417)
Based on LRT	(82.27760, 246.27774)
Boot-p	(82.07265, 220.99469)
Boot-t	(36.07920, 180.23492)
Bayes	(84.01076, 254.40808)

In this case, $n = 20$, $T = 150$, and $D = 13$. Further

$$S = \sum_{d=1}^n X_{i:n} + (n - D)T = 1761. \quad (29)$$

Hence,

$$\hat{\theta} = \frac{1761}{13} = 135.46154. \quad (30)$$

We have reported different 95% confidence and credible intervals of θ in Table 9.

6. Conclusion

In this section, we have considered the Type-I censoring scheme in case of exponential failure distributions. We have proposed two-sided exact confidence interval of θ based on the exact distribution of the conditional MLE of θ . We have compared different confidence intervals, namely, (a) exact confidence interval, (b) confidence interval based on the asymptotic distribution of $\hat{\theta}$, (c) confidence interval based on the asymptotic distribution of $\ln\hat{\theta}$, (d) confidence interval based on the LRT, (e) Boot-p confidence interval, and (f) Boot-t along with the credible interval based on the non-informative inverted

gamma prior, through extensive computer simulations. We have also analyzed one real dataset for illustrative purposes. It is observed that the Bayes credible interval with non-informative prior works quite well in terms of the length of the interval and coverage percentages. It is also observed that in this case, the Boot-t method does not work at all.

When the data are observed under Type-I censoring scheme, another important point is that at $D = 0$ which is typically ignored, the MLE does not exist. One may look at the interesting discussion by Bain and Engelhardt [14] in this aspect.

Appendix

A. Proof of Equation (7)

The conditional MGF of $\hat{\theta}$ given $D \geq 1$ is given by

$$\Phi_{\hat{\theta}}(\omega) = E(e^{\omega\hat{\theta}} | D \geq 1) = \sum_{d=1}^n E(e^{\omega\hat{\theta}} | D = d) \times P(D = d). \quad (A.1)$$

Now

$$\begin{aligned}
 E(e^{\omega\theta}|D=d) &= E(e^{\omega S/D}|D=d) = e^{[\omega(n-d)T/d]} E\left(e^{\left[\omega/d \sum_{i=1}^d y_{i:d}\right]}\right) \\
 &= \left[q^{-\frac{\omega\theta}{d}}\right]^{n-d} \times E\left(e^{\left[\omega/d \sum_{i=1}^d y_{i:d}\right]}\right),
 \end{aligned} \tag{A.2}$$

where $y_{i:d}, i = 1, \dots, d$ are the order statistics of a random sample of size d from the distribution with PDF

where $p = 1 - e^{-T/\theta}$. Let $q = 1 - p$; then,

$$f(y; \theta) = \begin{cases} \frac{1}{p\theta} e^{-y/\theta}, & 0 \leq y \leq T \\ 0, & \text{O.W} \end{cases}, \tag{A.3}$$

$$E\left(e^{\left[\omega/d \sum_{i=1}^d y_{i:d}\right]}\right) = \prod_{i=1}^d E\left(e^{\left[\omega y_{i:d}/d\right]}\right) = \prod_{i=1}^d \left[\int_0^T e^{\omega/d y_i} \frac{1}{p\theta} e^{-y_i/\theta} dy_i\right] = [1 - q^{(1-\omega\theta/d)}]^d \left[p\left(1 - \frac{\omega\theta}{d}\right)\right]^{-d}. \tag{A.4}$$

Again,

Hence, we get equation (7) as

$$P(D=d) = (1-q^n)^{-1} \binom{n}{d} p^d q^{n-d}. \tag{A.5}$$

$$\Phi_{\hat{\theta}}(\omega) = (1-q^n)^{-1} \sum_{d=1}^n \binom{n}{d} [q^{1-\omega\theta/d}]^{n-d} [1 - q^{1-\omega\theta/d}]^{n-d} \left[1 - \frac{\omega\theta}{d}\right]^{-d}. \tag{A.6}$$

B. Proof of Equation (15)

$$\begin{aligned}
 E(S|D \geq 1) &= \sum_{d=1}^n E(S|D=d)P(D=d) \\
 &= \sum_{d=1}^n E\left(\left[\sum_{i=1}^D x_{i:n} + (n-d)T\right]|D=d\right)P(D=d)
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{d=1}^n \left[E \left(\sum_{i=1}^D x_{i:n} \right) + (n-d)T \right] P(D=d) \\
&= \sum_{d=1}^n \left[d \left(\theta - \frac{Tq}{p} \right) + (n-d)T \right] (1-q^n)^{-1} \binom{n}{d} p^d q^{(n-d)} \\
&= (1-q^n)^{-1} \left[\left(\theta - \frac{Tq}{p} \right) \sum_{d=1}^n d \binom{n}{d} p^d q^{(n-d)} + nT \sum_{d=1}^n \binom{n}{d} p^d q^{(n-d)} - T \sum_{d=1}^n d \binom{n}{d} p^d q^{(n-d)} \right] \quad (\text{B.1}) \\
&= (1-q^n)^{-1} \left[\left(\theta - \frac{Tq}{p} \right) np + nT(1-q^n) - Tnp \right] \\
&= \frac{n(p\theta - Tq^n)}{1-q^n}.
\end{aligned}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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