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Estimation Method under Three-Parameter Generalized Exponential Model: Consistency, Uniqueness and its Applications

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Abstract:	In numerous instances, the generalized exponential distribution can be used as an alternative to the most widely used non-regular family of distributions: Weibull, gamma, lognormal with three-parameters when analyzing lifetime or any skewed continuous data. A non-regular family is a class of probability distributions that do not satisfy the regularity conditions typically assumed in classical statistical inference. Some key features of such family of distributions are: support of its probability density function depends on one its parameters; its likelihood function may not be bounded for a certain range of parameter space, hence maximum likelihood estimators do not exist; the likelihood function even may not be differentiable or integrable as needed, hence Fisher Information may not exist or be infinite. Moreover, standard results like MLE existence, consistency, asymptotic normality may fail. Therefore, specialized or robust inferential techniques are needed. This article offers a consistent method for estimating the parameters of a three-parameter generalized exponential distribution that sidesteps the issue of an unbounded likelihood function. The method is hinged on a maximum likelihood estimation of shape and scale parameters that uses a location-invariant statistic. Important estimator properties, such as uniqueness and consistency, are demonstrated for the first time under this approach. In addition, quantile estimates for the assumed distribution are provided. We present a Monte Carlo simulation study along with comparisons to a number of well-known estimation techniques in terms of bias and root mean square error. For illustrative purposes, two real data sets from different origins; medical sciences and reliability engineering, have been analyzed and the goodness of fit along with the bootstrap confidence intervals are compared with existing traditional methods.
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Newcastle, GB
December 15, 2025

Dear Editor-in-Chief,

I would like to provide a brief note regarding the supplementary material included with this revision. When uploading the compressed archive, the submission system appears to extract the individual R files rather than displaying them within their original folder structure. The intended organization and grouping of the files are described clearly in the accompanying README file.

All scripts are self-contained and can be used independently, the folder descriptions are provided only for clarity and ease of reference.

Please let me know if any further information is required.

Best regards.
Kiran Prajapat

Kiran Prajapat
Lecturer in Mathematical Sciences
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September 12, 2025

Dear Editor-in-Chief,

We would like to submit the enclosed work for possible publication in Japanese Journal of Statistics and Data Science. It is entitled as "Estimation Method under Three-Parameter Generalized Exponential Model: Consistency, Uniqueness and its Applications". This article is not being considered for publication anywhere.

We believe that this manuscript is a good fit for the Journal because the article covers some theoretical findings that are analytically challenging and adds robustness to the proposed method.

These are the most important aspects of the paper:
1. One of the main contributions of this study is a thorough discussion on the development of a location-parameter-free (LPF) method of estimation for the parameters, quantiles and confidence intervals of the three-parameter generalized exponential (GE) distribution. The approach employs a location-invariant statistic, effectively addressing the challenge of an unbounded likelihood function.
2. Second significant contribution lies in proving the uniqueness and consistency of LPF estimators, a feat not yet achieved analytically for non-regular families in existing literature. The paper tackles the complexity arising from the two-dimensional likelihood function of the hypothesized distribution and makes an attempt to establish these fundamental properties.
3. We also conduct a comparative study with some prominent approaches on the basis of the biases and the RMSEs of the estimators obtained using a Monte Carlo simulation.
4. In addition, we analyze two real-world data from different application field, indicating (1) mortality rates of COVID-19 disease observed over 25 days, and (2) observed lifetimes in months of 40 electrical components, to illustrate the LPF approach. For the data sets, bootstrap confidence intervals are provided.

Please note that a portion of this manuscript as a preprint has been deposited in arXiv and can be accessed at the following link: https://doi.org/10.48550/arXiv.2403.17609.

Thank you for your time and attention.

Warm regards.

	Kiran Prajapat
Additional Information:	
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Estimation Method under Three-Parameter Generalized Exponential Model: Consistency, Uniqueness and its Applications

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Abstract

In numerous instances, the generalized exponential distribution can be used as an alternative to the most widely used non-regular family of distributions: Weibull, gamma, lognormal with three-parameters when analyzing lifetime or any skewed continuous data. A non-regular family is a class of probability distributions that do not satisfy the regularity conditions typically assumed in classical statistical inference. Some key features of such family of distributions are: support of its probability density function depends on one its parameters; its likelihood function may not be bounded for a certain range of parameter space, hence maximum likelihood estimators do not exist; the likelihood function even may not be differentiable or integrable as needed, hence Fisher Information may not exist or be infinite. Moreover, standard results like MLE existence, consistency, asymptotic normality may fail. Therefore, specialized or robust inferential techniques are needed. This article offers a consistent method for estimating the parameters of a three-parameter generalized exponential distribution that sidesteps the issue of an unbounded likelihood function. The method is hinged on a maximum likelihood estimation of shape and scale parameters that uses a location-invariant statistic. Important estimator properties, such as uniqueness and consistency, are demonstrated for the first time under this approach. In addition, quantile estimates for the assumed distribution are provided. We present a Monte Carlo simulation study along with comparisons to a number of well-known estimation techniques in terms of bias and root mean square error. For illustrative purposes, two real data sets from different origins; medical sciences and reliability engineering, have been analyzed and the goodness of fit along with the bootstrap confidence intervals are compared with existing traditional methods.

Key words: non-regular family; location-invariant statistics; modified maximum likelihood estimation; consistency; unimodal; confidence interval

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1 Introduction

Statistical modeling of any skewed data, such as COVID-19 mortality data, financial weekly return data and electrical lifetime data using a probability model that fits the best is important. The most widely used and well-liked distributions for analyzing such skewed data or lifetime data are the gamma and Weibull distributions with three parameters. These three parameters, which stand for location, scale, and shape, give the distributions a lot of flexibility when it comes to analyzing skewed data. Unfortunately, both distributions have some flaws as well. Additionally, as a special case, the exponentiated Weibull distribution reduces to the three-parameter generalized exponential (GE) distribution when its location parameter is not present. The exponentiated Weibull distribution was first proposed by Mudholkar et al. (1995). In many instances, it has been demonstrated that the GE model is applicable as a replacement for the gamma model or the Weibull model; for more information, see Gupta and Kundu (1999, 2001a, 2007).

Consider a three-parameter GE distribution denoted by GE(α, β, γ), where α, β and γ represent, respectively, the scale, shape, and location parameters. For $\alpha > 0,\ \beta > 0,\ \gamma \in \mathbb{R}$, a random variable that follows the GE(α, β, γ) has the cumulative distribution function (CDF) $F(.;\alpha, \beta, \gamma)$,

$$F(x;\alpha,\beta,\gamma) = \begin{cases} 0, & \text{if } x < \gamma \\ \left(1 - e^{-\frac{x-\gamma}{\alpha}}\right)^{\beta}, & \text{if } x > \gamma. \end{cases} \tag{1}$$

Here, all three parameters are undetermined. For the other similar three-parameter distributions, such as the lognormal, gamma, Weibull, and inverse Gaussian, etc., in which the location parameter is unknown, it is well known that the regularity conditions are not satisfied for the estimation method of the widely recognized maximum likelihood (ML) because the support of the probability density function (PDF) depends on the unknown location parameter, and thus the ML estimation may encounter difficulties. In the majority of time, the maximum likelihood estimator (MLE) does not exist for a particular range of the parameter space. In such situations, the likelihood becomes unbounded. Some additional issues with MLEs for non-regular distributions (when they exist): Asymptotic normality of the MLE and its functions cannot be used; the Fisher information matrix cannot be used for asymptotic variances and covariances due to the failure of the regularity conditions. Several authors, including Cohen and Whitten (1980, 1982); Cohen et al. (1984); Smith (1985); Smith and Naylor (1987); Hirose and Lai (1997); Hall and Wang (2005); Nagatsuka and Balakrishnan (2012); Nagatsuka et al. (2013, 2014); Nagatsuka and Balakrishnan (2015); Prajapat et al. (2021); Basu and Kundu (2023) have explored this issue. The identical issue arises in the case of three-parameter GE distribution. Gupta and Kundu (1999) elaborated on the ML estimation for the three-parameter GE distribution. When the shape parameter $\beta < 1$, it has been demonstrated that the MLE does not exist because the likelihood function becomes unbounded when the location parameter γ is approximately equal to the smallest observation in the observed sample; whereas asymptotic results have been presented only for the range of $\beta > 2$. Taking this problem into account, Prajapat et al. (2021) and Basu and Kundu (2023) offered other procedures for estimating the parameters of three-parameter GE for its entire parameter space.

Most recent and newly proposed estimation methods in this direction include the location-scale-parameter-free (LSPF) and location-parameter-free (LPF) methods. As the names of the methods imply, the LSPF method is based on a location and scale invariant statistic, whereas the LPF method uses a location invariant statistic; consequently, the likelihood functions based on the invariant statistics for these methods are one-dimensional and two-dimensional, respectively. Uniqueness and the consistency properties of LPF estimators have not yet been demonstrated analytically for any of the well-known three-parameter distributions examined in the literature. We assume this since the resultant likelihood function has a complicated form. Consequently, an attempt has been made for the hypothesized GE distribution, and consistency is proved whereas unimodality has been established for some cases.

This study shows that the LPF approach has an advantage over the LSPF method in that it requires less time to execute the simulations and has less computational complexity. The main explanation of this is the reduction in the number of integration as compare to the LSPF method. One of the LSPF method's drawbacks over the LPF method is that it uses the estimates from the preceding sequence to construct estimates separately and sequentially. This might lead to a significant build-up of biases in the sequences. The LPF, on the other hand, produces estimates through two-dimensional optimization by reducing number of steps in the estimation procedure. Moreover, LPF method works with a reduced number of degenerate random variables as compared to the LSPF method, which possibly may encourage it to better perform in some situations.

Therefore, the major purpose of the study is to develop the LPF method of estimation in detail for the parameters and quantiles of the three-parameter GE distribution. The second key purpose of the paper is to demonstrate the estimators' uniqueness and consistency. Because proving the uniqueness analytically is challenging, therefore proofs are provided for some particular cases. To obtain bias and root mean square error (RMSE) of the estimators, a Monte Carlo simulation is implemented. On the basis of the biases and RMSE of the estimators, we also conduct a comparative study with some prominent methods.

The remaining sections are structured as follows. The estimation procedure based on the proposed LPF method for the three-parameter GE distribution is detailed in Section 2. In this section, we also discuss the estimators' properties, such as their uniqueness and consistency. In Section 3, a Monte Carlo simulation study is conducted for the purpose of evaluating the LPF method and making comparisons to some existing methods. In Section 4, the LPF method along with other methods are illustrated using two real-world data sets of mortality rates and electrical lifetimes. Section 5 presents concluding remarks while summarizing the study.

2 Proposed Estimators and their Properties

Assume that $X_1, X_2, \ldots, X_n$ are n independent and identically distributed (*i.i.d.*) random variables following the three-parameter GE distribution with the common CDF defined in equation (1). Throughout the paper, it is presumed that $n \geq 3$. Consider the order statistics of $X_1, X_2, \ldots, X_n$ to be $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$. To begin the estimation procedure, we will consider the following statistic:

$$V_{(i)} = X_{(i)} - X_{(1)},\ i = 1, 2, \ldots, n. \tag{2}$$

$V_{(i)}$'s probability distribution is independent of the location parameter. It is worth noting that $V_{(1)} = 0$. The scale and shape parameters are therefore estimated based on the transformed data $V_{(1)}, V_{(2)}, \ldots, V_{(n)}$, whose joint probability distribution primarily depends on scale and shape parameters.

In this estimating sequence according to the developed LPF approach, estimators of the scale and shape parameter are then being used to estimate the location parameter. The estimation of parameters is discussed in detail in Subsection 2.1, while the estimation of quantiles for the lifetime distribution is elaborated on in Subsection 2.2.

2.1 Parameter Estimation

We estimate the scale and shape parameters α and β based on the random variables $V_{(1)}, V_{(2)}, \ldots, V_{(n)}$. As the likelihood function of α and β based on the transformed data is not dependent on the location parameter, it is bounded. Let $v_1, v_2, \ldots, v_n$ represent the respective realizations of $V_{(1)}, V_{(2)}, \ldots, V_{(n)}$. Note that v_1 must be 0.

Theorem 2.1. *The likelihood function of α and β, given $v_2, v_3, \ldots, v_n$, is given by*

$$\ell_v(\alpha, \beta | v_2, \ldots, v_n) = n! \Big(\frac{\beta}{\alpha}\Big)^n \int_0^\infty e^{-\frac{1}{\alpha}\sum_{i=1}^n (u+v_i)} \prod_{i=1}^n \big(1 - e^{-\frac{u+v_i}{\alpha}}\big)^{\beta-1} du, \ \alpha > 0, \ \beta > 0, \tag{3}$$

with $0 < v_2 < \cdots < v_n < \infty, \ v_1 = 0.$

Proof. See Appendix A. □

In Theorem 2.1, the likelihood function is a bounded and differentiable function with respect to the parameters α and β. These properties' proofs are listed in Appendix B. In order to maintain simplicity, we will refer to $\ell_v(\alpha, \beta | v_2, \ldots, v_n)$ as $\ell_v(\alpha, \beta)$ from now on. Due to the complexity of the likelihood function, we were unable to prove the unimodality of the bivariate function $\ell_v(\alpha, \beta)$, but we could establish the unimodality of the likelihood function $\ell_v(\alpha, \beta)$ when one of the parameters α and β is fixed. Now, we present two main findings in the subsequent theorems, the first of which relates to unimodality and the second to the consistency of the unique maximum. In Theorem 2.2, the unimodality of the likelihood function is emphasized.

Theorem 2.2. *For every $0 < v_2 < \cdots < v_n < \infty, v_1 = 0$, the likelihood function $\ell_v(\alpha, \beta)$ is unimodal function of $\alpha > 0$ (or $\beta > 0$) whenever β (or α) is fixed.*

Proof. Let us recall the likelihood function in equation (14) from Appendix B and rewrite the function $h_v(\alpha, \beta; u)$ defined in there:

$$\begin{aligned}\ell_v(\alpha, \beta) =& n! \int_0^\infty e^{h_v(\alpha,\beta;u)}\, du,\ \ \alpha > 0,\ \ \beta > 0,\\ \text{where } h_v(\alpha, \beta; u) =& n\ln(\beta) - n\ln(\alpha) - \frac{1}{\alpha}\sum_{i=1}^{n} c_i + (\beta - 1)\sum_{i=1}^{n} \ln(1 - e^{-c_i/\alpha})\end{aligned}$$

with $c_i = u + v_i$. Since the likelihood is differentiable with respect to α and β, therefore let us recall its derivatives, from equations (16) and (21), which are as follows.

$$\frac{\partial \ell_v(\alpha, \beta)}{\partial \alpha} = n! \int_0^\infty \frac{\partial h_v(\alpha, \beta; u)}{\partial \alpha} e^{h_v(\alpha,\beta;u)}\, du, \tag{4}$$

with $\frac{\partial h_v(\alpha,\beta;u)}{\partial \alpha} = -\frac{n}{\alpha} + \frac{1}{\alpha^2}\sum_{i=1}^{n} c_i\Big(\frac{1-\beta e^{-c_i/\alpha}}{1-e^{-c_i/\alpha}}\Big) = -\frac{n}{\alpha} + \frac{1}{\alpha^2}\sum_{i=1}^{n} c_i\Big(1 + \frac{1-\beta}{e^{c_i/\alpha}-1}\Big)$, and

$$\frac{\partial \ell_v(\alpha, \beta)}{\partial \beta} = n! \int_0^\infty \frac{\partial h_v(\alpha, \beta; u)}{\partial \beta} e^{h_v(\alpha,\beta;u)}\, du, \tag{5}$$

with $\frac{\partial h_v(\alpha,\beta;u)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln(1 - e^{-c_i/\alpha})$.

First we show that $\ell_v(\alpha, \beta)$ is a unimodal function of β for a fixed α, and then its unimodality with respect to α when β is fixed. Therefore, let us first assume that α is fixed. Note that, for any choice of u and v_i's, $e^{h_v(\alpha,\beta;u)} > 0$ and also the integrals are on positive support, hence the change in the sign of $\frac{\partial \ell_v(\alpha,\beta)}{\partial \beta}$ in (5) directly depends on the change in the sign of $\frac{\partial h_v(\alpha,\beta;u)}{\partial \beta}$. $\frac{\partial h_v(\alpha,\beta;u)}{\partial \beta} \to \infty$ as $\beta \downarrow 0$ and $\frac{\partial h_v(\alpha,\beta;u)}{\partial \beta} < 0$ as $\beta \to \infty$. Moreover, $\frac{\partial^2 h_v(\alpha,\beta;u)}{\partial \beta^2} = -\frac{n}{\beta^2} < 0\ \forall\ \beta$, *i.e.*, $\frac{\partial h_v(\alpha,\beta;u)}{\partial \beta}$ changes sign from positive to negative and the change in sign is only once. Hence, for a fixed α, $\frac{\partial \ell_v(\alpha,\beta)}{\partial \beta}$ changes sign in the similar way and $\frac{\partial \ell_v(\alpha,\beta)}{\partial \beta} = 0$ has a unique solution which maximizes the likelihood function $\ell_v(\alpha, \beta)$ with respect to β. Now, we move forward to show that $\ell_v(\alpha, \beta)$ is a unimodal function of α for a fixed β. With the same argument as given above, sign of $\frac{\partial \ell_v(\alpha,\beta)}{\partial \alpha}$ in (4) directly depends on the sign of $\frac{\partial \ell_v(\alpha,\beta)}{\partial \alpha}$ or else, we can say that sign of $\frac{\partial \ell_v(\alpha,\beta)}{\partial \alpha}$ is inversely proportional to the sign of $\alpha - \frac{1}{n}\sum_{i=1}^{n} c_i\Big(1 + \frac{1-\beta}{e^{c_i/\alpha}-1}\Big)$. Say

$$H_{1,n}(\alpha) = \frac{1}{n}\sum_{i=1}^{n} c_i\Big(1 + \frac{1-\beta}{e^{c_i/\alpha} - 1}\Big). \tag{6}$$

When $\beta > 1$, $H_{1,n}(\alpha)$ is a strictly decreasing function in α that decreases from $\frac{1}{n}\sum_{i=1}^{n} c_i$ to $-\infty$. $H_{1,n}(\alpha)$ is a constant function of α whenever $\beta = 1$ taking value $\frac{1}{n}\sum_{i=1}^{n} c_i$. It is easy to see that

$H_{1,n}(\alpha)$ and α meet exactly once whenever $\beta \geq 1$. Therefore, $\alpha - H_{1,n}(\alpha)$ changes sign from negative to positive and change in sign is only once whenever $\beta \geq 1$. Again, when $\beta < 1$, $H_{1,n}(\alpha)$ is strictly increasing in α and it increases from $\frac{1}{n}\sum\limits_{i=1}^{n} c_i$ to ∞. Now, let us consider the quantity

$$\frac{\partial H_{1,n}(\alpha)}{\partial \alpha} = \frac{1-\beta}{n\alpha^2}\sum_{i=1}^{n}\left(\frac{c_i^2 e^{-c_i/\alpha}}{(1-e^{-c_i/\alpha})^2}\right).$$

Since $\frac{s^2 e^{-s}}{(1-e^{-s})^2} < 1 \ \forall \ s > 0$ (see part (i) of Lemma 2 in Ghitany et al. (2013)), we have $\frac{\partial H_{1,n}(\alpha)}{\partial \alpha} < 1$. Hence, α and $H_{1,n}$ have to meet exactly once. This implies that $\alpha - H_{1,n}(\alpha)$ changes its sign only once from negative to positive for $\beta < 1$ also. Since the sign of $\frac{\partial \ell_v(\alpha,\beta)}{\partial \alpha}$ is inversely proportional to the sign of $\alpha - H_{1,n}(\alpha)$, it is clear that change in sign of $\frac{\partial \ell_v(\alpha,\beta)}{\partial \alpha}$ from positive to negative and it is only once. It implies that $\ell_v(\alpha,\beta)$ is unimodal with respect to α for a fixed β. □

Now, in Theorem 2.3, we demonstrate the consistency of the estimators of α and β obtained by maximizing $\ell_v(\alpha,\beta)$.

Theorem 2.3. *Estimators based on the maximization of the likelihood $\ell_v(\alpha,\beta)$ are consistent estimators for $\alpha > 0$ and $\beta > 0$.*

Proof. To show the consistency of the estimators of (α,β) obtained by maximizing the likelihood function $\ell_v(\alpha,\beta)$, it suffices to prove the following result:
For any fixed $\alpha \neq \alpha_0$ and $\beta \neq \beta_0$, where α_0 and β_0 are true values of the parameters α and β,

$$\lim_{n\to\infty} P\left(\frac{\ell_v(\alpha,\beta;V_{(2)},\ldots,V_{(n)})}{\ell_v(\alpha_0,\beta_0;V_{(2)},\ldots,V_{(n)})} < 1\right) = 1.$$

To proceed further, we recall the conditional joint PDF of the order statistics of $V_{(2)},\ldots,V_{(n)}$ given $Z_{(1)} = u$ (See Appendix A). Here $Z_{(1)} = X_{(1)} - \gamma_0$ and γ_0 is true value of the parameter γ. The conditional joint PDF is as follows:

$$\begin{aligned} f_{V_{(2)},\ldots,V_{(n)}|Z_{(1)}=u}(v_2,\ldots,v_n|u) &= \frac{f_{Z_{(1)},V_{(2)},\ldots,V_{(n)}}(u,v_2,\ldots,v_n)}{f_{Z_{(1)}}(u)} \\ &= \frac{n!g(u;\alpha_0,\beta_0)\left\{\prod_{i=2}^{n} g(u+v_i;\alpha_0,\beta_0)\right\}}{ng(u;\alpha_0,\beta_0)(1-G(u;\alpha_0,\beta_0))^{n-1}} \\ &= (n-1)!\prod_{i=2}^{n}\frac{g(u+v_i;\alpha_0,\beta_0)}{1-G(u;\alpha_0,\beta_0)} \end{aligned}$$

$$= (n-1)! \prod_{i=2}^{n} f_{V_i|Z_{(1)}=u}(v_i|u),$$

where $f_{V_i|Z_{(1)}=u}(v_i|u) = \frac{g(u+v_i;\alpha_0,\beta_0)}{1-G(u;\alpha_0,\beta_0)}$. Here, $V_i^{\prime}s$ are *i.i.d.* conditional on $Z_{(1)} = u$, with a common conditional PDF $f_{V_i|Z_{(1)}=u}(.|u)$. Define

$$\ell_u(\alpha,\beta;V_{(2)},\dots,V_{(n)}) = (n-1)! \prod_{i=2}^{n} \frac{g(u+V_i;\alpha,\beta)}{1-G(u;\alpha,\beta)}$$

and then, consider the following quantity for fixed $u > 0$, conditioned on $Z_{(1)} = u$. For every $\alpha \neq \alpha_0$ and $\beta \neq \beta_0$,

$$\frac{1}{n-1}\ln\left\{\frac{\ell_u(\alpha,\beta;V_{(2)},\dots,V_{(n)})}{\ell_u(\alpha_0,\beta_0;V_{(2)},\dots,V_{(n)})}\right\} = \frac{1}{n-1}\sum_{i=2}^{n}\ln\left\{\frac{g(u+V_i;\alpha,\beta)/(1-G(u;\alpha,\beta))}{g(u+V_i;\alpha_0,\beta_0)/(1-G(u;\alpha_0,\beta_0))}\right\}. \tag{7}$$

By the law of large numbers, right hand side of the equation 7 converges to

$$E\left(\ln\left\{\frac{g(u+V;\alpha,\beta)/(1-G(u;\alpha,\beta))}{g(u+V;\alpha_0,\beta_0)/(1-G(u;\alpha_0,\beta_0))}\right\}\middle|Z_{(1)} = u\right).$$

Here, V has a conditional PDF $f_{V|Z_{(1)}=u}(.|u)$ given $Z_{(1)} = u$. Now, by Jensen's inequality,

$$\begin{aligned}
&E\left(\ln\left\{\frac{g(u+V;\alpha,\beta)/(1-G(u;\alpha,\beta))}{g(u+V;\alpha_0,\beta_0)/(1-G(u;\alpha_0,\beta_0))}\right\}\middle|Z_{(1)} = u\right)\\
&\leq \ln\left\{E\left(\frac{g(u+V;\alpha,\beta)/(1-G(u;\alpha,\beta))}{g(u+V;\alpha_0,\beta_0)/(1-G(u;\alpha_0,\beta_0))}\middle|Z_{(1)} = u\right)\right\}\\
&= \ln\left\{\int_0^\infty \frac{g(u+v;\alpha,\beta)}{(1-G(u;\alpha,\beta))}\,dv\right\} = 0.
\end{aligned}$$

Therefore, it implies that

$$\begin{aligned}
&\lim_{n\to\infty} P\left(\frac{1}{n-1}\ln\left\{\frac{\ell_u(\alpha,\beta;V_{(2)},\dots,V_{(n)})}{\ell_u(\alpha_0,\beta_0;V_{(2)},\dots,V_{(n)})}\right\} < 0 \;\middle|\; Z_{(1)} = u\right) = 1\\
&\implies \lim_{n\to\infty} P\left(\frac{\ell_u(\alpha,\beta;V_{(2)},\dots,V_{(n)})}{\ell_u(\alpha_0,\beta_0;V_{(2)},\dots,V_{(n)})} < 1 \;\middle|\; Z_{(1)} = u\right) = 1\\
&\implies \lim_{n\to\infty} P\big(\ell(\alpha,\beta;V_{(2)},\dots,V_{(n)}) < \ell(\alpha_0,\beta_0;V_{(2)},\dots,V_{(n)})|Z_{(1)} = u\big) = 1.
\end{aligned} \tag{8}$$

Moreover

$$
\begin{aligned}
&P(\ell(\alpha,\beta;V_{(2)},\ldots,V_{(n)}) < \ell(\alpha_0,\beta_0;V_{(2)},\ldots,V_{(n)})) \\
&= \int_0^\infty P\big(\ell(\alpha,\beta;V_{(2)},\ldots,V_{(n)}) < \ell(\alpha_0,\beta_0;V_{(2)},\ldots,V_{(n)})|Z_{(1)} = u\big) f_{Z_{(1)}}(u)\,du
\end{aligned}
$$

Therefore, using the Lebesgue's dominated convergence theorem and (8)

$$
\begin{aligned}
&\lim_{n\to\infty} P\big(\ell(\alpha,\beta;V_{(2)},\ldots,V_{(n)}) < \ell(\alpha_0,\beta_0;V_{(2)},\ldots,V_{(n)})\big) \\
&= \int_0^\infty \lim_{n\to\infty} P\big(\ell(\alpha,\beta;V_{(2)},\ldots,V_{(n)}) < \ell(\alpha_0,\beta_0;V_{(2)},\ldots,V_{(n)})|Z_{(1)} = u\big) f_{Z_{(1)}}(u)\,du \\
&= \int_0^\infty \lim_{n\to\infty} f_{Z_{(1)}}(u)\,du = \lim_{n\to\infty} \int_0^\infty f_{Z_{(1)}}(u)\,du = 1. \qquad (9)
\end{aligned}
$$

Hence the result is proved. □

We already have estimators based on the maximization of the likelihood function $\ell_v(\alpha,\beta)$ using the location-invariant data $v_2, v_3, \ldots, v_n$. Let us refer to the estimators of α and β based on the location-invariant data as $\widehat{\alpha}_v$ and $\widehat{\beta}_v$, respectively.

In addition, it is appropriate to use $X_{(1)}$ as an estimator for the location parameter γ. Let us represent it as $\widehat{\gamma}_{init}$ *i.e.*, $\widehat{\gamma}_{init} = X_{(1)}$. Because $E(X_{(1)}) = \gamma + \alpha\int_0^\infty (1 - F_Y(y;\beta))^n\,dy$ with $Y \sim \mathrm{GE}(1,\beta,0)$, it should be emphasized that $\widehat{\gamma}_{init}$ is a biased estimator of the parameter γ. It is therefore feasible to look at a bias-corrected estimate of γ, denoted by $\widehat{\gamma}_v$, as $X_{(1)} - \widehat{\alpha}_v \int_0^\infty (1 - F_Y(y;\widehat{\beta}_v))^n\,dy$.

Now, we establish that $\widehat{\gamma}_v$ is a consistent estimator of γ. To explain the consistency of $\widehat{\gamma}_{init}$, consider the following probability for an arbitrary $\epsilon > 0$:

$$
\mathrm{P}(|\widehat{\gamma}_{init} - \gamma| > \epsilon) = \mathrm{P}(X_{(1)} - \gamma > \epsilon) = \mathrm{P}(n^{1/\beta}\frac{X_{(1)} - \gamma}{\alpha} > n^{1/\beta}\frac{\epsilon}{\alpha}) \qquad (10)
$$

Using the Theorem 8 of Gupta and Kundu (1999), it can be observed that the probability aforesaid in (10) is less than $e^{-n(\epsilon/\alpha)^\beta}$ and converges to 0 as sample size n approaches ∞, proving that $\widehat{\gamma}_{init}$ is consistent for γ. Recall, in order to show the consistency of the bias-corrected estimator, that $\widehat{\gamma}_v = X_{(1)} - \widehat{\alpha}_v \int_0^\infty (1-(1-e^{-y})^{\widehat{\beta}_v})^n\,dy$. Using Slutsky's theorem and the facts that $\widehat{\alpha}_v$ and $\widehat{\beta}_v$ are consistent for α and β, respectively, it can be shown that $\widehat{\alpha}_v \int_0^\infty (1 - (1 - e^{-y})^{\widehat{\beta}_v})^n\,dy$

converges to 0 in probability. Consequently, applying Slutsky's theorem once more, $\widehat{\gamma}_v$ is consistent for γ.

The above-mentioned detailed estimation approach under the LPF method described in this section is quickly summarized as follows:

Step 1. Obtaining $\widehat{\alpha}_v$ and $\widehat{\beta}_v$, take $\widehat{\gamma}_{init} = X_{(1)}$.

Step 2. Utilize $\widehat{\alpha}_v$, $\widehat{\beta}_v$ and $\widehat{\gamma}_{init}$ of Step 1 to obtain the bias-corrected $\widehat{\gamma}_v$ as follows:

$$\widehat{\gamma}_v = X_{(1)} - \widehat{\alpha}_v \int_0^\infty \left(1 - (1 - e^{-y})^{\widehat{\beta}_v}\right)^n dy.$$

2.2 Quantile Estimation

We consider the estimation of the ζ-th quantile of the three-parameter GE distribution with CDF given in (1). For $0 < \zeta < 1$, the ζ-th quantile is the solution to the equation $F(x_\zeta; \alpha, \beta, \gamma) = \zeta$ with respect to x_ζ, which yields

$$x_\zeta = \gamma - \alpha \ln\left(1 - \zeta^{1/\beta}\right). \tag{11}$$

In this study, the quantile estimator is obtained by substituting the LPF estimators of α, β, and γ into (11). The following theorem establishes the consistency of the plug-in estimator $\widehat{x}_{\zeta,v}$.

Theorem 2.4. *For a fixed $\zeta \in (0,1)$, and parameters $\alpha > 0$, $\beta > 0$, and $\gamma \in \mathbb{R}$, the ζ-th quantile of the GE distribution is $x_\zeta(\alpha,\beta,\gamma) = \gamma - \alpha\ln\left(1 - \zeta^{1/\beta}\right)$ and its plug-in estimator based on the LPF estimators is*

$$\widehat{x}_{\zeta,v} = x_\zeta(\widehat{\alpha}_v, \widehat{\beta}_v, \widehat{\gamma}_v) = \widehat{\gamma}_v - \widehat{\alpha}_v \ln\left(1 - \zeta^{1/\widehat{\beta}_v}\right).$$

Then

$$\widehat{x}_{\zeta,v} \xrightarrow{P} x_\zeta(\alpha,\beta,\gamma),$$

where $\xrightarrow{P}$ denotes convergence in probability.

Proof. Define the mapping $g : (0,\infty) \times (0,\infty) \times \mathbb{R} \to \mathbb{R}$ by $g(\alpha,\beta,\gamma) = \gamma - \alpha\ln\left(1 - \zeta^{1/\beta}\right)$. For any $\beta > 0$ and $\zeta \in (0,1)$ we have $0 < \zeta^{1/\beta} < 1$, and therefore $1 - \zeta^{1/\beta} \in (0,1)$ and its logarithm is finite. Thus g is well defined and continuous for all (α, β, γ) with $\beta > 0$.

From the consistency results of the LPF estimators established in Section 2.1, we have

$$(\widehat{\alpha}_v, \widehat{\beta}_v, \widehat{\gamma}_v) \xrightarrow{P} (\alpha, \beta, \gamma),$$

and since g is continuous at (α, β, γ), the Continuous Mapping Theorem (Theorem 1.10 in Shao (2003)) implies

$$\widehat{x}_{\zeta,v} = g(\widehat{\alpha}_v, \widehat{\beta}_v, \widehat{\gamma}_v) \xrightarrow{P} g(\alpha, \beta, \gamma) = x_\zeta(\alpha, \beta, \gamma).$$

This proves the result. □

Remark 2.1. *The above theorem requires $\zeta \in (0, 1)$. For $\zeta = 0$, the quantile equals γ, so $\widehat{x}_{0,v} = \widehat{\gamma}_v \xrightarrow{P} \gamma$. For ζ very close to 1, numerical instability may arise, although the plug-in estimator remains consistent.*

3 Simulation Study

We evaluate the performance of the proposed estimator through a Monte Carlo simulation study. The proposed method of estimation is the LPF method. We compare the performance of the LPF method to the LSPF method and two other modified maximum likelihood estimation methods, because these methods provide estimators for the entire parameter space. Since the MLE does not exist when $\beta < 1$, the LPF method is compared to the MLE when the shape parameter is assumed to be greater than or equal to 1. The comparisons are based on the estimators' biases and the RMSE. These two modified maximum likelihood estimators (MMLEs) are MMLE I and MMLE III, that are discussed in Section 3 of Prajapat et al. (2021). Recall that MMLE I is the most convenient estimation method, which has been recently implemented by Pasari and Dikshit (2014) and Raqab et al. (2008), whereas the MMLE III method was proposed by Hall and Wang (2005). In this article, the MMLE I method is modified using the bias-corrected estimator of γ before being implemented.

All results are obtained using the Monte Carlo simulation with the shape parameter β set to $0.50,\ 0.75,\ 1.00,\ 1.50,\ 2.00,\ 3.00$ and the sample size n set to $20,\ 50, 100$ and 200. All simulation results in terms of biases and RMSEs are provided based on $10,000$ simulations. Because the GE model is a location-scale family, without loss of generality we equate the scale parameter α to 1 and the location parameter γ to 0. We present bias and RMSE for the estimators of location, scale, and shape parameters in Subsection 3.1 and for the estimators of quantiles in Subsection 3.2.

During the simulation study, a certain proportion of the generated data sets has to be rejected. This proportion is reported in the last column of the corresponding tables. These rejections occur when the resulting parameter estimates are clearly unreasonable or numerically unstable, for example, an estimate such as 10^7 for the shape parameter. Such values indicate that the corresponding data set is not suitable for evaluating estimators' performance and therefore must be excluded. To implement this, we choose an upper cutoff value for the shape parameter estimate, denoted by β_U. This threshold is determined, for instance, by examining the histogram of the simulated estimates and identifying a reasonable upper bound beyond which the estimates are implausible. Therefore, the proportion of rejection is defined as

$$p = \frac{\text{Number of samples with } \widehat{\beta}_v \geq \beta_U}{\text{Total number of simulations}}.$$

Since estimation in three-parameter models for very small samples (e.g., $n \leq 10$) is well known to produce highly unstable estimates with large biases, very high variability, and frequent non-convergence (see Gupta and Kundu (2001b)), we focus on moderate and larger sample sizes $n = 20, 50, 100, 200$, as we observe similar issues for such small samples. Moreover, for such very small sample sizes, the rejection proportion p also becomes extremely high. Furthermore, in the case of the LSPF method, numerical results for $n = 200$ are omitted due to a computational issue, where the likelihood calculations exceed machine precision and produce values that are out of bounds during the simulations.

We also evaluate the computational efficiency of the LPF method, since computation time is an important practical consideration. The total time required for any estimation method depends on several factors, such as the value of the shape parameter β, the sample size n, and the number of Monte Carlo replications. Among these, the sample size has the strongest influence, and therefore we report computation times for different values of n.

All computations were carried out in `R`, using the `foreach` and `doParallel` packages for parallel execution on a high-performance computing (HPC) cluster with 40 processing cores. For each method, we generate $25,000$ samples, and the first $10,000$ samples are used to obtain the estimates. Thus, the computational times reported in Table 1 are compared based on $25,000$ Monte Carlo iterations. The `R` implementations of the MLE, MMLE I, and MMLE III methods produce numerical results within seconds or minutes, and therefore their computational cost is negligible compared to that of the LSPF and LPF methods. The results in Table 1 clearly

Table 1: Computational time (LPF vs. LSPF) for different sample sizes n at $\beta = 1.50$.

Method	$n = 20$	$n = 50$	$n = 100$	$n = 200$
LSPF	18.50 hrs	27.07 hrs	33.90 hrs	–
LPF	2.80 mins	3.20 mins	3.60 mins	6.00 mins

show the substantial computational advantage of the LPF method. For instance, when $\beta = 1.50$ and $n = 100$, the LSPF method requires approximately 33.90 hours, whereas the LPF method completes the same number of replications in only 3.60 minutes. This significant reduction in computational time demonstrates the practical benefit of the LPF method, particularly for large sample sizes or simulation-intensive studies. All `R` source codes used to generate the simulation results and data analyses are provided as supplementary material to ensure full reproducibility.

3.1 Evaluation of Parameter Estimation

In Tables 8-9, all numerical simulations for estimating the parameters are presented. The shape parameter is assumed to be greater than 1 in Table 9 so that the MLEs are computed sensibly. To get a clear picture of the performance of the estimates of all three parameters, their biases and RMSEs are plotted in Figures 1-3 against varying β values. Now, we will attempt to summarize the results of the simulation study performed. The following observations are based on the simulation study reported in the first two tables:

- As the value of the shape parameter β approaches zero or as the sample grows in size, the performance of each method for estimating all parameters improves.
- MMLE I: When $\beta \leq 1$, it underestimates the scale parameter, and when $\beta > 1$, it overestimates the scale parameter. Also, the shape parameter is underestimated when $\beta > 1$.

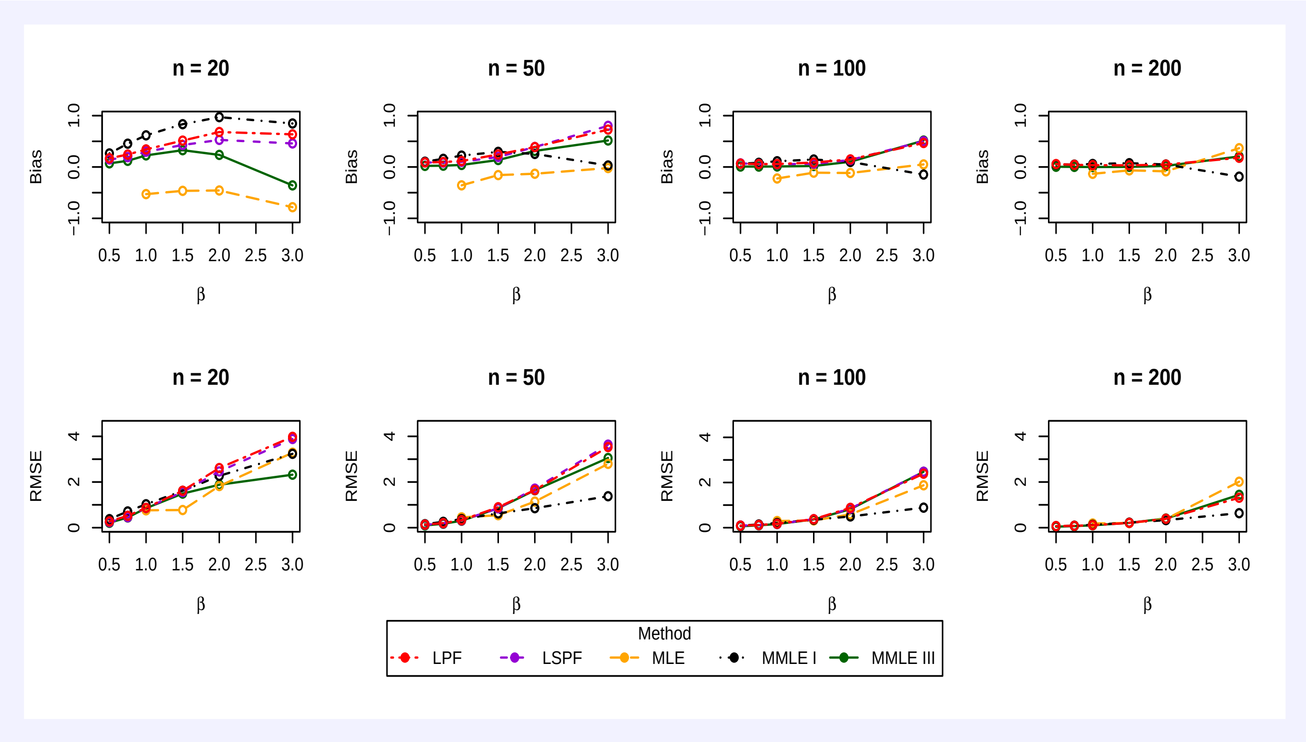


Figure 1: Plots for bias and RMSE of the estimator of shape parameter based on various estimation methods varying β values.

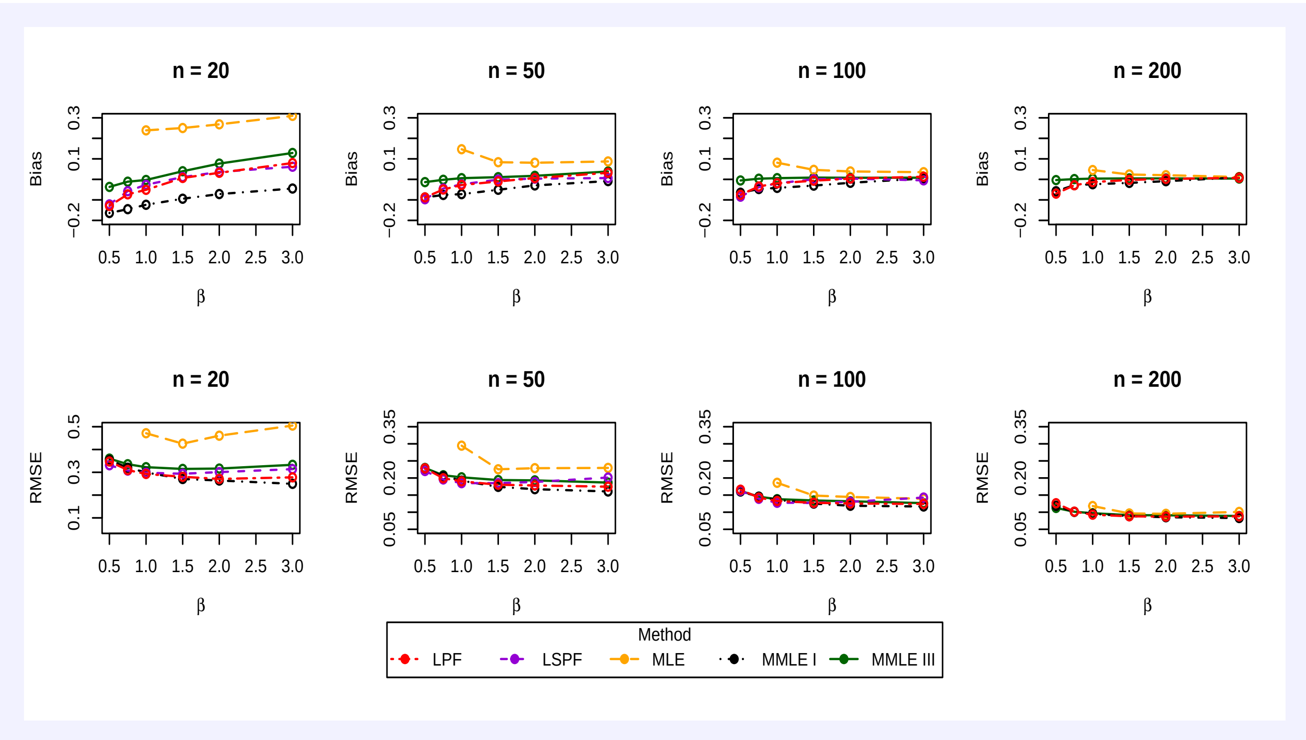


Figure 2: Plots for bias and RMSE of the estimator of scale parameter based on various estimation methods varying β values.

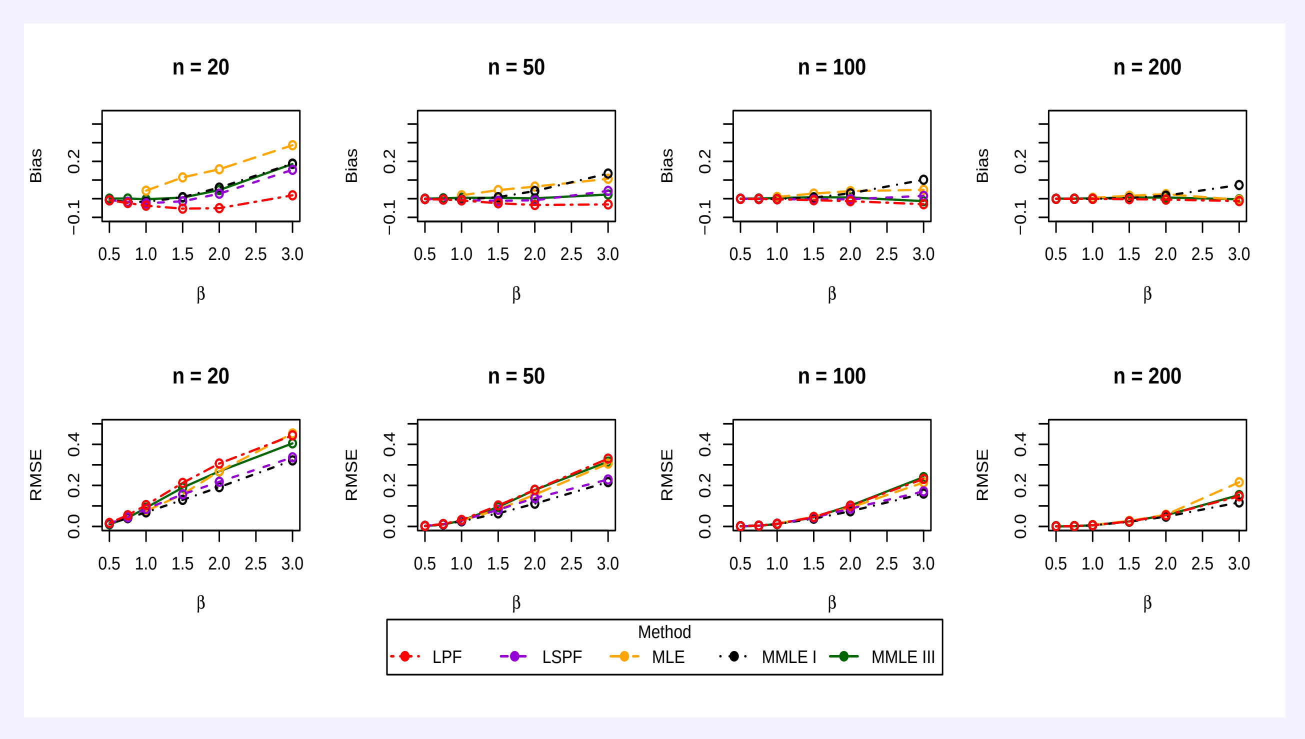


Figure 3: Plots for bias and RMSE of the estimator of location parameter based on various estimation methods varying β values.

- LSPF and LPF methods are nearly always similar in performance when estimating the shape and scale parameters.
  - To estimate the shape parameter:
    * If $\beta > 1$, it is recommended to use MMLE III for $n \leq 20$, MLE for $20 < n \leq 100$, and either LPF or MMLE III for $n > 100$.
    * If $\beta \leq 1$, it is best advised to use MMLE III.
  - To estimate the scale parameter:
    * When $\beta > 1$, LPF is suggested as the first choice due to its small bias and RMSE, followed by MMLE III as the second choice because it performs slightly worse than LPF.
    * If $\beta \leq 1$, it is best advised to use MMLE III.
    * For small sample sizes, such as $n \leq 20$, we strongly discourage the use of MLE. In this situation, LPF or MMLE I is preferable if RMSE is the prime concern while MMLE III is preferable if bias is the prime concern.
    * LSPF and LPF perform similar and outperform all other methods.

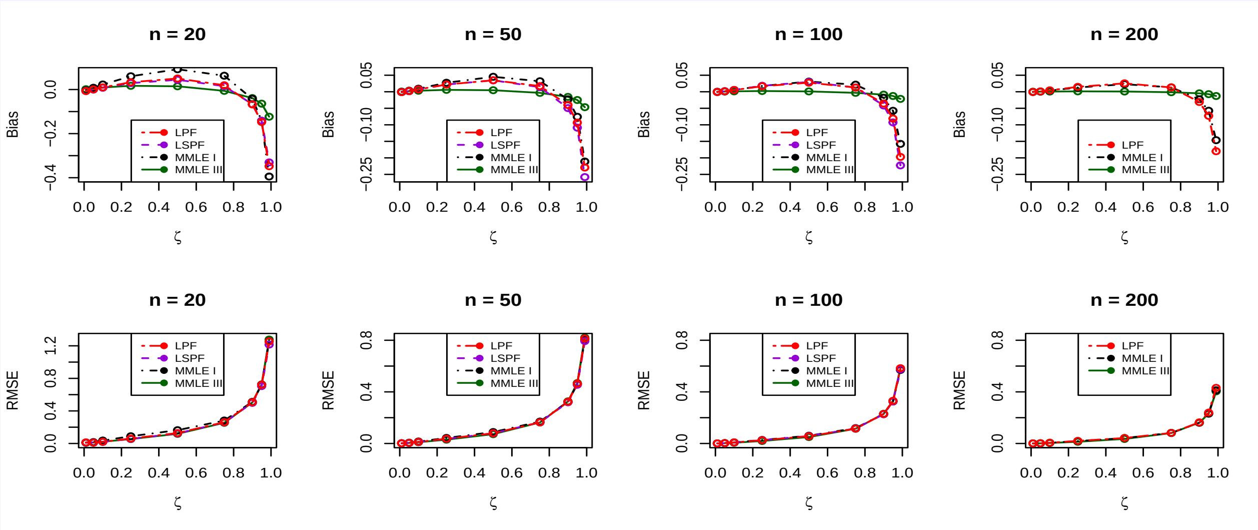
- To estimate the location parameter:
  - LPF and LSPF perform similar in terms of biases and RMSEs for $\beta \leq 2$. As β moves away from 2 when $n \leq 50$, LPF method surpasses the LSPF method in terms of bias whereas LSPF outperforms LPF in terms of RMSE. The explanation for this might be that the LSPF produces estimates individually and sequentially, using the estimates obtained in the previous sequence, resulting in a substantial accumulation of biases in the sequences. This is one disadvantage of the LSPF approach over the LPF method.
  - LPF consistently performs better than all other methods in terms of RMSE and bias, followed by MMLE III, when estimating location parameters. We do not advise MMLE I.

- MMLE I is the only method that estimates parameters with a proportion of rejection of approximately zero, regardless of β value and sample size. On the basis of the reported proportion of rejections, we recommend using this method as a first preference. If we only consider the LSPF and LPF methods, LSPF has a lower proportion of rejection than LPF. However, if we compare LPF to all other methods including LSPF, we notice the following: After MMLE I, the second preference is LPF for $0.5 \leq \beta < 3$ and MLE for relatively large values of ($\beta \geq 3$); when β is very small ($\beta \approx 0.5$), it has been observed that all the methods have a very small proportion of rejection (≈ 0) for sample size $n \geq 50$ and therefore any method can be recommended but when $n \leq 20$, we advise to use LPF.

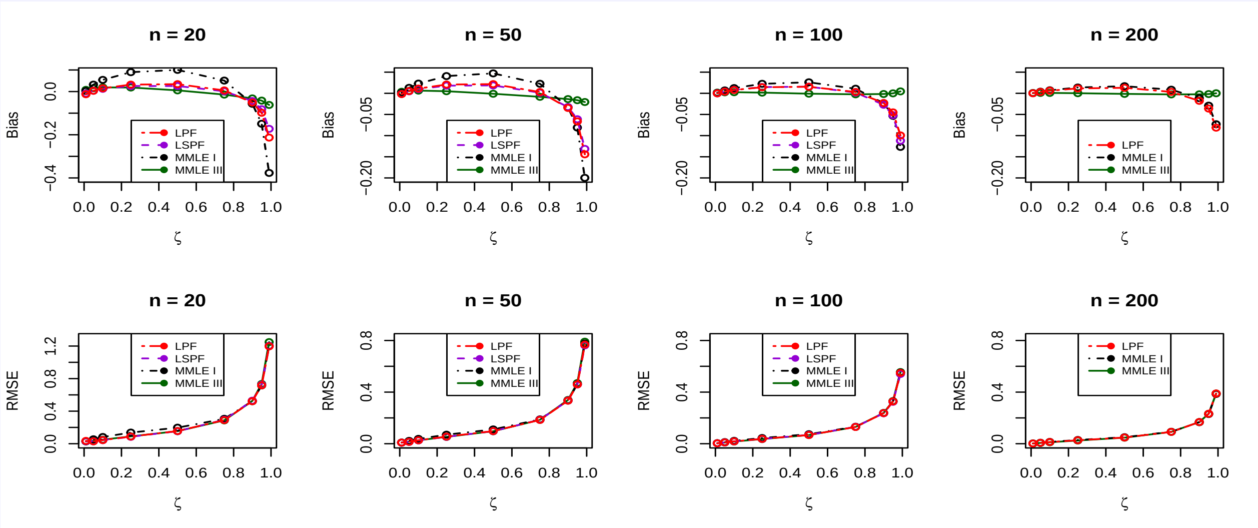
3.2 Evaluation of Quantile Estimation

In practice, it is essential to estimate all three parameters in such a way that their combined performance is satisfactory. One may consider the quantile estimation as a possible solution to this problem, in which all parameters are utilized to estimate a qantile of the distribution. We therefore present quantile estimates for $\zeta = 0.01,\ 0.05,\ 0.10,\ 0.25,\ 0.50,\ \ 0.75,\ \ 0.90,\ \ 0.95,\ \ 0.99$ using equation (11) based on Monte Carlo simulation for the same values of shape parameter and sample size as in Subsection 3.1.

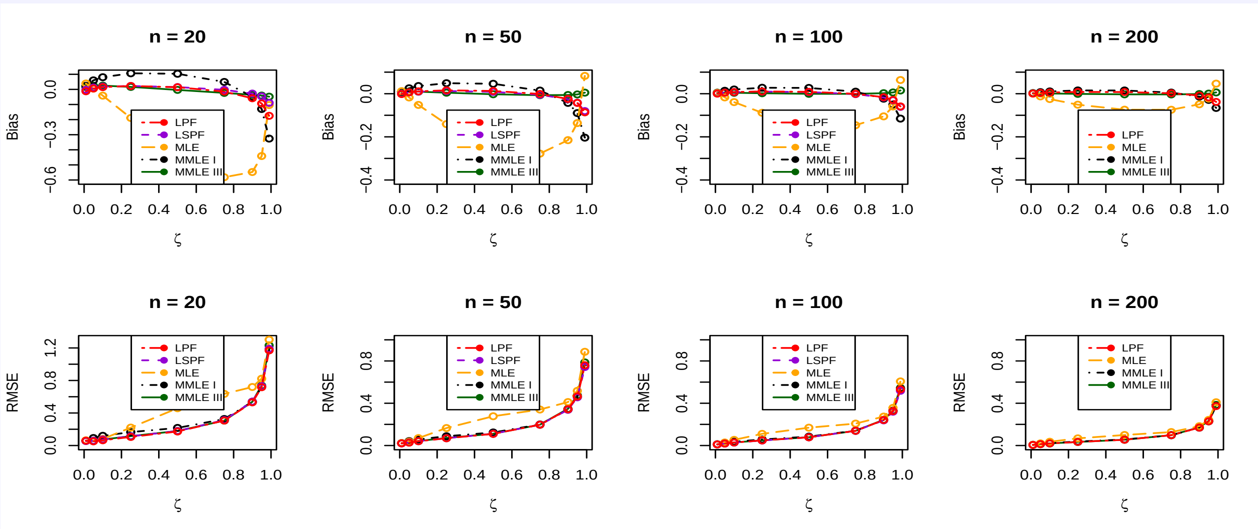
In Tables 10-13, the biases and RMSEs of the quantile estimation are shown. Figures 4-5 exhibit plots of the biases and RMSEs related to each method as the parameter value β



(a) β = 0.5

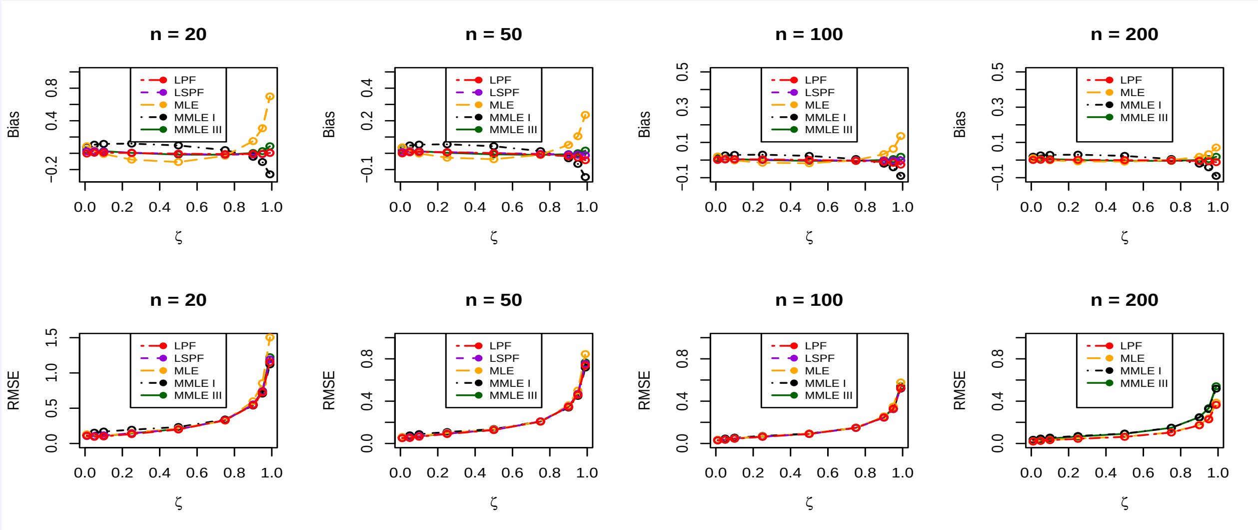


(b) β = 0.75

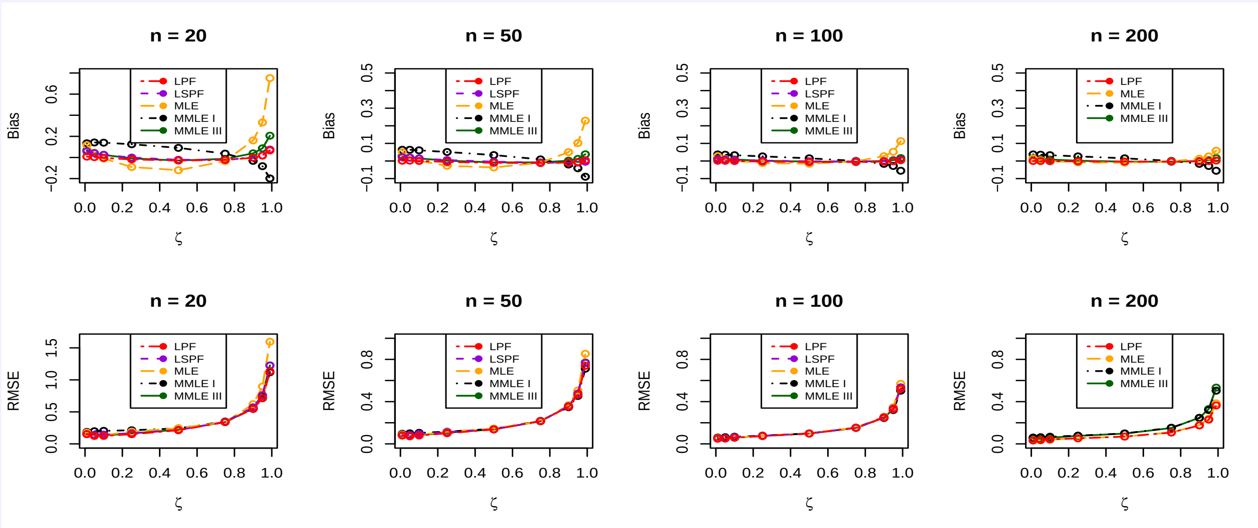


(c) β = 1.00

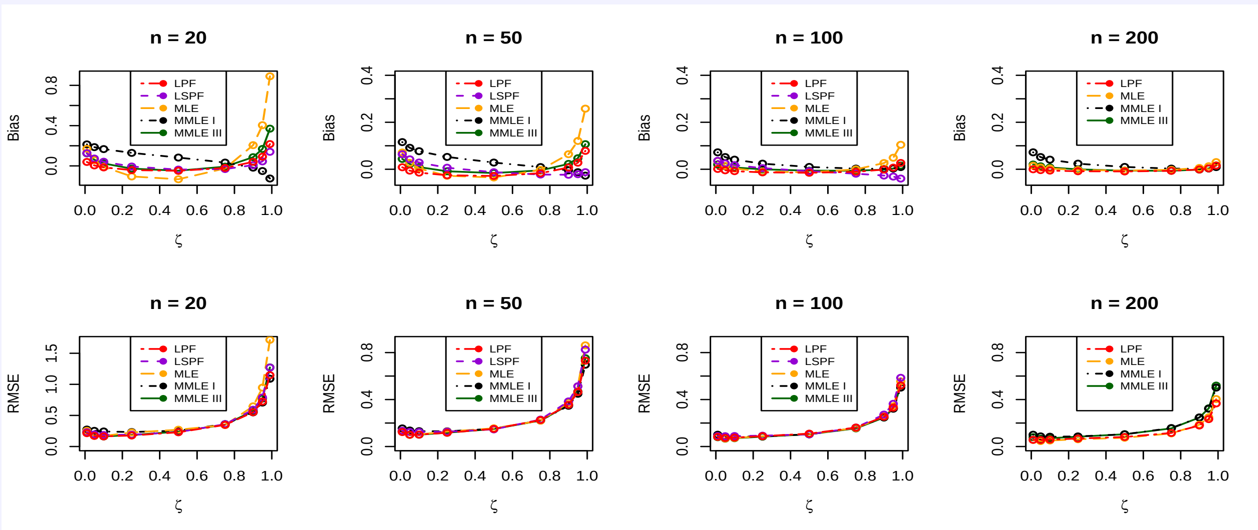
Figure 4: Plots for bias and RMSE based on various estimation methods when β = 0.50, 0.75, 1.00.



(a) β = 1.5



(b) β = 2.0



(c) β = 3.0

Figure 5: Plots for bias and RMSE based on various estimation methods when $\beta = 1.5,\ 2.0,\ 3.0$.

is varied for each of the sample sizes previously studied. The following can be deduced from these results:

- The absolute value of the bias grows as ζ approaches towards 0 or 1 from 0.5, whereas the RMSE increases as ζ approaches to 1 from 0.
- All methods are equivalent in terms of RMSE, particularly when $n > 20$.
- LPF and LSPF comparisons: LPF and LSPF methods yield similar performances almost everywhere except for the situation when $\beta > 2$ and $n < 100$. For this exception, we recommend LPF when ζ is small (< 0.2), LSPF when ζ is close to 1 (> 0.8) and any of the LPF or LSPF when $0.2 \leq \zeta \leq 0.8$.
- In terms of bias and RMSE, LPF performs much better than all other methods when $\beta \geq 1$. It has the lowest bias values of the quantile estimates. Consequently, LPF is strongly advised when $\beta \geq 1$.
- When $\beta \leq 1$, MMLE III provides the highest performance for any choice of $0 < \zeta < 1$, although LPF performs very well for $0 < \zeta < 0.7$ and much better as n grows. In this instance, we advise LPF for mainly sample sizes $n > 20$.

4 Illustrative Examples

The GE distribution is generally used for modeling a wide range of real-world skewed datasets. For instance, such data can be originated from the field of medical sciences, reliability engineering, insurance and economics, etc. In this section, we will consider two kind of data sets that are medical data and electrical failure data. We illustrate them using the three-parameter GE distribution in Example 1 and Example 2, respectively. In Example 2 (electrical lifetime data), the location parameter and higher quantiles are more important for engineering applications.

Example 1: This example concentrates on analyzing a dataset of size 25 representing mortality rates for COVID-19 disease where mortality rate is a quantity defined as a percentage of people dying among the total affected people by this disease. The data are reported by Nik et al. (2023) and Jeon et al. (2024) and have been used for modeling the well-known Pareto distribution. The data in Table 2 denotes the mortality rate of COVID-19 that is observed for 25 days (10 April, 2020 - 4 May, 2020) in Canada. A basic data analysis or summary yields the descriptive statistics as shown in Table 2 which confirm that the distribution of the mortality rate is positively skewed.

Now, before we proceed, let us summarize which method is recommended based on the simulation results. For the medical data, the primary quantities of interest are the shape parameter and the relevant quantiles. When the shape parameter is the main target, the MLE is highly recommended for sample sizes $20 < n \leq 100$ and $\beta \geq 1$. For quantile estimation, the preferred method depends on the quantile level: when $\beta > 2$ and $n < 100$, LPF is recommended for small quantiles ($\zeta < 0.2$), LSPF is preferred when ζ is close to 1 ($\zeta > 0.8$), and either LPF or LSPF performs well for intermediate quantiles ($0.2 \leq \zeta \leq 0.8$).

Table 2: Mortality data, its summary and estimates of the model parameters.

Data:
3.1091, 3.3825, 3.1444, 3.2135, 2.4946, 3.5146, 4.9274,
3.3769, 6.8686, 3.0914, 4.9378, 3.1091, 3.2823, 3.8594, 4.0480, 4.1685,
3.6426, 3.2110, 2.8636, 3.2218, 2.9078, 3.6346, 2.7957, 4.2781, 4.2202

Summary:

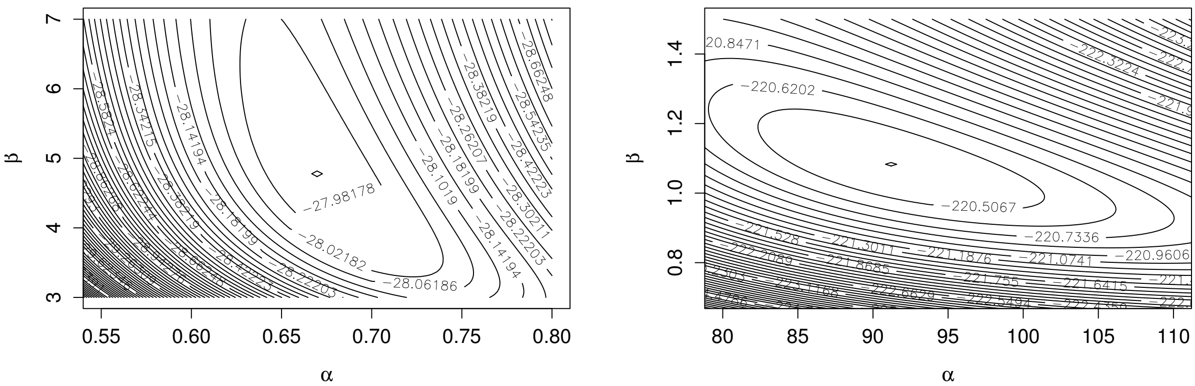
Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skew	Kurtosis
2.49	3.11	3.38	3.65	4.05	6.87	1.78	3.63

Method	Estimates			CvM		KS	
	Shape	Scale	Location	Statistic	p-value	Statistic	p-value
LPF	4.7657	0.6615	2.1600	0.0581	0.8260	0.1174	0.8593
LSPF	5.1209	0.6471	2.1469	0.0579	0.8270	0.1193	0.8458
MLE	3.4652	0.6870	2.2998	0.0542	0.8506	0.1111	0.8998
MMLE I	6.0763	0.5846	2.2365	0.1184	0.5018	0.1614	0.4961
MMLE III	14.6906	0.6018	1.6443	0.0653	0.7808	0.1281	0.7787

Table 3: Various quantile estimates of the COVID-19 mortality model for Example 1.

Method	ζ								
	0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
LPF	2.4767	2.6642	2.7946	3.0707	3.4829	4.0369	4.6888	5.1612	6.2365
LSPF	2.4850	2.6740	2.8042	3.0782	3.4844	4.0283	4.6669	5.1293	6.1815
MLE	2.5111	2.6755	2.7962	3.0620	3.4729	4.0378	4.7099	5.1991	6.3148
MMLE I	2.6062	2.7882	2.9111	3.1659	3.5387	4.0336	4.6121	5.0304	5.9813
MMLE III	2.4343	2.6615	2.8061	3.0931	3.4962	4.0171	4.6179	5.0500	6.0301

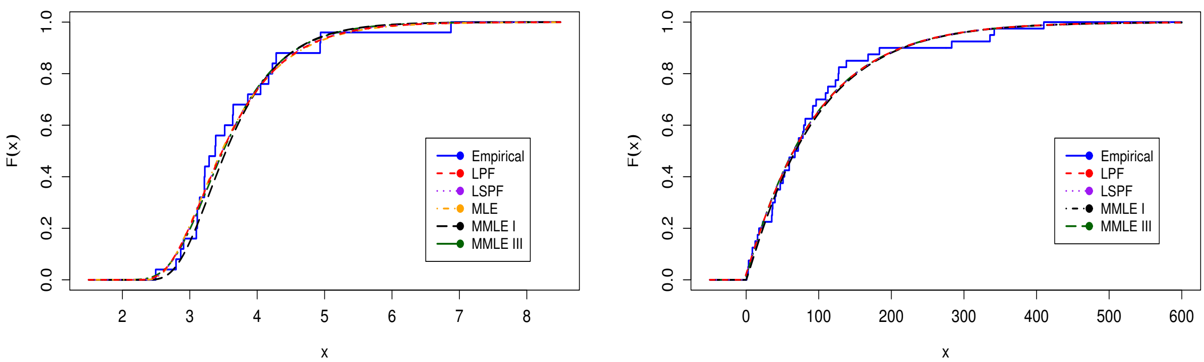
The parameter estimates for all methods considered in this study are shown in Table 2, together with the KS distance statistic, the CvM test statistic, and their associated p-values. To demonstrate the LPF approach, we maximize the likelihood function $\ell_v(\alpha, \beta)$ in equation



(a) Example 1

(b) Example 2

Figure 6: Plots of the log-likelihood function $\ln(\ell_v(\alpha, \beta))$ under LPF method.



(a) Example 1

(b) Example 2

Figure 7: Fitted CDF plots along with the empirical CDF.

(3) with respect to the parameters α and β. We estimate α and β to be 0.6615 and 4.7657, respectively. By following the steps described in Section 2, we estimate γ to be 2.1600 using the estimators of α and β. MLE performs remarkably well for this specific data set, whereas LPF performs second best, according to the findings shown in Table 2 based on the CvM test. MMLE I demonstrates worse findings with the p-value as 0.4961. For this specific data set, quantile estimates are given in Table 3. Subfigure 6a shows a graphic that may be used to determine reasonable initial parameter values. It also shows that the likelihood function is unimodal, meaning that the likelihood is globally maximized by the derived estimates. Plots of the empirical CDF and the fitted CDF are shown in Subfigure 7a. By setting $\beta_U = 50$, we may determine the bootstrap confidence intervals (CIs). Table 3 shows the 95% and 99% bootstrap CIs for each method. MMLE III performs poorly for this dataset, with an unacceptably high

Table 4: Bootstrap CIs for Example 1 based on 10,000 simulations.

CI	Method	Shape	Scale	Location	p
95%	LPF	(1.0586, 23.3770)	(0.3626, 1.0752)	(1.9052, 2.7931)	0.2495
	LSPF	(1.0259, 25.6883)	(0.3558, 1.0835)	(1.9063, 2.8115)	0.2794
	MLE	(0.3779, 14.3362)	(0.4817, 1.6542)	(1.7374, 2.8627)	0.1373
	MMLE I	(1.3515, 16.8792)	(0.3705, 0.8724)	(2.1230, 2.9181)	0.0021
	MMLE III	(0.8757, 12.8457)	(0.4675, 1.0958)	(1.6584, 2.8713)	0.5088
99%	LPF	(0.8713, 40.0100)	(0.3028, 1.2429)	(1.7511, 2.9021)	0.2495
	LSPF	(0.8277, 41.4136)	(0.2892, 1.2139)	(1.7802, 2.9335)	0.2794
	MLE	(0.3531, 33.9480)	(0.4160, 2.1448)	(1.2456, 2.9639)	0.1373
	MMLE I	(1.0755, 27.2214)	(0.3266, 0.9921)	(1.9981, 2.9997)	0.0021
	MMLE III	(0.6870, 20.1944)	(0.4086, 1.2930)	(1.3495, 2.9771)	0.5088

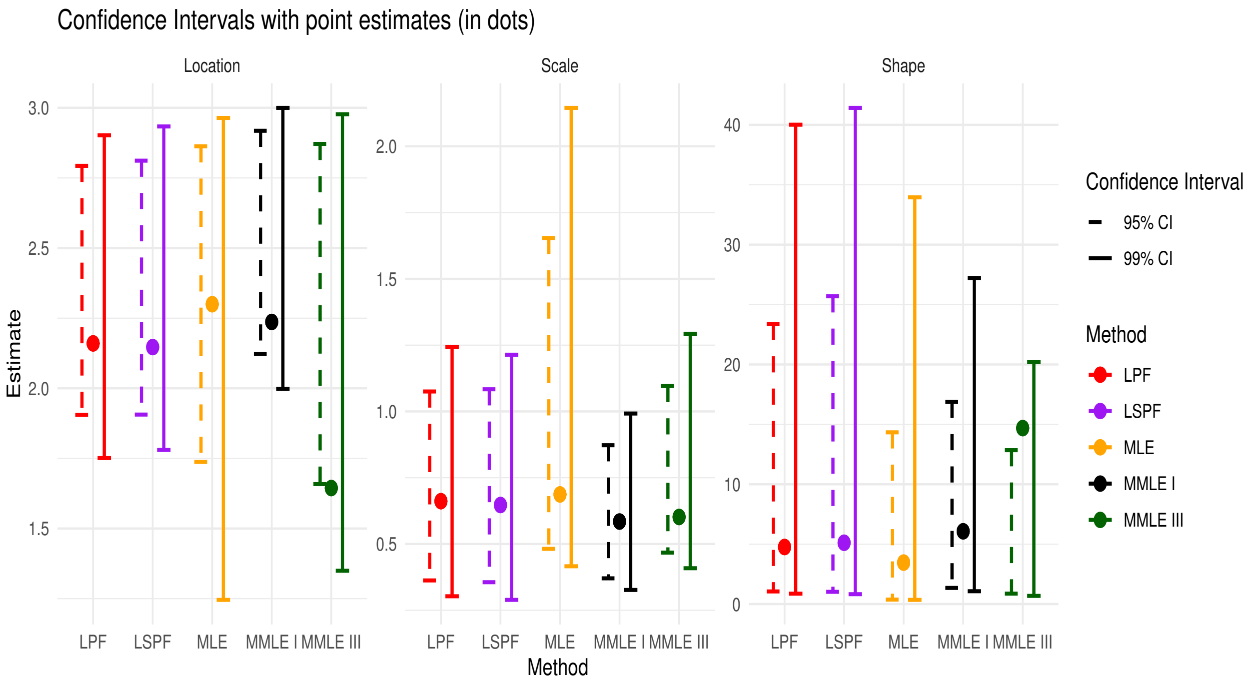


Figure 8: Graphical representation of the bootstrap CIs along with the point estimates for Example 1.

rejection proportion when this method is implemented, and the 95% bootstrap CI for MMLE III does not contain its own point estimates for the shape and location parameters. A graphical summary of the CIs is shown in Figure 8, confirming this behavior.

In summary, the behavior of the methods for this medical dataset is consistent with the simulation study, although MMLE III is not recommended for this dataset.

Example 2: We consider another data analysis for a set of real data that was initially reported by Bain and Engelhardt (1988) (see example 4.6.3 on page 162) and analyzed by Prajapat et al. (2021). The data indicate the observed lifetimes in months of a random sample of 40 electrical components. Histogram plot of the data set reveals that the shape of the density is reversed 'J' shaped and a simple data analysis provides its summary in Table 5. Based on the simulation study, for this type of electrical lifetime data, where the location parameter and higher quantiles are most relevant for engineering applications, either LPF or LSPF is recommended for estimating the location parameter when $\beta \leq 2$. For high-quantile estimation, the simulation results indicate that LPF generally yields lower bias and RMSE when $\beta \geq 1$. In this example, our aim is simply to illustrate how the different methods behave on a practical dataset. We now discuss estimates for the model parameters presented in the Table 5.

Table 5: Electrical lifetime data, its summary and estimates of the parameters.

Data:
0.15, 2.37, 2.90, 7.39, 7.99, 12.05, 15.17, 17.56, 22.40, 34.84, 35.39, 36.38, 39.52, 41.07, 46.50, 50.52, 52.54, 58.91, 58.93, 66.71, 71.48, 71.84, 77.66, 79.31, 80.90, 90.87, 91.22, 96.35, 108.92, 112.26, 122.71, 126.87, 127.05, 137.96, 167.59, 183.53, 282.49, 335.33, 341.19, 409.97

Summary:

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skew	Kurtosis
0.15	35.25	69.09	93.12	114.87	409.97	1.74	2.49

Method	Estimates			CvM		KS	
	Shape	Scale	Location	Statistic	p-value	Statistic	p-value
LPF	1.0821	91.1620	-2.7991	0.0836	0.9323	0.0414	0.9258
LSPF	1.0799	91.2659	-2.7673	0.0838	0.9311	0.0415	0.9252
MMLE I	1.2564	85.0422	-2.7578	0.0897	0.8899	0.0543	0.8495
MMLE III	1.0408	92.1752	-1.4577	0.0851	0.9231	0.0419	0.9229

Table 6: Various quantile estimates for the lifetime model of the electrical components under Example 2.

Method	ζ								
	0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
LPF	-1.4970	3.1096	8.7601	26.8607	65.4500	129.8222	213.9440	277.3149	424.1757
LSPF	-1.4750	3.1138	8.7517	26.8349	65.4270	129.8431	214.0460	277.4846	424.5093
MMLE I	-0.5531	5.4629	12.0672	31.5204	70.2034	132.1558	211.5682	270.9731	408.1978
MMLE III	-0.3469	3.8768	9.2265	26.7915	65.0078	129.5150	214.2825	278.2687	426.6927

The LPF estimates of α, β and γ are 91.1620, 1.0821 and -2.7991, respectively. According to the results reported in Table 5 based on the CvM test, LPF, LSPF and MMLE III perform really good for this particular data set, while LPF performs best. In Table 6, quantiles estimates are reported for this particular data set.

The plot in Subfigure 6b indicates that the likelihood function is unimodal. Subfigure 7b demonstrates the plots of the fitted CDF and the empirical CDF. To find the bootstrap confidence intervals (CIs), we set $\beta_U = 12$. Based on this data set, Table 3 and Figure 9 present

Table 7: Bootstrap CIs for Example 2 based on 10,000 simulations.

CI	Method	Shape	Scale	Location	p
95%	LPF	(0.7102, 2.3575)	(56.0280, 132.2849)	(-14.1289, 4.4647)	0.0046
	LSPF	(0.7462, 2.3137)	(56.7866, 130.8569)	(-12.8434, 4.5948)	0.0020
	MMLE I	(0.8165, 2.3900)	(53.5170, 123.5312)	(-6.4196, 7.4623)	0.0000
	MMLE III	(0.6568, 2.1126)	(57.0883, 137.9524)	(-8.8587, 6.7738)	0.0014
99%	LPF	(0.6400, 3.4396)	(48.6631, 149.7184)	(-22.8317, 8.7151)	0.0046
	LSPF	(0.6765, 3.7231)	(48.3591, 147.1999)	(-19.4118, 8.3977)	0.0020
	MMLE I	(0.7230, 3.1111)	(46.0003, 138.4216)	(-9.3808, 11.2007)	0.0000
	MMLE III	(0.5831, 3.1162)	(48.3295, 157.8311)	(-17.9056, 10.3762)	0.0014

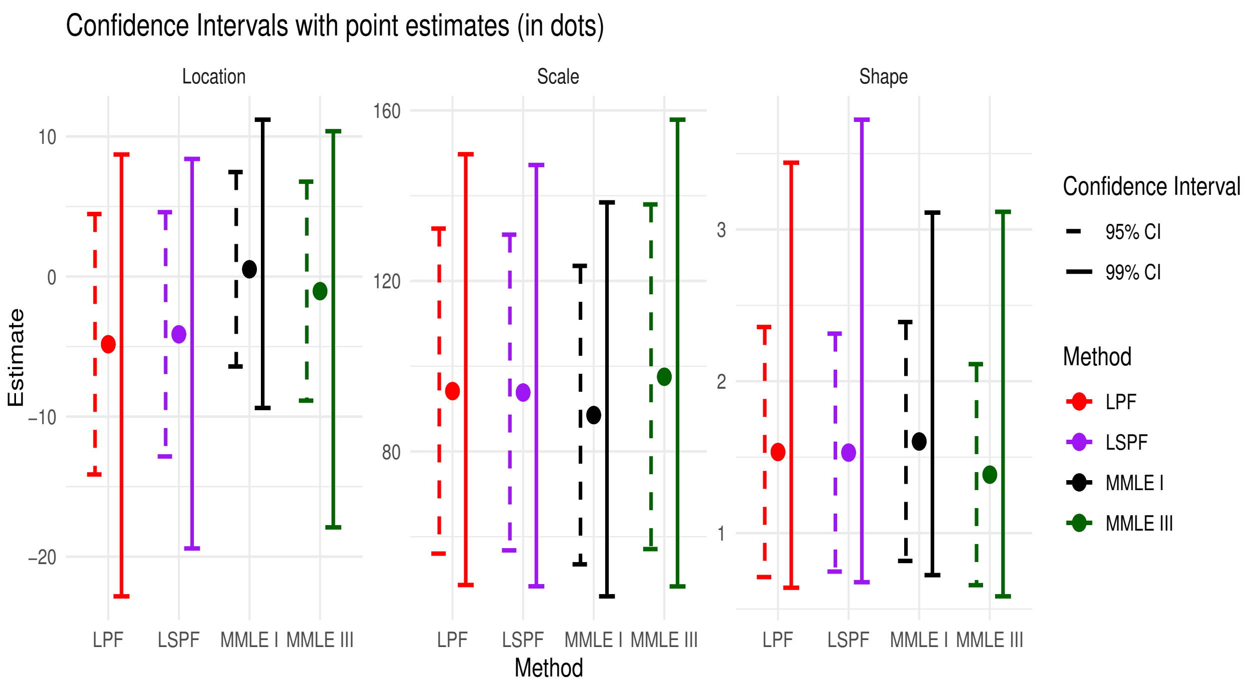


Figure 9: Graphical representation of the bootstrap CIs along with the point estimates for Example 2.

95% and 99% bootstrap CIs for each method where we found that the proportion of rejection is negligible for all the methods. For this data analysis, each method's CI contains its corresponding point estimate. Overall, the results for this dataset agree with the simulation study: LPF and LSPF give stable parameter and quantile estimates. All methods provide comparable fits based on the CDF and goodness-of-fit tests. It is to be noted that, the location parameter γ under this study belongs to the set of real numbers R, hence the estimation method and the algorithms are proposed accordingly. Although, the estimation of the location parameter can be constrained to $\gamma \geq 0$ if the application requires a physically interpretable location parameter.

5 Summary

In this paper, we consider an earlier-proposed estimation method for a three-parameter GE distribution known as the LPF method. We discussed some properties of the estimators, such as uniqueness and consistency. A Monte Carlo simulation study has been conducted to evaluate the performance of the LPF method comparative to other existing methods. In numerical simulations, we reported bias and RMSE for estimators of all three parameters and quantiles of the GE distribution. For $\beta > 1$, the LPF method performs better than all other methods for estimating scale parameters in terms of bias and RMSE. In addition to other methods such as MMLE I, MLE, etc., MMLE III has excellent and consistent performance when estimating shape parameter. When $\beta < 1$, the LPF method is recommended for the estimation of the shape parameter when the sample size is small. When $n \geq 50$, MMLE III is advisable if β is sufficiently small ($\beta \approx 0.5$) and any method is acceptable if $0.5 < \beta \leq 1$ as they all have similar performances for all parameters. The LPF method is recommended for estimating quantiles, especially when $\beta > 1$. Quantile estimation based on the LPF method appeared performing very good even for large values of the shape parameter. Based on the reported results of the proportion of rejected samples during the simulation study, it is recommended that LSPF and MMLE I be used for data analysis before any other methods. Two real data sets from different sources; medical sciences and reliability engineering, have been used to illustrate the LPF method. Surprisingly, the MMLE III method does not perform well for the first data set form COVID-19 disease. Moreover, the proposed method performs very good among existing methods for both the data sets.

According to this study, the LPF method is superior to the LSPF method in a number of aspects and offers certain advantages: (1) It consumes less time to run simulations and has a lower computational complexity. The primary cause for this is the decrease in the number of integration; (2) One disadvantage of the LSPF approach over the LPF method is that it employs estimates from the previous step in order to generate estimates separately in steps. This might result in a considerable accumulation of biases in the steps. The LPF, on the other hand, creates estimates by reducing the number of steps in the estimating approach; (3) the LPF method is based on $n-1$ observations, whereas the LSPF method makes use of just $n-2$ observations.

Supplementary Material

The supplementary material contains a ZIP archive (`Rcodes.zip`) including all `R` scripts used for the Monte Carlo simulations and the illustrative data analyses. Detailed instructions and folder descriptions are provided in an accompanying README file.

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Conflict of Interest

We hereby declare that the information provided here is accurate, and no apparent conflicts of interest that are relevant to the content of this article.

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A Proof of Theorem 2.1

Using the transformation of random variables, we find the joint PDF of the random variables $V_{(2)}, V_{(3)}, \dots, V_{(n)}$ for $\alpha > 0,\ \beta > 0$. Define $Z_{(i)} = X_{(i)} - \gamma, i = 1, 2, \dots, n$. $Z_1, Z_2, \dots, Z_n$ are *i.i.d.* random variables from the GE distribution GE$(\alpha, \beta, 0)$ with the shape parameter β and the scale parameter α. Denote the PDF and the CDF of GE$(\alpha, \beta, 0)$ by $g(.;\alpha,\beta)$ and $G(.;\alpha,\beta)$, respectively, for convenience. Now, $V_{(i)}$ in equation (2) can be rewritten in terms of $Z_{(i)}$'s as follows:

$$V_{(i)} = Z_{(i)} - Z_{(1)},\ i = 2, 3, \dots, n \implies Z_{(i)} = Z_{(1)} + V_{(i)},\ i = 2, 3, \dots, n. \qquad .$$

Let us assume that $U = Z_{(1)}$. Therefore $Z_{(i)} = U + V_{(i)},\ i = 2, 3, \dots, n$. It can be shown that the Jacobian of the transformation $J = \frac{\partial(Z_{(1)}, Z_{(2)}, \dots, Z_{(n-1)}, Z_{(n)})}{\partial(U, V_{(2)}, \dots, V_{(n-1)}, V_{(n)})} = 1$. If we use a notation $f_{Y_1, Y_2, \dots, Y_p}(.)$ to denote the joint PDF of $Y_1, Y_2, \dots, Y_p$, we have

$$\begin{aligned} f_{U, V_{(2)}, \dots, V_{(n)}}(u, v_2, \dots, v_n; \alpha, \beta) &= |J| f_{Z_{(1)}, Z_{(2)}, \dots, Z_{(n-1)}, Z_{(n)}}(u, u + v_2, \dots, u + v_n; \alpha, \beta) \\ &= n! g(u; \alpha, \beta) \Big\{ \prod_{i=2}^{n} g(u + v_i; \alpha, \beta) \Big\}, \end{aligned}$$

$0 < v_2 < \cdots < v_n < \infty$ and $0 < u < \infty$. Therefore

$$\begin{aligned} & f_{V_{(2)}, \dots, V_{(n)}}(v_2, \dots, v_n; \alpha, \beta) \\ &\quad = \int_0^\infty n! g(u; \alpha, \beta) \Big\{ \prod_{i=2}^{n} g(u + v_i; \alpha, \beta) \Big\}\, du \\ &\quad = n! \Big(\frac{\beta}{\alpha}\Big)^n \int_0^\infty e^{-\frac{1}{\alpha}\sum_{i=1}^{n}(u+v_i)} \prod_{i=1}^{n} \big(1 - e^{-\frac{u+v_i}{\alpha}}\big)^{\beta-1}\, du, \end{aligned} \tag{12}$$

with $v_1 = 0$ and hence, the likelihood function of (α, β) given $v_2, v_3, \dots, v_n$ is given by

$$\ell_v(\alpha, \beta | v_2, \dots, v_n) = f_{V_{(2)}, \dots, V_{(n)}}(v_2, \dots, v_n; \alpha, \beta),\ \alpha > 0,\ \beta > 0$$

which proves the theorem.

B Boundedness and Differentiablity of the Likelihood Function $\ell_v(\alpha, \beta)$

B.1 Boundedness

By equation (12), we have

$$\ell_v(\alpha, \beta) = n! \int_0^\infty g(u; \alpha, \beta) \prod_{i=2}^{n} g(u + v_i; \alpha, \beta)\, du, \ \alpha > 0, \ \beta > 0. \tag{13}$$

It can be shown for $\alpha > 0, \ \beta > 0, \ 0 < v_2 < \cdots < v_n < \infty$ and $0 < u < \infty$ that $\frac{(n-1)! \prod_{i=2}^{n} g(u+v_i;\alpha,\beta)}{(1-G(u;\alpha,\beta))^{n-1}}$ is bounded , *i.e.*, $\exists$ an $M > 0$ such that $(n-1)! \prod_{i=2}^{n} g(u + v_i; \alpha, \beta) < M(1 - G(u; \alpha, \beta))^{n-1}$ $\forall \ \alpha > 0, \ \beta > 0, \ 0 < v_2 < \cdots < v_n < \infty$ and $0 < u < \infty$. It implies:

$$\int_0^\infty n! g(u; \alpha, \beta) \prod_{i=2}^{n} g(u + v_i; \alpha, \beta)\, du < M \int_0^\infty n g(u; \alpha, \beta)(1 - G(u; \alpha, \beta))^{n-1}\, du = M,$$

B.2 Differentiablity

First we show differentiablity of the likelihood function $\ell_v(\alpha, \beta)$ in equation (3) with respect to β. In order to do so, let us rewrite the likelihood function as follows:

$$\ell_v(\alpha, \beta) = n! \int_0^\infty e^{h_v(\alpha,\beta;u)}\, du, \ \alpha > 0, \ \beta > 0, \tag{14}$$

where

$$h_v(\alpha, \beta; u) = n \ln(\beta) - n \ln(\alpha) - \frac{1}{\alpha} \sum_{i=1}^{n} (u + v_i) + (\beta - 1) \sum_{i=1}^{n} \ln(1 - e^{-\frac{u+v_i}{\alpha}}). \tag{15}$$

To show that the likelihood function is differentiable, we need to show that the partial derivative, with respect to β, can be taken inside the integral in the equation (3). Given $0 < v_2 < \cdots < v_n < \infty, \ v_1 = 0$ and $\alpha > 0$, we show:

1. $\frac{\partial}{\partial \beta} e^{h_v(\alpha,\beta;u)}$ exists,

2. $\left|\frac{\partial}{\partial\beta}e^{h_v(\alpha,\beta;u)}\right| < h_2(u)$ for some positive function h_2 and $\forall\ u \in (0,\infty)$ such that $\int_0^\infty h_2(u)\,du < \infty$, *i.e.* $\frac{\partial}{\partial\beta}e^{h_v(\alpha,\beta;u)}$ is an integrable function with respect to the variable u.

Since exponential, logarithmic and polynomials are well-known smooth functions, therefore $e^{h_v(\alpha,\beta;u)}$ is differentiable with respect to β and it is given by

$$\frac{\partial}{\partial\beta}e^{h_v(\alpha,\beta;u)} = e^{h_v(\alpha,\beta;u)}\frac{\partial}{\partial\beta}h_v(\alpha,\beta;u) = \Big(\frac{n}{\beta} + \sum_{i=1}^{n}\ln\big(1-e^{-\frac{u+v_i}{\alpha}}\big)\Big)e^{h_v(\alpha,\beta;u)}. \tag{16}$$

Equation (16) can be simplified as follows:

$$\frac{\partial}{\partial\beta}e^{h_v(\alpha,\beta;u)} = \frac{n}{\beta}e^{h_v(\alpha,\beta;u)} + \Big(\frac{\beta}{\alpha}\Big)^n\Big(\sum_{i=1}^{n}\ln\big(1-e^{-\frac{u+v_i}{\alpha}}\big)\Big)\prod_{i=1}^{n}\Big\{e^{-\frac{u+v_i}{\alpha}}\big(1-e^{-\frac{u+v_i}{\alpha}}\big)^{\beta-1}\Big\}. \tag{17}$$

It is clear that first term of the equation (17) is integrable with respect to the variable u. Now, we show integrability of the second term and in order to do so, let us assume that

$$\mathfrak{H}_1(u) = \Big(\sum_{i=1}^{n}\ln\big(1-e^{-\frac{u+v_i}{\alpha}}\big)\Big)\prod_{i=1}^{n}\Big\{e^{-\frac{u+v_i}{\alpha}}\big(1-e^{-\frac{u+v_i}{\alpha}}\big)^{\beta-1}\Big\}.$$

For every $\alpha > 0,\ \beta > 0,\ 0 < v_2 < \cdots < v_n < \infty,\ v_1 = 0$ and $0 < u < \infty$, $\mathfrak{H}_1(u)$ tends to zero as $u \to \infty$. Also, $\mathfrak{H}_1(u) < h_3(u)$, where $h_3(u) = C_1e^{-\frac{u}{\alpha}}\big(1-e^{-\frac{u}{\alpha}}\big)^{\beta-1}\big(C_2+\ln\big(1-e^{-\frac{u}{\alpha}}\big)\big)$ with some finite constants C_1 and C_2. Now it is enough to show that $h_3(u)$ is integrable. Therefore, consider the quantity

$$\begin{aligned}\int_0^\infty h_3(u)\,du &= \int_0^\infty C_1e^{-\frac{u}{\alpha}}\big(1-e^{-\frac{u}{\alpha}}\big)^{\beta-1}\big(C_2+\ln\big(1-e^{-\frac{u}{\alpha}}\big)\big)\,du\\ &= C_1C_2\int_0^\infty e^{-\frac{u}{\alpha}}\big(1-e^{-\frac{u}{\alpha}}\big)^{\beta-1}\,du + C_1\int_0^\infty e^{-\frac{u}{\alpha}}\big(1-e^{-\frac{u}{\alpha}}\big)^{\beta-1}\ln\big(1-e^{-\frac{u}{\alpha}}\big)\,du\\ &= C_1C_2\alpha\int_0^1 u_1^{\beta-1}\,du_1 + C_1\alpha\int_0^1 u_1^{\beta-1}\ln(u_1)\,du_1\end{aligned}$$

Now, when $\beta \le 1$, we see that the integrand of the second term has singularity point at the boundary of the integration domain, but the integrand is integrable because

$$\lim_{\epsilon\to 0^+}\int_\epsilon^1 u^{\beta-1}\ln(u)\,du = \lim_{\epsilon\to 0^+}\int_\epsilon^1 \ln(u)\,du = -1.$$

and hence, after simplification

$$\int_0^\infty h_3(u)\,du = C_1 C_2 \frac{\alpha}{\beta} - C_1 \frac{\alpha}{\beta^2} < \infty.$$

Hence, part (*ii*) of Theorem 16.8 of Billingsley (1994) implies that the likelihood function $\ell_v(\alpha, \beta)$ is differentiable with respect to β and its derivative is given by

$$\begin{aligned}\frac{\partial}{\partial\beta}\ell_v(\alpha,\beta) &= n! \int_0^\infty \frac{\partial}{\partial\beta} e^{h_v(\alpha,\beta;u)}\,du \\ &= n!\Big(\frac{\beta}{\alpha}\Big)^n \int_0^\infty \Big(\frac{n}{\beta} + \sum_{i=1}^n \ln\big(1 - e^{-\frac{u+v_i}{\alpha}}\big)\Big) e^{-\frac{1}{\alpha}\sum_{i=1}^n (u+v_i)} \prod_{i=1}^n \big(1 - e^{-\frac{u+v_i}{\alpha}}\big)^{\beta-1}. \end{aligned} \quad (18)$$

Now, in a similar manner, we step forward to show that $\ell_v(\alpha, \beta)$ is differentiable with respect to α by demonstrating that $\frac{\partial}{\partial\alpha} e^{h_v(\alpha,\beta;u)}$ exists and it is integrable with respect to the variable u provided $0 < v_2 < \cdots < v_n < \infty,\ v_1 = 0$ and $\beta > 0$. The first point is obvious to satisfy as it is a function of exponential, logarithmic and polynomial functions which are well-known smooth functions and the derivative is given by

$$\begin{aligned}\frac{\partial}{\partial\alpha} e^{h_v(\alpha,\beta;u)} &= e^{h_v(\alpha,\beta;u)} \frac{\partial}{\partial\alpha} h_v(\alpha,\beta;u) \\ &= \Big(-\frac{n}{\alpha} + \frac{1}{\alpha}\sum_{i=1}^n \frac{u+v_i}{\alpha} - \frac{(\beta-1)}{\alpha}\sum_{i=1}^n \frac{u+v_i}{\alpha}\Big(\frac{e^{-\frac{u+v_i}{\alpha}}}{1 - e^{-\frac{u+v_i}{\alpha}}}\Big)\Big) e^{h_v(\alpha,\beta;u)}. \end{aligned} \quad (19)$$

Equation (19) is simplified as follows:

$$\begin{aligned}\frac{\partial}{\partial\alpha} e^{h_v(\alpha,\beta;u)} &= -\frac{n}{\alpha} e^{h_v(\alpha,\beta;u)} + \frac{1}{\alpha}\Big(\sum_{i=1}^n \frac{u+v_i}{\alpha}\Big) e^{h_v(\alpha,\beta;u)} - \frac{(\beta-1)}{\alpha}\Big(\sum_{i=1}^n \frac{u+v_i}{\alpha}\Big(\frac{e^{-\frac{u+v_i}{\alpha}}}{1 - e^{-\frac{u+v_i}{\alpha}}}\Big)\Big) e^{h_v(\alpha,\beta;u)} \\ &= -\frac{n}{\alpha} e^{h_v(\alpha,\beta;u)} + \frac{\beta}{\alpha}\Big(\sum_{i=1}^n \frac{u+v_i}{\alpha}\Big) e^{h_v(\alpha,\beta;u)} - \frac{(\beta-1)}{\alpha}\Big(\sum_{i=1}^n \frac{u+v_i}{\alpha}\Big(\frac{1}{1 - e^{-\frac{u+v_i}{\alpha}}}\Big)\Big) e^{h_v(\alpha,\beta;u)} \\ &= -\frac{n}{\alpha} e^{h_v(\alpha,\beta;u)} + \frac{\beta}{\alpha^2}\Big(\sum_{i=1}^n v_i\Big) e^{h_v(\alpha,\beta;u)} + \frac{n\beta}{\alpha^2} u e^{h_v(\alpha,\beta;u)} \\ &\quad - \frac{(\beta-1)}{\alpha^2}\Big(\sum_{i=1}^n \Big(\frac{1}{1 - e^{-\frac{u+v_i}{\alpha}}}\Big)\Big) u e^{h_v(\alpha,\beta;u)} - \frac{(\beta-1)}{\alpha^2}\Big(\sum_{i=2}^n v_i \Big(\frac{1}{1 - e^{-\frac{u+v_i}{\alpha}}}\Big)\Big) e^{h_v(\alpha,\beta;u)}. \end{aligned} \quad (20)$$

It is clear that first two terms of equation (20) are integrable as $e^{h_v(\alpha,\beta;u)}$ is integrable with respect to u. Now, we show integrablity of the third term and the proofs for the remaining terms goes along the same line and hence proofs are omitted. Assume that

$$\mathfrak{H}_2(u) = ue^{h_v(\alpha,\beta;u)} = \Big(\frac{\beta}{\alpha}\Big)^n u \prod_{i=1}^{n} \Big\{ e^{-\frac{u+v_i}{\alpha}} \big(1 - e^{-\frac{u+v_i}{\alpha}}\big)^{\beta-1} \Big\}.$$

For every $\alpha > 0,\ \beta > 0,\ 0 < v_2 < \cdots < v_n < \infty,\ v_1 = 0$ and $0 < u < \infty$, $\mathfrak{H}_2(u)$ tends to zero as $u \to \infty$. Also, $\mathfrak{H}_2(u) < h_4(u)$, where $h_4(u) = C_3 u e^{-\frac{u}{\alpha}}\big(1 - e^{-\frac{u}{\alpha}}\big)^{\beta-1}$ with some finite constant C_3. Now it is enough to show that $h_4(u)$ is integrable. Therefore, we consider the quantity

$$\begin{aligned}
\int_0^\infty h_4(u)\, du &= \int_0^\infty C_3 u e^{-\frac{u}{\alpha}}\big(1 - e^{-\frac{u}{\alpha}}\big)^{\beta-1}\, du \\
&= -\, C_3\alpha^2 \int_0^1 u_1^{\beta-1} \ln(1 - u_1)\, du_1 \\
&= -\, C_3\alpha^2 \int_0^1 (1 - y)^{\beta-1} \ln(y)\, dy \\
&= -\, C_3\alpha^2 \Big(\int_0^{1/2} (1 - y)^{\beta-1} \ln(y)\, dy + \int_{1/2}^1 (1 - y)^{\beta-1} \ln(y)\, dy \Big).
\end{aligned}$$

We denote the first and the second terms of the above equation by I_1 and I_2, respectively. Here

$$I_1 = \int_0^{1/2} (1 - y)^{\beta-1} \ln(y)\, dy < \int_0^{1/2} \ln(y)\, dy = \frac{1}{2}(\ln(1/2) - 1) < \infty$$

and

$$\begin{aligned}
I_2 = \int_{1/2}^1 (1 - y)^{\beta-1} \ln(y)\, dy &= \frac{1}{2^\beta} \ln(1/2) + \int_{1/2}^1 \frac{(1 - y)^\beta}{y}\, dy \\
&< \frac{1}{2^\beta} \ln(1/2) + 2 \int_{1/2}^1 (1 - y)^\beta\, dy = \frac{1}{2^\beta}\big(\ln(1/2) - \frac{1}{\beta}\big) < \infty.
\end{aligned}$$

Therefore, $\int_0^\infty h_4(u)\, du < \infty$. Hence, it implies that the likelihood function $\ell_v(\alpha, \beta)$ is differentiable with respect to α and its derivative is given by

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\ell_v(\alpha, \beta) &= n! \int_0^\infty \frac{\partial}{\partial\alpha} e^{h_v(\alpha,\beta;u)}\, du \\
&= n! \Big(\frac{\beta}{\alpha}\Big)^n \int_0^\infty \Big(-\frac{n}{\alpha} + \frac{1}{\alpha} \sum_{i=1}^n \frac{u + v_i}{\alpha} - \frac{(\beta - 1)}{\alpha} \sum_{i=1}^n \frac{u + v_i}{\alpha} \Big(\frac{e^{-\frac{u+v_i}{\alpha}}}{1 - e^{-\frac{u+v_i}{\alpha}}} \Big)\Big) \\
&\quad \times e^{-\frac{1}{\alpha}\sum_{i=1}^n (u+v_i)} \prod_{i=1}^n \big(1 - e^{-\frac{u+v_i}{\alpha}}\big)^{\beta-1}. \qquad (21)
\end{aligned}$$

C Tables

[§] β_U is an upper bound for the estimates of shape parameter such that the estimates greater than β_U are rejected and the rejected proportion is p reported in the last column of the tables.

Table 8: Biases and RMSEs of the estimators while varying sample size n based on $10,000$ simulations.

β	β_U§	n	Method	Shape		Scale		Location		p
				Bias	RMSE	Bias	RMSE	Bias	RMSE	
0.50	2	20	LPF	0.1706	0.2750	-0.1296	0.3473	-0.0089	0.0171	0.0047
			LSPF	0.1468	0.2450	-0.1220	0.3310	-0.0078	0.0154	0.0036
			MMLE I	0.2642	0.3742	-0.1640	0.3538	-0.0080	0.0146	0.0076
			MMLE III	0.0709	0.2128	-0.0368	0.3601	0.0005	0.0103	0.0039
		50	LPF	0.0885	0.1335	-0.0879	0.2282	-0.0017	0.0026	0.0000
			LSPF	0.0885	0.1265	-0.0968	0.2204	-0.0016	0.0025	0.0000
			MMLE I	0.1029	0.1526	-0.0881	0.2272	-0.0015	0.0025	0.0000
			MMLE III	0.0217	0.0977	-0.0134	0.2294	0.0003	0.0017	0.0000
		100	LPF	0.0650	0.0898	-0.0752	0.1658	-0.0009	0.0010	0.0000
			LSPF	0.0701	0.0904	-0.0843	0.1631	-0.0009	0.0010	0.0000
			MMLE I	0.0646	0.0909	-0.0653	0.1605	-0.0009	0.0010	0.0000
			MMLE III	0.0086	0.0631	-0.0054	0.1623	0.0001	0.0004	0.0000
		200	LPF	0.0572	0.0705	-0.0695	0.1262	-0.0009	0.0009	0.0000
			MMLE I	0.0493	0.0643	-0.0579	0.1186	-0.0009	0.0009	0.0000
			MMLE III	0.0046	0.0427	-0.0035	0.1125	0.0000	0.0001	0.0000
0.75	5	20	LPF	0.2421	0.5214	-0.0730	0.3099	-0.0231	0.0547	0.0097
			LSPF	0.1916	0.4600	-0.0576	0.3076	-0.0174	0.0445	0.0094
			MMLE I	0.4540	0.7109	-0.1452	0.3201	-0.0171	0.0407	0.0023
			MMLE III	0.1202	0.4467	-0.0113	0.3354	0.0010	0.0413	0.0099
		50	LPF	0.0967	0.2004	-0.0495	0.1993	-0.0047	0.0118	0.0000
			LSPF	0.0857	0.1898	-0.0446	0.1958	-0.0042	0.0113	0.0000
			MMLE I	0.1587	0.2548	-0.0759	0.2056	-0.0036	0.0106	0.0000
			MMLE III	0.0253	0.1703	-0.0021	0.2080	0.0022	0.0096	0.0000
		100	LPF	0.0569	0.1189	-0.0347	0.1426	-0.0015	0.0041	0.0000
			LSPF	0.0586	0.1169	-0.0384	0.1392	-0.0015	0.0041	0.0000
			MMLE I	0.0813	0.1414	-0.0470	0.1454	-0.0012	0.0039	0.0000
			MMLE III	0.0076	0.1033	0.0032	0.1453	0.0011	0.0036	0.0000
		200	LPF	0.0420	0.0807	-0.0288	0.1016	-0.0008	0.0016	0.0000
			MMLE I	0.0466	0.0859	-0.0281	0.1010	-0.0007	0.0015	0.0000
			MMLE III	0.0024	0.0687	0.0014	0.1013	0.0005	0.0014	0.0000
1.00	8	20	LPF	0.3425	0.8685	-0.0510	0.2928	-0.0375	0.1042	0.0262
			LSPF	0.2975	0.8620	-0.0264	0.2980	-0.0246	0.0853	0.0252
			MLE	-0.5280	0.7605	0.2389	0.4714	0.0429	0.0747	0.0096
			MMLE I	0.6165	1.0248	-0.1239	0.2995	-0.0172	0.0687	0.0023
			MMLE III	0.2259	0.8532	-0.0030	0.3225	-0.0031	0.0911	0.0280
		50	LPF	0.1170	0.3318	-0.0275	0.1903	-0.0092	0.0316	0.0001
			LSPF	0.1001	0.3174	-0.0241	0.1849	-0.0072	0.0288	0.0002
			MLE	-0.3572	0.4509	0.1469	0.2946	0.0183	0.0287	0.0003
			MMLE I	0.2219	0.3751	-0.0721	0.1922	-0.0045	0.0250	0.0000
			MMLE III	0.0398	0.3037	0.0059	0.2020	0.0047	0.0281	0.0001
		100	LPF	0.0581	0.1722	-0.0191	0.1328	-0.0030	0.0126	0.0000
			LSPF	0.0534	0.1649	-0.0192	0.1276	-0.0023	0.013	0.0000
			MLE	-0.2239	0.2926	0.0813	0.1860	0.0093	0.0137	0.0000
			MMLE I	0.1119	0.2067	-0.0415	0.1362	-0.0015	0.0117	0.0000
			MMLE III	0.0104	0.1616	0.0060	0.1381	0.0033	0.0118	0.0000
		200	LPF	0.0360	0.1083	-0.0133	0.0928	-0.0011	0.0058	0.0000
			MLE	-0.1329	0.1827	0.0455	0.1181	0.0048	0.0070	0.0000
			MMLE I	0.0584	0.1252	-0.0237	0.0951	-0.0006	0.0055	0.0000
			MMLE III	-0.0006	0.1043	0.0036	0.0970	0.0020	0.0057	0.0000

Table 9: Biases and RMSEs of the estimators while varying sample size n based on $10,000$ simulations.

β	β_U	n	Method	Shape		Scale		Location		p
				Bias	RMSE	Bias	RMSE	Bias	RMSE	
1.5	12	20	LPF	0.5140	1.6165	0.0070	0.2802	-0.0533	0.2121	0.0752
			LSPF	0.4277	1.5597	0.0111	0.2937	-0.014	0.1566	0.0756
			MLE	-0.4644	0.7732	0.2503	0.4257	0.1141	0.1571	0.0700
			MMLE I	0.8335	1.6122	-0.0944	0.2706	0.0084	0.1295	0.0041
			MMLE III	0.3272	1.4925	0.0396	0.3148	0.0060	0.1893	0.1010
		50	LPF	0.2435	0.8945	-0.0101	0.1804	-0.0245	0.1024	0.0040
			LSPF	0.1952	0.8489	-0.0014	0.1851	-0.0116	0.0816	0.0036
			MLE	-0.1575	0.5464	0.0836	0.2255	0.0460	0.0820	0.0066
			MMLE I	0.2948	0.6252	-0.0507	0.1741	0.0081	0.0637	0.0000
			MMLE III	0.1359	0.8503	0.0107	0.1940	0.0045	0.0949	0.0047
		100	LPF	0.0865	0.3755	-0.0065	0.1275	-0.0081	0.0460	0.0000
			LSPF	0.0630	0.3743	0.0004	0.1274	-0.0041	0.0434	0.0000
			MLE	-0.1108	0.3225	0.0464	0.1486	0.0274	0.0469	0.0010
			MMLE I	0.1459	0.3493	-0.0304	0.1251	0.0055	0.0379	0.0000
			MMLE III	0.0196	0.3634	0.0086	0.1343	0.0081	0.0445	0.0000
		200	LPF	0.0318	0.2081	-0.0026	0.0880	-0.0025	0.0244	0.0000
			MLE	-0.0655	0.2048	0.0237	0.0963	0.0156	0.0269	0.0000
			MMLE I	0.0739	0.2185	-0.0172	0.0887	0.0035	0.0230	0.0000
			MMLE III	0.0026	0.2116	0.0046	0.0914	0.0059	0.0245	0.0000
2.0	20	20	LPF	0.6811	2.6117	0.0321	0.2710	-0.0507	0.3066	0.1209
			LSPF	0.5276	2.4641	0.0361	0.3009	0.0266	0.2175	0.1310
			MLE	-0.4570	1.8233	0.2683	0.4606	0.1577	0.2692	0.0748
			MMLE I	0.9708	2.2697	-0.0716	0.2640	0.0594	0.1915	0.0035
			MMLE III	0.2357	1.8799	0.0769	0.3164	0.0462	0.2685	0.1851
		50	LPF	0.3885	1.6485	0.0063	0.1782	-0.0338	0.1786	0.0175
			LSPF	0.3895	1.7103	0.0035	0.1884	-0.0086	0.1380	0.0198
			MLE	-0.1315	1.1409	0.0810	0.2287	0.0648	0.1552	0.0168
			MMLE I	0.2545	0.8535	-0.0296	0.1673	0.0411	0.1109	0.0000
			MMLE III	0.3126	1.6384	0.0172	0.1930	0.0018	0.1778	0.0238
		100	LPF	0.1481	0.8780	0.0039	0.1255	-0.0133	0.1010	0.0009
			LSPF	0.1358	0.8422	0.0040	0.1302	-0.0029	0.0867	0.0010
			MLE	-0.1168	0.5871	0.0386	0.1445	0.0405	0.0917	0.0050
			MMLE I	0.0990	0.4971	-0.0167	0.1188	0.0288	0.0739	0.0000
			MMLE III	0.1020	0.8177	0.0078	0.1319	0.0066	0.1012	0.0009
		200	LPF	0.0470	0.3897	0.0022	0.0871	-0.0049	0.0546	0.0000
			MLE	-0.0842	0.3770	0.0203	0.0952	0.0244	0.0570	0.0000
			MMLE I	0.0488	0.3317	-0.0088	0.0853	0.0163	0.0478	0.0000
			MMLE III	0.0210	0.3996	0.0045	0.0909	0.0061	0.0545	0.0000
3.0	30	20	LPF	0.6359	3.9790	0.0796	0.2779	0.0181	0.4438	0.2037
			LSPF	0.4588	3.8851	0.0617	0.3138	0.1546	0.3366	0.2088
			MLE	-0.7827	3.2839	0.3096	0.5051	0.2866	0.4529	0.1408
			MMLE I	0.8472	3.2347	-0.0447	0.2494	0.1879	0.3217	0.0031
			MMLE III	-0.3581	2.3205	0.1287	0.3328	0.1860	0.4048	0.3161
		50	LPF	0.7287	3.5317	0.0319	0.1746	-0.0307	0.3304	0.0715
			LSPF	0.8014	3.6369	0.0064	0.2014	0.0416	0.2287	0.0744
			MLE	-0.0205	2.7964	0.0873	0.2295	0.1066	0.3061	0.0554
			MMLE I	0.0295	1.3769	-0.0083	0.1606	0.1347	0.2161	0.0000
			MMLE III	0.5161	3.0517	0.0382	0.1866	0.0225	0.3155	0.1033
		100	LPF	0.4671	2.3920	0.0128	0.1241	-0.0296	0.2317	0.0151
			LSPF	0.5027	2.4756	-0.0048	0.1428	0.0150	0.1708	0.0174
			MLE	0.0504	1.8775	0.0350	0.1393	0.0475	0.2143	0.0156
			MMLE I	-0.1470	0.8864	0.0042	0.1167	0.1014	0.1600	0.0000
			MMLE III	0.5147	2.4750	0.0082	0.1266	-0.0131	0.2404	0.0191
		200	LPF	0.1794	1.3109	0.0075	0.0879	-0.0128	0.1468	0.0008
			MLE	0.3662	2.0130	0.0114	0.1008	-0.0062	0.2152	0.0011
			MMLE I	-0.1878	0.6348	0.0096	0.0830	0.0732	0.1170	0.0000
			MMLE III	0.2055	1.4385	0.0040	0.0890	-0.0031	0.1529	0.0013

Table 10: Biases of the quantile estimators while varying sample size n based on $10,000$ simulations.

β	n	Method	ζ								
			0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
0.50	20	LPF	-0.0070	0.0005	0.0098	0.0332	0.0501	0.0195	-0.0675	-0.1473	-0.3479
		LSPF	-0.0063	0.0001	0.0082	0.0292	0.0442	0.0153	-0.0666	-0.1417	-0.3306
		MMLE I	-0.0044	0.0081	0.0229	0.0602	0.0921	0.0627	-0.0424	-0.1418	-0.3946
		MMLE III	0.0015	0.0052	0.0094	0.0169	0.0147	-0.0060	-0.0387	-0.0641	-0.1233
	50	LPF	-0.0012	0.0021	0.0071	0.0218	0.0348	0.0170	-0.0401	-0.0935	-0.2292
		LSPF	-0.0012	0.0020	0.0070	0.0218	0.0345	0.0138	-0.0495	-0.1086	-0.2581
		MMLE I	-0.0008	0.0032	0.0092	0.0272	0.0452	0.0318	-0.0229	-0.0758	-0.2111
		MMLE III	0.0005	0.0016	0.0031	0.0057	0.0046	-0.0034	-0.0156	-0.0250	-0.0467
	100	LPF	-0.0007	0.0014	0.0051	0.0166	0.0274	0.0127	-0.0356	-0.0812	-0.1971
		LSPF	-0.0007	0.0016	0.0055	0.0180	0.0296	0.0128	-0.0415	-0.0927	-0.2227
		MMLE I	-0.0007	0.0015	0.0053	0.0175	0.0307	0.0214	-0.0186	-0.0576	-0.1577
		MMLE III	0.0002	0.0006	0.0012	0.0022	0.0010	-0.0033	-0.0088	-0.0128	-0.0216
	200	LPF	-0.0007	0.0010	0.0042	0.0148	0.0254	0.0130	-0.0310	-0.0729	-0.1799
		MMLE I	-0.0008	0.0007	0.0035	0.0128	0.0228	0.0133	-0.0228	-0.0575	-0.1465
		MMLE III	0.0001	0.0003	0.0006	0.0012	0.0007	-0.0015	-0.0048	-0.0073	-0.0129
0.75	20	LPF	-0.0124	0.0037	0.0152	0.0323	0.0343	0.0055	-0.0507	-0.0981	-0.2131
		LSPF	-0.009	0.0038	0.0129	0.0257	0.0257	0.0014	-0.0439	-0.0816	-0.1726
		MMLE I	0.0027	0.0321	0.0541	0.0898	0.1004	0.0509	-0.0565	-0.1492	-0.3768
		MMLE III	0.0079	0.0158	0.0194	0.0188	0.0056	-0.0140	-0.0316	-0.0419	-0.0619
	50	LPF	-0.0018	0.0053	0.0112	0.0207	0.0217	0.0030	-0.0346	-0.0664	-0.1442
		LSPF	-0.0016	0.0047	0.0098	0.0181	0.0186	0.0014	-0.0326	-0.0613	-0.1314
		MMLE I	0.0012	0.0129	0.0229	0.0407	0.0471	0.0225	-0.0328	-0.0810	-0.1998
		MMLE III	0.0036	0.0057	0.0064	0.0049	-0.0011	-0.0083	-0.0137	-0.0163	-0.0206
	100	LPF	-0.0001	0.0040	0.0078	0.0143	0.0155	0.0031	-0.0229	-0.0451	-0.0995
		LSPF	-0.0001	0.0041	0.0080	0.0147	0.0156	0.0017	-0.0271	-0.0517	-0.112
		MMLE I	0.0008	0.0068	0.0123	0.0224	0.0260	0.0108	-0.0234	-0.0532	-0.1267
		MMLE III	0.0016	0.0024	0.0026	0.0016	-0.0009	-0.0024	-0.0017	-0.0001	0.0045
	200	LPF	0.0001	0.0031	0.0061	0.0115	0.0130	0.0033	-0.0178	-0.0361	-0.0812
		MMLE I	0.0003	0.0038	0.0071	0.0137	0.0168	0.0085	-0.0115	-0.0292	-0.0731
		MMLE III	0.0007	0.0010	0.0009	0.0002	-0.0014	-0.0026	-0.0024	-0.0018	0.0001
1.00	20	LPF	-0.0124	0.0061	0.0151	0.0227	0.0143	-0.0146	-0.0587	-0.0933	-0.1749
		LSPF	-0.0026	0.0121	0.0185	0.0226	0.0154	-0.0029	-0.0277	-0.0462	-0.0889
		MLE	0.0390	0.0051	-0.0414	-0.1891	-0.4239	-0.5816	-0.5472	-0.4413	-0.1038
		MMLE I	0.0236	0.0598	0.0807	0.1067	0.1035	0.0494	-0.0487	-0.1299	-0.3257
		MMLE III	0.0159	0.0244	0.0250	0.0164	-0.0033	-0.0231	-0.0357	-0.0409	-0.0482
	50	LPF	-0.0016	0.0066	0.0111	0.0157	0.0128	-0.0012	-0.0241	-0.0425	-0.0863
		LSPF	-0.0005	0.0064	0.0098	0.0126	0.0085	-0.0049	-0.0256	-0.0419	-0.0804
		MLE	0.0118	-0.0174	-0.0520	-0.1409	-0.2437	-0.2777	-0.2152	-0.1361	0.0826
		MMLE I	0.0086	0.0252	0.0356	0.0488	0.0461	0.0145	-0.0425	-0.0898	-0.2037
		MMLE III	0.0087	0.0103	0.0095	0.0047	-0.0025	-0.0068	-0.0060	-0.0033	0.0050
	100	LPF	0.0004	0.0049	0.0076	0.0105	0.0088	-0.0005	-0.0161	-0.0287	-0.0589
		LSPF	0.0008	0.0049	0.0073	0.0097	0.0076	-0.0019	-0.0177	-0.0304	-0.0609
		MLE	0.0036	-0.0171	-0.0389	-0.0884	-0.1375	-0.1460	-0.1050	-0.0592	0.0635
		MMLE I	0.0045	0.0136	0.0196	0.0276	0.0269	0.0095	-0.0230	-0.0500	-0.1154
		MMLE III	0.0046	0.0050	0.0044	0.0022	-0.0005	-0.0007	0.0025	0.0059	0.0150
	200	LPF	0.0008	0.0037	0.0057	0.0082	0.0078	0.0021	-0.0084	-0.0170	-0.0380
		MLE	0.0006	-0.0124	-0.0248	-0.0509	-0.0741	-0.0744	-0.0490	-0.0226	0.0467
		MMLE I	0.0024	0.0073	0.0106	0.0152	0.0150	0.0052	-0.0132	-0.0286	-0.0659
		MMLE III	0.0023	0.0019	0.0012	-0.0008	-0.0029	-0.0033	-0.0015	0.0005	0.0060

Table 11: Biases of the quantile estimators while varying sample size n based on $10,000$ simulations.

β	n	Method	ζ								
			0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
1.5	20	LPF	-0.0030	0.0068	0.0077	0.0033	-0.0050	-0.0098	-0.0086	-0.0054	0.0046
		LSPF	0.0235	0.0251	0.0208	0.0077	-0.0082	-0.0165	-0.0147	-0.0096	0.0063
		MLE	0.0873	0.0333	-0.0083	-0.0783	-0.1076	-0.0311	0.1494	0.3078	0.6989
		MMLE I	0.0788	0.1057	0.1156	0.1189	0.0961	0.0395	-0.0431	-0.1074	-0.2584
		MMLE III	0.0399	0.0354	0.0263	0.0045	-0.0164	-0.0177	0.0037	0.0265	0.0866
	50	LPF	-0.0007	0.0060	0.0076	0.0065	0.0009	-0.0080	-0.0184	-0.0258	-0.0423
		LSPF	0.0070	0.0105	0.0103	0.0071	0.0015	-0.0038	-0.0073	-0.0090	-0.0118
		MLE	0.0366	0.0145	-0.0016	-0.0271	-0.0365	-0.0096	0.0516	0.1048	0.2358
		MMLE I	0.0351	0.0483	0.0534	0.0552	0.0434	0.0134	-0.0307	-0.0652	-0.1462
		MMLE III	0.0189	0.0167	0.0127	0.0030	-0.0066	-0.0098	-0.0054	0.0004	0.0164
	100	LPF	0.0008	0.0038	0.0045	0.0038	0.0007	-0.0045	-0.0108	-0.0154	-0.0259
		LSPF	0.0026	0.0039	0.0036	0.0019	-0.0005	-0.0024	-0.0031	-0.0032	-0.0028
		MLE	0.0196	0.0066	-0.0022	-0.0151	-0.0182	-0.0014	0.0336	0.0635	0.1364
		MMLE I	0.0189	0.0264	0.0293	0.0305	0.0237	0.0059	-0.0204	-0.0410	-0.0896
		MMLE III	0.0111	0.0084	0.0055	0.0000	-0.0045	-0.0044	0.0006	0.0056	0.0187
	200	LPF	0.0008	0.0018	0.0019	0.0013	-0.0004	-0.0027	-0.0055	-0.0074	-0.0117
		MLE	0.0106	0.0034	-0.0012	-0.0076	-0.0088	0.0001	0.0182	0.0335	0.0708
		MMLE I	0.0102	0.0142	0.0157	0.0163	0.0123	0.0022	-0.0128	-0.0244	-0.0520
		MMLE III	0.0067	0.0051	0.0036	0.0010	-0.0010	-0.0007	0.0021	0.0049	0.0119
2.0	20	LPF	0.0081	0.0030	-0.0042	-0.0185	-0.0286	-0.0233	-0.0024	0.0172	0.0668
		LSPF	0.0583	0.0404	0.0257	-0.0010	-0.0218	-0.0221	-0.0015	0.0197	0.0748
		MLE	0.1168	0.0418	-0.0099	-0.0906	-0.1206	-0.0347	0.1614	0.3320	0.7519
		MMLE I	0.1345	0.1422	0.1401	0.1253	0.0909	0.0364	-0.0319	-0.0824	-0.1984
		MMLE III	0.0685	0.0440	0.0246	-0.0086	-0.0288	-0.0127	0.0393	0.0869	0.2062
	50	LPF	0.0017	0.0021	0.0001	-0.0052	-0.0104	-0.0118	-0.0089	-0.0054	0.0041
		LSPF	0.0215	0.0186	0.0144	0.0054	-0.0040	-0.0094	-0.0101	-0.0088	-0.0041
		MLE	0.0501	0.0199	0.0007	-0.0272	-0.0366	-0.0097	0.0504	0.1022	0.2292
		MMLE I	0.0637	0.0636	0.0607	0.0512	0.0338	0.0091	-0.0201	-0.0413	-0.0895
		MMLE III	0.0285	0.0212	0.0144	0.0018	-0.0080	-0.0080	0.0018	0.0119	0.0382
	100	LPF	0.0013	0.0010	-0.0003	-0.0031	-0.0056	-0.0059	-0.0038	-0.0016	0.0043
		LSPF	0.0096	0.0080	0.0060	0.0019	-0.0018	-0.0028	-0.0010	0.0012	0.0071
		MLE	0.0277	0.0113	0.0015	-0.0118	-0.0154	-0.0016	0.0275	0.0523	0.1129
		MMLE I	0.0374	0.0357	0.0331	0.0263	0.0153	6e-04	-0.0162	-0.0282	-0.0554
		MMLE III	0.0163	0.0119	0.0082	0.0016	-0.0036	-0.0040	0.0002	0.0048	0.0166
	200	LPF	0.0002	-0.0002	-0.0010	-0.0024	-0.0036	-0.0035	-0.0022	-0.0008	0.0026
		MLE	0.0149	0.0052	-0.0003	-0.0075	-0.0095	-0.0021	0.0133	0.0264	0.0583
		MMLE I	0.0206	0.0196	0.0183	0.0147	0.0089	0.0012	-0.0077	-0.0140	-0.0283
		MMLE III	0.0081	0.0053	0.0033	-0.0001	-0.0025	-0.0022	0.0006	0.0033	0.0102
3.0	20	LPF	0.0392	0.005	-0.0149	-0.0416	-0.0486	-0.0199	0.0403	0.0917	0.2168
		LSPF	0.1266	0.0730	0.0414	-0.0063	-0.0368	-0.0311	0.0076	0.0449	0.1401
		MLE	0.1699	0.0577	-0.0095	-0.1049	-0.1326	-0.0251	0.2060	0.4044	0.8903
		MMLE I	0.2137	0.1864	0.1670	0.1293	0.0829	0.0328	-0.0173	-0.0511	-0.1252
		MMLE III	0.1247	0.0610	0.0243	-0.0272	-0.0474	-0.0075	0.0865	0.1684	0.3699
	50	LPF	0.0085	-0.0059	-0.0144	-0.0256	-0.0285	-0.0169	0.0074	0.0280	0.0783
		LSPF	0.0640	0.0423	0.0287	0.0065	-0.0126	-0.0225	-0.0235	-0.0211	-0.0124
		MLE	0.0712	0.0270	0.0031	-0.0277	-0.0350	-0.0026	0.0641	0.1205	0.2580
		MMLE I	0.1156	0.0914	0.0769	0.0525	0.0285	0.0093	-0.0048	-0.0125	-0.0273
		MMLE III	0.0441	0.0222	0.0093	-0.0086	-0.0161	-0.0049	0.0228	0.0470	0.1068
	100	LPF	0.0013	-0.0044	-0.0081	-0.0132	-0.0150	-0.0108	-0.0013	0.0068	0.0268
		LSPF	0.0359	0.0248	0.0175	0.0048	-0.0083	-0.0189	-0.0268	-0.0312	-0.0396
		MLE	0.0349	0.0143	0.0035	-0.0100	-0.0134	-0.0004	0.0264	0.0490	0.1041
		MMLE I	0.0721	0.0519	0.0408	0.0238	0.0099	0.0028	0.0020	0.0035	0.0092
		MMLE III	0.0201	0.0126	0.0076	-0.0004	-0.0060	-0.0062	-0.0015	0.0033	0.0159
	200	LPF	-0.0001	-0.0036	-0.0057	-0.0085	-0.0093	-0.0067	-0.0010	0.0038	0.0155
		MLE	0.0149	0.0073	0.0030	-0.0031	-0.0057	-0.0026	0.0056	0.0128	0.0305
		MMLE I	0.0441	0.0286	0.0206	0.0094	0.0021	0.0018	0.0073	0.0129	0.0276
		MMLE III	0.0106	0.0065	0.0040	0.0000	-0.0028	-0.0029	-0.0006	0.0017	0.0079

Table 12: RMSEs of the quantile estimators while varying sample size n based on $10,000$ simulations.

β	n	Method	ζ 0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
0.50	20	LPF	0.0134	0.0096	0.0207	0.0609	0.1284	0.2614	0.5069	0.7214	1.2550
		LSPF	0.0126	0.0096	0.0192	0.0566	0.1243	0.2596	0.4993	0.7054	1.2152
		MMLE I	0.0101	0.0171	0.0362	0.0889	0.1635	0.2822	0.5117	0.7217	1.2559
		MMLE III	0.0089	0.0121	0.0217	0.0543	0.1185	0.2540	0.5049	0.7255	1.2771
	50	LPF	0.0020	0.0045	0.0122	0.0367	0.0794	0.1638	0.3232	0.4635	0.8136
		LSPF	0.0020	0.0042	0.0115	0.0356	0.0789	0.1638	0.3194	0.4552	0.7936
		MMLE I	0.0018	0.0062	0.0152	0.0430	0.0886	0.1697	0.3232	0.4606	0.8065
		MMLE III	0.0017	0.0043	0.0097	0.0295	0.0722	0.1629	0.3263	0.4681	0.8209
	100	LPF	0.0008	0.0029	0.0080	0.0255	0.0561	0.1152	0.2288	0.3300	0.5837
		LSPF	0.0008	0.0029	0.0082	0.0262	0.0577	0.1159	0.2267	0.3258	0.5748
		MMLE I	0.0008	0.0032	0.0086	0.0272	0.0602	0.1191	0.2281	0.3253	0.5697
		MMLE III	0.0005	0.0022	0.0057	0.0194	0.0498	0.1142	0.2300	0.3304	0.5800
	200	LPF	0.0008	0.0019	0.0059	0.0198	0.0425	0.0819	0.1637	0.2391	0.4309
		MMLE I	0.0008	0.0018	0.0054	0.0187	0.0418	0.0822	0.1607	0.2320	0.4123
		MMLE III	0.0002	0.0013	0.0037	0.0134	0.0352	0.0806	0.1613	0.2311	0.4042
0.75	20	LPF	0.0347	0.0290	0.0440	0.0881	0.1556	0.2904	0.5222	0.7176	1.1965
		LSPF	0.0307	0.0300	0.0441	0.0852	0.1539	0.2939	0.5283	0.7239	1.2006
		MMLE I	0.0285	0.0544	0.0825	0.1369	0.1971	0.3060	0.5227	0.7164	1.2023
		MMLE III	0.0301	0.0350	0.0495	0.0895	0.1546	0.2868	0.5265	0.7340	1.2485
	50	LPF	0.0085	0.0147	0.0268	0.0551	0.0979	0.1844	0.3338	0.4596	0.7678
		LSPF	0.0083	0.0143	0.0258	0.0534	0.0973	0.1858	0.3347	0.4591	0.7625
		MMLE I	0.0085	0.0220	0.0374	0.0699	0.1108	0.1887	0.3345	0.4614	0.7763
		MMLE III	0.0092	0.0161	0.0264	0.0529	0.0966	0.1852	0.3391	0.4697	0.7908
	100	LPF	0.0035	0.0098	0.0181	0.0374	0.0674	0.1297	0.2373	0.3277	0.5486
		LSPF	0.0035	0.0098	0.0180	0.0374	0.0683	0.1308	0.2366	0.3250	0.5407
		MMLE I	0.0038	0.0126	0.0224	0.0441	0.0737	0.1317	0.2371	0.3277	0.5513
		MMLE III	0.0040	0.0095	0.0169	0.0359	0.0671	0.1302	0.2388	0.3305	0.5551
	200	LPF	0.0017	0.0069	0.0129	0.0267	0.0479	0.0917	0.1679	0.2322	0.3895
		MMLE I	0.0018	0.0076	0.0141	0.0289	0.0504	0.0926	0.1669	0.2301	0.3857
		MMLE III	0.0019	0.0061	0.0114	0.0248	0.0470	0.0918	0.1679	0.2320	0.3888
1.00	20	LPF	0.0595	0.0514	0.0646	0.1056	0.1725	0.3072	0.5310	0.7171	1.1701
		LSPF	0.0551	0.0589	0.0744	0.1139	0.1784	0.3129	0.5396	0.7286	1.1893
		MLE	0.0638	0.0566	0.0812	0.2203	0.4586	0.6350	0.7197	0.8220	1.3018
		MMLE I	0.0585	0.0923	0.1193	0.1651	0.2165	0.3261	0.5375	0.7216	1.1781
		MMLE III	0.0569	0.0588	0.0737	0.1151	0.1788	0.3065	0.5376	0.7370	1.2308
	50	LPF	0.0202	0.0272	0.0404	0.0685	0.1102	0.1969	0.3430	0.4644	0.7594
		LSPF	0.0202	0.0284	0.0415	0.0695	0.1116	0.1967	0.3385	0.4562	0.7425
		MLE	0.0237	0.0360	0.0704	0.1653	0.2769	0.3405	0.4112	0.5172	0.8867
		MMLE I	0.0227	0.0417	0.0583	0.0879	0.1235	0.1993	0.3394	0.4594	0.7546
		MMLE III	0.0221	0.0305	0.0430	0.0710	0.1117	0.1970	0.3474	0.4746	0.7863
	100	LPF	0.0101	0.0185	0.0283	0.0482	0.0778	0.1390	0.2413	0.3261	0.5321
		LSPF	0.0107	0.0184	0.0278	0.0477	0.0782	0.1391	0.2382	0.3198	0.5177
		MLE	0.0117	0.0288	0.0535	0.1103	0.1688	0.2074	0.2738	0.3564	0.6066
		MMLE I	0.0118	0.0248	0.0362	0.0573	0.0842	0.1406	0.2415	0.3271	0.5369
		MMLE III	0.0113	0.0196	0.0290	0.0492	0.0786	0.1392	0.2434	0.3308	0.5443
	200	LPF	0.0055	0.0127	0.0197	0.0337	0.0545	0.0977	0.1695	0.2289	0.3729
		MLE	0.0065	0.0203	0.0356	0.0678	0.0996	0.1273	0.1828	0.2426	0.4092
		MMLE I	0.0063	0.0154	0.0232	0.0378	0.0571	0.0976	0.1689	0.2289	0.3757
		MMLE III	0.0062	0.0130	0.0198	0.0339	0.0547	0.0981	0.1720	0.2337	0.3841

Table 13: RMSEs of the quantile estimators while varying sample size n based on $10,000$ simulations.

β	n	Method	ζ								
			0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
1.5	20	LPF	0.1122	0.0944	0.1000	0.1327	0.1979	0.3294	0.5437	0.7210	1.1528
		LSPF	0.1055	0.1052	0.1146	0.1438	0.2004	0.3301	0.5523	0.7378	1.1904
		MLE	0.1284	0.1014	0.1050	0.1531	0.2118	0.3227	0.5982	0.8551	1.5038
		MMLE I	0.1260	0.1527	0.1692	0.195	0.2337	0.3402	0.5386	0.7073	1.1213
		MMLE III	0.1132	0.0999	0.1070	0.1432	0.2060	0.3293	0.5518	0.7447	1.2240
	50	LPF	0.0508	0.0521	0.0637	0.0893	0.1269	0.2075	0.3447	0.4591	0.7374
		LSPF	0.0507	0.0581	0.0697	0.0930	0.1278	0.2083	0.3489	0.4663	0.7520
		MLE	0.0605	0.0571	0.0686	0.0992	0.1329	0.2061	0.3618	0.5003	0.8451
		MMLE I	0.0601	0.0766	0.0885	0.1081	0.1365	0.2089	0.3396	0.4494	0.7172
		MMLE III	0.0529	0.0554	0.0673	0.0941	0.1295	0.2066	0.3481	0.4691	0.7668
	100	LPF	0.0288	0.0352	0.0447	0.0630	0.0898	0.1482	0.2466	0.3279	0.5252
		LSPF	0.0296	0.0368	0.0461	0.0638	0.0894	0.1467	0.2444	0.3255	0.5225
		MLE	0.0348	0.0376	0.0475	0.0684	0.0932	0.1488	0.2561	0.3487	0.5769
		MMLE I	0.0352	0.0467	0.0556	0.0708	0.0937	0.1491	0.2452	0.3249	0.5184
		MMLE III	0.0309	0.0374	0.0472	0.0660	0.0915	0.1484	0.2491	0.3339	0.5409
	200	LPF	0.0179	0.0243	0.0312	0.0438	0.0626	0.1036	0.1719	0.2282	0.3645
		MLE	0.0207	0.0257	0.0330	0.0469	0.0649	0.1048	0.1767	0.2374	0.3858
		MMLE I	0.0211	0.0296	0.0364	0.0477	0.064	0.1032	0.1714	0.2279	0.3652
		MMLE III	0.0188	0.0256	0.0328	0.0459	0.0639	0.1041	0.1737	0.2317	0.3729
2.0	20	LPF	0.1511	0.1243	0.1243	0.1504	0.2125	0.3394	0.5453	0.7159	1.1317
		LSPF	0.1524	0.1423	0.1452	0.1652	0.2158	0.3449	0.5712	0.7608	1.2238
		MLE	0.1762	0.1314	0.1359	0.1917	0.2451	0.3377	0.6211	0.8959	1.5959
		MMLE I	0.1868	0.1976	0.2033	0.2126	0.2422	0.3490	0.5470	0.7133	1.1187
		MMLE III	0.1610	0.1318	0.1316	0.1625	0.2223	0.3391	0.5563	0.7475	1.2263
	50	LPF	0.0788	0.0717	0.0791	0.1006	0.1365	0.2163	0.3519	0.4647	0.7393
		LSPF	0.0816	0.0852	0.0932	0.1097	0.1373	0.2149	0.3569	0.4761	0.7666
		MLE	0.0888	0.0745	0.0842	0.1144	0.1452	0.2127	0.3663	0.5053	0.8535
		MMLE I	0.0956	0.0997	0.1046	0.1157	0.1425	0.2177	0.3470	0.4537	0.7120
		MMLE III	0.0829	0.0744	0.0822	0.1048	0.1378	0.2138	0.3546	0.4749	0.7707
	100	LPF	0.0492	0.0501	0.0577	0.0737	0.0977	0.1529	0.2482	0.3277	0.5211
		LSPF	0.0503	0.0552	0.0629	0.0771	0.0983	0.1532	0.2517	0.3341	0.5348
		MLE	0.0544	0.0509	0.0587	0.0769	0.0991	0.1527	0.2572	0.3472	0.5689
		MMLE I	0.0599	0.0631	0.0677	0.0779	0.0988	0.1528	0.2447	0.3204	0.5038
		MMLE III	0.0509	0.0507	0.0587	0.0754	0.0976	0.1503	0.2475	0.3301	0.5326
	200	LPF	0.0321	0.0355	0.0414	0.0524	0.0689	0.1076	0.1741	0.2294	0.3638
		MLE	0.0335	0.0346	0.0410	0.0534	0.0692	0.1074	0.1779	0.2377	0.3842
		MMLE I	0.0369	0.0407	0.0450	0.0536	0.069	0.1081	0.1744	0.2289	0.3608
		MMLE III	0.0311	0.0347	0.0410	0.0526	0.0687	0.1076	0.1765	0.2340	0.3741
3.0	20	LPF	0.2147	0.1710	0.1625	0.1772	0.2294	0.3479	0.5517	0.7238	1.1473
		LSPF	0.2298	0.1940	0.1845	0.1899	0.2317	0.3590	0.5915	0.7881	1.2697
		MLE	0.2525	0.1743	0.1704	0.2255	0.2704	0.3452	0.6464	0.9475	1.7161
		MMLE I	0.2731	0.2520	0.2420	0.2326	0.2552	0.3600	0.5496	0.7072	1.0899
		MMLE III	0.2350	0.1757	0.1627	0.1839	0.2396	0.3516	0.5726	0.7712	1.2724
	50	LPF	0.1230	0.1004	0.1013	0.1170	0.1503	0.2255	0.3553	0.4642	0.7310
		LSPF	0.1340	0.1240	0.1233	0.1263	0.1460	0.2293	0.3841	0.5130	0.8249
		MLE	0.1335	0.0997	0.1026	0.1273	0.1550	0.2204	0.3733	0.5125	0.8614
		MMLE I	0.1543	0.1364	0.1307	0.1297	0.1510	0.2229	0.3470	0.4492	0.6965
		MMLE III	0.1268	0.1001	0.1006	0.1181	0.1496	0.2210	0.3538	0.4685	0.7522
	100	LPF	0.0824	0.0726	0.0762	0.0883	0.1096	0.1614	0.2533	0.3308	0.5205
		LSPF	0.0894	0.0875	0.0893	0.0922	0.1038	0.1617	0.2720	0.3637	0.5852
		MLE	0.0829	0.0683	0.0728	0.0880	0.1079	0.1574	0.2562	0.3421	0.5546
		MMLE I	0.1004	0.0863	0.0834	0.0860	0.1042	0.1574	0.2477	0.3220	0.5017
		MMLE III	0.0792	0.0667	0.0707	0.0839	0.1044	0.1544	0.2467	0.3254	0.5188
	200	LPF	0.0570	0.0547	0.0589	0.0680	0.0814	0.1155	0.1790	0.2333	0.3671
		MLE	0.0561	0.0466	0.0513	0.0619	0.0747	0.1120	0.1859	0.2492	0.4043
		MMLE I	0.0650	0.0549	0.0543	0.0590	0.0735	0.1109	0.1745	0.2271	0.3547
		MMLE III	0.0523	0.0466	0.0504	0.0598	0.0745	0.1114	0.1778	0.2337	0.3703

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Response to the Reviewers’ Comments

We sincerely thank the reviewer for their careful reading and constructive comments. We have revised the manuscript extensively and addressed all concerns. Below are our responses to the comments.

Response to Reviewer # 1

Major comments

Comment 1: p.4, l.5. The only proposal related to quantiles appears in expression (11), and the paper does not, in fact, present a detailed LFP method for quantiles. If there are additional ideas or methodological innovations not described in the manuscript, they should be stated explicitly.

Response : We thank the reviewer for pointing this out. We agree that our description of quantile estimation was brief. We have now expanded Section 2.2 substantially to clearly define how quantile estimation follows directly from the LPF parameter estimates and prove the consistency property of the quantile estimator in additional Theorem 2.2 and Remark 2.1. We do not require a separate estimation procedure because the quantiles are well defined continuous function of (α, β, γ) and this fact is also used to prove the consistency of the quantile estimates. Moreover, the contribution lies in proving consistency of LPF estimators, which in turn justifies using them inside the quantile expression. (Present Manuscript: Pages 10-11).

Comment 2: Reporting only the average bias and RMSE does not address whether the coverage probability behaves correctly. In particular, three-parameter models such as the one studied here often become unstable near the boundary, so the coverage probability should also be investigated.

Response : Since the generalized exponential family is not a regular family, we could not establish the asymptotic distribution of the proposed estimators. Therefore the asymptotic normality of the MLE and its functions cannot be used, even approximate confidence intervals cannot be constructed. Since the regularity conditions fail, the Fisher information matrix cannot be used for asymptotic variances and covariances. However, we have provided the bootstrap confidence intervals for the illustrative examples which it seems work quite well. (Present Manuscript: Figures 8-9 or Tables 4 and 7. (See Pages 22-24)).

Comment 3: p.11, l.23. Lifetime data often have small sample sizes. The numerical experiments should therefore also evaluate performance when the sample size is small (e.g., $n = 5$ or 10).

Response: We thank the reviewer for raising this point. In three-parameter lifetime models such as the GE distribution, sample sizes as small as $n \leq 10$ typically lead to numerical instability, and extremely large variability in the parameter estimates. We note that only a few papers on three-parameter lifetime models report simulations with such very small sample sizes. Most of the literature avoids these cases because it is well documented that estimation in three-parameter families becomes highly unstable when n is very small: the location and shape parameters interact in a way that produces estimates with large biases and a very high variability, and frequent numerical non-convergence/instability. This in turn leads to a very high rejection proportion (p) in our case. We have now added a short discussion in the manuscript to make this clear. If the reviewer considers it essential, we can report additional results for $n = 5$ and $n = 10$ and update all relevant figures and tables accordingly. (Present Manuscript: Page 12, 2nd last Paragraph).

Comment 4: p.3, l.45. The authors state, “This study shows … that it requires less time to execute the simulations and has less computational complexity.” Thus, the simulation study should also examine computation time. In addition, to demonstrate that the method is not overly complex, it is recommended to provide the source code used for computation.

Response : We appreciate this comment. The computational time for any method depends on the following factors, such as the value of the shape parameter β, the sample size n, the number of iterations/simulations used to obtain the estimates, etc. Majorly, the computational time depends on the sample size n, therefore we provide the computational time for various values of sample sizes. For the computations, we have applied parallel computing using `R` language. The `R` codes are run on the computers of High performance computing (HPC) using 40 processing cores. We have used the first $10,000$ iterations/samples out of $25,000$ generated samples to find the estimates. Thus, the computational times reported in Table 4 are compared based on $25,000$ iterations. Moreover, the `R` codes of the MLE, MMLE I, and MMLE III methods provide numerical results in seconds/minutes of time which is less that the computational times used by LSPF and LPF methods. All computations are carried out in `R`, employing the `foreach` and

Table 1: Computational time (LPF vs. LSPF) for different sample sizes n and $\beta = 1.50$.

Method	$n = 20$	$n = 50$	$n = 100$	$n = 200$
LSPF	18.50 hrs	27.07 hrs	33.90 hrs	–
LPF	2.80 mins	3.20 mins	3.60 mins	6.00 mins

`doParallel` packages for parallel execution on a high-performance computing (HPC) cluster utilizing 40 processing cores. To further demonstrate transparency, we have now provided complete reproducible `R` source codes for all simulation studies and data analyses as supplementary material, referenced in Section 3 of the revised manuscript (Present Manuscript: Page 12, Paragraph in bottom; Page 13, Paragraphs 1-2).

Minor comments

Comment 1: What does "p" in Tables 8-9 represent?

Response : We thank the reviewer for catching this ambiguity. We now state clearly in the present manuscript that "p" denotes the proportion of rejected samples (non-convergent or invalid estimates), although it was briefly defined earlier (Old Manuscript: Pages 31-32) in Tabel 8. (Present Manuscript: Page 12, Paragraph 2).

Comment 2: p.11, l.27. The authors set $\alpha = 1$ and $\gamma = 0$. If this does not reduce generality, this point should be stated clearly.

Response : We now explain this explicitly. Because the LPF method relies on location-invariant transformations and because the GE model is a scale family, this setting for simulations involves no loss of generality. A justification has been added in Section 3. (Present Manuscript: Page 12 Lines 2-3).

Comment 3: For three-parameter models of this kind, selecting initial values for numerical optimization is generally difficult. How were the initial values chosen?

Response: We agree with the reviewer, and this point was already discussed in the manuscript when describing Subfigure 6a (Old Manuscript: Page 19, Lines 36-40). The contour plots (Figures 6a and 6b) show the unimodality of the objective function in the LPF and LSPF methods and can be used to choose reasonable initial values for the numerical optimization. (Present Manuscript: Pages 21 Lines 7-9).

Comment 4: The numerical experiments compare estimation accuracy for each parameter, but which parameters are of primary interest? Clarifying this point would also help in interpreting the results of the real-data analysis.

Response : For case of simulation study, the results were already discussed based on the primary interests in the old manuscript. Moreover, we find this comment very helpful for improving the illustrations. In the present manuscript, we have now added brief explanations in Section 4 to indicate the primary inferential targets in each real-data example. In Example 1 (medical data), the quantities of primary interest are the shape parameter and relevant quantiles. In Example 2 (electrical lifetime data), the location parameter and higher quantiles are more important for engineering applications. Short justifications are added in each example. (Present Manuscript: Page 20 Paragraph 2; Page 23 Paragraph 1 of Example 2.)

Response to Reviewer # 2

Theoretical Aspects

The authors propose the LFP method for the GE distribution. While the LFP method appears similar to LFPS, the key distinction between the two lies in whether the data are standardized. The statistical properties and theoretical foundations of the GE distribution have already been well established. Although the main theorems - such as those concerning consistency, boundedness, and uniqueness - are important, their proofs appear straightforward and intuitive to implement.

Comment 1. LFP seems similar to LFPS; distinction is mainly standardization. Proofs appear straightforward and intuitive.

Response : We thank the reviewer for the comment. The differences between the LSPF and LPF methods were already discussed in the original manuscript (see Introduction, Page 3, Paragraph 2; Page 4, Paragraph 1). Specifically: (1) The LSPF estimator is location–scale invariant, while the LPF estimator is only location-invariant; (2) LPF runs simulations faster and is less computationally complex because it uses fewer integrations. The LSPF likelihood requires double integration, whereas LPF only needs one; (3) LSPF involves a one-dimensional likelihood, while LPF involves a two-dimensional likelihood, making the theoretical proofs, especially for unimodality, more challenging.

This results in fundamentally different likelihood structures: LPF's bivariate likelihood is analytically distinct from LSPF's one-dimensional likelihood, and thus it was studied separately. We summarize this in the table below.

Additionally, the main theoretical contribution of this work is not just the consistency

Table 2: Comparison of LSPF and LPF Methods

LSPF	**LPF**
1. One-dimensional likelihood and optimization	Two-dimensional likelihood and optimization
2. Two-dimensional integration in likelihood	One-dimensional integration in likelihood
3. More computationally involved and time-consuming (due to point 2)	Less computationally involved and faster
4. Easier to prove theoretical results	More challenging theoretical proofs (unimodality) due to point 1

proof, but also the uniqueness of the LPF estimator for the three-parameter GE model. Proving uniqueness requires careful analysis of the bivariate likelihood, which is more complex than proving consistency. These details were already included in the original submission (see Page 4, Paragraph 1; Page 5, Paragraph before Theorem 2.2).

Simulation Study

The authors point out that the advantage of their proposed method is the reduced computational cost, and while they say that this is due to the reduction in integral calculations, they do not clarify the extent to which this has been achieved. The computing time should be presented. The advantage of the LPF method over the LSPF method is that bias accumulates during the calculation process, but the extent of this is not clear. This is said to be due to the change in the number of data used from $n-2$ to $n-1$, but even when $n = 20$, the results of two methods are almost the same.

Comment 2. Authors claim reduced computational cost but do not quantify its extent.

Response : We thank the reviewer for this helpful comment. In the revised manuscript, we now provide a clear quantitative comparison of the computational cost of the LPF and LSPF methods. A dedicated paragraph and a new table report the computation times for several sample sizes, based on 25,000 Monte Carlo replications computed using parallel processing on an HPC system. These results show that the LPF method is substantially faster than LSPF, for example, at $n = 100$ the LSPF method requires approximately

33.9 hours, whereas the LPF method completes the same number of iterations in about 3.6 minutes. (Present Manuscript: Page 12, Ending Paragraph; Page 13 Paragraphs 1-2 with Table 1).

Comment 3. Advantage of LPF over LSPF due to biases is unclear and for n=20 results are almost same.

Respoense : We thank the reviewer for this observation. We would like to note that the distinction in bias behaviour between the LPF and LSPF estimators was already documented in the original manuscript. In particular, the simulation results on Page 16, Paragraph 1, and Figure 3 of the present manuscript clearly show that the LSPF estimator exhibits larger bias for the location parameter, especially when $n < 50$, which reflects the difference between using $n - 1$ versus $n - 2$ transformed observations.

Applications: Example 1 (COVID-19)

It is noteworthy that relatively recent COVID-19 data were employed as medical data. However, the 25 data analyzed were partially extracted from the 36 data listed in the references. The basic descriptive statistics of the two datasets (e.g., skewness values of 1.78 and 1.16, and means of 3.65 and 3.28, respectively) differ considerably, raising questions about why certain data were excluded. Unless it is explicitly acknowledged that both datasets originate from the same source, any statistical comparison of estimation accuracy becomes meaningless. A more critical issue is that, while the presented data pertain to the mortality rate, its definition is unclear. For instance, the calculation process leading to the reported mortality rate of 3.1091 should be clearly explained, as this value cannot be readily derived from the original WHO data (i.e., the daily numbers of confirmed cases and deaths) alone. Explain why mortality rates exceed 1 although the definition of the mortality rate is a quantity defined as a proportion of people dying among the total affected people by this disease. Is the mortality rate different from the case fatality ratio?

Comment 4. Data used (25 points) is a subset of the 36-point dataset; descriptive statistics differ; why excluded points?

Response : The 36-day Canada mortality-rate dataset (10 April–15 May 2020) was originally reported by Almetwally et al. (2021)[1]. The two later studies, Nik et al. (2023)[2] and Jeon et al. (2024)[3], used only the first 25 days (10 April–4 May 2020) from the same dataset. We could not find any published explanation for why the additional days were not included in these later analyses. Although, we note that the extra days (5–15 May) are removed simply chronologically, and not due to any unusual observations or outliers. We use the 25-day data because both Nik et al. (2023) and Jeon et al. (2024) analysed this dataset. To avoid confusion, we have removed the citation to Almetwally et al. (2021) from this discussion (Present Manuscript: Page 19, bottom lines). Moreover, if needed, the analysis can also be repeated for the 36-day dataset; however, this would not materially affect our conclusions, as the data are used only for illustrative purposes.

Comment 5. Definition of mortality rate unclear. How is 3.1091 obtained? Why can mortality rate exceed 1? Is this different from case fatality ratio?

Response : We appreciate this important observation. We have now corrected the definition of the mortality rate used in the cited dataset. The reported values correspond to the WHO-style mortality rate defined as deaths per 100 diagnosed cases (i.e., a percentage), not a proportion. For this reason, values can exceed 1. This is therefore not the case fatality ratio. I hope this resolves the apparent contradiction. (Present Manuscript: Page 19, bottom lines).

Applications: Example 2 (Electrical data)

[1]Ehab M Almetwally et al. "A new inverted topp-leone distribution: applications to the COVID-19 mortality rate in two different countries". In: *Axioms* 10.1 (2021), p. 25.

[2]Ali Saadati Nik, Akbar Asgharzadeh, and Ayman Baklizi. "Inference Based on New Pareto-Type Records With Applications to Precipitation and Covid-19 Data". In: *Statistics, Optimization & Information Computing* 11.2 (2023), pp. 243–257.

[3]Young Eun Jeon, Suk-Bok Kang, and Jung-In Seo. "Pivotal-based inference for a Pareto distribution under the adaptive progressive Type-II censoring scheme". In: *AIMS Mathematics* 9.3 (2024), pp. 6041–6059.

The authors employ a well-known example to argue that the LFP method outperforms existing approaches in summary section. However, evaluating the effectiveness of a proposed method based on a single example appears insufficient and potentially misleading. At best, such an example can only demonstrate that the proposed method performs comparably to others, rather than conclusively proving its superiority. Moreover, the dataset used is outdated and thus lacks the relevance or granularity needed to yield meaningful insights. Explain why you adopted the negative value of -2.7991 for the estimate of location parameter. The lifetime is defined in non-negative space physically.

Comment 6. Evaluating superiority using a single example is insufficient.

Response : We agree with the reviewer that a single real-data example cannot establish superiority. In the revised manuscript, we have now rewritten the discussion to avoid any such implication. Example 2 is used only to illustrate how the different estimators behave for a practical engineering dataset, and the illustrative results are aligned with the simulation results (Section 3), where we identify which method is recommended under comparable sample-size and shape-parameter settings. The purpose of the example is therefore illustrative rather than evidential. (Present Manuscript: Page 19-24 Highlighted paragraphs in Section 4).

Comment 7. Why is the estimate of the location parameter negative (e.g., -2.7991), since lifetime is non-negative?

Response : In this study, the location parameter γ belongs to the set of real numbers R, hence the estimation method and the algorithms are proposed accordingly. But, we agree with reviewer's concern that the proposed method provides γ estimates that are not reasonable for this particular set of data. In order to overcome this, we can restrict the range of the location parameter γ for non-negative real numbers and provide the point and interval estimates of the parameters. Many three-parameter lifetime distributions (Weibull, Gamma, Lognormal) also allow negative location estimates even when observed data are non-negative. In the revision, we note this explicitly and explain that the estimation can be constrained to $\gamma \geq 0$ if the application requires a physically interpretable location parameter. (Present Manuscript: Page 24,Lines 1-4 from the bottom).

Literature

Comment 8. In the discussion of the regularity condition, it is somewhat surprising that the work of Richard L. Smith (Biometrika, 1985) is not cited. Approaches to addressing non-regularity in maximum likelihood estimation (MLE) include not only the method of moments and the Nagatuka-Balakrishnan method, but also the Hirose-Lai method (Technometrics, 1997).

Response : We thank the reviewer for bringing our attention to these literature. We have now cited: Smith (1985, Biometrika)[4], Hirose & Lai (1997, Technometrics)[5] and one additional citation to Smith and Naylor (1987, Journal of the Royal Statistical Society Series C: Applied Statistics)[6] in the discussion. These references appear in the revised Introduction. (Present Manuscript: Page 3, Line 2).

[4] Richard L Smith. "Maximum likelihood estimation in a class of nonregular cases". In: *Biometrika* 72.1 (1985), pp. 67–90.

[5] Hideo Hirose and Tze Leung Lai. "Inference from grouped data in three-parameter Weibull models with applications to breakdown-voltage experiments". In: *Technometrics* 39.2 (1997), pp. 199–210.

[6] Richard L Smith and JC918854 Naylor. "A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution". In: *Journal of the Royal Statistical Society Series C: Applied Statistics* 36.3 (1987), pp. 358–369.