# Testing the Goodness-of-fit of the Stable Distributions with applications to German Stock Index data and Bitcoin cryptocurrency data

Ruhul Ali Khan*[1], Ayan Pal†[2] and Debasis Kundu‡[3]

[1]Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA

[2]Department of Statistics, The University of Burdwan, Burdwan-713104, India

[3]Department of Mathematics & Statistics, Indian Institute of Technology Kanpur, Kanpur - 208016, India

**Abstract**

Outlier-prone data sets are of immense interest in diverse areas including economics, finance, statistical physics, signal processing, telecommunications and so on. Stable laws (also known as $\alpha$- stable laws) are often found to be useful in modeling outlier-prone data containing important information and exhibiting heavy tailed phenomenon. In this article, an asymptotic distribution of a unbiased and consistent estimator of the stability index $\alpha$ is proposed based on jackknife empirical likelihood (JEL) and adjusted JEL method. Next, using the sum-preserving property of stable random variables and exploiting $U$-statistic theory, we have developed a goodness-of-fit test procedure for $\alpha$-stable distributions where the stability index $\alpha$ is specified. Extensive simulation studies are performed in order to assess the finite sample performance of the proposed test. Finally, two appealing real life data examples related to the daily closing price of German Stock Index and Bitcoin cryptocurrency are analysed in detail for illustration purposes.

**Keywords:** Stable distribution, Nonparametric test, Asymptotic normality, Empirical likelihood, Wilks' theorem.

**AMS Subject Classification:** Primary 62G10, Secondary 90B25.

*ruhul@math.arizona.edu

†Corresponding author, email : ayan.pal33@gmail.com

‡kundu@iitk.ac.in

# 1 Introduction

Suppose $X_1$, $X_2 \dots X_n$ are independent and identically distributed (i.i.d) random variables with the associated cumulative distribution function (CDF) $F$. $F$ is said to have a stable distribution with the stability index $\alpha$ (or $\alpha$-stable distribution) if corresponding to each $n$, there exists constants $a_n$ such that $X_1+X_2+\cdots+X_n$ has the same distribution as $a_n+n^{1/\alpha}X_1$, $0<\alpha\leq 2$. Moreover, $F$ is called strictly stable if and only if $a_n=0$ for all values of $n$. When $\alpha=2$, $F$ is the CDF of a normal distribution and consequently possesses moments of all orders. For $0<\alpha<2$, $F$ possesses all moments of order $p$ where $p<\alpha$ and the tail probabilities approximately resemble those of a Pareto distribution. The stability index $\alpha$ plays an important role in the sense that it measures the tail heaviness of the distribution. As $\alpha$ decreases, the tail heaviness of the distribution increases (see Feller (2008) for more details). From a practical point of view, stable distributions are an attractive option for modeling data that exhibit heavy tails. Heavy tailed phenomena is quite common in many areas including signal processing, finance, biology, physics and condition monitoring. Some real life data instances of heavy tailed phenomena are $(i)$ data file traffic on the internet (Crovella and Bestavros (1997)), $(ii)$ returns on financial markets (Rachev (2003)) and $(iii)$ magnitudes of earthquakes and floods (Latchman et al. (2008)) and so on. For a detailed discussion on the history of heavy tailed distributions, see Mandelbrot and Hudson (2010) and the references cited therein. In addition, for applications of stable distributions in signal processing, one may refer to Nikias and Shao (1995).

A random variable $X$ is said to have a $\alpha$- stable distribution (or stable distribution with the stability index $\alpha$) with parameters $\alpha\in(0,2]$, $\sigma>0$, $\beta\in[-1,1]$ and $\mu\in\mathbb{R}$ if its characteristic function has the following form:

$$E(e^{itX})=\begin{cases}\exp\left(-\sigma^{\alpha}|t|^{\alpha}\left(1-i\beta(sign(t))\tan\frac{\pi\alpha}{2}\right)+i\mu t\right) & \text{for } \alpha\neq 1\\ \exp\left(-\sigma|t|\left(1+i\beta\frac{2}{\pi}(sign(t))\log|t|\right)+i\mu t\right) & \text{for } \alpha=1\end{cases} \tag{1}$$

where

$$sign(t)=\begin{cases}1 & \text{for } t>0\\ 0 & \text{for } t=0\\ -1 & \text{for } t<0.\end{cases} \tag{2}$$

It is to note here that $\alpha$-stable distributions are defined by the four parameters: $\alpha$ is the stability index (also known as the tail index, tail exponent, or characteristic exponent) , $\beta$ is the skewness parameter, $\sigma$ is the scale parameter and $\mu$ is the location or shift parameter.

The stability index determines the rate at which the distribution tails taper off. Smaller the value of $\alpha$, greater is the size and frequency of the extreme events. In case when $\alpha > 1$, the expected value of the distribution exists and is given by $\mu$. For $\beta > 0$, the distribution is skewed to the right, while the distribution is skewed to the left when $\beta < 0$. If $\beta = 0$, the distribution is symmetric about $\mu$. $\sigma$ controls the width and hence the dispersion of the distribution. $\mu$ measures the shift of the mode of the distribution. The condition $\alpha \neq 1$, $\mu = 0$ or $\alpha = 1$, $\beta = 0$ corresponds to the situation that $X$ has a strictly stable distribution. However, there are practical difficulties with stable laws, primarily stemming from the fact that for most of the stable distributions, no closed form expression exists for the associated probability density functions (PDFs). For $\alpha = 0.5, 1$ and 2, closed form expressions are available for PDFs and the corresponding distributions are well known as Lévy, Cauchy and normal distributions respectively. Setting $\alpha = 2$, (1) corresponds to the characteristic function of a Normal$(\mu, 2\sigma^2)$ distribution. Again, upon setting $(\alpha, \beta) = (1, 0)$, and $(\alpha, \beta) = (0.5, 1)$, one can obtain respectively the Cauchy $(\sigma, \mu)$ distribution and the Lévy distribution with scale parameter $\sigma$ and location parameter $\sigma + \mu$. Other heavy tailed distributions including the Student's-t distribution, lognormal distribution, and Pareto family of distributions are also among the popular candidate distributions to model data with high kurtosis.

Parameter estimation of a stable distribution is well studied in literature based on different approaches. Fama and Roll (1968, 1971) did early work on estimating $\alpha$ using order statistics, but their method can only be applied to symmetric stable distributions for $\alpha \in [1, 2]$. The method of maximum likelihood estimation is studied in this regard by DuMouchel (1973, 1975). Press (1972), Paulson et al. (1975), Koutrouvelis (1980) and Brockwell and Brown (1981) have considered estimation techniques in stable distributions directly using the characteristic function. McCulloch (1986) developed the quantile method now popular in application which works for both symmetric and skew stable distributions and $\alpha \in [0.5, 2]$. Mittnik et al. (1999) proposed an estimation methodology using the fast Fourier transform while Lombardi (2007) developed a MCMC method to estimate the stable distribution parameters. Buckle (1995) and Peters et al. (2012) have constructed estimation methodologies for stable models under Bayesian framework. However, in practice, direct maximization of the likelihood function still involves many challenges and as a consequence it has not emerged as a popular approach for estimation of the stablity index. Brcich et al. (2005) proposed a goodness-of-fit hypothesis test for symmetric $\alpha$- stable distributions. Additionally, the Hill estimator can also be adapted to estimate $\alpha$ but it exhibits large bias in small to moderately sized samples.

Lack of a closed-form density function together with the non-existence of moments has made parameter estimation a historically challenging task. While some applications require the estimation of all the four model parameters, in many instances, the parameter of greatest interest is the stability index $\alpha$ determining the tail heaviness of the distribution. The first part of the paper focuses only on the estimation procedure of the parameter $\alpha$ motivated by the works of DuMouchel (1983) and Fan (2006). Based on the empirical likelihood (EL) method, an asymptotic distribution of an estimator (proposed by Fan (2006)) of $\alpha$ is derived. Our proposed methodology performs significantly better than that of Fan (2006) for relatively small sample sizes. In the second part of the paper, a goodness-of-fit test procedure is developed for strictly stable distributions which can be applicable for each $\alpha \in (0, 2)$. The associated hypothesis testing problem is as follows: Suppose $X_1, X_2, \ldots, X_n$ are random variables with an unknown distribution $F$. Now, the problem is to ascertain whether the data is coming from a specified strictly stable distribution or not.

To the best of our knowledge, any nonparametric statistical methodology is yet to be addressed explicitly in the literature for the above mentioned hypothesis testing problem.

Rest of the paper is organized as follows. We start out by recalling an estimator of $\alpha$ proposed by Fan (2006) in Section 2. Analogous to the Wilks' theorem, an asymptotic distribution of the likelihood ratio of the parameter of interest is derived then. In Section 3, based on the random sample $X_1, X_2, \ldots, X_n$ from an unknown distribution $F$, a nonparametric goodness-of-fit test procedure is developed in order to test $H_0 : F \in F_\alpha$ vs. $H_1 : F \notin F_\alpha$ where $F_\alpha = \{$ F : F is a strictly stable distribution having specified stability index $\alpha\}$. Using the sum-preserving property of strictly stable random variables, the measure of departure is developed first and based on U-statistic theory, we propose a test statistic whose asymptotic distribution is formulated exploiting jackknife empirical likelihood (JEL) method. Additionally, adjusted JEL ratio test is also proposed and explored in this context for similar testing of hypothesis problem. In Section 4, we assess the performance of the estimator and the proposed tests under various scenarios by means of a simulation study and the associated salient features are reported. Two real-life data sets have been analysed in Section 5. Finally, we conclude the paper and provide a thorough summary in Section 6.

## 2 Estimation of Stable Parameter

Let us first recall an estimator of $\dfrac{1}{\alpha}$ and its asymptotic distribution proposed by Fan (2006). In this Section, we will also derive an asymptotic distribution of this estimator which will perform significantly better than that of Fan (2006). Suppose $X_1, X_2, \ldots, X_n$ are iid random variables with strictly $\alpha$-stable distribution.

**Theorem 2.1** (Theorem 3 of Feller (2008, Page no. 172))**.** *Let $X_1$, $X_2$ and $X_3$ be i.i.d. r.v.'s with strictly stable distribution $F$ having the index of stability $\alpha$ (or exponent $\alpha$), then*

$$s^{1/\alpha}X_1 + t^{1/\alpha}X_2 \stackrel{d}{=} (s+t)^{1/\alpha}X_3 \qquad \forall s,t>0 \tag{3}$$

*whenever the ratio $s/t$ is rational.*

Note that (3) holds for any particular choices of $s$ and $t$. Taking $s=t=1$, (3) boils down to

$$X_1 + X_2 = 2^{1/\alpha}X_3 \tag{4}$$

From (4), we have

$$\begin{aligned} X_1 + X_2 =& 2^{1/\alpha}X_1 \\ \implies E\left[\ln|X_1+X_2|\right] =& \frac{1}{\alpha}\ln 2 + E\left[\ln|X_1|\right] \\ \implies E\left[\frac{\ln|X_1+X_2| - \ln|X_1|}{\ln 2}\right] =& \frac{1}{\alpha}. \end{aligned} \tag{5}$$

It is to mention here that Zolotarev (1986) showed that logarithm of the absolute value of a stable random variable has finite moments of any order. Let us define a symmetric kernel $h(x_1,x_2)$ as

$$h(x_1,x_2) = \frac{\ln|x_1+x_2| - \frac{1}{2}\left(\ln|x_1| + \ln|x_2|\right)}{\ln 2}. \tag{6}$$

Now based on $h(x_1,x_2)$, Fan (2006) proposed a consistent unbiased estimator of $\frac{1}{\alpha}$ given by

$$\frac{1}{\widehat{\alpha}} = \binom{n}{2}^{-1}\sum\sum_{1\le i<j\le n} h\left(X_i,X_j\right). \tag{7}$$

It is to note here that $\frac{1}{\widehat{\alpha}}$ is an U-statistic of degree 2 and $E(\frac{1}{\widehat{\alpha}}) = \frac{1}{\alpha}$. An application of Theorem 1 of Lee (1990) (page no. 76) yields the following theorem.

**Theorem 2.2.** *Suppose $X_1, X_2, \ldots, X_n$ are i.i.d r.v.'s with distribution $F$. If $E[h^2\left(X_i,X_j\right)|X_1] < \infty$, then*

$$\sqrt{n}\left(\frac{1}{\widehat{\alpha}} - \frac{1}{\alpha}\right) \xrightarrow{d} N\left(0, 4\zeta^2\right)$$

*where $\zeta^2 = Var[E\left\{h\left(X_i,X_j\right)|X_1\right\}]$.*

However, in practice, it is very difficult to find a consistent estimator of $\zeta^2$ in order to apply Theorem 2.2 and hence for calculating $(1-\gamma)$-level confidence interval (CI) for $\frac{1}{\alpha}$. To address this issue, Fan (2006) established the following theorem.

**Theorem 2.3** (Theorem 2.1 of Fan (2006)). *Suppose $X_1, X_2, \ldots, X_n$ are i.i.d r.v.'s from a strictly stable distributed population $F$. Suppose further,*

$$S_1^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2,$$

*where* $Y_i = \dfrac{1}{n-1}\sum_{j\neq i} h(X_i, X_j);\ \ i = 1, 2, \ldots, n$ *and* $\bar{Y} = \dfrac{1}{n}\sum_{i=1}^{n} Y_i$. *Then,*

$$\frac{\sqrt{n}}{2S_1}\left(\frac{1}{\widehat{\alpha}} - \frac{1}{\alpha}\right) \xrightarrow{d} Z, \ \ as\ n \to \infty$$

*where $Z$ is a random variable with standard normal distribution.*

In this Section, an asymptotic distribution of $\dfrac{1}{\widehat{\alpha}}$ is developed in order to calculate $(1-\gamma)$-level CI for $\dfrac{1}{\alpha}$ using JEL method and adjusted JEL method. By means of a simulation study, our aim is to show that the proposed JEL and adjusted JEL method will perform significantly better than the method proposed by Fan (2006) in terms of coverage probabilities (CPs).

The empirical likelihood (EL) technique, a nonparametric approach, offers broad utility due to its robustness and the flexibility similar to parametric methods. Its application in survival analysis was pioneered by Thomas and Grunkemeier (1975), enhancing confidence intervals for Kaplan-Meier estimators. For a deeper understanding, seminal works by Owen (1988, 1990, 1991, 2001) are recommended. EL exhibits an appealing property: its ratio test statistic follows a chi-squared distribution asymptotically, like in parametric settings. However, its computational advantage diminishes when applied to nonlinear statistics like U-statistics, as solving simultaneous nonlinear equations becomes challenging (Jing et al. (2009)). Exploiting the asymptotic independence of jackknife pseudo-values, Jing et al. (2009) introduced the JEL method, combining jackknife and Owen's EL approaches. This method, straightforwardly proposed, is accompanied by the development of Wilks' theorem (Jing et al. (2009)). Recently, this approach has gained popularity among researchers, as evidenced by Liu et al. (2015), Zhao et al. (2015), Zardasht (2021), Liu et al. (2023), and Khan (2023). In this paper, we adapt the method of Jing et al. (2009) for comparative analysis.

Let $X_1, X_2, \ldots, X_n$ be a random sample from an unknown distribution $F$. Note that $\dfrac{1}{\widehat{\alpha}}$ given in (7) is an U-statistic of degree 2. Now (7) can be expressed as

$$\begin{aligned}\frac{1}{\widehat{\alpha}} =& \binom{n}{2}^{-1}\sum\sum_{1\leq i<j\leq n} h\left(X_i, X_j\right)\\ :=& T_n(X_1, X_2, \ldots, X_n). \end{aligned} \tag{8}$$

Thus, the associated jackknife pseudo-values are given by

$$\widehat{V}_i = nT_n - (n-1)T_{(n-1)}^{(-i)}, \tag{9}$$

where $T_{(n-1)}^{(-i)}$ is the computed value of $T_n$ based on the observations $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n$. Hence, it can be easily shown that $T_n = \dfrac{1}{n}\sum_i^n \widehat{V}_i$. In general, $V_i$'s are dependent r.v.'s, but asymptotically independent under mild conditions (see Shi (1984)). Now, the JEL method exploiting Owen's EL approach is as follows.

## 2.1 Jackknife empirical likelihood method

Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ be a probability vector, assigning probability $p_i$ to $\widehat{V}_i$. JEL, evaluated at $\dfrac{1}{\alpha}$, becomes

$$L(\frac{1}{\alpha}) = \max\left\{\prod_{i=1}^n p_i : \sum_{i=1}^n p_i = 1, \sum_{i=1}^n p_i \widehat{V}_i = \frac{1}{\alpha}\right\}. \tag{10}$$

By the standard Lagrange multiplier argument, the above maximization is achieved at

$$p_i = \frac{1}{n}\frac{1}{1+\lambda(\widehat{V}_i - \frac{1}{\alpha})}$$

where $\lambda = \lambda\Big(\dfrac{1}{\alpha}\Big)$ satisfies

$$\frac{1}{n}\sum_{i=1}^n \frac{\widehat{V}_i - \frac{1}{\alpha}}{1+\lambda(\widehat{V}_i - \frac{1}{\alpha})} = 0. \tag{11}$$

The resulting log empirical likelihood ratio is given as

$$\log R\Big(\frac{1}{\alpha}\Big) = -\sum_{i=1}^n \log\left[1+\lambda\Big(\widehat{V}_i - \frac{1}{\alpha}\Big)\right]. \tag{12}$$

Following Theorem 2 of Jing et al. (2009), we have the following theorem which is an analogue of Wilks' theorem.

**Theorem 2.4.** *Assume that*

1. $E\left[h^2(X_1, X_2)\right] < \infty$, *and*

2. $\sigma^2 = Var[E\left\{h\left(X_i, X_j\right)|X_1\right\}] > 0$.

*Then, we have,* $-2\log(R(\frac{1}{\alpha}))$ *converges in distribution to* $\chi^2$ *random variable with one degree of freedom.*

In view of Theorem 2.4, a CI with level $\gamma$ for $\frac{1}{\alpha}$ can be constructed as

$$R_{1/\alpha} = \left\{\frac{1}{\alpha} : -2\ln\left(R\Big(\frac{1}{\alpha}\Big)\right) \leq c\right\}$$

where $c$ is chosen to satisfy $P(\chi_1^2 \leq c) = 1-\gamma$.

## 2.2 Adjusted jackknife empirical likelihood method

When the sample size is small, the convex hull of $\widehat{V}_i - E\widehat{V}_i$ may not contain 0. To remove this difficulty, Chen and Ning (2016) combined the idea of the jackknife and adjusted empirical likelihood and proposed adjusted jackknife empirical likelihood (AJEL) method. Moreover, it can be observed that AJEL method usually gives better CPs than that of JEL. In AJEL method, Chen and Ning (2016) suggested to add one more jackknife pseudo-value with the jackknife pseudo-values given in (9) defined as

$$\widehat{V}_{n+1} = -\frac{a_n}{n}\sum_{i=1}^{n}\widehat{V}_i.$$

The most popular choice of $a_n$ is $\max\{1, \log(n)/2\}$ proposed by Chen et al. (2008) in the context of adjusted EL method and applying EL method to these $n+1$ jackknife pseudo-values, the AJEL ratio at $1/\alpha$ is defined as follows,

$$R^*(1/\alpha) = \max\left\{\prod_{i=1}^{n+1}((n+1)p_i) : \sum_{i=1}^{n+1}p_i = 1, \sum_{i=1}^{n+1}p_i\left(\widehat{V}_i - E\widehat{V}_i\right) = 0\right\}. \tag{13}$$

The adjusted jackknife empirical log-likelihood ratio is

$$\log R^*(1/\alpha) = -\sum_{i=1}^{n+1}\log\left[1+\lambda(\widehat{V}_i - E\widehat{V}_i)\right], \tag{14}$$

where the Lagrange multiplier $\lambda = \lambda(1/\alpha)$ satisfies

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{\widehat{V}_i - E\widehat{V}_i}{1+\lambda(\widehat{V}_i - E\widehat{V}_i)} = 0. \tag{15}$$

With the same conditions given by Jing et al. (2009), Wilks' theorem of the AJEL has been established by Chen and Ning (2016). Thus, as a special case, the following theorem holds for the above AJEL ratio.

**Theorem 2.5.** *Under the assumptions of Theorem 2.4, $-2\log(R^*(1/\alpha))$ converges in distribution to a $\chi^2$ random variable with one degree of freedom.*

Thus, in view of Theorem 2.5, a CI with level $\gamma$ for $\dfrac{1}{\alpha}$ can be constructed as

$$R^*_{1/\alpha} = \left\{ \frac{1}{\alpha} : -2\ln\left( R^*(\frac{1}{\alpha}) \right) \leq c \right\}$$

where $c$ is chosen to satisfy $P(\chi^2_1 \leq c) = 1 - \gamma$.

**Remark 2.1.** *From (7), we get a consistent estimator of $\alpha$ given by*

$$\widehat{\alpha} = \frac{\binom{n}{2}}{\sum\sum_{1\leq i<j\leq n} h\left(X_i, X_j\right)}. \tag{16}$$

*An application of Jensen inequality and the fact that $g(x) = 1/x$ is a convex function on $(0, 2]$ yields $E(\widehat{\alpha}) \geq \alpha$. Thus, $\widehat{\alpha}$ is a biased estimator for $\alpha$. Applying Delta method on Theorem 2.2, asymptotic normality of $\widehat{\alpha}$ can be obtained.*

# 3 Goodness-of-fit tests for the $\alpha$-stable distributions

The goodness-of-fit tests are used to determine whether a given sample comes from a population with a defined theoretical distribution. It is very interesting to develop another goodness-of-fit test procedure for strictly stable distributions which can be applicable for each $\alpha \in (0, 2)$. This test enables us to detect heavy tail behaviour of a given random sample. In this Section, we will proceed to develop a goodness-of-fit test procedure for a specified strictly $\alpha$-stable distribution. Let $F_\alpha = \{F : F \text{ is a stable distribution having specified stability index } \alpha\}$. Based on a random sample $X_1, X_2, \ldots, X_n$ from an unknown distribution $F$, we are interested in the following hypothesis testing problem:

$$H_0 : F \in F_\alpha \quad \text{vs} \quad H_1 : F \notin F_\alpha.$$

In order to develop our test, we use the sum-preserving property of stable random variables given in (4). From (4), we get a measure of departure given as follows:

$$\begin{aligned}
\Delta_\alpha(F) =& \int_{-\infty}^{\infty} \left[ P\left( \frac{X_1 + X_2}{2^{1/\alpha}} \leq t \right) - P(X_3 \leq t) \right] dF(t) \\
=& \int_{-\infty}^{\infty} P\left( \frac{X_1 + X_2}{2^{1/\alpha}} \leq t \right) dF(t) - \int_{-\infty}^{\infty} F(t) dF(t) \qquad (17)\\
=& \int_{-\infty}^{\infty} P\left( \frac{X_1 + X_2}{2^{1/\alpha}} \leq t | X_3 = t \right) dF(t) - \frac{1}{2} \\
=& P\left( \frac{X_1 + X_2}{2^{1/\alpha}} \leq X_3 \right) - \frac{1}{2}. \qquad (18)
\end{aligned}$$

Now, let us define a a kernel as

$$f(x_1, x_2, x_3) = I\left( \frac{x_1 + x_2}{2^{1/\alpha}} \leq x_3 \right) - \frac{1}{2} \tag{19}$$

where $I(.)$ is an indicator function. Note that $E(f(X_1, X_2, X_3)) = \Delta_\alpha(F)$. A symmetric kernel based on the kernel $f(x_1, x_2, x_3)$ is given by

$$\tau(x_1, x_2, x_3) = \frac{1}{3}\left[I\left(\frac{x_1+x_2}{2^{1/\alpha}} \leq x_3\right) + I\left(\frac{x_1+x_3}{2^{1/\alpha}} \leq x_2\right) + I\left(\frac{x_2+x_3}{2^{1/\alpha}} \leq x_1\right)\right] - \frac{1}{2}. \quad (20)$$

Based on a random sample $X_1, X_2, \ldots, X_n$ from $F$, an unbiased consistent estimator of $\Delta_\alpha(F)$ is given by

$$\widehat{\Delta}_\alpha(F) = \binom{n}{3}^{-1} \sum\sum\sum_{1\leq i<j<k\leq n} \tau\left(X_i, X_j, X_k\right). \quad (21)$$

Note that $\widehat{\Delta}_\alpha(F)$ is an U-statistic of degree 3 and $E(\widehat{\Delta}_\alpha(F)) = 0$. An application of Theorem 1 of Lee (1990, page no. 76) yields the following theorem.

**Theorem 3.1.** *Let $X_1, X_2, \ldots X_n$ be i.i.d. r.v.'s with distribution $F$. If $E[\tau^2\left(X_i, X_j, X_k\right)|X_1] < \infty$, then*

$$\sqrt{n}\left(\widehat{\Delta}_\alpha(F) - \Delta_\alpha(F)\right) \xrightarrow{d} N\left(0, 9\sigma^2\right)$$

*where $\sigma^2 = Var[E\left\{\tau\left(X_i, X_j, X_k\right)|X_1\right\}]$.*

Thus, in view of Theorem 3.1, one can reject the null hypothesis $H_0$ in favour of the alternative $H_1$ if $\dfrac{\sqrt{n}|\widehat{\Delta}_\alpha(F)|}{3\widehat{\sigma}} \geq Z_{\gamma/2}$ where $Z_{\gamma/2}$ is the upper $\gamma$-th quantile point of the standard normal distribution and $\widehat{\sigma}$ is a consistent estimator of $\sigma$. But in practice, it is difficult to find a consistent estimator of $\sigma$. Consequently, implementation of the test based on the above mentioned criteria is far away from simple. This motivate us to develop an nonparametric empirical likelihood-based test.

Let $X_1, X_2, \ldots, X_n$ be a random sample from an unknown distribution $F$. Now (21) can be expressed as

$$\begin{aligned}\widehat{\Delta}_\alpha(F) &= \binom{n}{3}^{-1} \sum\sum\sum_{1\leq i<j<k\leq n} \tau\left(X_i, X_j, X_k\right) \\ &:= U_n(X_1, X_2, \ldots, X_n). \end{aligned} \quad (22)$$

The jackknife pseudo-values are given by

$$\widehat{Z}_i = nU_n - (n-1)U_{(n-1)}^{(-i)}, \quad (23)$$

where $U_{(n-1)}^{(-i)}$ is the computed value of $U_n$ based on the observations $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n$. Next, we discuss the JEL and AJEL based test.

## 3.1 Jackknife empirical likelihood test

In this Subsection, we develop jackknife empirical likelihood (JEL) test. We omit here the methodology and proof since the details of JEL method is given in Subsection 2.1. The following theorem shows that the asymptotic distribution of the EL ratio test statistic follows a chi-squared distribution.

**Theorem 3.2.** *Assume that*

1. $E\left[h^2(X_1, X_2, X_3)\right] < \infty$, *and*

2. $\sigma^2 = Var[E\{h(X_i, X_j, X_k)\,|X_1\}] > 0.$

*Then, under $H_0$, $-2\log(R(\Delta))$ converges in distribution to a $\chi^2$ random variable with one degree of freedom where the log empirical likelihood ratio is obtained as*

$$\log R(\Delta) = -\sum_{i=1}^{n} \log\left[1 + \lambda \widehat{Z}_i\right] \tag{24}$$

*by plugging in the solution of $\lambda = \lambda(\Delta)$ satisfying*

$$\frac{1}{n}\sum_{i=1}^{n} \frac{\widehat{Z}_i}{1 + \lambda \widehat{Z}_i} = 0. \tag{25}$$

□

Thus, in view of Theorem 3.2, we reject the null hypothesis $H_0$ against the alternative $H_1$ at a level of significance $\gamma$ if

$$U_{JEL} := -2\log(R(\Delta)) > \chi^2_{1,\gamma}$$

where $\chi^2_{1,\gamma}$ is the $(1-\gamma)-$ th quantile of the $\chi^2$ distribution with one degree of freedom.

## 3.2 Adjusted jackknife empirical likelihood test

The adjusted jackknife empirical likelihood (AJEL) test is proposed following the methodology given in the subsection 2.2. For the sake of brevity, we omit the details of AJEL method. Now, by adding one more jackknife pseudo-value with $\widehat{Z}_i$'s, the considered jackknife pseudo-values for AJEL ratio test are given by

$$\widehat{W}_i = \begin{cases} \widehat{Z}_i & \text{for } i = 1, 2, \ldots, n \\ -\max(1, \frac{1}{2}\log(n))\frac{1}{n}\sum_{i=1}^{n} \widehat{Z}_i & \text{for } i = n+1 \end{cases} \tag{26}$$

The following theorem also establishes the fact that the asymptotic distribution of the empirical likelihood ratio test statistic follows a chi-squared distribution which is an analogue of Wilks' theorem.

**Theorem 3.3.** *Following the assumptions of Theorem 3.2, under $H_0$, $-2\log(R^*(\Delta))$ converges in distribution to $\chi^2$ random variable with one degree of freedom where the log empirical likelihood ratio is obtained as*

$$\log R^*(\Delta) = -\sum_{i=1}^{n+1} \log\left[1 + \lambda \widehat{W}_i\right] \tag{27}$$

*by plugging in the solution of $\lambda = \lambda(\Delta)$ that satisfies*

$$\frac{1}{n+1}\sum_{i=1}^{n+1} \frac{\widehat{W}_i}{1 + \lambda \widehat{W}_i} = 0. \tag{28}$$

Thus in view of Theorem 3.3, we reject the null hypothesis $H_0$ against the alternative $H_1$ at a level of significance $\gamma$ if $U_{AJEL} := -2\log(R^*(\Delta)) > \chi^2_{1,\gamma}$ where $\chi^2_{1,\gamma}$ is the $1-\gamma$ quantile of the $\chi^2$ distribution with one degree of freedom.

# 4 Simulation study

In this Section, Monte Carlo simulation studies have been carried out in order to assess the finite sample performance of the estimator $\frac{1}{\widehat{\alpha}}$ and the proposed tests ($U_{JEL}$ and $U_{AJEL}$). All the simulation experiments are performed in R statistical software (R Core Team (2021)) on PC platform. In particular, the CP of the estimator $\frac{1}{\widehat{\alpha}}$, empirical Type-I error and power of the proposed tests based on $U_{JEL}$ and $U_{AJEL}$ are calculated using the "emplik" package of Zhou (2020).

To perform the simulation study associated with the performance of the the estimator of the stable index defined by (7), we generate samples from $\alpha$- stable distributions using the "stabledist" package of Wuertz et al. (2016) following the method of Chambers et al. (1976). We have taken the values of $\alpha$ as $\alpha$ = 0.1, 0.3, 0.5, 0.7, 1.0, 1.3, 1.6 and 1.9. For each of the values of $\alpha$, we generate samples of size $n = 20, 30, 40$ from the stable distribution with $\beta = 0, \sigma = 1$ and $\mu = 0$. The performance of the estimator is investigated in terms of average estimates(AEs), root mean squared errors (RMSEs), coverage probabilities (CPs) and average lengths(ALs) of $\theta$ based on 2,000 replications. The results are reported in Table 1. The computed CPs and ALs are based on significance level $\gamma = 0.05$. The proposed tests based on JEL and AJEL methods generally outperform the method proposed by Fan (2006) in terms of higher CP but at a cost of longer AL. When it comes in terms of CP, the AJEL performs marginally better than JEL in case $\alpha \geq 0.5$. In terms of AL, the JEL generally performs better than the AJEL. As expected, CPs have improved for both the methods when sample size increases. Moreover, it seems that better CPs are achieved by AJEL at a cost of having longer confidence intervals than JEL when $\alpha \geq 0.5$.

Table 1: Estimated values for different choices of $\alpha$. Results are reported based on 2000 replications for each method.

| $\alpha$ | $\frac{1}{\alpha}$ | $n$ | AE $(1/\widehat{\alpha})$ | RMSE $(1/\widehat{\alpha})$ | CP and AL $(\frac{1}{\widehat{\alpha}})$ by Fan (2006) | | CP and AL $(\frac{1}{\widehat{\alpha}})$ by JEL | | CP and AL $(\frac{1}{\widehat{\alpha}})$ by AJEL | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | CP | AL | CP | AL | CP | AL |
| 0.1 | 10.0000 | 20 | 10.045 | 2.0695 | 0.8245 | 6.1177 | 0.8965 | 7.6833 | 0.8895 | 9.4576 |
| | | 30 | 10.006 | 1.6653 | 0.8410 | 5.0976 | 0.9115 | 6.4903 | 0.9095 | 7.7099 |
| | | 40 | 9.9793 | 1.4646 | 0.8575 | 4.4890 | 0.9060 | 5.6272 | 0.9020 | 6.5397 |
| 0.3 | 3.3333 | 20 | 3.3203 | 0.6888 | 0.8325 | 2.1117 | 0.9015 | 2.6665 | 0.8965 | 3.2084 |
| | | 30 | 3.3350 | 0.5844 | 0.8575 | 1.7741 | 0.9035 | 2.2354 | 0.9100 | 2.6164 |
| | | 40 | 3.3252 | 0.4964 | 0.8665 | 1.5341 | 0.9200 | 1.9453 | 0.9159 | 2.2304 |
| 0.5 | 2.0000 | 20 | 1.9998 | 0.4734 | 0.8425 | 1.4075 | 0.8940 | 1.7725 | 0.9010 | 2.0446 |
| | | 30 | 1.9948 | 0.3769 | 0.8685 | 1.1473 | 0.9115 | 1.4532 | 0.9185 | 1.6476 |
| | | 40 | 1.9901 | 0.3232 | 0.8740 | 1.0125 | 0.9135 | 1.2632 | 0.9185 | 1.4096 |
| 0.7 | 1.4286 | 20 | 1.4362 | 0.3676 | 0.8620 | 1.1365 | 0.9140 | 1.4367 | 0.9225 | 1.5859 |
| | | 30 | 1.4375 | 0.2977 | 0.8645 | 0.9250 | 0.9250 | 1.1792 | 0.9340 | 1.2892 |
| | | 40 | 1.4310 | 0.2528 | 0.8790 | 0.8077 | 0.9285 | 1.0125 | 0.9410 | 1.0969 |
| 1.0 | 1.0000 | 20 | 1.0020 | 0.3213 | 0.8555 | 0.9702 | 0.9150 | 1.2221 | 0.9330 | 1.2822 |
| | | 30 | 0.9996 | 0.2414 | 0.8720 | 0.7865 | 0.9480 | 0.9840 | 0.9550 | 1.0318 |
| | | 40 | 1.0009 | 0.2176 | 0.8745 | 0.6803 | 0.9300 | 0.8523 | 0.9480 | 0.8901 |
| 1.3 | 0.7692 | 20 | 0.7662 | 0.2846 | 0.8850 | 0.9098 | 0.9300 | 1.1585 | 0.9425 | 1.1730 |
| | | 30 | 0.7709 | 0.2267 | 0.8675 | 0.7259 | 0.9405 | 0.9189 | 0.9595 | 0.9377 |
| | | 40 | 0.7617 | 0.1944 | 0.8860 | 0.6254 | 0.9335 | 0.7844 | 0.9500 | 0.8002 |
| 1.6 | 0.6250 | 20 | 0.6269 | 0.2746 | 0.8705 | 0.8769 | 0.9340 | 1.1183 | 0.9585 | 1.1147 |
| | | 30 | 0.6294 | 0.2173 | 0.8895 | 0.7031 | 0.9275 | 0.8792 | 0.9455 | 0.8841 |
| | | 40 | 0.6299 | 0.1787 | 0.8750 | 0.6026 | 0.9435 | 0.7601 | 0.9565 | 0.7657 |
| 1.9 | 0.5263 | 20 | 0.5263 | 0.2665 | 0.8860 | 0.8518 | 0.9320 | 1.0916 | 0.9475 | 1.0770 |
| | | 30 | 0.5286 | 0.2069 | 0.8865 | 0.6846 | 0.9395 | 0.8599 | 0.9570 | 0.8563 |
| | | 40 | 0.5249 | 0.1815 | 0.8790 | 0.5795 | 0.9365 | 0.7371 | 0.9525 | 0.7364 |

In order to obtain empirical Type-I error, we resort to the method of Chambers et al. (1976)

to generate samples from corresponding stable distributions using the "stabledist" package of Wuertz et al. (2016). The values of $\alpha$ are taken as $\alpha = 0.1$, 0.3, 0.5, 0.7, 1.0, 1.3, 1.6 and 1.9. For each value of $\alpha$, we generate samples of size $n = 20, 40$ from the stable distribution with $\beta = 0, \sigma = 1$ and $\mu = 0$. We will denote the method proposed by Fan (2006) as $U_{FAN}$. The empirical Type-I error of the tests based on $U_{JEL}$, $U_{AJEL}$ and $U_{FAN}$ are calculated based on 10,000 replications. We compute the proportion of times the null hypothesis is rejected for 5% and 1% levels of significance. The results are reported in Table 2 indicating that the empirical Type-I error approaches the specified level of significance when the sample size increases. It is to observe here that usually $U_{AJEL}$ performs marginally better than $U_{JEL}$ and $U_{JEL}$ performs marginally better than $U_{FAN}$.

Next, we evaluate the empirical power of the proposed test based on $U_{JEL}$, $U_{AJEL}$ and $U_{FAN}$ by considering the alternatives from the following distributions:

1. *Weibull distribution*:
   PDF: $f(x, a, k) = \frac{k}{a^k} x^{k-1} e^{-(\frac{x}{a})^k}$, $x \geq 0$, $a > 0$, $k > 0$ and is denoted by Weibull$(a, k)$,

2. *Log-normal distribution*:
   PDF : $f(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{\log(x)-\mu}{\sigma}\right)^2}$, $x \geq 0$, $\mu \in \mathbb{R}$, $\sigma > 0$ and is denoted by Log-normal$(\mu, \sigma)$,

3. *Chi-square distribution*:
   PDF: $f(x, k) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}$, $x \geq 0$, $k \in \mathbb{N}$ and denoted by $\chi^2(k)$.

Table 2: The empirical Type-I error for different choices of $\alpha$.

| $\alpha$ | $n=20$ | | | | | | $n=40$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $U_{JEL}$ | | $U_{AJEL}$ | | $U_{FAN}$ | | $U_{JEL}$ | | $U_{AJEL}$ | | $U_{FAN}$ | |
| | $\gamma=.05$ | $\gamma=.01$ | $\gamma=.05$ | $\gamma=.01$ | $\gamma=.05$ | $\gamma=.01$ | $\gamma=.05$ | $\gamma=.01$ | $\gamma=.05$ | $\gamma=.01$ | $\gamma=.05$ | $\gamma=.01$ |
| 0.1 | 0.0755 | 0.0343 | 0.0578 | 0.0213 | 0.1259 | 0.0670 | 0.0496 | 0.014 | 0.0394 | 0.0117 | 0.0934 | 0.0435 |
| 0.3 | 0.0806 | 0.0364 | 0.0628 | 0.0254 | 0.1089 | 0.0540 | 0.0521 | 0.0169 | 0.0435 | 0.0126 | 0.0903 | 0.0402 |
| 0.5 | 0.0874 | 0.0420 | 0.0679 | 0.0299 | 0.1031 | 0.0487 | 0.0533 | 0.0194 | 0.0453 | 0.0164 | 0.0814 | 0.0330 |
| 0.7 | 0.0998 | 0.0517 | 0.0804 | 0.0349 | 0.0934 | 0.0411 | 0.0567 | 0.0198 | 0.0481 | 0.0162 | 0.0750 | 0.0280 |
| 1.0 | 0.1161 | 0.0649 | 0.0946 | 0.0471 | 0.0788 | 0.0303 | 0.0657 | 0.0252 | 0.0578 | 0.021 | 0.0666 | 0.0237 |
| 1.3 | 0.1202 | 0.0645 | 0.0979 | 0.0456 | 0.0747 | 0.0266 | 0.0669 | 0.0268 | 0.0583 | 0.0211 | 0.0692 | 0.0244 |
| 1.6 | 0.0913 | 0.0410 | 0.0712 | 0.0290 | 0.0750 | 0.0270 | 0.0619 | 0.0203 | 0.0524 | 0.0153 | 0.0654 | 0.0200 |
| 1.9 | 0.0601 | 0.0213 | 0.0440 | 0.0127 | 0.0735 | 0.0285 | 0.0416 | 0.0087 | 0.0326 | 0.0055 | 0.0687 | 0.0238 |

Table 3: Empirical power of the tests for different choices of $\alpha$.

| $\alpha$ | $n=20$ | | | | | | $n=40$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $U_{JEL}$ | | $U_{AJEL}$ | | $U_{FAN}$ | | $U_{JEL}$ | | $U_{AJEL}$ | | $U_{FAN}$ | |
| | $\gamma=.05$ | $\gamma=.01$ | $\gamma=.05$ | $\gamma=.01$ | $\gamma=.05$ | $\gamma=.01$ | $\gamma=.05$ | $\gamma=.01$ | $\gamma=.05$ | $\gamma=.01$ | $\gamma=.05$ | $\gamma=.01$ |
| | Model: Weibul(1, 2) | | | | | | | | | | | |
| 0.1 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 0.3 | 1.0000 | 0.9993 | 1.0000 | 0.9943 | 1.0000 | 0.9988 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 |
| 0.5 | 0.8386 | 0.5405 | 0.7546 | 0.3704 | 0.8653 | 0.7897 | 0.9991 | 0.9794 | 0.9988 | 0.9646 | 0.9652 | 0.9232 |
| 0.7 | 0.0926 | 0.0269 | 0.0632 | 0.0126 | 0.1459 | 0.0908 | 0.2114 | 0.0627 | 0.1791 | 0.0438 | 0.1036 | 0.0542 |
| 1.0 | 0.7456 | 0.5874 | 0.6949 | 0.4988 | 0.9802 | 0.8777 | 0.9733 | 0.9188 | 0.9655 | 0.8971 | 0.9997 | 0.9894 |
| 1.3 | 1.0000 | 0.9999 | 1.0000 | 0.9983 | 0.9995 | 0.9899 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9998 |
| 1.6 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 1.0000 | 0.9972 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 1.9 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| | Model: Log-normal(0, 1) | | | | | | | | | | | |
| 0.1 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 0.3 | 1.0000 | 0.9831 | 0.9997 | 0.9674 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 0.5 | 0.7552 | 0.4974 | 0.6820 | 0.3936 | 0.9997 | 0.9980 | 0.9993 | 0.9475 | 0.9983 | 0.9120 | 1.0000 | 1.0000 |
| 0.7 | 0.0853 | 0.0282 | 0.0597 | 0.0145 | 0.4450 | 0.3174 | 0.1496 | 0.0439 | 0.1245 | 0.0312 | 0.5975 | 0.4312 |
| 1.0 | 0.8691 | 0.7483 | 0.8330 | 0.6646 | 0.9988 | 0.9740 | 0.9961 | 0.9815 | 0.9948 | 0.9728 | 1.0000 | 0.9999 |
| 1.3 | 1.0000 | 0.9998 | 1.0000 | 0.9993 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 1.6 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 1.9 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| | Model: $\chi^2(2)$ | | | | | | | | | | | |
| 0.1 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 0.3 | 1.0000 | 0.9990 | 1.0000 | 0.9934 | 1.0000 | 0.9992 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 0.5 | 0.8369 | 0.5456 | 0.7553 | 0.3773 | 0.8671 | 0.7922 | 0.9993 | 0.9790 | 0.9986 | 0.9630 | 0.9673 | 0.9260 |
| 0.7 | 0.0933 | 0.0300 | 0.0654 | 0.0150 | 0.1414 | 0.0869 | 0.2013 | 0.0597 | 0.1664 | 0.0440 | 0.1060 | 0.0570 |
| 1.0 | 0.7386 | 0.5804 | 0.6910 | 0.4922 | 0.9799 | 0.8738 | 0.9732 | 0.9194 | 0.9658 | 0.9006 | 0.9996 | 0.9891 |
| 1.3 | 1.0000 | 0.9996 | 0.9999 | 0.9984 | 0.9996 | 0.9903 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9998 |
| 1.6 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 1.9 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

For each alternatives, we have considered the specified values of $\alpha$ as $\alpha =$ as 0.1, 0.3, 0.5, 0.7, 1.0, 1.3, 1.6 and 1.9 in order to apply the proposed test. The considered alternatives are given by Weibull$(1, 2)$, Log-normal$(0, 1)$ and $\chi^2(2)$. Now, we generate samples of size $n = 20, 40$ from every considered alternatives. For each fixed value of $\alpha$, the power of the tests based on $U_{JEL}$, $U_{AJEL}$ and $U_{FAN}$ are calculated based on 10,000 replications. We compute the proportion of counts the null hypothesis is rejected for 5% and 1% levels of significance. The results are reported in Table 3. Table 3 suggests that all the proposed tests have reasonably good power for even relatively small sample sizes except the neighbourhood of $\alpha = 0.7$ in all the considered scenarios. At this juncture it is pertinent to mention that all the proposed tests exhibit good power for relatively large values of $n$, i.e., $n \geq 100$ when the specified value of $\alpha$ is 0.7. Thus one can remark that convergence rate of all the proposed tests is slow in terms of empirical power for relatively small values of $n$ when the specified value of $\alpha$ is 0.7. Moreover, $U_{JEL}$ test has marginally better power for the chosen parameters when compared with the $U_{AJEL}$ test. For Weibull$(1, 2)$ and $\chi^2(2)$, $U_{FAN}$ has better power for $\alpha = 0.3, 0.5, 0.7$ and 1.0 when compared with the $U_{JEL}$ and $U_{AJEL}$ test. For Log-normal$(0, 1)$, $U_{FAN}$ has always better power than $U_{JEL}$ and $U_{AJEL}$. Finally, as expected, the empirical power increases when sample size increases.

## 5 Analysis of Real Life Data Examples

Here, we proceed to analyse two real life data sets using all the proposed methodologies.

**I. German Stock Index Data:**

More often than not, the stock market return distribution exhibits heavy tails. The normal distribution usually does not enjoy these attributes. So, it is quite reasonable to analyse stock market return data using stable distributions. Here, we consider returns of closing prices of the German Stock Index. The DAX (Deutscher Aktienindex (German stock index)) is a stock market index consisting of the 40 major German blue chip companies trading on the Frankfurt Stock Exchange. The data are observed daily from 1st November 2018 to 4th February 2019, excluding weekends and public holidays. We use the daily returns $r_d$ of closing prices of DAX for analysis:

$$r_d = (P_d - P_{d-1}) / P_{d-1}. \tag{29}$$

where $P_d$ is the closing price of the $d-$ th day. The skewness and kurtosis measures give some idea about the shape of the stock return distribution. The calculated coefficients of skewness and kurtosis for the considered data are 0.1506 and 4.5854 respectively. Note that the estimates of higher order moments of returns are sensitive to outliers which frequently affect stock returns, see Harvey and Siddique (2000). Now the $p$-value for the considered data is 0.0360 based on Shapiro-Wilk test and hence the hypotheses of normality is rejected. Moreover, we apply the Jarque-Bera Test (Jarque and Bera (1987)) to check whether the considered data have the skewness and kurtosis matching to a normal distribution. The computed $p$-value is given by 0.0328 indicating the rejection of the hypotheses of normality. These observations naturally motivate one to analyse the considered data through a stable distribution. The estimated value of $\alpha$ and $1/\alpha$ are obtained as 1.2944 and 0.7726 respectively. 95% confidence intervals of $1/\alpha$ are computed based on both JEL and AJEL methods and are given by $(0.4147, 0.9444)$ and $(0.3866, 0.9195)$ respectively. Table 4 presents the computed $p$-values for different choices of $\alpha$ based on JEL method, AJEL method and Fan's method.

Table 4: $p$-values corresponding to different choices of $\alpha$ for the DAX data

| Test | $\alpha$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 0.1 | 0.3 | 0.5 | 0.7 | 1.0 | 1.3 | 1.6 | 1.9 |
| $U_{JEL}$ | 0.1647 | 0.1822 | 0.1455 | 0.1688 | 0.4738 | 0.8200 | 0.9797 | 0.9255 |
| $U_{AJEL}$ | 0.1794 | 0.1974 | 0.1597 | 0.1830 | 0.4878 | 0.8258 | 0.9804 | 0.9279 |
| $U_{FAN}$ | 0.0000 | 0.0000 | 0.0000 | 7.21e-07 | 0.0858 | 0.9800 | 0.2650 | 0.0629 |

Based on both $U_{JEL}$ and $U_{AJEL}$ values in Table 4, one fails to reject the null hypothesis ($H_0$) at 5% significance level for all choices of $\alpha$, suggesting that the considered data is likely sourced from a strictly stable distribution. Additionally, one fails to reject $H_0$ at the 5% significance level for $\alpha = 1.0, 1.3, 1.6,$ and 1.9 based on $U_{FAN}$ values. Now, we model the data using the Cauchy distribution for which $\alpha = 1$ to validate the test results with the Kolmogorov–Smirnov (K-S) test, accompanied by visualization through a histogram with a fitted curve and a P-P plot. The estimated maximum likelihood estimators (MLEs) of the location and scale parameters, assuming the Cauchy distribution as the model, are $-0.00115 \pm 0.00082$ and $0.00456 \pm 0.00075$, respectively. Both the P-P plot and the histogram with the fitted Cauchy distribution, presented in Figure 1, demonstrate a reasonable good fit. Additionally, based on the MLEs, applying the K-S test yields $D = 0.0593$ and a $p$-value of 0.9704. Therefore, the conclusions of both the proposed tests align with the conclusion of the K-S test.

**II. Bitcoin Data:**

Now, we apply the proposed estimation procedure to the log-returns of Bitcoin cryptocurrency.

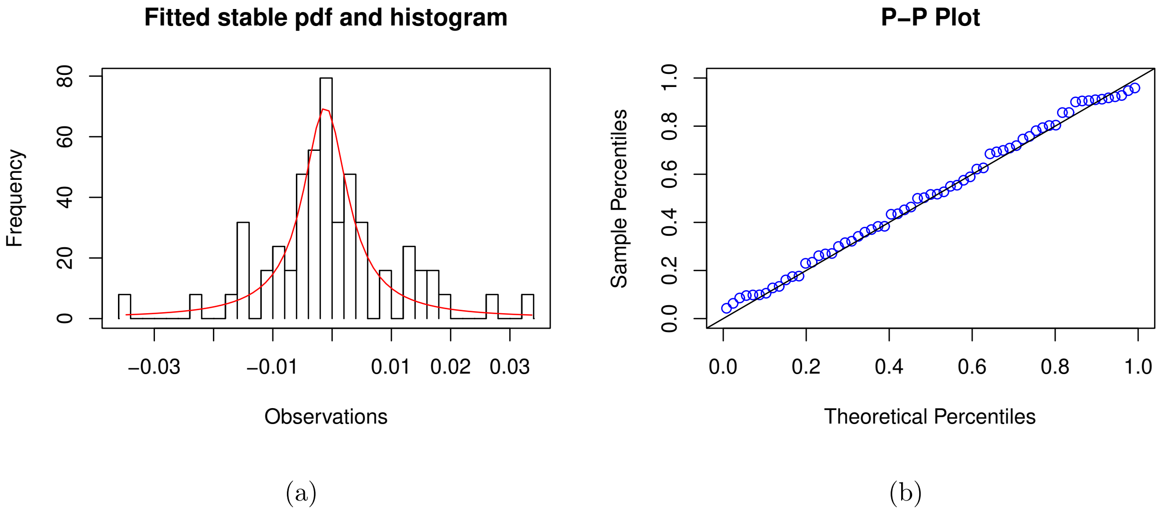



Figure 1: Histogram and P-P plot

Stable distributions are found to be a comparably good model for such data sets, see Wu and Chen (2020). The data analysed in this paper are daily closing prices of Bitcoin in USD from 1st January 2020 to 31st May 2020 which are obtained from CoinMarketCap (https://coinmarketcap.com/). We use the daily returns $r_d$ of closing prices for analysis:

$$r_d = (P_d - P_{d-1}) / P_{d-1}. \tag{30}$$

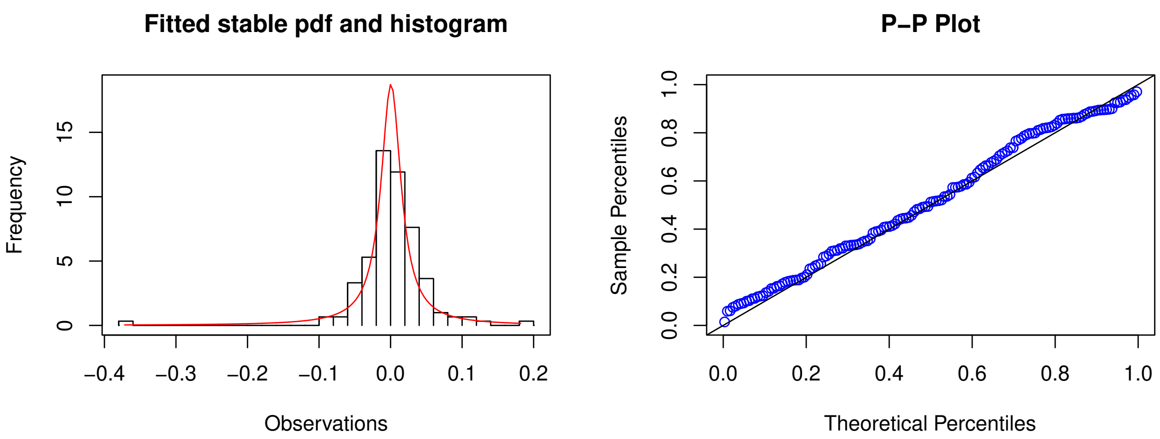
where $P_d$ is the closing price of the $d-$ th day. Based on the considered data, some measures of the shape of the Bitcoin cryptocurrency return distribution are first computed. The measures of skewness and kurtosis are obtained as -2.4984 and 25.9107 respectively. Now, the calculated $p$-values of Shapiro-Wilk test and Jarque-Bera test are given by $5.605e-14$ and $2.2e-16$ respectively. Thus both the tests reject the hypotheses of normality for the considered data. These observations naturally motivate one to analyse the considered data through a stable distribution. The estimated value of $\alpha$ and $1/\alpha$ are 1.2291 and 0.8136 respectively. 95% confidence intervals of $1/\alpha$ are computed based on both JEL and AJEL methods and are given by $(0.6497, 1.0249)$ and $(0.6262, 1.0075)$ respectively. Table 5 presents the computed $p$-values for different choices of $\alpha$ based on JEL method, AJEL method and Fan's method.

Based on the results in Table 5, both $U_{JEL}$ and $U_{AJEL}$ fail to reject the null hypothesis ($H_0$) at the 5% significance level for all choices of $\alpha$, indicating that the considered data follows a strictly stable distribution. Similarly, $U_{FAN}$ fails to reject $H_0$ at the 5% significance level for $\alpha = 1.3$. but rejects $H_0$ marginally at the 5% significance level for $\alpha = 1.0$. In particular, for $\alpha = 1$, while $U_{JEL}$ and $U_{AJEL}$ fail to reject the null hypothesis, $U_{FAN}$ marginally reject $H_0$ at the 5% significance level. For comparative analysis and illustration purpose, we model the

Table 5: $p$-values corresponding to different choices of $\alpha$ for the Bitcoin cryptocurrency data

| Test | $\alpha$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 0.1 | 0.3 | 0.5 | 0.7 | 1.0 | 1.3 | 1.6 | 1.9 |
| $U_{JEL}$ | 0.5740 | 0.7843 | 0.9213 | 0.7084 | 0.2723 | 0.1844 | 0.1715 | 0.1707 |
| $U_{AJEL}$ | 0.5804 | 0.7878 | 0.9226 | 0.7130 | 0.2800 | 0.1915 | 0.1786 | 0.1778 |
| $U_{FAN}$ | 0.0000 | 0.0000 | 0.0000 | 5.5938e-11 | 0.0470 | 0.6365 | 0.0445 | 0.0022 |

data set using the Cauchy distribution and perform a K-S test, accompanied by visualization through a histogram and P-P plot. The calculated MLEs of the location and scale parameters were $0.00058 \pm 0.00202$ and $0.01698 \pm 0.00191$ respectively. Figure 2 displays the histogram of the dataset along with the fitted Cauchy distribution densities. Visually, the Cauchy model appears to be a reasonable approximation. Furthermore, based on the MLEs, the K-S test yield $D = 0.0646$ and $p$-value $= 0.5541$. Consequently, the conclusions of both the $U_{JEL}$ and $U_{AJEL}$ tests align with the K-S test, while $U_{FAN}$ does not agree with the K-S test conclusion.




(a) (b)

Figure 2: Histogram and P-P plot

# 6 Summary

In this paper, we have derived the asymptotic distribution of an estimator of stability index $\alpha$ proposed by Fan (2006) using JEL and AJEL method. It is observed that our proposed methodologies perform better than that of Fan (2006) in terms of coverage probability. Based on the sum-preserving property of stable random variables, we also develop a goodness-of-fit test procedure for strictly $\alpha$-stable distributions exploiting $U$-statistic theory where $\alpha$ is specified. Motivated by the seminal work of Jing et al. (2009), we have derived the asymptotic result of the empirical likelihood ratio test statistic based on JEL method which is analogous to the Wilks' theorem. Moreover, we further extend the asymptotic result of empirical likelihood ratio test statistic for the AJEL method. A simulation study has been carried out to assess the merit of the proposed test procedures. Finally, we have applied all the proposed methodologies to some interesting real data scenarios in the context of daily closing price of German stock index and Bitcoin cryptocurrency. Our proposed methodologies suggest that both the data sets exhibit heavy tailed phenomenon and in both the instances, Cauchy distribution can be adopted as a reasonable model. One possible extension of our work can be to generalize our framework assuming the general JEL method proposed in Liu et al. (2023). Work is in progress and we hope to report our finding in upcoming article.


## Acknowledgements

The authors would like to thank the editor and the anonymous reviewer for their careful reading and constructive comments which have helped to improve the manuscript significantly.


## References


Brcich, R. F., Iskander, D. R., and Zoubir, A. M. (2005). The stability test for symmetric alpha-stable distributions. *IEEE Transactions on Signal Processing*, 53(3):977–986.

Brockwell, P. and Brown, B. (1981). High-efficiency estimation for the positive stable laws. *Journal of the American Statistical Association*, 76(375):626–631.

Buckle, D. (1995). Bayesian inference for stable distributions. *Journal of the American Statistical Association*, 90(430):605–613.

Chambers, J. M., Mallows, C. L., and Stuck, B. (1976). A method for simulating stable random variables. *Journal of the American Statistical Association*, 71(354):340–344.

Chen, J., Variyath, A. M., and Abraham, B. (2008). Adjusted empirical likelihood and its properties. *Journal of Computational and Graphical Statistics*, 17(2):426–443.

Chen, Y.-J. and Ning, W. (2016). Adjusted jackknife empirical likelihood. *arXiv preprint arXiv:1603.04093*.

Crovella, M. E. and Bestavros, A. (1997). Self-similarity in world wide web traffic: Evidence and possible causes. *IEEE/ACM Transactions on networking*, 5(6):835–846.

DuMouchel, W. H. (1973). Stable distributions in statistical inference: 1. symmetric stable distributions compared to other symmetric long-tailed distributions. *Journal of the American Statistical Association*, 68(342):469–477.

DuMouchel, W. H. (1975). Stable distributions in statistical inference: 2. information from stably distributed samples. *Journal of the American Statistical Association*, 70(350):386–393.

DuMouchel, W. H. (1983). Estimating the stable index *alpha* in order to measure tail thickness: A critique. *Annals of Statistics*, 11(4):1019–1031.

Fama, E. F. and Roll, R. (1968). Some properties of symmetric stable distributions. *Journal of the American Statistical Association*, 63(323):817–836.

Fama, E. F. and Roll, R. (1971). Parameter estimates for symmetric stable distributions. *Journal of the American Statistical Association*, 66(334):331–338.

Fan, Z. (2006). Parameter estimation of stable distributions. *Communications in Statistics—Theory and Methods*, 35(2):245–255.

Feller, W. (2008). *An introduction to probability theory and its applications, vol 2.* John Wiley & Sons.

Harvey, C. R. and Siddique, A. (2000). Conditional skewness in asset pricing tests. *The Journal of finance*, 55(3):1263–1295.

Jarque, C. M. and Bera, A. K. (1987). A test for normality of observations and regression residuals. *International Statistical Review/Revue Internationale de Statistique*, pages 163–172.

Jing, B.-Y., Yuan, J., and Zhou, W. (2009). Jackknife empirical likelihood. *Journal of the American Statistical Association*, 104(487):1224–1232.

Khan, R. A. (2023). Two-sample nonparametric test for proportional reversed hazards. *Computational Statistics & Data Analysis*, 182:107708.

Koutrouvelis, I. A. (1980). Regression-type estimation of the parameters of stable laws. *Journal of the American statistical association*, 75(372):918–928.

Latchman, J., Morgan, F., and Aspinall, W. (2008). Temporal changes in the cumulative piecewise gradient of a variant of the gutenberg–richter relationship, and the imminence of extreme events. *Earth-Science Reviews*, 87(3-4):94–112.

Lee, A. (1990). *U-Statistics: Theory and Practice*. Marcel Dekker: New York.

Liu, Y., Wang, S., and Zhou, W. (2023). General jackknife empirical likelihood and its applications. *Statistics and Computing*, 33(3):74.

Liu, Z., Xia, X., and Zhou, W. (2015). A test for equality of two distributions via jackknife empirical likelihood and characteristic functions. *Computational Statistics & Data Analysis*, 92:97–114.

Lombardi, M. J. (2007). Bayesian inference for $\alpha$-stable distributions: A random walk mcmc approach. *Computational statistics & data analysis*, 51(5):2688–2700.

Mandelbrot, B. B. and Hudson, R. L. (2010). *The (mis) behaviour of markets: a fractal view of risk, ruin and reward*. Profile books.

McCulloch, J. H. (1986). Simple consistent estimators of stable distribution parameters. *Communications in Statistics-Simulation and Computation*, 15(4):1109–1136.

Mittnik, S., Doganoglu, T., and Chenyao, D. (1999). Computing the probability density function of the stable paretian distribution. *Mathematical and Computer Modelling*, 29(10-12):235–240.

Nikias, C. L. and Shao, M. (1995). *Signal processing with alpha-stable distributions and applications*. Wiley-Interscience.

Owen, A. (1990). Empirical likelihood ratio confidence regions. *The Annals of Statistics*, 18(1):90–120.

Owen, A. (1991). Empirical likelihood for linear models. *The Annals of Statistics*, pages 1725–1747.

Owen, A. B. (1988). Empirical likelihood ratio confidence intervals for a single functional. *Biometrika*, 75(2):237–249.

Owen, A. B. (2001). *Empirical likelihood*. Chapman and Hall/CRC.

Paulson, A. S., Holcomb, E. W., and Leitch, R. A. (1975). The estimation of the parameters of the stable laws. *Biometrika*, 62(1):163–170.

Peters, G. W., Sisson, S. A., and Fan, Y. (2012). Likelihood-free bayesian inference for $\alpha$-stable models. *Computational Statistics & Data Analysis*, 56(11):3743–3756.

Press, S. J. (1972). Estimation in univariate and multivariate stable distributions. *Journal of the American statistical association*, 67(340):842–846.

R Core Team (2021). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria.

Rachev, S. T. (2003). *Handbook of heavy tailed distributions in finance: Handbooks in finance, Book 1*. Elsevier.

Shi, X. (1984). The approximate independence of jackknife pseudo-values and the bootstrap methods. *Journal of Wuhan Institute Hydra-Electric Engineering*, 2:83–90.

Thomas, D. R. and Grunkemeier, G. L. (1975). Confidence interval estimation of survival probabilities for censored data. *Journal of the American Statistical Association*, 70(352):865–871.

Wu, L. and Chen, S. (2020). Long memory and efficiency of bitcoin under heavy tails. *Applied Economics*, 52(48):5298–5309.

Wuertz, D., Maechler, M., and core team members., R. (2016). *stabledist: Stable Distribution Functions*. R package version 0.7-1.

Zardasht, V. (2021). Jackknife empirical likelihood inference for the variance residual life function. *REVSTAT–Statistical Journal*, 19(1):23–34.

Zhao, Y., Meng, X., and Yang, H. (2015). Jackknife empirical likelihood inference for the mean absolute deviation. *Computational Statistics & Data Analysis*, 91:92–101.

Zhou, M. (2020). *emplik: Empirical Likelihood Ratio for Censored/Truncated Data*. R package version 1.1-1.

Zolotarev, V. M. (1986). *One-dimensional stable distributions*, volume 65. American Mathematical Soc.