Semi-parametric Inference of Stage Life Testing Model

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Abstract

In this article we consider the classical inference of a semi-parametric stage life testing model under time constraint. We assume only two stress levels of a stage life testing experiment and a simple step-stress life testing experiment becomes a special case under this set up. We do not assume any specific parametric form of the lifetime of experimental units, rather we assume a piece wise increasing, decreasing or a constant hazard rate for the experimental units. The Weibull distribution becomes a special case under this model assumption. It is well known that due to its flexibility the Weibull distribution is a widely used lifetime distribution for analyzing time to event data. The proposed semi-parametric model is more flexible than the Weibull model and based on the data obtained from a stage life testing experiment we assume the piece wise increasing or decreasing or a constant hazard rate function. We have obtained the maximum likelihood estimators of the model parameters. Though the model involves significant number of unknown parameters, we just need to solve one or two dimensional optimization problem. Since the small sample properties of the maximum likelihood estimators are difficult to obtain we propose to use asymptotic properties for the construction of the confidence intervals of the unknown parameters. An extensive simulation study has been performed to asses the performance of the propose estimator. One simulated data from stage life testing experiment and one real data from step-stress life testing experiment have been analyzed for illustrative purpose. In real data analysis, it has been observed that, proposed semi-parametric model fits the data better than the Weibull model.

Key Words and Phrases: stage life testing, censoring, maximum likelihood estimates, piece wise hazard, confidence interval.

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1 Introduction

Accelerated life testing (ALT) experiment has been introduced in the reliability experiment to reduce the experimental time of highly reliable products. In an ALT experiment, the experimental units are put into higher stress level (i.e., by increasing load, voltage, temperature etc.) than the normal operating conditions. A particular type of ALT experiment is step-stress life testing experiment, where the stress level has been increased to its higher level after a pre-fixed time or after a pre-fixed number of failures. If there are only two stress levels then the experiment is known as simple step-stress life testing experiment. A significant amount of work can be found in the literature dealing with the different parametric models based on step-stress life testing experiment, see for example Xiong and Milliken [20], Balakrishnan [2], Kateri and Kamps [9], Kundu and Ganguly [10] and the references cited therein.

Censoring is another technique to reduce the experimental time, however all the censoring leads to loss of information. If the experiment is terminated at a pre-fixed time then it is termed as Type-I censoring, whereas, in Type-II censoring the experimenter stops the experiment after a pre-fixed number of failures. The combination of Type-I and Type-II censoring is known as hybrid censoring. In recent times the progressive censoring scheme becomes very popular as it allows to withdraw live experimental units during the experiment. In a progressive censoring scheme, the experimenter may withdraw the units at pre-fixed time during the experiment. An extensive survey on different progressive censoring schemes is available in Balakrishnan and Cramer [3].

Laumen and Cramer [11] first introduced the stage life testing experiment with two stress levels, where the withdrawal units of a progressive censoring scheme are again tested at a higher stress level. Later Laumen and Cramer [12] have extended the stage life testing experiment for multiple stress levels. The stage life testing experiment can also be viewed as a combination of step-stress and progressively censored life testing experiment. An order restricted inference and an optimal design of a stage life testing experiment is developed by Samanta et al. [15]. For more references of stage life testing experiment one is referred to Laumen and Cramer [13] and Samanta and Kundu [17, 18].

The main objective of this paper is to consider a semi-parametric approach for the inference of a stage life testing model with two stress levels when the data are Type-I censored. If we withdraw all the survive units at a prefixed time and put into higher stress level then the simple stage life testing experiment becomes a simple step-stress life testing experiment. Therefore, simple step-stress model is a special case of simple stage life testing experiment. Most of the semi-parametric inference in the literature is based on the assumption on piece-wise constant hazard function, see for example Ibrahim et al. [7], Balakrishnan et al. [4] and see the references cited therein. The main advantage of a semi-parametric model is that we do not need any parametric model assumption on the lifetime of the experimental units. Semi-parametric model assuming piecewise constant hazard function is very flexible model and exponential distribution is a special case under this assumption. Recently Pal et al. [14] developed the semi-parametric step-stress life testing experiment assuming piece-wise constant hazard function. However, the assumption of piecewise constant hazard rate may not hold always. Hence, we propose a more flexible model assumption where the piece-wise hazard rate may be increasing, decreasing or it may be constant depending on the model shape parameter. The exponential or Weibull distribution can be obtained as a special case of the proposed model. The Weibull model is a very popular model to analyze lifetime data due to its flexibility, see for reference Bai and Kim [1], Kateri and Balakrishnan [8] and Ganguly et al. [5]. The proposed semi-parametric model is more flexible than the Weibull lifetime model. In real data analysis, it has been observed that the proposed semi-parametric model fits better than the Weibull model.

The key points in this paper is to consider the maximum likelihood (ML) estimation of the parameters of a semi-parametric stage life testing model based on Type-I censored data. Here we have considered only two stress levels and developed the inference procedures under two different cases depending on the stress changing time point and the termination time of the experiment. Though the proposed model has significant number of model parameters, in one case we just need to solve two dimensional optimization problem and in other case we need to solve one dimensional optimization problem. Since the closed form maximum likelihood estimators do not exists, we propose to use asymptotic confidence intervals for the interval estimation. An extensive simulation study has been performed to check the consistency of the proposed estimator. One simulated data set from the stage life testing experiment and a solar light data set from a simple step-stress life testing experiment have been analyzed to show how the proposed model and the method can be used in practice. We have also proposed a method, as suggested by Samanta and Kundu [19], to select the number of cut points and the position of the cut points in case of a real data analysis.

The rest of the paper is organized as follows. In Section 2, we have provided the model description and classical inference of the model parameters. In section 3, we have provided the detailed experimental results. The analyses of two data sets have been presented in Section 4. In Section 5, we have presented some sensitivity analysis and finally we conclude the paper in Section 6.

2 Model Description and Classical Inference

Consider a stage life testing experiment with two stress levels. Suppose n number of identical experimental units are put into a life testing experiment at stress level S_1. Let D_1 be the number of failures before a pre-fixed time τ at stress level S_1. At the time point τ, out of the $n - D_1$ surviving units, R units are randomly drawn and put on stress level S_2 from stress level S_1. Note that, if $R > n - D_1$, then only $n - D_1$ units will be put at the stress level S_2 from S_1. Therefore, $R^* = min(R, n - D_1)$, denoting the effective number of the censored units at time point τ. Remaining units are continue to test in the same stress level S_1. Therefore, after time τ, two parallel experiment is running under two different stress levels. Both the experiment is terminated at time T. The failure time for each of the failed unit is recorded. The total number of failures at stress level S_i $(i = 1, 2)$ is denoted by n_i. The scheme is illustrated through the schematic diagram in Figure 1.

2.1 Semiparametric Model

In this subsection, we propose formally a semiparametric approach for a stage life testing model under Type-I censoring. It is assumed that n units are put in a stage life testing experiment at the

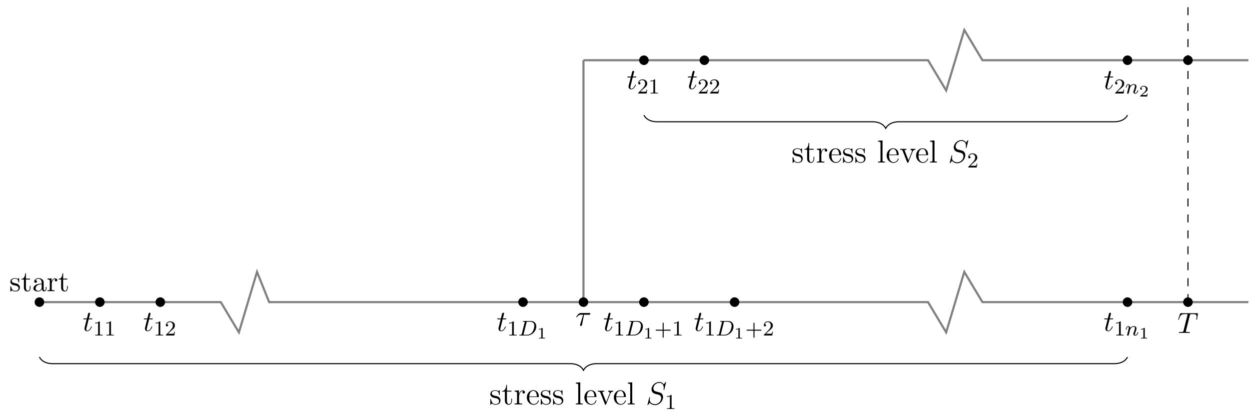

Figure 1: Illustration of stage stress life testing with two stress level and test termination time τ .

initial stress level S_1. At the time point τ, R^* units are put into a higher stress level S_2 and the remaining units continue to run at the initial stress level S_1. Both the experiment is terminated at time T. We consider two possible scenarios depending on the termination time of the experiment. In Case I we consider $T = 2\tau$, while that in Case II T is arbitrary.

2.1.1 Case I : $T = 2\tau$

Here, we propose a semiparametric simple stage life testing model when $T = 2\tau$. It is enough to define the HF till 2τ as the experiment stops at 2τ. Hence, we first propose the following model for the experimental units put in the experiment at time 0 at stress level S_1 and then shifted to stress level S_2 at time τ.

$$h_1(t) = \begin{cases} h_{01}(t;\alpha) & \text{if} \quad 0 < t \leq \tau \\ \beta h_{01}(t-\tau;\alpha) & \text{if} \quad \tau < t \leq 2\tau \\ \beta h_{01}(\tau;\alpha) & \text{if} \quad t > 2\tau. \end{cases} \tag{1}$$

Note that for $t > 2\tau$, it may not be unique. We have just used this for completeness purposes and the analysis is not going to be affected due to this. The corresponding cumulative HF $H_1(t) = \int_0^t h_1(u)du$ is as follows:

$$H_1(t) = \begin{cases} H_{01}(t;\alpha) & \text{if} \quad 0 < t \leq \tau \\ H_{01}(\tau;\alpha) + \beta H_{01}(t-\tau;\alpha) & \text{if} \quad \tau < t \leq 2\tau \\ (1+\beta)H_{01}(\tau;\alpha) + \beta(t^\alpha - (2\tau)^\alpha)h_{01}(\tau;\alpha) & \text{if} \quad t > 2\tau, \end{cases}$$

where $H_{01}(t;\alpha) = \int_0^t h_{01}(u;\alpha)du$.

The corresponding survival function and the probability density function (PDF) become

$$S_1(t) = e^{-H_1(t)} \quad \text{and} \quad f_1(t) = h_1(t)e^{-H_1(t)},$$

respectively. No specific parametric form is assumed for $h_{01}(t)$. Instead, it is assumed that $h_{01}(t;\alpha)$ is defined based on (a) the number of cut points M and (b) the position of the cut points $0 = \tau_0 < \tau_1 < \ldots < \tau_{M-1} < \tau_M = \tau$. It is defined as follows;

$$h_{01}(t) = \sum_{j=1}^{M} \alpha\lambda_{0j}t^{\alpha-1} I_{[\tau_{j-1},\tau_j]}(t),$$

where $\alpha > 0$, $\lambda_{0j} > 0$, and for $j = 1, 2, \ldots, M$,

$$I_{[\tau_{j-1},\tau_j]}(t) = \begin{cases} 1 & \text{if } t \in [\tau_{j-1}, \tau_j] \\ 0 & \text{otherwise.} \end{cases}$$

Note that when $M = 1$, it becomes a Weibull model, and it coincides with the KH model assumptions. Further, when $\alpha = 1$, it becomes the piecewise constant hazard function which has been considered by several authors in the literature.

Now let us compute $S_1(t)$ and $f_1(t)$, which will be needed later. Note that

$$H_{01}(t;\alpha) = \sum_{j=1}^{M} \lambda_{0j}\left\{\min\{t^\alpha, \tau_j^\alpha\} - \tau_{j-1}^\alpha\right\} I_{[\tau_{j-1},\tau]}(t) \quad \text{and} \quad H_{01}(\tau) = \sum_{j=1}^{M} \lambda_{0j}(\tau_j^\alpha - \tau_{j-1}^\alpha).$$

Hence,

$$H_1(t) = \begin{cases} \displaystyle\sum_{j=1}^{M} \lambda_{0j}\left\{\min\{t^\alpha, \tau_j^\alpha\} - \tau_{j-1}^\alpha\right\} I_{[\tau_{j-1},\tau]}(t) & \text{if } \ 0 < t \le \tau \\ \displaystyle\sum_{j=1}^{M} \lambda_{0j}(\tau_j^\alpha - \tau_{j-1}^\alpha) + \beta\sum_{j=1}^{M} \lambda_{0j}\left\{\min\{(t-\tau)^\alpha, \tau_j^\alpha\} - \tau_{j-1}^\alpha\right\} I_{[\tau_{j-1},\tau]}(t-\tau) & \text{if } \ \tau < t \le 2\tau \\ \displaystyle(1+\beta)\sum_{j=1}^{M} \lambda_{0j}(\tau_j^\alpha - \tau_{j-1}^\alpha) + \beta(t^\alpha - (2\tau)^\alpha)\lambda_{0M} & \text{if } \ t > 2\tau. \end{cases}$$

Now let us define the hazard function for the experimental units which run at stress level S_1 even after time τ.

$$h_2(t) = \begin{cases} h_{01}(t;\alpha) & \text{if } \ 0 < t \le \tau \\ h_{02}(t;\alpha) & \text{if } \ \tau < t \le 2\tau \end{cases} \tag{2}$$

The corresponding cumulative HF, $H_2(t) = \displaystyle\int_0^t h_2(u)du$ is as follows:

$$H_2(t) = \begin{cases} H_{01}(t;\alpha) & \text{if } \ 0 < t \le \tau \\ H_{01}(\tau;\alpha) + H_{02}(t;\alpha) & \text{if } \ \tau < t \le 2\tau \end{cases}$$

where $H_{01}(t;\alpha) = \int_0^t h_{01}(u;\alpha)du$ and $H_{02}(t;\alpha) = \int_\tau^t h_{02}(u;\alpha)du$.

The corresponding survival function and the probability density function (PDF) becomes

$$S_2(t) = e^{-H_2(t)} \quad \text{and} \quad f_2(t) = h_2(t)e^{-H_2(t)},$$

and hence, they can be easily obtained.

We have already defined $h_{01}(t;\alpha)$. Similarly, $h_{02}(t;\alpha)$ is defined based on the (a) the number of cut points Q and (b) the position of the cut points $\tau < \tau_{M+1} < \ldots < \tau_{M+Q-1} < \tau_{M+Q} = 2\tau$, as follows;

$$h_{02}(t) = \sum_{j=M+1}^{M+Q} \alpha\lambda_{0j}t^{\alpha-1}I_{[\tau_{j-1},\tau_j]}(t).$$

Hence,

$$H_2(t) = \begin{cases} \sum_{j=1}^{M} \lambda_{0j}\left\{\min\{t^\alpha, \tau_j^\alpha\} - \tau_{j-1}^\alpha\right\} I_{[\tau_{j-1},\tau]}(t) & \text{if} \quad 0 < t \le \tau \\ \sum_{j=1}^{M} \lambda_{0j}(\tau_j^\alpha - \tau_{j-1}^\alpha) + \sum_{j=M+1}^{M+Q} \lambda_{0j}\{\min\{t^\alpha, \tau_j^\alpha\} - \tau_{j-1}^\alpha\}I_{[\tau_{j-1},2\tau]}(t) & \text{if} \quad \tau < t \le 2\tau \end{cases}$$

Now let us look at the log-likelihood contribution for each data point. Suppose the failure time point $t \in [0,\tau] \cap [\tau_{j-1}, \tau_j]$, for $j = 1, \ldots, M$, then the log-likelihood contribution is

$$\begin{aligned} \ln f(t) &= \ln h_1(t) - H_1(t) \\ &= \ln(\alpha\lambda_{0j}t^{\alpha-1}) - \sum_{u=1}^{j-1}\lambda_{0u}(\tau_u^\alpha - \tau_{u-1}^\alpha) - \lambda_{0j}(t^\alpha - \tau_{j-1}^\alpha). \end{aligned}$$

Suppose the failure time point $t \in [\tau, 2\tau] \cap [\tau + \tau_{j-1}, \tau + \tau_j]$ and the failure occur at stress level S_2, for $j = 1, \ldots, M$, then the log-likelihood contribution is

$$\begin{aligned} \ln f(t) &= \ln h_1(t) - H_1(t) \\ &= \ln(\alpha\beta\lambda_{0j}(t-\tau)^{\alpha-1}) - \sum_{u=1}^{M}\lambda_{0u}(\tau_u^\alpha - \tau_{u-1}^\alpha) - \beta\sum_{u=1}^{j-1}\lambda_{0u}(\tau_u^\alpha - \tau_{u-1}^\alpha) - \beta\lambda_{0j}((t-\tau)^\alpha - \tau_{j-1}^\alpha). \end{aligned}$$

Suppose the failure time point $t \in [\tau, 2\tau] \cap [\tau_{j-1}, \tau_j]$ and the failure occur at stress level S_1, for $j = M+1, \ldots, M+Q$, then the log-likelihood contribution is

$$\begin{aligned} \ln f(t) &= \ln h_2(t) - H_2(t) \\ &= \ln(\alpha\lambda_{0j}t^{\alpha-1}) - \sum_{u=1}^{M}\lambda_{0u}(\tau_u^\alpha - \tau_{u-1}^\alpha) - \sum_{u=M+1}^{j-1}\lambda_{0u}(\tau_u^\alpha - \tau_{u-1}^\alpha) - \lambda_{0j}(t^\alpha - \tau_{j-1}^\alpha)]. \end{aligned}$$

Let t_{ijp}, denotes the p-th observation at the i-th stress level, i.e. $i = 1, 2$, and at j-th cut level $[\tau_{j-1}, \tau_j]$, i.e. $j = 1, \dots, M+Q$, if $i = 1$ and $j = 1, \dots, M$, if $i = 2$. Further m_{ij} denotes the number of observations at the i-th stress level and at the j-th cut level ($j = 1, \dots, M+Q$, if $i = 1$ and $j = 1, \dots, M$, if $i = 2$). We will denote the set of all unknown model parameters as $\boldsymbol{\eta} = (\alpha,\, \beta,\, \lambda_{01},\, \lambda_{02}, \dots, \lambda_{0M+Q})^\top$. Hence, the log-likelihood of the observed data becomes

$$l(\boldsymbol{\eta}|Data) = \sum_{i=1}^{2}\sum_{j=1}^{M}\sum_{p=1}^{m_{ij}} \ln f(t_{ijp}) + \sum_{j=M+1}^{M+Q}\sum_{p=1}^{m_{1j}} \ln f(t_{1jp}) + (n-R-n_1)\ln S_1(2\tau) + (R-n_2)\ln(S_2(2\tau)), \tag{3}$$

where,

$$\sum_{j=1}^{M+Q}\sum_{p=1}^{m_{1j}} \ln f(t_{1jp}) = \sum_{j=1}^{M+Q}\sum_{p=1}^{m_{1j}} \left[\ln(\alpha\lambda_{0j} t_{1jp}^{\alpha-1}) - \sum_{u=1}^{j-1} \lambda_{0u}(\tau_u^\alpha - \tau_{u-1}^\alpha) - \lambda_{0j}(t_{1jp}^\alpha - \tau_{j-1}^\alpha) \right]$$

$$\begin{aligned} \sum_{j=1}^{M}\sum_{p=1}^{m_{2j}} \ln f(t_{2jp}) &= \sum_{j=1}^{M}\sum_{p=1}^{m_{2j}} \Bigg[\ln(\alpha\beta\lambda_{0j}(t_{2jp}-\tau)^{\alpha-1}) - \sum_{u=1}^{M} \lambda_{0u}(\tau_u^\alpha - \tau_{u-1}^\alpha) \\ &\quad -\beta \sum_{u=1}^{j-1} \lambda_{0u}(\tau_u^\alpha - \tau_{u-1}^\alpha) - \beta\lambda_{0j}((t_{2jp}-\tau)^\alpha - \tau_{j-1}^\alpha) \Bigg] \end{aligned}$$

$$(n-R-n_1)\ln S_1(2\tau) = -(n-R-n_1)\sum_{u=1}^{M+Q} \lambda_{0u}(\tau_u^\alpha - \tau_{u-1}^\alpha),$$

and

$$(R-n_2)\ln S_2(2\tau) = -(R-n_2)(1+\beta)\sum_{u=1}^{M} \lambda_{0u}(\tau_u^\alpha - \tau_{u-1}^\alpha).$$

Therefore, the log-likelihood function (3) can be written as

$$\begin{aligned} l(\boldsymbol{\eta}|Data) &= (n_1+n_2)\ln\alpha + n_2\ln(\beta) + \sum_{j=1}^{M}(m_{1j}+m_{2j})\ln\lambda_{0j} + \sum_{j=M+1}^{M+Q} m_{1j}\ln\lambda_{0j} \\ &\quad - \sum_{j=1}^{M}\lambda_{0j}\delta_j(\alpha,\beta) - \sum_{j=M+1}^{M+Q}\lambda_{0j}\delta_j(\alpha) - \alpha\sum_{j=1}^{M+Q}\sum_{p=1}^{m_{1j}}\ln(t_{1jp}) - \alpha\sum_{j=1}^{M}\sum_{p=1}^{m_{2j}}\ln(t_{2jp}). \end{aligned} \tag{4}$$

Here, for $j = 1, \ldots, M-1$,

$$\begin{aligned}
\delta_j(\alpha,\beta) &= (\tau_j^\alpha - \tau_{j-1}^\alpha)(\sum_{w=1+j}^{M}(m_{1w}+\beta m_{2w}) + (n-D_1) + (R-n_2)\beta) + \sum_{p=1}^{m_{1j}}(t_{1jp}^\alpha - \tau_{j-1}^\alpha) \\
&\quad + \beta\sum_{p=1}^{m_{2j}}((t_{2jp}-\tau)^\alpha - \tau_{j-1}^\alpha) \qquad (5) \\
\delta_M(\alpha,\beta) &= \sum_{p=1}^{m_{1M}}(t_{1Mp}^\alpha - \tau_{M-1}^\alpha) + ((n-D_1)+(R-n_2)\beta)(\tau_M^\alpha - \tau_{M-1}^\alpha) \\
&\quad + \beta\sum_{p=1}^{m_{2M}}((t_{2Mp}-\tau)^\alpha - \tau_{M-1}^\alpha). \qquad (6)
\end{aligned}$$

For $j = M+1, \ldots, M+Q$,

$$\delta_j(\alpha) = (\tau_j^\alpha - \tau_{j-1}^\alpha)(\sum_{w=1+j}^{M+Q} m_{1w} + (n-R-n_1)) + \sum_{p=1}^{m_{1j}}(t_{1jp}^\alpha - \tau_{j-1}^\alpha). \qquad (7)$$

Hence, for a given α and β, the MLE of λ_{0j}, for $j = 1, \ldots, M$

$$\widehat{\lambda}_{0j}(\alpha,\beta) = \frac{m_{1j}+m_{2j}}{\delta_j(\alpha,\beta)},$$

for $j = M+1, \ldots, M+Q$

$$\widehat{\lambda}_{0j}(\alpha) = \frac{m_{1j}}{\delta_j(\alpha)}.$$

The profile log-likelihood of α and β without the additive constant is given by

$$\begin{aligned}
l_1(\alpha,\beta) &= (n_1+n_2)\ln\alpha + n_2\ln(\beta) - \sum_{j=1}^{M}(m_{1j}+m_{2j})\ln(\delta_j(\alpha,\beta)) - \sum_{j=M+1}^{M+Q} m_{1j}\ln(\delta_j(\alpha)) \\
&\quad - \alpha\sum_{j=1}^{M+Q}\sum_{p=1}^{m_{1j}}\ln(t_{1jp}) - \alpha\sum_{j=1}^{M}\sum_{p=1}^{m_{2j}}\ln(t_{2jp}). \qquad (8)
\end{aligned}$$

The MLEs of α and β can be obtained by solving two dimensional non-linear equations.

$$\begin{aligned}
\frac{n_1+n_2}{\alpha} - \sum_{j=1}^{M}\frac{(m_{1j}+m_{2j})\delta_j^{'(\alpha)}(\alpha,\beta)}{\delta_j(\alpha,\beta)} - \sum_{j=M+1}^{M+Q}\frac{m_{1j}\delta_j^{'(\alpha)}(\alpha)}{\delta_j(\alpha)} &= 0 \\
\frac{n_2}{\beta} - \sum_{j=1}^{M}\frac{(m_{1j}+m_{2j})\delta_j^{'(\beta)}(\alpha,\beta)}{\delta_j(\alpha,\beta)} &= 0, \qquad (9)
\end{aligned}$$

where, for $j = 1, \ldots, M$, $\delta_j^{'(\alpha)}(\alpha,\beta)$ and $\delta_j^{'(\beta)}(\alpha,\beta)$ are first order derivative of $\delta_j(\alpha,\beta)$ with respect to α and β respectively. For $j = M+1, \ldots, M+Q$, $\delta_j^{'(\alpha)}(\alpha)$ is first order derivative of $\delta_j(\alpha)$ with respect to α. One may adopt Newton-Raphson method or any other numerical technique to solve equation 9.

Now we propose to use asymptotic confidence interval for interval estimation since we do not have the closed form of the MLEs and hence it is not possible to construct the exact confidence intervals. Asymptotic confidence interval can be obtained by assuming the asymptotic normality of the MLEs. The asymptotic variance of the MLEs are obtained using observed Fisher information matrix which can be obtained as

$$F \;=\; ((f_{ij})) = ((-\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial\eta_i\partial\eta_j})).$$

The elements of the Fisher information matrix are given in Appendix 7.1. The asymptotic distribution of $\widehat{\boldsymbol{\eta}}$ is given by $(\widehat{\boldsymbol{\eta}} - \boldsymbol{\eta}) \sim N_{M+Q+2}(0, F^{-1})$. Therefore $100(1-\delta)\%$ asymptotic CI of $\boldsymbol{\eta}$ is given by

$$\left[\widehat{\eta}_i \pm z_{1-\frac{\delta}{2}}\sqrt{V_{ii}}\right],$$

where V_{ii} is $(i,i)^{th}$ element of the matrix F^{-1}. We can apply this model even when the termination time of the experiment is less than 2τ.

Case II: T is Arbitrary

In this case, we propose a semiparametric simple stage step-stress model. In this case also, it is enough to define the hazard function till the time point T. First propose the following model for the experimental units put in the experiment at time 0 at stress level S_1 and then shifted to stress level S_2 at time τ. It is defined as follows:

$$h_1(t) = \begin{cases} h_{01}(t) & \text{if} \quad 0 < t \leq \tau \\ h_{11}(t) & \text{if} \quad \tau < t \leq T. \end{cases} \tag{10}$$

As before, it is assumed here that both $h_{01}(t)$ and $h_{11}(t)$ are piecewise increasing, decreasing or constants depending on the shape parameter α. $h_{01}(t)$ and $h_{11}(t)$ are given by

$$h_{01}(t) = \sum_{j=1}^{M} \alpha\lambda_{0j}t^{\alpha-1}I_{[\tau_{j-1},\tau_j]}(t),$$

$$h_{11}(t) = \sum_{j=1}^{S} \alpha\lambda_{1j}t^{\alpha-1}I_{[\gamma_{j-1},\gamma_j]}(t),$$

where $\alpha > 0$, $\lambda_{0j} > 0$, $\lambda_{1j} > 0$, and

$$I_A(t) = \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{otherwise.} \end{cases}$$

Note that here $\tau_M = \tau = \gamma_0 < \gamma_1 < \ldots < \gamma_S = T$. Hence, the cumulative hazard function is given by

$$H_1(t) = \begin{cases} \sum_{j=1}^{M} \lambda_{0j} \left\{ \min\{t^\alpha, \tau_j^\alpha\} - \tau_{j-1}^\alpha \right\} I_{[\tau_{j-1},\tau]}(t) & \text{if } \; 0 < t \leq \tau \\ \sum_{j=1}^{M} \lambda_{0j}(\tau_j^\alpha - \tau_{j-1}^\alpha) + \sum_{j=1}^{S} \lambda_{1j} \left\{ \min\{t^\alpha, \gamma_j^\alpha\} - \gamma_{j-1}^\alpha \right\} I_{[\gamma_{j-1},T]}(t) & \text{if } \; \tau < t \leq T. \end{cases}$$

Now we will define the hazard function for the experimental units put in the experiment at time 0 at stress level S_1 and remains at stress level S_1 even after time τ. It is defined as follows:

$$h_2(t) = \begin{cases} h_{01}(t) & \text{if } \; 0 < t \leq \tau \\ h_{02}(t) & \text{if } \; \tau < t \leq T, \end{cases} , \tag{11}$$

where, $h_{01}(t)$ is already defined above and $h_{02}(t)$ is defined as

$$h_{02}(t) = \sum_{j=M+1}^{M+Q} \alpha \lambda_{0j} t^{\alpha-1} I_{[\tau_{j-1},\tau_j]}(t).$$

Hence, the cumulative hazard function is given by

$$H_2(t) = \begin{cases} \sum_{j=1}^{M} \lambda_{0j} \left\{ \min\{t^\alpha, \tau_j^\alpha\} - \tau_{j-1}^\alpha \right\} I_{[\tau_{j-1},\tau]}(t) & \text{if } \; 0 < t \leq \tau \\ \sum_{j=1}^{M} \lambda_{0j}(\tau_j^\alpha - \tau_{j-1}^\alpha) + \sum_{j=M+1}^{M+Q} \lambda_{0j} \left\{ \min\{t^\alpha, \tau_j^\alpha\} - \tau_{j-1}^\alpha \right\} I_{[\tau_{j-1},T]}(t) & \text{if } \; \tau < t \leq T. \end{cases}$$

Now, suppose the failure time point $t \in [0, T] \cap [\tau_{j-1}, \tau_j]$ and the failure occur at S_1, then for $j = 1, \ldots, M+Q$, then the log-likelihood contribution is

$$\ln f(t) \;=\; \ln(\alpha \lambda_{0j} t^{\alpha-1}) - \sum_{u=1}^{j-1} \lambda_{0u}(\tau_u^\alpha - \tau_{u-1}^\alpha) - \lambda_{0j}(t^\alpha - \tau_{j-1}^\alpha).$$

Suppose the failure time point $t \in [\tau, T] \cap [\gamma_{j-1}, \gamma_j]$ and the failure occur at S_2, for $j = 1, \ldots, S$, then the log-likelihood contribution is

$$\ln f(t) \;=\; \ln(\alpha \lambda_{1j} t^{\alpha-1}) - \sum_{u=1}^{M} \lambda_{0u}(\tau_u^\alpha - \tau_{u-1}^\alpha) - \sum_{u=1}^{j-1} \lambda_{1u}(\gamma_u^\alpha - \gamma_{u-1}^\alpha) - \lambda_{1j}(t^\alpha - \gamma_{j-1}^\alpha).$$

If an observation is censored at time T after passing through stress level S_1 the log-likelihood contribution is

$$\ln S(t) = -\sum_{u=1}^{M+Q} \lambda_{0u}(\tau_u^\alpha - \tau_{u-1}^\alpha).$$

If an observation is censored at time T after passing through stress level S_1 upto time τ and through S_2 between time τ and T then the log-likelihood contribution is

$$\ln S(t) = -\sum_{u=1}^{M} \lambda_{0u}(\tau_u^\alpha - \tau_{u-1}^\alpha) - \sum_{u=1}^{S} \lambda_{1u}(\delta_u^\alpha - \delta_{u-1}^\alpha).$$

Therefore, the log-likelihood function of the observed data can be written as

$$\begin{aligned} l(\boldsymbol{\eta}|Data) &= (n_1+n_2)\ln\alpha + \sum_{j=1}^{M+Q} m_{1j}\ln\lambda_{0j} + \sum_{j=1}^{S} m_{2j}\ln\lambda_{1j} - \sum_{j=1}^{M+Q}\lambda_{0j}\delta_{0j}(\alpha) \\ &\quad - \sum_{j=1}^{S}\lambda_{1j}\delta_{1j}(\alpha) + (\alpha-1)\sum_{j=1}^{M+Q}\sum_{p=1}^{m_{1j}}\ln(t_{1jp}) + (\alpha-1)\sum_{j=1}^{S}\sum_{p=1}^{m_{2j}}\ln(t_{2jp}). \end{aligned} \tag{12}$$

Here, for $j = 1, \ldots, M$,

$$\delta_{0j}(\alpha) = (\tau_j^\alpha - \tau_{j-1}^\alpha)(\sum_{w=1+j}^{M+Q} m_{1w} + (n-n_1)) + \sum_{p=1}^{m_{1j}}(t_{1jp}^\alpha - \tau_{j-1}^\alpha),$$

for $j = M+1, \ldots, M+Q-1$,

$$\delta_{0j}(\alpha) = (\tau_j^\alpha - \tau_{j-1}^\alpha)(\sum_{w=1+j}^{M+Q} m_{1w} + (n-r-n_1)) + \sum_{p=1}^{m_{1j}}(t_{1jp}^\alpha - \tau_{j-1}^\alpha),$$

for $j = M+Q$,

$$\delta_{0j}(\alpha) = (\tau_j^\alpha - \tau_{j-1}^\alpha)(n-r-n_1) + \sum_{p=1}^{m_{1j}}(t_{1jp}^\alpha - \tau_{j-1}^\alpha),$$

for $j = 1, \ldots, S-1$,

$$\delta_{1j}(\alpha) = (\gamma_j^\alpha - \gamma_{j-1}^\alpha)(\sum_{w=1+j}^{S} m_{2w} + (r-n_2)) + \sum_{p=1}^{m_{2j}}(t_{2jp}^\alpha - \gamma_{j-1}^\alpha),$$

and for $j = S$,

$$\delta_{1j}(\alpha) = (\gamma_j^\alpha - \gamma_{j-1}^\alpha)(r-n_2) + \sum_{p=1}^{m_{2j}}(t_{2jp}^\alpha - \gamma_{j-1}^\alpha).$$

Hence, for a given α, the MLE of λ_{0j}, for $j = 1, \ldots, M+Q$ is given by

$$\widehat{\lambda}_{0j}(\alpha) = \frac{m_{1j}}{\delta_{0j}(\alpha)},$$

and for $j = 1, \ldots, S$ the MLE of λ_{1j} for given α is

$$\widehat{\lambda}_{1j}(\alpha) = \frac{m_{2j}}{\delta_{1j}(\alpha)}.$$

The profile log-likelihood of α without the additive constant is given by

$$\begin{aligned} l_2(\alpha) &= (n_1+n_2)\ln\alpha - \sum_{j=1}^{M+Q} m_{1j}\ln(\delta_{0j}(\alpha)) - \sum_{j=1}^{S} m_{2j}\ln(\delta_{1j}(\alpha)) \\ &\quad +(\alpha-1)\sum_{j=1}^{M+Q}\sum_{p=1}^{m_{1j}}\ln(t_{1jp}) + (\alpha-1)\sum_{j=1}^{S}\sum_{p=1}^{m_{2j}}\ln(t_{2jp}). \end{aligned} \tag{13}$$

The MLEs of α say $\widehat{\alpha}$ can be obtained by solving one dimensional non-linear equations.

$$\frac{n_1+n_2}{\alpha} - \sum_{j=1}^{M+Q}\frac{(m_{1j})\delta'_{0j}(\alpha)}{\delta_{0j}(\alpha)} - \sum_{j=1}^{S}\frac{m_{2j}\delta'_{1j}(\alpha)}{\delta_{1j}(\alpha)} + \sum_{j=1}^{M+Q}\sum_{p=1}^{m_{1j}}\ln(t_{1jp}) + \sum_{j=1}^{S}\sum_{p=1}^{m_{2j}}\ln(t_{2jp}) = 0. \tag{14}$$

In this case also one may adopt Newton-Raphson method or any other numerical technique to solve equation 14. Once we have the MLE of α, the MLEs of λ_{0j} and λ_{1j} can be obtained by replacing $\widehat{\alpha}$ in $\widehat{\lambda}_{0j}(\alpha)$ and $\widehat{\lambda}_{1j}(\alpha)$ respectively. Similar to Case I, the asymptotic confidence intervals can be obtained assuming the asymptotic normality of the MLEs. The elements of the Fisher information matrix are given in Appendix 7.1. It has been observed in simulation study that the assumption of asymptotic normality may not hold in some cases. The log-transformation can be used to improve the accuracy of the confidence intervals. Here we assume that $\frac{\ln(\widehat{\eta}_i)-\ln(\eta_i)}{\sqrt{Var(\ln(\widehat{\eta}_i))}}$ follows approximately standard normal distribution. Therefore the asymptotic confidence interval of η_i can be obtained as $\left(\widehat{\eta}_i exp\big(-\frac{z_{1-\frac{\delta}{2}}\sqrt{V_{ii}}}{\widehat{\eta}_i}\big), \widehat{\eta}_i exp\big(\frac{z_{1-\frac{\delta}{2}}\sqrt{V_{ii}}}{\widehat{\eta}_i}\big)\right)$.

3 Simulation Results and Data Analysis

3.1 Simulation Results

We have performed an extensive simulation study to establish the proposed method for both the cases. For Case I simulation study we have taken $M = 2$ and $Q = 2$, i.e., we have two cut points before τ and two cut points between τ and T. The true parameter values are assumed as $\alpha = 1.3$, $\beta = 2$, $\lambda_{01} = 0.10$, $\lambda_{02} = 0.12$, $\lambda_{03} = 0.15$ and $\lambda_{04} = 0.18$. The cut points are taken as $\tau_1 = 1$, $\tau_2 = 2$, $\tau_3 = 3$ and the termination time of the experiment is 4. Simulation results are provided for different sample sizes and R is taken as the 30% and 40% of the sample size. In simulation study, we have provided the average estimates (AEs) of the MLEs and the associated mean squared errors (MSEs). We have also obtained the average lengths (ALs) and coverage percentages (CPs) of 95% asymptotic confidence intervals. Simulation results are based of 2000 replications. Simulation results for Case I are provided in Table 1 and 2.

It has been observed that MSEs decrease as sample size increases. Coverage percentages of the asymptotic confidence intervals are very close to the nominal value and the average lengths are decreases as sample size increases. All the points established the consistency of the proposed estimator.

For Case II simulation study also we have considered $m = 2$, $Q = 2$ and $S = 2$. The true parameter values are taken as $\alpha = 1.3$, $\lambda_{01} = 0.10$, $\lambda_{02} = 0.12$, $\lambda_{03} = 0.15$, $\lambda_{04} = 0.18$ $\lambda_{11} = 0.20$ and $\lambda_{12} = 0.25$. In this case the cut points are taken as $\tau_1 = 1$, $\tau_2 = 2$, $\tau_3 = 3.5$, $\gamma_1 = 3$ and the termination point of the experiment is $T = 5$. Here also the results are provided for different sample sizes and for different values of R. In this case, to construct the asymptotic confidence intervals we have used the logarithmic transformation for λ_{01}, λ_{02}, λ_{03}, λ_{04}, λ_{11} and λ_{12} since the coverage percentages of CIs were not closed to the nominal values without the transformation. We have observed better results in terms of the coverage percentages of the model parameters after considering the logarithmic transformation. The AEs along with the MSEs of the MLEs are provided in Table 3 and the CPs along with the ALs of 95% asymptotic confidence intervals are provided in Table 4. In this case also we have observed all the consistency properties of the proposed estimator.

Table 1: MLEs of the parameters along with the corresponding Mean Square Errors.

		α		β		λ_{01}		λ_{02}		λ_{03}		λ_{04}	
n	$\mathscr{R}$	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
70	21	1.4129	0.1932	2.2265	1.0281	0.1008	0.001268	0.1207	0.004058	0.1630	0.015722	0.2056	0.032788
70	28	1.4135	0.1823	2.1523	0.8357	0.1030	0.001206	0.1198	0.003784	0.1582	0.013580	0.1999	0.030802
100	30	1.3907	0.1364	2.1386	0.6142	0.1003	0.000819	0.1200	0.002991	0.1545	0.009311	0.1939	0.020510
100	40	1.3696	0.1078	2.0861	0.4713	0.1002	0.000801	0.1221	0.002674	0.1567	0.009834	0.1966	0.019913
140	42	1.3482	0.0788	2.0883	0.4183	0.1005	0.000605	0.1220	0.002118	0.1566	0.007022	0.1948	0.013704
140	56	1.3408	0.0744	2.0684	0.3201	0.1007	0.000580	0.1226	0.001909	0.1565	0.006947	0.1968	0.014438
170	51	1.3381	0.0646	2.0838	0.3330	0.0993	0.000497	0.1219	0.001698	0.1574	0.005423	0.1920	0.010476
170	68	1.3351	0.0535	2.0688	0.2756	0.1008	0.000460	0.1213	0.001475	0.1552	0.005214	0.1914	0.011018
200	60	1.3448	0.0543	2.0385	0.2620	0.0999	0.000398	0.1201	0.001400	0.1516	0.004339	0.1881	0.009617
200	80	1.3298	0.0443	2.0788	0.2308	0.0997	0.000409	0.1203	0.001193	0.1544	0.004335	0.1871	0.007985
230	69	1.3380	0.0496	2.0560	0.2228	0.1002	0.000374	0.1204	0.001217	0.1532	0.004018	0.1871	0.007906
230	92	1.3314	0.0414	2.0390	0.1891	0.1005	0.000354	0.1203	0.001136	0.1531	0.003897	0.1867	0.007651
270	81	1.3216	0.0397	2.0508	0.1967	0.1001	0.000313	0.1212	0.001023	0.1553	0.003510	0.1882	0.006402
270	108	1.3249	0.0342	2.0349	0.1623	0.1006	0.000299	0.1205	0.000916	0.1522	0.003067	0.1852	0.006061
300	90	1.3258	0.0369	2.0248	0.1683	0.1003	0.000270	0.1209	0.000965	0.1524	0.003141	0.1862	0.006078
300	120	1.3242	0.0336	2.0192	0.1461	0.1003	0.000263	0.1212	0.000906	0.1523	0.002898	0.1853	0.006149
330	99	1.3249	0.0322	2.0451	0.1655	0.1001	0.000249	0.1203	0.000879	0.1510	0.002563	0.1846	0.005379
330	132	1.3211	0.0292	2.0361	0.1346	0.1000	0.000240	0.1209	0.000816	0.1535	0.002834	0.1852	0.005343
400	120	1.3202	0.0253	2.0217	0.1211	0.1001	0.000209	0.1198	0.000648	0.1518	0.002155	0.1838	0.004259
400	160	1.3164	0.0243	2.0274	0.1086	0.1004	0.000203	0.1206	0.000680	0.1529	0.002145	0.1844	0.004333

Table 2: AL and CP of 95% asymptotic confidence interval.

		α		β		λ_{01}		λ_{02}		λ_{03}		λ_{04}	
n	$\mathscr{R}$	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
70	21	1.5896	95.40	3.6451	92.90	0.1339	91.45	0.2409	88.70	0.3919	83.25	0.5328	83.50
70	28	1.4973	95.25	3.2594	92.65	0.1325	93.15	0.2313	86.95	0.3803	84.60	0.5154	82.20
100	30	1.3088	94.60	2.9244	93.35	0.1122	93.20	0.2064	88.95	0.3372	85.45	0.4558	85.10
100	40	1.2281	95.25	2.6253	93.95	0.1094	93.40	0.2018	90.60	0.3426	87.05	0.4602	85.80
140	42	1.0756	95.65	2.4109	93.30	0.0951	93.45	0.1769	91.35	0.3073	90.15	0.4171	88.45
140	56	1.0142	94.90	2.1881	94.50	0.0928	93.75	0.1703	91.95	0.3074	89.85	0.4181	88.50
170	51	0.9724	95.50	2.1816	94.40	0.0858	93.30	0.1608	92.15	0.2857	91.00	0.3890	89.50
170	68	0.9138	95.30	1.9818	94.50	0.0844	94.25	0.1527	93.15	0.2807	90.65	0.3834	88.75
200	60	0.8996	95.35	1.9694	93.70	0.0795	94.60	0.1461	91.65	0.2554	90.30	0.3588	89.10
200	80	0.8420	96.35	1.8369	94.85	0.0772	93.30	0.1400	93.10	0.2587	91.35	0.3534	91.40
230	69	0.8326	94.95	1.8458	94.20	0.0742	92.95	0.1362	92.20	0.2397	90.25	0.3355	89.20
230	92	0.7850	95.45	1.6807	94.90	0.0725	93.95	0.1304	91.95	0.2395	90.95	0.3310	90.40
270	81	0.7598	94.70	1.7008	93.60	0.0685	94.20	0.1264	93.05	0.2241	91.40	0.3122	90.95
270	108	0.7203	95.60	1.5461	94.30	0.0669	94.15	0.1205	92.65	0.2201	92.15	0.3039	91.35
300	90	0.7228	94.60	1.5937	94.75	0.0651	94.70	0.1196	92.90	0.2093	91.45	0.2935	90.50
300	120	0.6843	94.65	1.4551	93.30	0.0634	94.20	0.1149	92.45	0.2090	92.25	0.2886	90.50
330	99	0.6879	95.05	1.5321	94.70	0.0620	94.00	0.1134	93.50	0.1980	91.90	0.2776	91.20
330	132	0.6504	94.90	1.3965	94.60	0.0603	93.65	0.1094	93.10	0.2006	92.35	0.2754	91.70
400	120	0.6235	95.15	1.3783	95.20	0.0564	94.30	0.1028	94.05	0.1807	92.60	0.2519	92.50
400	160	0.5880	94.40	1.2634	94.60	0.0550	94.55	0.0989	92.55	0.1814	93.05	0.2489	91.15

Table 3: MLEs of the parameters along with the corresponding Mean Square Errors.

		α		λ_{01}		λ_{02}		λ_{03}	
n	$\mathscr{R}$	AE	MSE	AE	MSE	AE	MSE	AE	MSE
70	21	1.4464	0.2798	0.1021	0.001583	0.1246	0.006535	0.1721	0.025906
70	28	1.4372	0.2774	0.1010	0.001480	0.1269	0.006560	0.1732	0.024806
100	30	1.3975	0.1745	0.1002	0.001062	0.1226	0.004112	0.1646	0.014018
100	40	1.3980	0.1668	0.1010	0.001071	0.1218	0.004031	0.1645	0.015107
130	39	1.3826	0.1290	0.1009	0.000804	0.1217	0.002944	0.1606	0.010307
130	52	1.3774	0.1385	0.1015	0.000806	0.1230	0.003158	0.1628	0.011803
170	51	1.3655	0.0988	0.1005	0.000634	0.1212	0.002384	0.1583	0.007775
170	68	1.3722	0.1020	0.1007	0.000639	0.1213	0.002364	0.1564	0.008019
200	60	1.3452	0.0774	0.1001	0.000512	0.1214	0.001920	0.1564	0.006095
200	80	1.3590	0.0791	0.1005	0.000516	0.1202	0.001894	0.1547	0.006563
230	69	1.3440	0.0704	0.1004	0.000448	0.1211	0.001619	0.1558	0.005306
230	92	1.3491	0.0680	0.0997	0.000478	0.1195	0.001607	0.1544	0.005559
270	81	1.3337	0.0555	0.1001	0.000390	0.1214	0.001434	0.1549	0.004336
270	108	1.3347	0.0559	0.1008	0.000378	0.1210	0.001333	0.1556	0.004544
300	90	1.3403	0.0516	0.1004	0.000336	0.1193	0.001276	0.1528	0.004006
300	120	1.3287	0.0501	0.1000	0.000339	0.1217	0.001286	0.1577	0.004599
330	99	1.3290	0.0444	0.1008	0.000335	0.1202	0.001084	0.1539	0.003421
330	132	1.3325	0.0454	0.1002	0.000323	0.1203	0.001104	0.1537	0.003859
400	120	1.3215	0.0358	0.1000	0.000257	0.1214	0.000983	0.1546	0.002933
400	160	1.3172	0.0359	0.1006	0.000267	0.1220	0.000976	0.1554	0.003121

		λ_{04}		λ_{11}		λ_{12}	
n	$\mathscr{R}$	AE	MSE	AE	MSE	AE	MSE
70	21	0.2280	0.070741	0.2284	0.048130	0.3156	0.130093
70	28	0.2354	0.071331	0.2305	0.041383	0.3186	0.122304
100	30	0.2134	0.040382	0.2176	0.026030	0.2940	0.064059
100	40	0.2143	0.040920	0.2128	0.023627	0.2894	0.057618
130	39	0.2028	0.024537	0.2134	0.018074	0.2786	0.041609
130	52	0.2095	0.029906	0.2159	0.018501	0.2867	0.047470
170	51	0.1974	0.018189	0.2095	0.014310	0.2755	0.032691
170	68	0.1956	0.019195	0.2058	0.012825	0.2702	0.031084
200	60	0.1968	0.014763	0.2094	0.010933	0.2716	0.025185
200	80	0.1936	0.015397	0.2039	0.010235	0.2659	0.024655
230	69	0.1938	0.012834	0.2061	0.010203	0.2695	0.022964
230	92	0.1903	0.012515	0.2048	0.008897	0.2645	0.021126
270	81	0.1934	0.010505	0.2065	0.008152	0.2664	0.018298
270	108	0.1928	0.010514	0.2063	0.007416	0.2656	0.017338
300	90	0.1876	0.009199	0.2049	0.007321	0.2629	0.016263
300	120	0.1915	0.009458	0.2072	0.007208	0.2661	0.016420
330	99	0.1894	0.008476	0.2069	0.006732	0.2636	0.014743
330	132	0.1904	0.008984	0.2035	0.006020	0.2608	0.014060
400	120	0.1889	0.006685	0.2038	0.005309	0.2631	0.012049
400	160	0.1918	0.007507	0.2066	0.005173	0.2637	0.012026

Table 4: AL and CP of 95% asymptotic confidence interval.

		α		λ_{01}		λ_{02}		λ_{03}	
n	$\mathscr{R}$	AE	MSE	AE	MSE	AE	MSE	AE	MSE
70	21	1.8735	94.25	0.1670	96.25	0.3802	94.30	0.8632	94.10
70	28	1.8649	94.85	0.1662	97.35	0.3873	95.45	0.8976	95.85
100	30	1.5415	94.30	0.1348	96.20	0.2892	95.35	0.6054	95.25
100	40	1.5361	95.45	0.1354	95.60	0.2870	95.30	0.6247	95.20
130	39	1.3366	94.90	0.1170	94.80	0.2402	94.95	0.4836	95.25
130	52	1.3251	93.35	0.1174	94.65	0.2417	93.70	0.4986	94.05
170	51	1.1597	94.35	0.1010	94.65	0.2029	94.10	0.3954	94.30
170	68	1.1599	93.95	0.1012	94.30	0.2027	94.30	0.4008	94.40
200	60	1.0582	95.25	0.0926	96.10	0.1845	95.00	0.3512	95.25
200	80	1.0632	95.00	0.0927	95.80	0.1823	94.75	0.3555	94.85
230	69	0.9840	94.60	0.0861	95.60	0.1694	94.75	0.3196	95.05
230	92	0.9897	95.10	0.0857	94.75	0.1678	94.90	0.3262	95.00
270	81	0.9046	95.05	0.0790	95.45	0.1553	95.50	0.2888	95.05
270	108	0.9009	94.50	0.0793	95.35	0.1541	95.10	0.2949	95.00
300	90	0.8602	94.55	0.0749	95.75	0.1438	94.20	0.2671	94.80
300	120	0.8541	94.00	0.0748	95.30	0.1465	95.20	0.2809	94.75
330	99	0.8140	94.75	0.0714	94.30	0.1374	95.10	0.2539	95.10
330	132	0.8158	94.90	0.0712	94.90	0.1374	94.95	0.2595	95.30
400	120	0.7376	94.45	0.0645	95.10	0.1248	94.60	0.2282	94.95
400	160	0.7333	95.10	0.0646	94.95	0.1250	94.45	0.2335	95.30

		λ_{04}		λ_{11}		λ_{12}	
n	$\mathscr{R}$	AE	MSE	AE	MSE	AE	MSE
70	21	1.6416	95.10	1.1784	94.90	2.1203	94.85
70	28	1.7525	95.45	1.1222	95.35	2.0808	95.25
100	30	1.0772	94.95	0.8148	95.05	1.3928	95.20
100	40	1.1159	95.25	0.7664	95.45	1.3421	95.75
130	39	0.8220	95.45	0.6517	94.75	1.0634	95.30
130	52	0.8535	94.55	0.6287	94.10	1.0543	94.00
170	51	0.6496	94.90	0.5308	94.10	0.8584	94.95
170	68	0.6582	94.85	0.5012	94.15	0.8252	94.55
200	60	0.5755	95.55	0.4769	95.45	0.7553	95.50
200	80	0.5781	95.20	0.4466	94.50	0.7242	95.10
230	69	0.5143	95.25	0.4287	94.60	0.6799	95.25
230	92	0.5197	95.70	0.4119	94.85	0.6603	95.30
270	81	0.4633	94.75	0.3895	94.35	0.6079	94.95
270	108	0.4678	94.80	0.3732	94.50	0.5912	94.90
300	90	0.4197	94.60	0.3620	94.05	0.5608	95.15
300	120	0.4362	94.55	0.3525	94.10	0.5559	94.70
330	99	0.3981	95.20	0.3449	94.95	0.5294	95.65
330	132	0.4081	94.35	0.3284	95.40	0.5149	95.05
400	120	0.3536	95.00	0.3045	94.35	0.4703	94.85
400	160	0.3634	94.80	0.2969	94.90	0.4611	95.20

4 Data Analysis

In this subsection we have provided the analyses of one simulated data analysis set and one real life data set. The details are provided below.

4.1 Simulated Data Analysis

A simulated data set from a stage life testing experiment with $n = 150$ and $R = 60$, provided in Table 12 in the Appendix 7.2, has been analyzed in this section. We have simulated a data set from the proposed model of Case II with $M = 2$, $Q = 2$ and $S = 2$. Therefore we have seven model parameters in the assumed model and they are assumed as $\alpha = 1.3$, $\lambda_{01} = 0.10$, $\lambda_{02} = 0.12$, $\lambda_{03} = 0.15$, $\lambda_{04} = 0.18$ $\lambda_{11} = 0.20$ and $\lambda_{12} = 0.25$. In this case the cut points are taken as $\tau_1 = 1$, $\tau_2 = 2$, $\tau_3 = 3.5$, $\gamma_1 = 3$ and the termination point of the experiment is $T = 6$. It has been observed from the data that 76 observations ($m_{11} = 9$, $m_{12} = 22$, $m_{13} = 18$ and $m_{14} = 27$) are failed at first stress level and 53 observations ($m_{21} = 20$ and $m_{22} = 33$) are failed at second stress level. The MLEs of the model parameters are estimated as $\widehat{\alpha} = 1.29967414$, $\widehat{\lambda}_{01} = 0.06190049$, $\widehat{\lambda}_{02} = 0.11528346$, $\widehat{\lambda}_{03} = 0.13318167$, $\widehat{\lambda}_{04} = 0.21226634$ $\widehat{\lambda}_{11} = 0.23093186$ and $\widehat{\lambda}_{12} = 0.24630603$. The asymptotic confidence intervals are provided in Table 5.

Table 5: Lower limit and upper limit of asymptotic CIs of simulated data.

	α		λ_{01}		λ_{02}		λ_{03}	
Level	LL	UL	LL	UL	LL	UL	LL	UL
90%	0.7474	1.8519	0.0358	0.1069	0.0556	0.2390	0.0468	0.3788
95%	0.6396	1.9597	0.0322	0.1190	0.0482	0.2755	0.0382	0.4645
99%	0.4342	2.1651	0.0263	0.1458	0.0368	0.3613	0.0259	0.6853

	λ_{04}		λ_{11}		λ_{12}	
	LL	UL	LL	UL	LL	UL
90%	0.0584	0.7720	0.0857	0.6226	0.0709	0.8559
95%	0.0454	0.9932	0.0706	0.7555	0.0556	1.0914
99%	0.0281	1.6054	0.0488	1.0926	0.0350	1.7345

4.2 Solar Light Data Analysis

In this subsection we have analyzed a simple step-stress data set taken from Han and Kundu [6]. Simple step-stress life testing experiment is a special case of simple stage life testing experiment. If we withdraw all the survived units at a pre-fixed time from a stage life testing experiment and put them in the higher stress level then we will have a simple step-stress life testing experiment. Hence we have implemented our proposed model to a data obtained from a simple step-stress life testing experiment. Thirty five identical solar light devices are put into a life testing experiment at the initial temperature $293K$ and the temperature has been increased to $353K$ after five hundred hours. The experiment is terminated at time six hundred hours. Thirty one units are failed before the experiment is terminated and among which sixteen units are failed at first stress level and fifteen units are failed at second stress level. Though in the original data set, the exact failure times along with the causes of failures for each of the units are provided, we have not considered the causes of failures for our analysis. For more details of the data set, see Han and Kundu [6] or Samanta et al. [16].

4.3 Choice of Cut-Points

In this subsection we will discuss a method proposed by Samanta and Kundu [19] for selection of number of cut-points and the position of cut-points. Here we have two stress levels and the number of cut-points at first and second stress level are respectively denoted by M and Q. First it is assumed that both M and Q are known. Here τ is the stress changing time and T is the termination time of the experiment. We will discuss the method of selecting the position of the cut-points for the first stress level and it will be very similar for the second stress level. As discussed in Samanta and Kundu [19], the position of the cut points $\{\tau_1, \ldots, \tau_{M-1}\}$ is modeled as the arrival times of the events in $[0, \tau]$, from a non-homogeneous Poisson process $\{N(t); t \geq 0\}$ with intensity function $u(t) = \alpha_1 t^{\alpha_1 - 1}$, for $\alpha_1 > 0$. Here, the arrival times of the events from $N(t)$ are not observed, hence they are being treated as missing values. For the given number of cut points $M - 1$ in $[0, \tau]$, the likelihood function of the observation is treated as a function of the unknown parameters $\alpha, \lambda_{01}, \ldots, \lambda_{0M}, \alpha_1$ when the arrival times of the events from $N(t)$, which are missing, are replaced by their corresponding expected values. Suppose $\{N(t); t \geq 0\}$ is a non-homogeneous Poisson process with intensity function $u(t) = \alpha_1 t^{\alpha_1 - 1}$, for $\alpha_1 > 0$. Let $T_1, T_2, \ldots$ denote the arrival times of the first, second etc. arrival times of the events from $N(t)$. Then the joint PDF of $T_1, T_2, \ldots, T_n$, given that $N(\tau) = n$, can be written as

$$f_{T_1,\ldots,T_n|N(\tau)=n}(t_1, \ldots, t_n) = \tfrac{n!\alpha_1^n}{\tau^{n\alpha_1}} t_1^{\alpha_1-1} \ldots t_n^{\alpha_1-1} \qquad 0 < t_1 < t_2 < \ldots < t_n < \tau, \tag{15}$$

and zero, otherwise. Hence, for $j = 1, 2, \ldots, n$,

$$\begin{aligned} E(T_j|N(\tau) = n) &= \frac{n!\alpha_1^n}{\tau^{n\alpha_1}} \int_0^{t_2} \int_0^{t_3} \cdots \int_0^{t_n} \int_0^{\tau} t_1^{\alpha_1-1} \ldots t_{j-1}^{\alpha_1-1} t_j^{\alpha_1} t_{j+1}^{\alpha_1-1} \ldots t_n^{\alpha_1-1} dt_1 dt_2 \ldots dt_n \\ &= \frac{n!\alpha_1^n}{\tau^{n\alpha_1}} \times \frac{\tau^{n\alpha_1+1}}{\alpha_1(2\alpha_1)\ldots((j-1)\alpha_1)(j\alpha_1+1)((j+1)\alpha_1+1)\ldots(n\alpha_1+1)} \\ &= \frac{n!}{(j-1)!} \frac{\alpha_1^{n-j+1}\tau}{(j\alpha_1+1)((j+1)\alpha_1+1)\ldots(n\alpha_1+1)} \\ &= \frac{\tau}{(1+\frac{1}{j\alpha_1})\ldots(1+\frac{1}{n\alpha_1})} \end{aligned}$$

In case of second stress level, the position of the cut points are $\{\gamma_1, \ldots, \gamma_{Q-1}\}$ and $\{\gamma_1, \ldots, \gamma_{Q-1}\}$ denotes the arrival times of the events between $[\tau, T]$, form a new non-homogeneous Poisson process which starts at τ, $\{N(\tau + t); t \geq 0\}$ with intensity function $u(t) = \alpha_2 t^{\alpha_2 - 1}$, for $\alpha_2 > 0$. In this case, for given number of cut points $N(T) = n$, the position of the $j-$th cut-point will be

$$E(T_j|N(T) = n) = \tau + \frac{T - \tau}{(1+\frac{1}{j\alpha_2})\ldots(1+\frac{1}{n\alpha_2})}$$

For a given M, and Q we compute the maximum likelihood estimators of the unknown parameters, and then based on some information theoretic criteria, like Akike Information Criterion (AIC) or Bayes Information Criterion (BIC), we estimate M and Q also.

In practice for given M, Q and for fixed (α_1, α_2) obtain the MLEs of the model parameters $\alpha, \lambda_{01}, \ldots, \lambda_{0M}, \lambda_{11}, \ldots, \lambda_{1Q}$ and the corresponding log-likelihood values. For fixed M and Q, choose the combination of (α_1, α_2) for which the log-likelihood is maximum. Then based on AIC or BIC we will estimate M and Q. The method is implemented in data analysis part and has been discussed in detail.

4.4 MLEs and Goodness-of-Fit

We have analyze the solar light data assuming a semi-parametric model. We have considered three combinations of M and Q, i.e., $(M = 2, Q = 1)$, $(M = 1, Q = 2)$ and $(M = 2, Q = 2)$. For fixed M and Q, we have obtained the MLEs of the model parameters along with the log-likelihood values and AIC for different combinations of (α_1, α_2). The results are provided in Table 6, Table 7 and Table 8. Based on the results from Table 6 to Table 8, we conclude that the model with $M = 2$, $Q = 2$, $\alpha_1 = 1.2$ and $\alpha_2 = 1$ is the best model (since it has the minimum AIC) to fit the data among the candidate models. The proposed semi-parametric model with $M = 1$ and $Q = 1$, becomes the step-stress model under the assumption of Weibull distribution with same shape parameter and different scale parameter for different stress level. We have compared the best fit semi-parametric model with the Weibull step-stress model. The MLEs of the model parameters with $M = 1$, $Q = 1$ are $\widehat{\alpha} = 1.2593$, $\widehat{\lambda}_{01} = 0.08074$, $\widehat{\lambda}_{11} = 0.94114$ and the corresponding AIC of the model is 117.3249. Hence, the performance of the best fit semi-parametric model is better than Weibull step-stress model. The profile log-likelihood of α for the best fitted model has been provided in Figure 2, which is a unimodal function of α. The asymptotic confidence intervals of the model parameters for the best model is provided in Table 9. Next we have performed Kolmogorov-Smirnov (KS) test to check the goodness-of-fit of the data. The KS distance between empirical and fitted CDF is 0.06837677 and the p-value of the test is 0.655. Therefore, we conclude that the data fit the proposed model quite good. The graphical representation of empirical and fitted CDF is provided in Figure 3.

Table 6: Parameter estimates along with log-likelihood and AIC for different choice of α_1 and for $M = 2, Q = 1$.

α_1	α	λ_{01}	λ_{02}	λ_{11}	Log-likelihood	AIC
0.8	1.2565	0.0808	0.0813	0.9477	-55.6624	119.3248
0.9	1.3750	0.0782	0.0602	0.7101	-55.5993	119.1985
1.0	1.2653	0.0806	0.0795	0.9274	-55.6623	119.3245
1.1	1.1808	0.0827	0.1004	1.1449	-55.6246	119.2493
1.2	1.7400	0.0642	0.0212	0.3043	-54.4334	116.8669
1.3	1.6456	0.0670	0.0264	0.3768	-54.7744	117.5488
1.4	1.5685	0.0695	0.0320	0.4499	-55.0335	118.0671
1.5	1.5045	0.0716	0.0379	0.5223	-55.2298	118.4596
1.6	1.4505	0.0735	0.0439	0.5930	-55.3769	118.7538
1.7	1.5791	0.0680	0.0280	0.4391	-54.8371	117.6742

Table 7: Parameter estimates along with log-likelihood and AIC for different choice of α_2 and for $M = 1, Q = 2$.

α_2	α	λ_{01}	λ_{11}	λ_{12}	Log-likelihood	AIC
0.7	1.2765	0.0787	1.0981	0.5263	-54.9212	117.8424
0.8	1.2734	0.0791	1.0612	0.5777	-55.1664	118.3329
0.9	1.2794	0.0784	1.0984	0.4073	-54.5898	117.1796
1.0	1.2861	0.0776	1.1359	0.2128	-53.5312	115.0624
1.1	1.2844	0.0778	1.1178	0.2257	-53.7572	115.5144
1.2	1.2829	0.0780	1.1021	0.2387	-53.9553	115.9107
1.3	1.2815	0.0781	1.0882	0.2520	-54.1300	116.2601
1.4	1.2802	0.0783	1.0760	0.2654	-54.2848	116.5696
1.5	1.2789	0.0784	1.0652	0.2791	-54.4226	116.8451
1.6	1.2777	0.0786	1.0554	0.2929	-54.5456	117.0912

Table 8: Parameter estimates along with log-likelihood and AIC for different choice of α_1 and α_2 and for $M = 2, Q = 2$.

α_1	α_2	α	λ_{01}	λ_{02}	λ_{11}	λ_{12}	Log-likelihood	AIC
1.0	0.8	1.2950	0.0785	0.0747	1.0069	0.5471	-55.1640	120.3281
1.0	0.9	1.3078	0.0777	0.0727	1.0252	0.3791	-54.5857	119.1713
1.0	1.0	1.3224	0.0767	0.0706	1.0405	0.1942	-53.5246	117.0493
1.0	1.1	1.3188	0.0769	0.0711	1.0286	0.2070	-53.7513	117.5026
1.0	1.2	1.3155	0.0772	0.0716	1.0184	0.2199	-53.9500	117.9000
1.1	0.8	1.2066	0.0808	0.0949	1.2501	0.6849	-55.1404	120.2808
1.1	0.9	1.2178	0.0799	0.0927	1.2769	0.4763	-54.5681	119.1361
1.1	1.0	1.2306	0.0790	0.0902	1.3008	0.2450	-53.5139	117.0277
1.1	1.1	1.2275	0.0792	0.0908	1.2849	0.2609	-53.7388	117.4777
1.1	1.2	1.2246	0.0794	0.0914	1.2712	0.2770	-53.9360	117.8720
1.2	0.8	1.7845	0.0615	0.0194	0.3264	0.1693	-53.8534	117.7068
1.2	0.9	1.8022	0.0605	0.0188	0.3294	0.1162	-53.2417	116.4835
1.2	1.0	1.8222	0.0593	0.0181	0.3311	0.0589	-52.1434	114.2868
1.2	1.1	1.8172	0.0596	0.0183	0.3279	0.0628	-52.3792	114.7585
1.2	1.2	1.8125	0.0599	0.0184	0.3252	0.0668	-52.5865	115.1731
1.3	0.8	1.6850	0.0645	0.0245	0.4072	0.2132	-54.2114	118.4227
1.3	0.9	1.7009	0.0635	0.0237	0.4124	0.1469	-53.6065	117.2130
1.3	1.0	1.7188	0.0623	0.0229	0.4163	0.0748	-52.5158	115.0316
1.3	1.1	1.7144	0.0626	0.0231	0.4119	0.0797	-52.7497	115.4995
1.3	1.2	1.7102	0.0629	0.0233	0.4082	0.0847	-52.9553	115.9105
1.4	0.8	1.6039	0.0670	0.0298	0.4889	0.2580	-54.4842	118.9684
1.4	0.9	1.6184	0.0661	0.0290	0.4966	0.1783	-53.8848	117.7697
1.4	1.0	1.6347	0.0650	0.0281	0.5029	0.0911	-52.8004	115.6008
1.4	1.1	1.6306	0.0652	0.0283	0.4972	0.0970	-53.0328	116.0656
1.4	1.2	1.6269	0.0655	0.0285	0.4924	0.1031	-53.2369	116.4737

Table 9: Lower limit and upper limit of asymptotic CIs of solar light data.

	α		λ_{01}		λ_{02}		λ_{11}		λ_{12}	
Level	LL	UL	LL	UL	LL	UL	LL	UL	LL	UL
90%	0.970891	2.673516	0.022823	0.154187	0.003063	0.106806	0.048435	2.263322	0.004591	0.755012
95%	0.804781	2.839626	0.018942	0.185777	0.002166	0.151033	0.033287	3.293311	0.002790	1.242098
99%	0.488135	3.156273	0.013278	0.265028	0.001119	0.292361	0.016284	6.731855	0.001080	3.208422

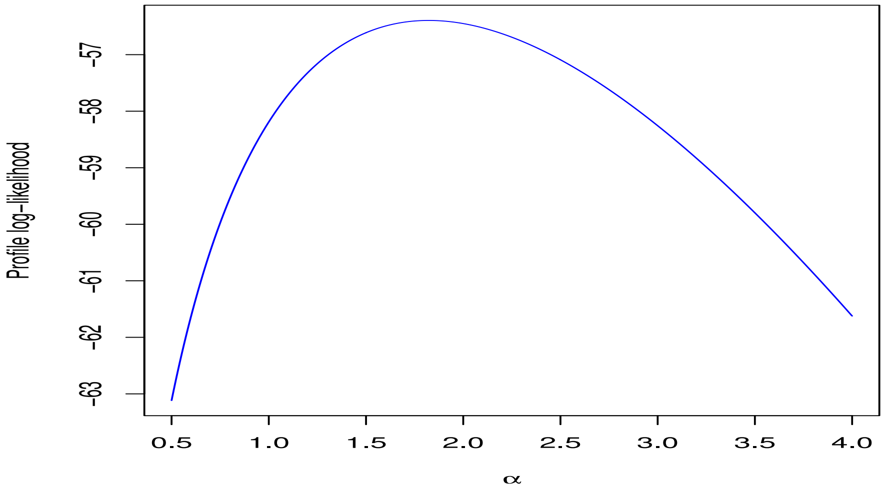

Figure 2: Profile log-likelihood of α for the best-fit model.

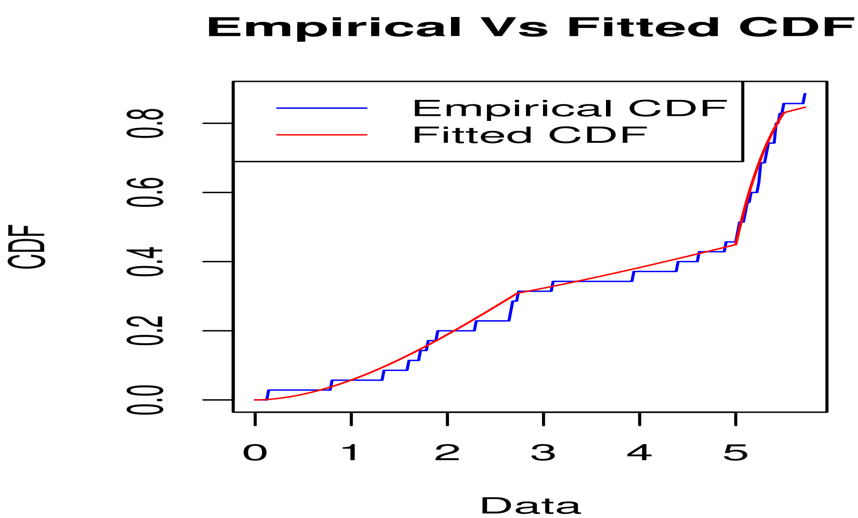

Figure 3: Empirical and the fitted CDFs for the Solar Light data.

5 Sensitivity Analysis of Cut-points

In this section we have performed the sensitivity analysis of the number of cut-points and the position of the cut-points of both the stress level. We have proposed to use a non-homogeneous Poisson process approach to select the position of cut-points and for selection of the number of cut-points we have proposed to use minimum AIC approach. In this section we have performed a simulation study to evaluate how the proposed method works for different sample sizes. We have performed this analysis in two ways. Firstly, we assume that the number of cut-points are known. Since the position of cut-points depends on the correct selection of α_1 and α_2, we have performed the simulation study for average estimates and mean square error of the estimates of α_1 and α_2. We have generated samples from simple step-stress stage life testing experiment by taking the parameter values as we have obtained in case of best fitted model for the real data. Since $M = 2$ and $Q = 2$ with $\alpha_1 = 1.2$, $\alpha_2 = 1$ provide the best fitted model for the data, we have taken $\alpha = 1.8222$, $\lambda_{01} = 0.0593$, $\lambda_{02} = 0.0181$, $\lambda_{11} = 0.3311$ and $\lambda_{12} = 0.0589$. We have estimated α_1 and α_2 along with corresponding MSEs for different sample sizes. Results are reported in Table 10. It has been observed that MSEs for both α_1 and α_2 decrease as sample size increases. Secondly, to evaluate the performance of selection of cut-points based on minimum AIC approach, we have simulated the data from model with $M = 2$ and $Q = 2$ with the same set of parameter values as above. Next we have fitted the data taking different combination of (M, Q) as $(2, 2)$, $(2, 1)$ and $(1, 2)$. We have noted the percentage of cases when the minimum AIC is the model with $M = 2$ and $Q = 2$. Results are reported in Table 11. It have been observed that if sample size increases then the percentage of correct selection of (M, Q) also increases and it is very close to 100% for $n = 200$. Therefore this non-homogeneous Poisson process and minimum AIC approach works well to select the position and the number of cut-points.

6 Conclusion

In this article we have proposed a very flexible semi-parametric model to analyze the data obtained from a simple stage life testing experiment. Depending on the data we have assumed a piecewise

Table 10: Average estimates and MSEs of α_1 and α_2.

	α_1		α_2	
n	AE	MSE	AE	MSE
35	1.1569	0.0188	0.9155	0.0195
70	1.1837	0.0138	0.9626	0.0130
100	1.1817	0.0148	0.9789	0.0118
150	1.1937	0.0098	0.9944	0.0093
200	1.1918	0.0091	0.9939	0.0087
250	1.1971	0.0065	0.9885	0.0071
300	1.1978	0.0064	0.9938	0.0054
350	1.1954	0.0045	0.9949	0.0051
400	1.1999	0.0041	0.9994	0.0046

Table 11: Percentages of correct selection of M and Q.

n	35	70	100	150	200
Percentage	72.05	90.05	94.65	98.85	99.80

increasing or decreasing or a constant hazard rate function. We have obtained the MLEs of the model parameters and propose to use asymptotic confidence intervals. An extensive simulation study suggest the consistency of the propose estimator. The method can be easily implemented to a real data set obtain from either a stage life testing experiment or from a simple step-stress life testing experiment. The proposed model is a more flexible model than the widely used Weibull model. Here we have only considered the classical inference of the model parameters, it will be of interest to provide Bayesian inference of the model parameters also. Further, the model can be extended for multiple stress levels also. More work in needed along this direction.

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Data Availability Declaration

We have used one real data set which is available in Han and Kundu [6].

Competing Interest Declaration

None of the author has any competing interests in preparation of this manuscript.

Author Contributions Declaration

Both the authors have contributed equally.

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7 Appendix:

7.1 Elements of Fisher information Matrix

Case I:

$$
\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial \alpha^2} &= -\frac{n_1+n_2}{\alpha^2} - \sum_{j=1}^{M} \lambda_{0j} \delta_j^{''(\alpha)}(\alpha,\beta) - \sum_{j=M+1}^{M+Q} \lambda_{0j} \delta_j^{''\alpha}(\alpha), \\
\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial \alpha \partial \beta} &= -\sum_{j=1}^{M} \lambda_{0j} \delta_j^{''(\alpha,\beta)}(\alpha,\beta), \\
\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial \alpha \partial \lambda_{0j}} &= -\delta_j^{'(\alpha)}(\alpha,\beta), \qquad\qquad j = 1,\ldots,M \\
\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial \alpha \partial \lambda_{0j}} &= -\delta_j^{'(\alpha)}(\alpha), \qquad\qquad j = M+1,\ldots,M+Q \\
\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial \beta^2} &= -\frac{n_2}{\beta^2},
\end{aligned}
$$

$$\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial\beta\partial\lambda_{0j}} = -\delta_j^{'(\beta)}(\alpha,\beta), \qquad j = 1,\ldots,M$$

$$\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial\beta\partial\lambda_{0j}} = 0, \qquad j = M+1,\ldots,M+Q$$

$$\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial\lambda_{0j}^2} = \frac{m_{1j}+m_{2j}}{\lambda_{0j}^2}, \qquad j = 1,\ldots,M$$

$$\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial\lambda_{0j}^2} = \frac{m_{1j}}{\lambda_{0j}^2}, \qquad j = M+1,\ldots,M+Q$$

$$\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial\lambda_{0j}\partial\lambda_{0k}} = 0, \qquad j = 1,\ldots,M+Q-1 \quad k = j+1,\ldots,M+Q,$$

where $\delta_j^{'(\alpha)}(\alpha,\beta)$ and $\delta_j^{''(\alpha)}(\alpha,\beta)$ are respectively first and second order partial derivative of $\delta_j(\alpha,\beta)$ with respect to α ($j = 1,\ldots M$). $\delta_j^{''(\alpha,\beta)}(\alpha,\beta)$ is second order partial derivative of $\delta_j(\alpha,\beta)$ with respect to α and β ($j = 1,\ldots M$). $\delta_j^{'(\alpha)}(\alpha)$ and $\delta_j^{''(\alpha)}(\alpha)$ are respectively first and second order partial derivative of $\delta_j(\alpha)$ with respect to α ($j = M+1,\ldots M+Q$).

Case II:

$$\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial\alpha^2} = -\frac{n_1+n_2}{\alpha^2} - \sum_{j=1}^{M+Q}\lambda_{0j}\delta_{0j}^{''}(\alpha) - \sum_{j=1}^{S}\lambda_{1j}\delta_{1j}^{''}(\alpha),$$

$$\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial\alpha\partial\lambda_{0j}} = -\delta_{0j}^{'}(\alpha), \qquad j = 1,\ldots,M+Q$$

$$\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial\alpha\partial\lambda_{1j}} = -\delta_{1j}^{'}(\alpha), \qquad j = 1,\ldots,S$$

$$\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial\lambda_{0j}^2} = -\frac{m_{1j}}{\lambda_{0j}^2}, \qquad j = 1,\ldots,M+Q$$

$$\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial\lambda_{1j}^2} = -\frac{m_{2j}}{\lambda_{1j}^2}, \qquad j = 1,\ldots,S$$

$$\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial\lambda_{0j}\partial\lambda_{0k}} = 0, \qquad j=1,\ldots,M+Q-1, \quad k=j+1,\ldots M+Q$$

$$\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial\lambda_{0j}\partial\lambda_{1k}} = 0, \qquad j=1,\ldots,M+Q, \quad k=1,\ldots S$$

$$\frac{\partial^2 l(\boldsymbol{\eta}|Data)}{\partial\lambda_{1j}\partial\lambda_{1k}} = 0, \qquad j=1,\ldots,S-1, \quad k=j+1,\ldots S,$$

where $\delta_{0j}^{'}(\alpha)$ and $\delta_{0j}^{''}(\alpha)$ are respectively first and second order partial derivative of $\delta_{0j}(\alpha)$ with respect to α $(j=1,\ldots M+Q)$. $\delta_{1j}^{'}(\alpha)$ and $\delta_{1j}^{''}(\alpha)$ are respectively first and second order partial derivative of $\delta_{1j}(\alpha)$ with respect to α $(j=1,\ldots S)$.

7.2 Simulated Data Set

Table 12: Simulated data obtained from two stage life testing experiment.

Stress Level	Observed Data								
S_1	0.0296	0.0515	0.2083	0.2218	0.7706	0.8395	0.8666	0.8764	0.9297
	1.0044	1.0561	1.0664	1.2162	1.2709	1.2764	1.4051	1.4139	1.4164
	1.5072	1.5562	1.5653	1.6340	1.6563	1.7318	1.7810	1.8191	1.8239
	1.9058	1.9241	1.9383	1.9420	2.2123	2.3487	2.4163	2.4245	2.5518
	2.7329	2.7430	2.8526	2.8863	2.9201	2.9988	3.0423	3.0506	3.2321
	3.2494	3.3742	3.3913	3.4735	3.5624	3.7338	3.7645	3.8385	3.8509
	3.8679	3.9473	3.9613	3.9832	4.3183	4.4100	4.4528	4.5919	4.6012
	4.6594	4.6740	4.7775	4.8501	4.9302	4.9328	4.9472	5.0728	5.1063
	5.1342	5.2115	5.3598	5.5059					
S_2	2.0500	2.1342	2.2179	2.3289	2.3705	2.3919	2.4222	2.4391	2.4840
	2.5420	2.5525	2.5683	2.6130	2.6910	2.7070	2.7676	2.8826	2.8891
	2.9364	2.9399	3.0175	3.0246	3.0625	3.0681	3.1636	3.2353	3.2558
	3.4867	3.5463	3.5612	3.6919	3.8825	3.9352	4.0227	4.0712	4.1247
	4.3696	4.3832	4.4792	4.6960	4.7171	4.9941	5.1475	5.2012	5.2015
	5.2390	5.5305	5.6680	5.8680	5.8691	5.8977	5.9608	5.9802	