Statistical Inference for the Jointly Adaptive Progressive Type-II Censored Weibull Distributions

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Abstract

In a life-testing experiment, the joint progressive censoring scheme for more than one exponential population was proposed by Rasouli and Balakrishnan [21]. In this paper, we focus on the joint adaptive progressive Type-II (JAPC-II) censoring scheme for two populations to reduce the experimental time as well as cost of the experiment. The methodology is developed for the Weibull distribution with common shape parameter β and different scale parameters α & λ, respectively. We consider the maximum likelihood estimators of the unknown scale and shape parameters along with their asymptotic and bootstrap confidence intervals. Further, the Bayesian inference procedure is provided with the corresponding credible intervals. Theoretical studies are illustrated based on simulation study. We also provide some connection between our proposed model with the balanced joint adaptive progressive censoring scheme, proposed by Sultana et. al. [23]. Finally, the methods are illustrated with the analysis of a real data set.

Keywords: Adaptive progressive censoring, Bayesian estimation, Weibull distribution, Jointly censored populations, Maximum likelihood estimation

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1 Introduction

In reliability theory, censored data are playing an important role. The scenario of censoring comes in the life testing experiments when the lifetime of the components or units of the experiment may not always be recorded exactly. In many cases, components or units are lost or removed from the experiments before they fail and censored datasets are observed. Otherwise, to provide time and cost efficiency, censoring schemes are preferable for the experimenters. In literature, there are many different types of censoring scheme available for different types of lifetime experiments. The most basic censoring schemes are Type-I and Type-II censoring scheme. In the Type-I censoring scheme we pre-fixed the experimental time before starting the experiment and will stop the experiment at that time. Whereas, in the Type-II censoring scheme, we pre-determined the number of failures, say, m. The test will terminate when a random sample of size m is observed.

Also, in the progressive censoring schemes, the experimenters are allowed to remove live units from the experiment during the test. The progressive Type-II censoring scheme can be briefly described as follows. Suppose a life test starts with n components. Prior to the experiment, an integer $m < n$ is determined. Again, the progressive censoring scheme $R = (R_1, R_2, \cdots, R_m)$ with $R_i \geq 0$ and $\sum_{i=1}^{m} R_i + m = n$ is pre-specified. Now, at the time of the first failure, R_1 number of live units are removed from the experiment. At the time of the second failure, R_2 number of live units are removed from the remaining $n - R_1 - 1$ units and it goes on until the m^{th} failure occurs. Finally, at the m^{th} failure, all remaining, $R_m = n - m - \sum_{i=1}^{m-1} R_i$, units are excluded from the test and terminate the test. For more details on progressive censoring scheme, one can see Balakrishnan and Cramer [4]. Getting m^{th} failure at a reasonable time, for highly reliable products, is the major drawback of this censoring scheme.

On the other hand, Ng et al. [20] proposed an adaptive progressive Type-II censoring scheme which is a mixture of Type-I and progressive Type-II censoring. This censoring was proposed, to save both the total test time and the cost induced by the failure of the units and increase the efficiency of statistical analysis. Thus this censoring scheme actually overcomes the major difficulties of the Progressive Type-II censoring scheme in a reasonable experimental time. In this censoring scheme, the number of failures m is fixed in advance and the progressive censoring scheme $R = (R_1, R_2, \cdots, R_m)$ is also pre-determined. However, some R_i values may change during the experiment. The experimenter firstly determines the ideal test time T which is also pre-fixed. This scheme is described as follows.

At the time of the first failure, we remove R_1, randomly selected units from the test. We continue the test in a similar manner and if the time T occurred before the m^{th} failure, we want to continue the experiment until the m^{th} failure occurred. In this case after time T, we do not withdraw any items from the experiment and at the time of the m^{th} failure we will remove the remaining items from the test. On the other hand, if the m^{th} failure occurred before the time point T, we will stop the experiment by removing the remaining items from the test. Which is actually the Progressive Type-II censoring scheme as discussed before. The purpose of doing such kind of experiment can be viewed as a design in which we are assured of getting m observed failure times for efficiency of statistical inference within an affordable time bound. From the basic properties of order statistics, we know that the fewer operating items are withdrawn (i.e., the larger the number of items on the test), the smaller the expected total test time (See Ng and Chan [19]). Therefore, we should leave as many surviving items on the test as possible, to terminate the experiment as soon as possible to get a fixed number of failure m. This idea reduces the total time of the experiment as well as cost of the experiment. Thus, we can save time and cost in this censoring scheme to get the m number of failures.

In the literature, many authors considered this censoring scheme for many inferential problems. For instance, the adaptive progressively Type-II censoring scheme is handled by Sobhi and Soliman [22], Mohie El-Din et al. [16] and Ye et al. [27] for the exponentiated Weibull, generalized exponential and extreme value distributions, respectively. Further, Ismail [14] is considered a step-stress partially accelerated life test model based on adaptive Type-II progressive censoring scheme. Furthermore, most of the conventional censoring scheme deals with one sample problem. Whereas, when the experimenters decide to conduct comparative life tests of units, joint censoring schemes are needed. For this purpose, joint censoring schemes have been suggested in the literature.

Rasouli and Balakrishnan [21] proposed the joint progressive Type-II censoring scheme for two exponential populations. The experiment can be described as follows. Suppose two independent samples (say Sample-I and Sample-II) of sizes n_1 and n_2 such that $N = n_1 + n_2$, are selected from two populations and placed on a life-testing experiment. Let m and R_i, $i = 1, \cdots, m$ are the same as defined above. The experiment is terminated as soon as we get a total m number of failures from the samples. Now when first failure occurred from Sample-I (or Sample-II), s_1 surviving units are randomly removed from the Sample-I and from Sample-II q_1 number of units are removed at the same time such that $R_1 = s_1 + q_1$. Similarly, at time of second failure from Sample-II (say), then R_2 units are randomly removed from the experiment such that s_2 and q_2 units are removed from the experiment from Sample-I and Sample-II, respectively. We continue the experiment till the occurrence of the m^{th} failure. Finally, at the time of the m^{th} failure (which may be from Sample-I or Sample-II), all the remaining surviving units are removed from the experiment. Note that here the progressive censoring scheme $R = (R_1, R_2, \cdots, R_m)$ has the decomposition $S + Q = (s_1, s_2, \cdots, s_m) + (q_1, q_2, \cdots, q_m)$. Further, note that if $R_1 = R_2 = \cdots = R_{m-1} = 0$ then $R_m = N - m$, which corresponds to the joint Type-II censoring scheme proposed by Balakrishnan and Rasouli[5]

Several works have been done in the jointly censored samples in the literature. For example, Ashour and Abo-Kasem [2]-[3] studied joint type-II and joint progressive Type-I censoring schemes for different distributions. Balakrishnan and Feng [6] studied exact likelihood inference for k-exponential populations under joint Type-II censoring and also Balakrishnan et al. [7] studied the same problem under joint progressive Type-II censoring scheme. Doostparast et al. [12] studied Bayes estimation based on joint progressive Type-II censored data under LINEX loss function. Abo-Kasem et al. [1] studied exponential distributions under the joint Type-I progressive hybrid censoring scheme. The Lindley distribution is handled by Krishna and Goel [15] under jointly Type-II censoring.

Mondal and Kundu [17] studied the Weibull distribution based on the balanced joint progressive censoring scheme. The balanced adaptive progressive censoring scheme, by Sultana et. al. [23], is an alternative of the usual balanced joint progressive censoring scheme. The author considered two independent exponential samples with mean θ_1 and θ_2 and obtained the MLEs of the unknown means. They have also obtained the exact distribution of the MLEs. Further they considered the Bayesian inferences and some optimality criteria for optimum R_i's over all possible R_i, for $i = 1, \cdots, m$. Again, Sultana et. al. [23] studied the advantage of the experimental time for the balanced joint adaptive censoring scheme over the balanced progressive censoring scheme by Mondal and Kundu [18]. It is seen that these studies are mostly concentrated on Type-II, progressive Type-II and related censoring schemes.

In this paper, we consider two independent Weibull populations with common shapes and different scale parameters when the data is coming from jointly adaptive progressive Type-II censored (JAPC-II) for two Weibull samples are combined together in Section 2. We obtain the maximum likelihood estimators (MLEs) of the unknown parameters in Section 3. In Section 4, we also obtain the asymptotic confidence intervals as well as bootstrap confidence intervals for the unknown model parameters. We further consider the Bayes estimators, using the squared error loss function, of the unknown parameters under the assumptions of independent gamma priors in Section 5. Further, we obtain Bayesian confidence intervals. To assess the efficiency of the estimates, simulation studies are performed in Section 6. We illustrate the proposed methods through the analysis of a real life data set in Section 7. Further, we describe the proposed modeling with existing balanced JAPC scheme in Section 8, while Section 9 ends with some concluding remarks.

2 Model Description

Suppose that $X = (X_1, X_2, \cdots, X_{n_1})$ are independently and identically distributed (i.i.d.) lifetimes of the test units from Sample-I of size n_1, similarly, $Y = (Y_1, Y_2, \cdots, Y_{n_2})$, test units form Sample-II of size n_2, are also i.i.d. and X_i's and Y_j's are independently distributed. Let X and Y be Weibull distributed with common shape parameter β and different scale parameters α and λ, respectively. The probability density function (PDF) and cumulative distribution function (CDF), for both Sample-I and Sample-II, which are given by

$$\begin{aligned} f(x;\alpha,\beta) &= \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}} \quad \text{and} \quad F(x;\alpha,\beta) = 1 - e^{-\alpha x^{\beta}} \quad \text{for} \quad x > 0, \alpha, \beta > 0 \\ g(y;\lambda,\beta) &= \lambda\beta y^{\beta-1}e^{-\lambda y^{\beta}} \quad \text{and} \quad G(y;\lambda,\beta) = 1 - e^{-\lambda y^{\beta}} \quad \text{for} \quad y > 0, \lambda, \beta > 0. \end{aligned} \tag{2.1}$$

Then, it is assumed that $W_{(1)}, W_{(2)}, \cdots, W_{(N)}$ where $N = n_1 + n_2$, denote the combined ordered samples form of test samples $(X_1, X_2, \cdots, X_{n_1}, Y_1, Y_2, \cdots, Y_{n_2})$. Now the proposed JAPC-II can be described as follows.

We pre-fixed an integer m, total number of failure, the time point T, and removals $R = (R_1, R_2, \cdots, R_m)$ are also fixed in advance. Firstly, at the time of the first failure (from either Sample-I or Sample-II), R_1 units are randomly withdrawn from the remaining $(N - 1)$ surviving units. At this stage, removals from the Sample-I are denoted by s_1 and the removals from the Sample-II are denoted by q_1 such a way that $R_1 = s_1 + q_1$. Similarly, at the time of the second failure, $R_2 = s_2 + q_2$ units are randomly removed from the experiment such that there are s_2 and q_2 units are removed from the Sample-I and Sample-II, respectively. We continue the test until either of m^{th} failure or time T occurred. Suppose the time T occurred before the m^{th} failure, we continue the test until the m^{th} failure occur and we will not remove any samples from both of the samples, Sample-I or Sample-II. Unlike the progressive Type-II censoring scheme, number of remaining units are adjusted due to the case of $m-$th failure is before the time T or not. If the $m-$th failure occurs before the time point T, the experiment stops at the time point $W_{(m)}$ and the progressive censoring scheme $R_1, R_2, \cdots, R_m$ is provided as shown in Figures (1)-(2). On the other hand, if $T < W_{(m)}$, that is the number of observed failures has not reached m , we would want to terminate the experiment as soon as possible for fixed value of m. For this purpose, we do not withdraw any items after passed time point T except for the time $m-$th of the failure where all remaining surviving items are removed as seen in Figure (2). In this case, we denote the numbers of failures before T as J as

$$W_{(J)} < T < W_{(J+1)} \quad J = 1, 2, \cdots, m.$$

where $W_{(0)} \equiv 0$ and $W_{(m+1)} \equiv \infty$. Then, $W_{(J+1)} = \cdots W_{(m-1)} = 0$ and $R_m = N - m - \sum_{i=1}^{J} R_i$. In this case, and the progressive censoring scheme is obtained as

$$\left(R_1, R_2, \cdots R_{J^*}, 0, 0, \cdots, 0, N - m - \sum_{i=1}^{J^*} R_i\right)$$

where $J^* = \max\{J : W_{(J)} < T\}$. This censoring scheme aims to speed up the test as much as possible in the case of when the test duration exceeds the prefixed time threshold T. Note that if the pre-fixed time $T \to \infty$ the proposed JAPC-II scheme coincides with the joint progressive censoring scheme proposed by Rasouli and Balakrishnan [21]

Let Z_i, $(\forall i = 1, 2, \cdots, m)$ take two values either 1 or 0 depending on whether $W_{(i)}$ is an X or Y failure, respectively. Then, suppose $R_i = s_i + q_i$, $i = 1, 2, \cdots, m$ where s_i and q_i denotes the number of units withdrawn at the time of $i-$th failure that belongs to sample X and Y, respectively. Then, under JAPC-II the likelihood function of $(W, Z, S + Q)$ where $S =$

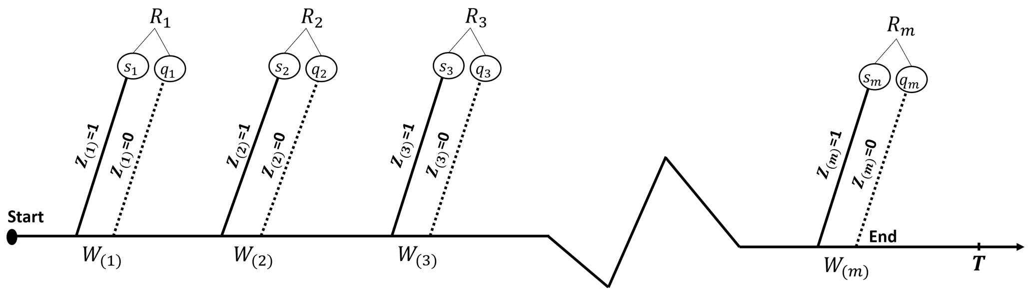

Figure 1: The schematic representation of experiment when it terminates before time T (solid line denotes the Sample-I while dotted line denotes the Sample-II).

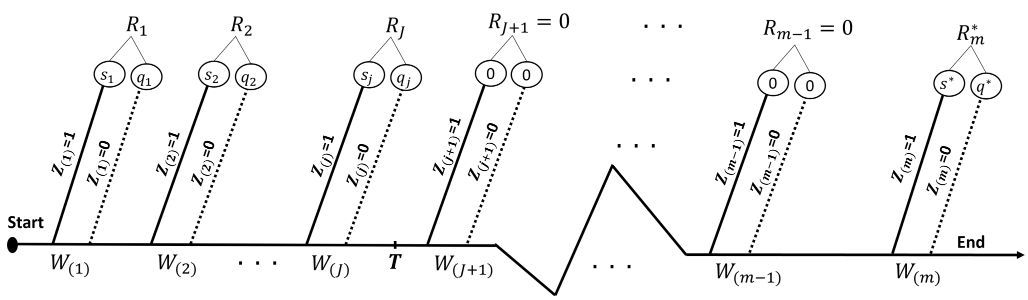

Figure 2: The schematic representation of experiment when it terminates after time T (solid line denotes the Sample-I while dotted line denotes the Sample-II).

$s_1, s_2, \cdots, s_m$ and $Q = q_1, q_2, \cdots, q_m$ can be written as

$$L(\mathbf{w},\mathbf{z}) = B_J \Bigg[\prod_{i=1}^{m} [f(w_i)]^{z_i} [g(w_i)]^{1-z_i} \Bigg] \Bigg[\prod_{i=1}^{J} [\bar{F}(w_i)]^{s_i} [\bar{G}(w_i)]^{q_i} \Bigg] [\bar{F}(w_m)]^{s^*} [\bar{G}(w_m)]^{q^*}, \tag{2.2}$$

where

$$B_J = \prod_{i=1}^{m} \Bigg[N - i + 1 - \sum_{k=1}^{min\{i-1,J\}} R_k \Bigg].$$

Note that $\bar{F}(.) = 1 - F(.)$, $\bar{G}(.) = 1 - G(.)$, $s^* = \sum_{i=J+1}^{m} s_i$ for $J < m$ or

$s^* = 0$ otherwise. Similarly, $q^* = \sum_{i=J+1}^{m} q_i$ for $J < m$ or $q^* = 0$ otherwise. As a special case, for $J = m$, we obtain joint progressive Type-II censoring scheme which is defined by Rasouli and Balakrishnan [21].

3 Maximum Likelihood Estimation

In this section, we consider maximum likelihood estimations of unknown model parameters of the Weibull distributions. Using equations (2.1) and (2.2), the likelihood function of α, λ, and β is obtained as

$$\begin{aligned} L(\alpha, \lambda, \beta|\mathbf{w,z}) &\propto \left[\prod_{i=1}^{m} \beta\alpha^{z_i}\lambda^{1-z_i}w_i^{\beta-1}e^{-\alpha z_i w_i^{\beta}}e^{-\lambda(1-z_i)w_i^{\beta}}\right] \times \left[\prod_{i=1}^{J} e^{-\alpha s_i w_i^{\beta}}e^{-\lambda q_i w_i^{\beta}}\right] \\ &\times e^{-\alpha s^* w_m^{\beta}}e^{-\lambda q^* w_m^{\beta}} \\ &\propto \beta^m\alpha^{d_1}\alpha^{d_2}e^{(\beta-1)\sum_{i=1}^{m}\ln(w_i)}e^{-\alpha\xi_1(\beta)}e^{-\lambda\xi_2(\beta)}, \end{aligned} \tag{3.3}$$

where $d_1 = \sum_{i=1}^{m} z_i$, $d_2 = m - \sum_{i=1}^{m} z_i$, and

$$\xi_1(\beta) = \sum_{i=1}^{m} z_i w_i^{\beta} + \sum_{i=1}^{J} s_i w_i^{\beta} + s^* w_m^{\beta}$$

$$\xi_2(\beta) = \sum_{i=1}^{m}(1 - z_i)w_i^{\beta} + \sum_{i=1}^{J} q_i w_i^{\beta} + q^* w_m^{\beta}.$$

Then the log-likelihood function may then be written as

$$\ell(\alpha, \lambda, \beta|\mathbf{w,z}) \propto m\ln(\beta)+d_1\ln(\alpha)+d_2\ln(\lambda)+(\beta-1)\sum_{i=1}^{m}\ln(w_i)-\alpha\xi_1(\beta)-\lambda\xi_2(\beta). \tag{3.4}$$

By differentiating equation (3.4) partially with respect to α, λ, β and equating them to zero, we get the following equations.

$$\frac{\partial\ell}{\partial\alpha} = \frac{d_1}{\alpha} - \xi_1(\beta) = 0 \quad \text{and} \quad \frac{\partial\ell}{\partial\lambda} = \frac{d_2}{\lambda} - \xi_2(\beta) = 0 \tag{3.5}$$

and

$$\frac{\partial\ell}{\partial\beta} = \frac{m}{\beta} + \sum_{i=1}^{m}\ln(w_i) - \alpha\xi_1^{'}(\beta) - \lambda\xi_2^{'}(\beta) = 0,$$

where $\xi_1'(\beta)$ and $\xi_2'(\beta)$ denotes the first order derivatives, with respect to β, of the $\xi_1(\beta)$ and $\xi_2(\beta)$, respectively, as given in the following

$$\xi_1'(\beta) = \sum_{i=1}^{m} z_i w_i^{\beta} \ln(w_i) + \sum_{i=1}^{J} s_i w_i^{\beta} \ln(w_i) + s^* w_m^{\beta} \ln(w_m),$$
$$\xi_2'(\beta) = \sum_{i=1}^{m} (1 - z_i) w_i^{\beta} \ln(w_i) + \sum_{i=1}^{J} q_i w_i^{\beta} \ln(w_i) + q^* w_m^{\beta} \ln(w_m).$$

Theorem 1. *If it is assumed that $X = (X_1, X_2, \cdots, X_{n_1})$ are i.i.d. by the Weibull(α, β) and $Y = (Y_1, Y_2, \cdots, Y_{n_2})$ are i.i.d. by the Weibull(λ, β), then under the JAPC-II censoring scheme, the MLEs of α and λ, denoted by $\hat{\alpha}$ and $\hat{\lambda}$, exist and they are given by*

$$\hat{\alpha} = \frac{d_1}{\xi_1(\beta)} \quad \text{and} \quad \hat{\lambda} = \frac{d_2}{\xi_2(\beta)}$$

Theorem 2. *If it is assumed that $X = (X_1, X_2, \cdots, X_{n_1})$ are i.i.d. by the Weibull(α, β) and $Y = (Y_1, Y_2, \cdots, Y_{n_2})$ are i.i.d. by the Weibull(λ, β), then under the JAPC-II censoring scheme, the MLE of β, denoted by $\hat{\beta}$, exists and it can be obtained as the solution of the following non-linear equation*

$$H(\beta) = \frac{m}{\beta} + \sum_{i=1}^{m} \ln(w_i) - \hat{\alpha}\xi_1'(\beta) - \hat{\lambda}\xi_2'(\beta). \tag{3.6}$$

The proofs of the existence and uniqueness of the $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\beta}$ are given in the Appendix section.

It is clear that the MLEs of the parameters exist only in the case that at least one failure is observed from both samples. In this censoring scheme, it is suggested that $1 \leq \sum_{i=1}^{m} s_i \leq m - 1$, the numbers of failures from the Sample-I and Sample-II can not exceed n_1 and n_2, respectively. Thus, the restriction $m - n_2 \leq \sum_{i=1}^{m} s_i \leq n_1$ is obtained. Consequently, the MLEs of the parameters exist when $\max(1, m - n_2) \leq \sum_{i=1}^{m} s_i \leq min(r - 1, n_1)$. It is clearly seen that the existence of the MLEs are guaranteed if $m > \max(n_1, n_2)$.

4 Asymptotic Confidence Intervals

Here, we consider the approximate confidence intervals for the unknown parameters α, λ, and β based on the large sample approximations of the maximum likelihood estimators (may be referred as asymptotic theory). For this purpose, we can use the observed Fisher information matrix, $I(\theta)$, where $\theta = (\alpha, \lambda, \beta)$, to obtain the asymptotic variance of MLEs of the unknown parameters. Then the observed Fisher information matrix is given by

$$I(\theta) = -\begin{pmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 L}{\partial \alpha \partial \lambda} & \frac{\partial^2 L}{\partial \alpha \partial \beta} \\ \frac{\partial^2 L}{\partial \lambda \partial \alpha} & \frac{\partial^2 L}{\partial \lambda^2} & \frac{\partial^2 L}{\partial \lambda \partial \beta} \\ \frac{\partial^2 L}{\partial \beta \partial \alpha} & \frac{\partial^2 L}{\partial \beta \partial \lambda} & \frac{\partial^2 L}{\partial \beta^2} \end{pmatrix} \quad and \quad I^{-1}(\theta) = \begin{pmatrix} \mathrm{Var}(\hat{\alpha}) & \mathrm{Cov}(\hat{\alpha}, \hat{\lambda}) & \mathrm{Cov}(\hat{\alpha}, \hat{\beta}) \\ \mathrm{Cov}(\hat{\lambda}, \hat{\alpha}) & \mathrm{Var}(\hat{\lambda}) & \mathrm{Cov}(\hat{\lambda}, \hat{\beta}) \\ \mathrm{Cov}(\hat{\beta}, \hat{\alpha}) & \mathrm{Cov}(\widehat{\beta}, \hat{\lambda}) & \mathrm{Var}(\hat{\beta}) \end{pmatrix}.$$

From the log-likelihood function in (3.4), we have

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{d_1}{\alpha^2}, \ \frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{d_2}{\lambda^2}, \ \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = 0, \ \frac{\partial^2 \ell}{\partial \alpha \partial \beta} = -\xi_1(\beta), \ \frac{\partial^2 \ell}{\partial \lambda \partial \beta} = -\xi_2(\beta),$$

and

$$\frac{\partial^2 \ell}{\partial \beta^2} = -\frac{m}{\beta^2} - \xi_1^{''}(\beta) - \xi_2^{''}(\beta)$$

where

$$\xi_1^{''}(\beta) = \sum_{i=1}^{m} z_i w_i^{\beta} \ln^2(w_i) + \sum_{i=1}^{J} s_i w_i^{\beta} \ln^2(w_i) + s^* w_m^{\beta} \ln^2(w_m)$$
$$\xi_2^{''}(\beta) = \sum_{i=1}^{m} (1 - z_i) w_i^{\beta} \ln^2(w_i) + \sum_{i=1}^{J} q_i w_i^{\beta} \ln^2(w_i) + q^* w_m^{\beta} \ln^2(w_m).$$

Note that $\xi_1^{''}(\beta)$ and $\xi_2^{''}(\beta)$ are the second order derivatives of $\xi_1(\beta)$ and $\xi_2(\beta)$ with respect to β, respectively. The asymptotic normality of the MLE tells us that $(\hat{\alpha}, \hat{\lambda}, \hat{\beta})$ is approximately multivariate normal with mean (α, λ, β) and the covariance matrix $I(\alpha, \lambda, \beta)$. That is, $(\hat{\alpha}, \hat{\lambda}, \hat{\beta}) \sim N\big\{(\alpha, \lambda, \beta), I^{-1}(\hat{\alpha}, \hat{\lambda}, \hat{\beta})\big\}$. Then, the approximate $100(1-\gamma)\%$ confidence intervals for α, λ, and β are obtained as

$$\left(\hat{\alpha} - z_{\frac{\gamma}{2}} \sqrt{Var(\hat{\alpha})}, \hat{\alpha} + z_{\frac{\gamma}{2}} \sqrt{Var(\hat{\alpha})}\right),$$

$$\left(\hat{\lambda} - z_{\frac{\gamma}{2}} \sqrt{Var(\hat{\lambda})}, \hat{\lambda} + z_{\frac{\gamma}{2}} \sqrt{Var(\hat{\lambda})}\right),$$

and

$$\left(\hat{\beta} - z_{\frac{\gamma}{2}}\sqrt{Var(\hat{\beta})}, \hat{\beta} + z_{\frac{\gamma}{2}}\sqrt{Var(\hat{\beta})}\right),$$

where $z_{\frac{\gamma}{2}}$ denotes the upper $\frac{\gamma}{2}^{th}$ percentage point of the standard normal distribution.

4.1 Bootstrap Confidence Intervals

Following asymptotic confidence intervals (ACI) for MLEs, bootstrap confidence intervals are obtained in this subsection. For this purpose, we obtain parametric percentile bootstrap method (boot-p) and studentized bootstrap method (boot-t) to obtain approximate confidence intervals for estimation. The corresponding algorithms which is defined by Efron and Tibshirani [13] can be described as follows.
For boot-p confidence intervals:

Step 1: Generate two samples X and Y of size n_1 and n_2 from Weibull populations, with corresponding scale parameters α, λ and shape parameter β, respectively.

Step 2: Evaluate the censored sample $(w_{(1)}, w_{(2)}, \cdots, w_{(m)}; z_{(1)}, z_{(2)}, \cdots, z_{(m)})$ where m is the number of failures. Then, calculate the MLEs of α, λ, and β.

Step 3: Generate bootstrap samples $(X_1^*, X_2^*, \cdots, X_{n_1}^*)$ and $(Y_1^*, Y_2^*, \cdots, Y_{n_2}^*)$ based on Weibull$(\hat{\alpha}, \beta)$ and Weibull$(\hat{\lambda}, \beta)$ then compute the bootstrap estimates of α, λ, and β which is denoted by $\hat{\alpha}^*$, $\hat{\lambda}^*$, and $\hat{\beta}^*$.

Step 4: Repeat steps 2-3 for B times and obtain bootstrap estimates $\hat{\alpha}_j^*$, $\hat{\lambda}_j^*$, $\hat{\beta}_j^*$, $j = 1, 2, \cdots, B$.

Step 5: Let $G(z) = P(\hat{\theta}_i^* \leq z)$ for $i = 1, 2$ be the CDF of $\hat{\theta}_i^*$ where $\theta = (\alpha, \lambda, \beta)$. Define $\hat{\theta}_{i_{Boot-p}} = G^{-1}(z)$ for a given z. Then, the approximate $100(1-\gamma)\%$ confidence interval of θ_i is given by

$$\left(\hat{\theta}^{*(\gamma/2)}_{i_{Boot-p}}, \hat{\theta}^{*(1-\gamma/2)}_{i_{Boot-p}}\right). \tag{4.7}$$

where $\hat{\theta}^{*(\gamma)}_{i_{Boot-p}}$ is the γ percentile of $\hat{\theta}^*_{i_j}$,$i = 1, 2$ and $j = 1, 2, \cdots, B$.

For boot-t confidence intervals, after generating the bootstrap samples and calculating the bootstrap estimates in the first four steps:

Step 1: Compute estimates of $Var(\hat{\theta}_i^*)$ from the observed Fisher information matrix in equation (4).

Step 2: Calculate

$$T^* = \frac{\hat{\theta}_i^* - \hat{\theta}_i}{\sqrt{Var(\hat{\theta}_i^*)}}, i = 1, 2.$$

Step 3: Let $H(z) = P(t^* \leq z)$ be the CDF t^*. From the obtained t^* values, the approximate $100(1-\gamma)\%$ confidence interval of θ_i can be obtained as follows. For a given z, define

$$\hat{\theta}_{iBoot-t} = \hat{\theta}_i + H^{-1}(z)\sqrt{Var(\hat{\theta}_i)}.$$

Then, the approximate $100(1-\gamma)\%$ confidence interval of θ_i is given by

$$\left(\hat{\theta}^{*(\gamma/2)}_{iBoot-t}, \hat{\theta}^{*(1-\gamma/2)}_{iBoot-t}\right). \tag{4.8}$$

5 Bayesian Inference

In this section, we obtain Bayes estimates of the unknown model parameters and the corresponding credible intervals based on the JAPC-II data. In this problem, we assume independent gamma priors for all unknown parameters α, λ, and β. Suppose, α, λ, and β have Gamma(a_1, b_1), Gamma(a_2, b_2), and Gamma(a_3, b_3) priors with non-negative hyper-parameters a_1, b_1, a_2, b_2, a_3, $b_3 (> 0)$. Thus, the joint prior distribution for α, λ, and β is given by

$$\pi_0(\alpha, \lambda, \beta) = \frac{b_1^{a_1} b_2^{a_2} b_3^{a_3}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \alpha^{a_1-1} e^{-\alpha b_1} \lambda^{a_2-1} e^{-\lambda b_2} \beta^{a_3-1} e^{-\beta b_3}, \quad \alpha, \lambda, \beta > 0.$$

Using the observed censored samples and the prior distributions for the parameters, the joint posterior density function of parameters α, λ, and β are obtained as

$$\pi(\alpha, \lambda, \beta | \mathbf{w,z}) \propto \alpha^{d_1+a_1-1} \lambda^{d_2+a_2-1} \beta^{m+a_3-1} e^{-\beta\left(b_3 - \sum_{i=1}^m \ln(w_i)\right)} e^{-\alpha\left(b_1+\xi_1(\beta)\right)} e^{-\lambda\left(b_2+\xi_2(\beta)\right)}. \tag{5.9}$$

From equation (5.9), it is clear that the posterior densities of α and λ are the gamma distribution as

$$\pi(\alpha) \sim \text{Gamma}\Big(d_1{+}a_1, b_1{+}\xi_1(\beta)\Big) \quad \text{and} \quad \pi(\lambda) \sim \text{Gamma}\Big(d_2{+}a_2, b_2{+}\xi_2(\beta)\Big)$$

and the posterior density of β is obtained as

$$\pi(\beta|\alpha,\lambda) \propto \beta^{m+a_3-1} e^{-\beta\left(b_3-\sum_{i=1}^{m}\ln(w_i)\right)} e^{-\alpha\left(b_1+\xi_1(\beta)\right)} e^{-\lambda\left(b_2+\xi_2(\beta)\right)}$$

It is clear that random samples of $\hat{\alpha}$ and $\hat{\lambda}$ can be easily obtained from gamma densities. However, the density of $\pi(\beta|\alpha,\lambda)$ can not be reduced analytically to well known distributions and therefore it is not possible to get sample directly by standard methods. It is observed that density plot of the conditional posterior density of β is like to Gaussian distribution (see Fig. 3).

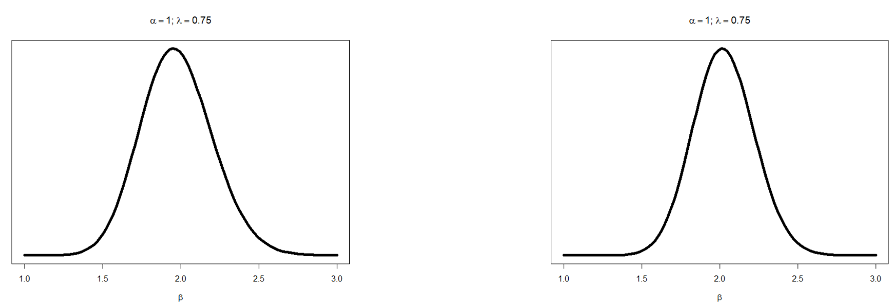

Figure 3: Plots for the posterior probability density function of β in the case of $n_1 = 40$, $n_2 = 34$ with $m = 55$(left) and $m = 65$(right).

In this case, we propose Metropolis-Hasting (M-H) sampling in Gibbs algorithm with normal proposal distribution as suggested by Tierney [25]. Thus, we consider Markov Chain Monte Carlo (MCMC) method and generate a sample by using Gibbs sampling with M-H algorithm under normal proposal as given in the following.

- **Step 1:** Start by using the initial values of $\alpha^{(0)}, \lambda^{(0)}$, and $\beta^{(0)}$.
- **Step 2:** Set $i = 1$.

- **Step 3:** Generate $\alpha^{(i)}$ from $\text{Gamma}\bigg(d_1 + a_1, b_1 + \xi_1(\beta)\bigg)$.

- **Step 4:** Generate $\lambda^{(i)}$ from $\text{Gamma}\bigg(d_2 + a_2, b_2 + \xi_2(\beta)\bigg)$.

- **Step 5:** Generate $\beta^{(i)}$ from $\pi(\beta|\alpha, \lambda)$ by using the M-H algorithm with normal proposal $N(\beta^{(i-1)}, \sigma_{\hat{\beta}})$ where $\sigma_{\hat{\beta}}$ is the asymptotic variance of the MLE of β.

  – Let $v = \beta^{(i-1)}$ and generate w from the proposal as $w = N(\beta^{(i-1)}, \sigma_{\hat{\beta}})$.

  – Let $p(v, w) = \min\bigg\{1, \frac{\pi(w|\alpha^{(i)},\lambda^{(i)})N(v,\sigma_{\hat{\beta}})}{\pi(v|\alpha^{(i)},\lambda^{(i)})N(w,\sigma_{\hat{\beta}})}\bigg\}$.

  – Generate U from $U(0, 1)$, then accept proposal if $U \leq p(v, w)$ and set $\beta^{(i)} = w$ or otherwise $\beta^{(i)} = v$.

- **Step 6:** Set $i = i + 1$.

- **Step 7:** Repeat 2-6, for M times.

Thus, the approximate posterior means of α, λ, and β, denoted by $\hat{\theta}^B = (\hat{\alpha}^B, \hat{\lambda}^B, \hat{\beta}^B)$, under the squared error loss function, can be derived as

$$\hat{\theta}^B = \frac{1}{M-K} \sum_{i=K+1}^{M} \theta^{(i)}$$

where K denotes the burn-in period. Further, we can construct the $100(1-\gamma)\%$ HPD credible intervals for the Bayesian estimations of the parameters by using the method proposed by Chen and Shao [9] as

$$\bigg(\hat{\theta}^B_{[\frac{\gamma}{2}(M-K)]}, \hat{\theta}^B_{[(1-\frac{\gamma}{2})(M-K)]}\bigg)$$

where $\theta = (\alpha, \lambda, \beta)$ with $[\frac{\gamma}{2}(M-B)]$ and $[(1-\frac{\gamma}{2})(M-B)]$ are the smallest integers less than or equal to $\frac{\gamma}{2}(M-B)$ and $(1-\frac{\gamma}{2})(M-B)$, respectively.

6 Simulation Study

In this section, some simulation studies are performed for evaluating the performance of estimation methods which are developed in the previous sections. We consider different sample sizes for the two populations such as $(n_1, n_2) = (25, 25), (40, 34), (50, 60)$ and different number of failure for each sample sizes as $(34, 42), (55, 65), (85, 95)$, respectively. We take parameter values for two populations as $(\alpha, \lambda, \beta) = (1, 0.75, 2)$ and two different time points T as 0.75 and 1.25. We compare the estimates with their mean squared errors (MSE) and approximate confidence intervals with their average lengths (AL) and coverage probabilities (CP). In Bayesian estimations, we use two different priors by using informative and non-informative hyper-parameters. Firstly, we consider the small and non-negative hyper-parameters as $a_1 = a_2 = b_1 = b_2 = 0.0001$ suggested by Congdon ([10], page 69) which are almost like Jeffrey's priors but they are proper, inversely. Alternatively, we use non-negative and informative hyper-parameters as $a_1 = 1, b_1 = 1$, $a_2 = 1.5, b_2 = 2$ and $a_1 = 2, b_1 = 1$ to provide the actual values of the parameters. We run a Monte Carlo process for 2000 times and computed the average values of the estimates as well as the average widths of 95% simultaneous confidence intervals. For bootstrap computations, we use 500 replications for each iteration. In Bayesian estimations, we run Gibbs sampling M-H algorithm and discard the first 500 values in burn-in period. Then, we take every third observation for the thinning procedure to obtain an uncorrelated Markov chain. Then we perform Markov Chain Monte Carlo (MCMC) which runs for 2000 replications. For all computations, R (R Core Team,[24]) software is used. We use three types of progressive censoring schemes as described in the following

$$\begin{aligned}
\text{CS-I} &\longrightarrow (R_1, R_2, \cdots, R_m) = (0_{(m-1)}, N-m) \\
\text{CS-II} &\longrightarrow (R_1, R_2, \cdots, R_m) = (0_{(\frac{m}{2}-1)}, N-m, 0_{(\frac{m}{2})}) \quad \text{if} \quad m \quad \text{is even,} \\
& = (0_{(\frac{m+1}{2}-1)}, N-m, 0_{(\frac{m+1}{2})}) \quad \text{if} \quad m \quad \text{is odd.} \\
\text{CS-III} &\longrightarrow (R_1, R_2, \cdots, R_m) = (1_{(N-m)}, 0_{(2m-N)})
\end{aligned}$$

where $0_{(k)}$ means repeated 0 values k times. Results for $T = 0.75$ are reported in Tables 1-3 and the results for $T = 1.25$ are reported in Tables 4-6. It is clearly seen that we obtain consistent results and all expectations are provided. It is known that Bayesian estimations perform better than MLE in small sample sizes. We see this observation in all cases. The performances of the Bayes estimates under informative priors are clearly better than the estimations under non-informative priors. It is observed that non-informative priors make Bayes estimation very close to the MLEs. However, using the informative priors promote the Bayesian method to have superiority over the MLEs. The differences of the performances between the Bayes and MLE estimations decrease in parallel to increasing sample sizes. Also, the mean squared errors of the estimators and the average lengths of the approximate confidence intervals decrease parallel to increasing sample sizes. Among all methods, boot-p methods have the worst performances. On the contrary, the boot-t method showed good performances and we obtain shorter credible intervals than ACIs. We observe that the coverage probabilities of the confidence intervals almost equal to their actual value 0.95 in most cases. Rarely, some intervals by bootstrap methods, especially boot-p methods obtain smaller CPs than expected to be. As expected, the MSE and ALs decrease if the observed sample size increases. In all schemes, we met these expectations. Since all theoretical findings perform very well, differences in the simulation schemes do not have any adverse effect on the performances of the estimates.

Table 1: In the case of $CS-I$ and $\alpha=1$, $\lambda=0.75$, $\beta=2$, $T=0.75$, the average estimates (AE) and the corresponding mean squared errors (MSE) of the parameters with average lengths (AL) and coverage probabilities (CP) of their approximate confidence intervals. (The Bayes and HPD columns present the results based on the non-informative and informative priors, respectively.)

			AE			MSE			AL					CP				
n_1,n_2	m		MLE	Bayes		MLE	Bayes		ACI	Boot-p	Boot-t	HPD		ACI	Boot-p	Boot-t	HPD	
25,25	34	α	1.0555	1.0609	1.0481	0.0906	0.0884	0.0721	1.0294	1.2148	1.0673	1.0224	0.9805	93.9000	92.3500	91.9500	94.3500	95.5500
		λ	0.7773	0.7835	0.7542	0.0491	0.0497	0.0349	0.7890	0.8952	0.8429	0.7830	0.7291	93.4500	92.4000	91.3500	93.6000	95.1500
		β	2.1037	2.0751	2.0651	0.0913	0.0835	0.0766	1.0792	1.1815	1.0536	1.0577	1.0310	95.1000	88.3500	93.0000	93.1500	94.5500
	42	α	1.0382	1.0484	1.0355	0.0684	0.0687	0.0583	0.9119	1.0306	0.9445	0.9121	0.8776	93.2000	91.5500	91.3000	93.8500	94.2500
		λ	0.7737	0.7868	0.7574	0.0361	0.0385	0.0275	0.7074	0.7780	0.7489	0.7089	0.6648	94.3500	93.7500	92.1000	94.7000	95.8500
		β	2.0999	2.0572	2.0678	0.0754	0.0722	0.0642	0.9862	1.0670	0.9651	0.9565	0.9428	95.5000	88.2000	92.7500	93.2500	94.8000
40,34	55	α	1.0320	1.0378	1.0302	0.0448	0.0431	0.0401	0.7619	0.8284	0.7809	0.7586	0.7402	94.1500	93.0000	92.9000	94.1500	94.5500
		λ	0.7653	0.7820	0.7517	0.0300	0.0307	0.0242	0.6354	0.6841	0.6618	0.6395	0.6026	93.6500	94.1000	92.0000	94.2000	95.0500
		β	2.0659	2.0481	2.0441	0.0518	0.0540	0.0475	0.8388	0.8875	0.8232	0.8179	0.8055	95.1500	90.2500	92.8500	92.8000	93.8000
	65	α	1.0282	1.0237	1.0277	0.0355	0.0376	0.0325	0.6991	0.7475	0.7147	0.6907	0.6824	95.0500	93.4000	94.0500	93.4500	95.1000
		λ	0.7672	0.7657	0.7563	0.0257	0.0248	0.0213	0.5860	0.6229	0.6074	0.5790	0.5597	94.3000	94.0500	93.0500	93.8000	95.0000
		β	2.0521	2.0459	2.0328	0.0436	0.0442	0.0408	0.7767	0.8146	0.7640	0.7576	0.7442	95.3500	91.6500	93.9500	92.1000	93.5000
50,60	85	α	1.0259	1.0209	1.0249	0.0308	0.0323	0.0285	0.6616	0.7006	0.6717	0.6547	0.6472	94.7500	93.6500	93.8500	94.4000	95.1000
		λ	0.7578	0.7627	0.7515	0.0152	0.0153	0.0136	0.4697	0.4854	0.4807	0.4670	0.4545	93.8500	94.1500	92.5000	94.0000	94.5500
		β	2.0459	2.0270	2.0330	0.0345	0.0319	0.0333	0.6711	0.6941	0.6608	0.6452	0.6414	93.8000	91.2500	93.2000	93.3000	91.6500
	95	α	1.0221	1.0204	1.0219	0.0300	0.0266	0.0281	0.6244	0.6556	0.6323	0.6184	0.6115	93.4000	92.9000	93.2000	93.6000	94.0500
		λ	0.7561	0.7591	0.7511	0.0132	0.0130	0.0120	0.4439	0.4558	0.4519	0.4409	0.4305	94.5500	94.2000	93.1500	93.5000	95.3000
		β	2.0448	2.0304	2.0318	0.0291	0.0316	0.0286	0.6384	0.6575	0.6289	0.6102	0.6060	94.4500	90.5500	93.0000	93.1000	91.5500

Table 2: In the case of $CS-II$ and $\alpha=1$, $\lambda=0.75$, $\beta=2$, $T=0.75$, the average estimates (AE) and the corresponding mean squared errors (MSE) of the parameters with average lengths (AL) and coverage probabilities (CP) of their approximate confidence intervals. (The Bayes and HPD columns present the results based on the non-informative and informative priors, respectively.)

			AE			MSE			AL					CP				
n_1,n_2	m		MLE	Bayes		MLE	Bayes		ACI	Boot-p	Boot-t	HPD		ACI	Boot-p	Boot-t	HPD	
25,25	34	α	1.0725	1.0563	1.0585	0.0865	0.0790	0.0692	1.0133	1.2007	0.9923	1.0218	0.9622	94.9500	90.7000	93.8000	94.6500	95.2500
		λ	0.7822	0.7995	0.7535	0.0481	0.0511	0.0340	0.7714	0.8850	0.7775	0.7928	0.7126	94.4500	92.9000	92.4500	94.1000	95.7000
		β	2.1223	2.0819	2.0716	0.1011	0.0889	0.0806	1.1023	1.2477	1.0866	1.0629	1.0610	95.2000	87.7500	93.0000	94.5000	94.4000
	42	α	1.0421	1.0423	1.0313	0.0617	0.0602	0.0529	0.8812	0.9958	0.9033	0.8794	0.8514	94.0500	92.0500	92.7500	94.9000	94.8500
		λ	0.7710	0.7746	0.7563	0.0362	0.0344	0.0276	0.6927	0.7596	0.7177	0.6854	0.6506	93.7500	93.2000	92.1000	94.3500	94.8000
		β	2.1048	2.0649	2.0627	0.0917	0.0825	0.0770	1.0434	1.1253	1.0138	1.0203	1.0029	95.2000	87.5000	93.3500	92.5000	93.8500
40,34	55	α	1.0326	1.0317	1.0285	0.0412	0.0376	0.0369	0.7341	0.8011	0.7398	0.7278	0.7133	94.8500	92.8500	93.5500	94.6500	95.3500
		λ	0.7743	0.7769	0.7570	0.0292	0.0292	0.0234	0.6266	0.6743	0.6367	0.6226	0.5943	94.6000	92.7500	93.0500	94.9000	95.2000
		β	2.0779	2.0412	2.0474	0.0599	0.0564	0.0528	0.8790	0.9404	0.8556	0.8492	0.8440	95.8000	89.6500	94.0500	93.1000	94.4000
	65	α	1.0178	1.0184	1.0229	0.0343	0.0307	0.0312	0.6758	0.7219	0.6860	0.6662	0.6609	94.8000	93.4000	94.2500	95.0500	94.9000
		λ	0.7665	0.7693	0.7541	0.0231	0.0245	0.0194	0.5758	0.6085	0.5903	0.5725	0.5512	94.7500	93.8500	93.0500	94.5500	95.2500
		β	2.0704	2.0449	2.0397	0.0508	0.0495	0.0469	0.8266	0.8614	0.8097	0.8033	0.7924	95.0000	91.5500	93.8000	93.2000	93.7500
50,60	85	α	1.0266	1.0257	1.0243	0.0287	0.0298	0.0266	0.6364	0.6723	0.6385	0.6309	0.6228	95.3000	93.4000	94.7500	94.5500	95.5000
		λ	0.7614	0.7622	0.7532	0.0138	0.0156	0.0123	0.4557	0.4717	0.4591	0.4527	0.4419	95.1500	94.0000	93.5500	93.0500	95.3000
		β	2.0485	2.0222	2.0302	0.0369	0.0359	0.0349	0.7079	0.7368	0.6929	0.6811	0.6783	95.1000	91.7000	94.2500	92.9000	93.5500
	95	α	1.0182	1.0142	1.0174	0.0258	0.0251	0.0242	0.6009	0.6278	0.6043	0.5946	0.5899	94.5500	93.7000	93.8500	94.7500	94.9500
		λ	0.7550	0.7599	0.7485	0.0129	0.0128	0.0118	0.4317	0.4441	0.4370	0.4303	0.4198	94.6500	94.1500	93.5500	95.0500	94.6000
		β	2.0437	2.0294	2.0287	0.0341	0.0346	0.0332	0.6825	0.6981	0.6706	0.6587	0.6519	95.1000	91.9500	94.2000	91.9000	92.7500

Table 3: In the case of $CS-III$ and $\alpha=1$, $\lambda=0.75$, $\beta=2$, $T=0.75$, the average estimates (AE) and the corresponding mean squared errors (MSE) of the parameters with average lengths (AL) and coverage probabilities (CP) of their approximate confidence intervals. (The Bayes and HPD columns present the results based on the non-informative and informative priors, respectively.)

			AE			MSE			AL					CP				
n_1,n_2	m		MLE	Bayes		MLE	Bayes		ACI	Boot-p	Boot-t	HPD		ACI	Boot-p	Boot-t	HPD	
25,25	34	α	1.0592	1.0535	1.0455	0.0748	0.0787	0.0598	0.9777	1.1207	0.9861	0.9636	0.9307	95.2500	92.3000	93.7500	93.7500	95.5500
		λ	0.7882	0.7813	0.7553	0.0491	0.0418	0.0339	0.7833	0.8844	0.7973	0.7687	0.7195	94.3500	93.1500	92.6500	95.3000	95.6000
		β	2.1281	2.0652	2.0631	0.1325	0.1090	0.1031	1.2698	1.3898	1.2390	1.2318	1.2055	94.8500	89.7000	93.3000	94.5000	94.6500
	42	α	1.0440	1.0355	1.0396	0.0574	0.0558	0.0495	0.8783	0.9691	0.8984	0.8639	0.8481	94.6000	93.4500	92.8000	94.8500	95.6500
		λ	0.7740	0.7693	0.7543	0.0362	0.0332	0.0276	0.6913	0.7510	0.7157	0.6824	0.6498	94.3500	93.5000	92.3000	94.4000	95.2000
		β	2.0919	2.0860	2.0509	0.0860	0.0819	0.0727	1.0736	1.1562	1.0527	1.0439	1.0292	95.5000	90.1500	94.0000	93.7500	94.9500
40,34	55	α	1.0294	1.0301	1.0254	0.0406	0.0382	0.0365	0.7208	0.7718	0.7255	0.7150	0.7020	94.0500	92.5500	92.8500	94.4000	94.0500
		λ	0.7770	0.7717	0.7584	0.0273	0.0286	0.0219	0.6298	0.6696	0.6388	0.6216	0.5968	95.3500	93.7000	94.3000	93.7500	96.0000
		β	2.0815	2.0424	2.0478	0.0739	0.0702	0.0645	0.9620	1.0121	0.9447	0.9325	0.9234	94.1000	89.1000	92.7500	92.7000	93.0000
	65	α	1.0174	1.0221	1.0165	0.0312	0.0316	0.0287	0.6686	0.7050	0.6785	0.6665	0.6534	94.9000	94.0000	93.7000	94.7000	95.0500
		λ	0.7625	0.7732	0.7502	0.0218	0.0223	0.0185	0.5730	0.6010	0.5857	0.5754	0.5485	94.6000	93.9500	92.7500	95.0000	95.3000
		β	2.0574	2.0311	2.0341	0.0507	0.0494	0.0464	0.8373	0.8751	0.8236	0.8167	0.8066	95.0000	91.2000	94.2000	92.4500	94.1000
50,60	85	α	1.0253	1.0164	1.0233	0.0293	0.0274	0.0271	0.6272	0.6526	0.6297	0.6168	0.6141	94.9000	93.0500	94.4000	94.4500	94.9000
		λ	0.7669	0.7648	0.7579	0.0149	0.0146	0.0133	0.4581	0.4733	0.4618	0.4532	0.4439	94.9000	93.9000	93.8500	94.4000	95.0000
		β	2.0444	2.0215	2.0241	0.0397	0.0384	0.0381	0.7537	0.7774	0.7429	0.7266	0.7219	95.6000	93.0000	94.5000	92.5000	93.4000
	95	α	1.0164	1.0193	1.0158	0.0252	0.0250	0.0236	0.5977	0.6209	0.6027	0.5951	0.5863	94.5000	94.1000	93.6000	94.7500	95.2500
		λ	0.7575	0.7667	0.7510	0.0127	0.0133	0.0116	0.4326	0.4434	0.4372	0.4332	0.4203	94.5500	94.2500	93.7000	95.1500	95.0500
		β	2.0399	2.0216	2.0236	0.0355	0.0326	0.0340	0.6904	0.7094	0.6809	0.6637	0.6602	94.2000	91.5000	93.8000	93.3000	91.9000

Table 4: In the case of $CS - I$ and $\alpha = 1$, $\lambda = 0.75$, $\beta = 2$, $T = 1.25$, the average estimates (AE) and the corresponding mean squared errors (MSE) of the parameters with average lengths (AL) and coverage probabilities (CP) of their approximate confidence intervals. (The Bayes and HPD columns present the results based on the non-informative and informative priors, respectively.)

			AE			MSE			AL					CP				
n_1, n_2	m		MLE	Bayes		MLE	Bayes		ACI	Boot-p	Boot-t	HPD		ACI	Boot-p	Boot-t	HPD	
25,25	34	α	1.0555	1.0609	1.0481	0.0906	0.0884	0.0721	1.0294	1.2148	1.0673	1.0224	0.9805	93.9000	92.3500	91.9500	94.4000	95.5500
		λ	0.7773	0.7835	0.7542	0.0491	0.0497	0.0349	0.7890	0.8952	0.8429	0.7830	0.7291	93.4500	92.4000	91.3500	93.6000	95.1500
		β	2.1037	2.0751	2.0651	0.0913	0.0835	0.0766	1.0792	1.1815	1.0536	1.0577	1.0310	95.1000	88.3500	93.0000	93.1500	94.5500
	42	α	1.0382	1.0484	1.0355	0.0684	0.0687	0.0583	0.9119	1.0306	0.9445	0.9121	0.8776	93.2000	91.5500	91.3000	93.8500	94.2500
		λ	0.7737	0.7868	0.7574	0.0361	0.0385	0.0275	0.7074	0.7780	0.7489	0.7089	0.6648	94.3500	93.7500	92.1000	94.7000	95.8500
		β	2.0999	2.0572	2.0678	0.0754	0.0722	0.0642	0.9862	1.0670	0.9651	0.9565	0.9428	95.5000	88.2000	92.7500	93.2500	94.8000
40,34	55	α	1.0320	1.0378	1.0302	0.0448	0.0431	0.0401	0.7619	0.8284	0.7809	0.7586	0.7402	94.1500	93.0000	92.9000	94.1500	94.5500
		λ	0.7653	0.7820	0.7517	0.0300	0.0307	0.0242	0.6354	0.6841	0.6618	0.6395	0.6026	93.6500	94.1000	92.0000	94.2000	95.0500
		β	2.0659	2.0481	2.0441	0.0518	0.0540	0.0475	0.8388	0.8875	0.8232	0.8179	0.8055	95.1500	90.2500	92.8500	92.8000	93.8000
	65	α	1.0282	1.0237	1.0277	0.0355	0.0376	0.0325	0.6991	0.7475	0.7147	0.6907	0.6824	95.0500	93.4000	94.0500	93.4500	95.1000
		λ	0.7672	0.7657	0.7563	0.0257	0.0248	0.0213	0.5860	0.6229	0.6074	0.5790	0.5597	94.3000	94.0500	93.0500	93.8000	95.0000
		β	2.0521	2.0459	2.0328	0.0436	0.0442	0.0408	0.7767	0.8146	0.7640	0.7577	0.7442	95.3500	91.6500	93.9500	92.1000	93.5000
50,60	85	α	1.0259	1.0209	1.0249	0.0308	0.0323	0.0285	0.6616	0.7006	0.6717	0.6547	0.6472	94.7500	93.6500	93.8500	94.4000	95.1000
		λ	0.7578	0.7627	0.7515	0.0152	0.0153	0.0136	0.4697	0.4854	0.4807	0.4670	0.4545	93.8500	94.1500	92.5000	94.0000	94.5500
		β	2.0459	2.0270	2.0330	0.0345	0.0319	0.0333	0.6711	0.6941	0.6608	0.6452	0.6414	93.8000	91.2500	93.2000	93.3000	91.6500
	95	α	1.0221	1.0204	1.0219	0.0300	0.0266	0.0281	0.6244	0.6556	0.6323	0.6184	0.6115	93.4000	92.9000	93.2000	94.3500	94.0500
		λ	0.7561	0.7591	0.7511	0.0132	0.0130	0.0120	0.4439	0.4558	0.4519	0.4409	0.4305	94.5500	94.2000	93.1500	94.9500	95.3000
		β	2.0448	2.0304	2.0318	0.0291	0.0316	0.0286	0.6384	0.6575	0.6289	0.6102	0.6060	94.4500	90.5500	93.0000	91.4500	91.5500

Table 5: In the case of $CS-II$ and $\alpha=1$, $\lambda=0.75$, $\beta=2$, $T=1.25$, the average estimates (AE) and the corresponding mean squared errors (MSE) of the parameters with average lengths (AL) and coverage probabilities (CP) of their approximate confidence intervals. (The Bayes and HPD columns present the results based on the non-informative and informative priors, respectively.)

			AE			MSE			AL					CP				
n_1,n_2	m		MLE	Bayes		MLE	Bayes		ACI	Boot-p	Boot-t	HPD		ACI	Boot-p	Boot-t	HPD	
25,25	34	α	1.0690	1.0537	1.0557	0.0871	0.0818	0.0689	1.0144	1.2038	1.0298	0.9938	0.9634	95.1000	92.5500	93.7000	94.9500	95.9000
		λ	0.7858	0.7943	0.7568	0.0494	0.0478	0.0343	0.7779	0.8940	0.8107	0.7750	0.7177	93.7000	92.1000	92.6500	94.6500	95.3500
		β	2.1037	2.0696	2.0546	0.0910	0.0849	0.0756	1.0486	1.1630	1.0262	1.0338	1.0081	94.6000	87.6000	92.1500	93.8000	94.0000
	42	α	1.0430	1.0456	1.0379	0.0644	0.0627	0.0543	0.8995	1.0171	0.9252	0.8947	0.8665	94.3500	93.1500	92.6000	94.2500	95.4500
		λ	0.7724	0.7888	0.7542	0.0369	0.0379	0.0282	0.6945	0.7663	0.7280	0.6999	0.6525	93.2500	92.6500	92.1500	94.3000	94.8000
		β	2.1078	2.0552	2.0711	0.0792	0.0697	0.0672	0.9790	1.0628	0.9584	0.9480	0.9384	94.5500	88.2500	92.2500	92.9000	93.6500
40,34	55	α	1.0312	1.0403	1.0271	0.0404	0.0407	0.0360	0.7411	0.8090	0.7522	0.7406	0.7207	94.5500	93.0000	93.9500	95.0500	95.4000
		λ	0.7693	0.7780	0.7529	0.0296	0.0306	0.0238	0.6231	0.6747	0.6401	0.6251	0.5908	94.1500	93.6500	92.9500	93.9000	95.0000
		β	2.0674	2.0508	2.0402	0.0520	0.0529	0.0477	0.8205	0.8722	0.8069	0.8029	0.7906	94.4000	89.7500	92.2000	92.5500	93.1500
	65	α	1.0194	1.0198	1.0183	0.0361	0.0336	0.0330	0.6821	0.7281	0.6928	0.6773	0.6655	93.1500	93.1000	91.8500	94.2000	94.2000
		λ	0.7633	0.7712	0.7513	0.0249	0.0254	0.0208	0.5761	0.6111	0.5942	0.5760	0.5507	93.5000	93.4500	91.5000	94.0500	94.6500
		β	2.0605	2.0409	2.0397	0.0459	0.0417	0.0427	0.7722	0.8123	0.7597	0.7487	0.7400	94.9000	90.0000	92.8000	93.1500	92.6000
50,60	85	α	1.0228	1.0166	1.0205	0.0277	0.0276	0.0257	0.6429	0.6787	0.6477	0.6340	0.6291	95.5500	94.0000	94.7500	95.0000	95.8500
		λ	0.7602	0.7638	0.7526	0.0140	0.0146	0.0125	0.4569	0.4745	0.4640	0.4547	0.4426	95.2500	93.9500	93.8500	94.3500	95.6000
		β	2.0382	2.0288	2.0216	0.0298	0.0297	0.0288	0.6527	0.6762	0.6435	0.6279	0.6234	95.4500	91.3000	94.0500	92.7500	93.2500
	95	α	1.0162	1.0159	1.0152	0.0254	0.0261	0.0237	0.6097	0.6401	0.6157	0.6051	0.5973	94.9500	94.5000	93.7000	94.5000	95.3000
		λ	0.7575	0.7575	0.7514	0.0137	0.0132	0.0124	0.4360	0.4496	0.4436	0.4322	0.4235	94.0500	93.3500	92.6500	94.2500	93.7500
		β	2.0453	2.0250	2.0310	0.0288	0.0277	0.0281	0.6310	0.6508	0.6226	0.6024	0.6003	94.7500	92.1500	93.8500	93.3000	92.7000

Table 6: In the case of $CS-III$ and $\alpha=1$, $\lambda=0.75$, $\beta=2$, $T=1.25$, the average estimates (AE) and the corresponding mean squared errors (MSE) of the parameters with average lengths (AL) and coverage probabilities (CP) of their approximate confidence intervals. (The Bayes and HPD columns present the results based on the non-informative and informative priors, respectively.)

			AE			MSE			AL					CP				
n_1,n_2	m		MLE	Bayes		MLE	Bayes		ACI	Boot-p	Boot-t	HPD		ACI	Boot-p	Boot-t	HPD	
25,25	34	α	1.0592	1.0535	1.0455	0.0748	0.0787	0.0598	0.9777	1.1207	0.9861	0.9636	0.9307	95.2500	92.3000	93.7500	93.7500	95.5500
		λ	0.7882	0.7813	0.7553	0.0491	0.0418	0.0339	0.7833	0.8844	0.7973	0.7687	0.7195	94.3500	93.1500	92.6500	95.3000	95.6000
		β	2.1281	2.0652	2.0631	0.1325	0.1090	0.1031	1.2698	1.3898	1.2390	1.2318	1.2055	94.8500	89.7000	93.3000	94.5000	94.6500
	42	α	1.0440	1.0350	1.0396	0.0574	0.0558	0.0495	0.8783	0.9691	0.8984	0.8639	0.8481	94.6000	93.4500	92.8000	94.8500	95.6500
		λ	0.7740	0.7719	0.7543	0.0362	0.0332	0.0276	0.6913	0.7510	0.7157	0.6824	0.6498	94.3500	93.5000	92.3000	94.4000	95.2000
		β	2.0919	2.0459	2.0509	0.0860	0.0819	0.0727	1.0736	1.1562	1.0527	1.0439	1.0292	95.5000	90.1500	94.0000	93.7500	94.9500
40,34	55	α	1.0294	1.0301	1.0254	0.0406	0.0382	0.0365	0.7208	0.7718	0.7255	0.7150	0.7020	94.0500	92.5500	92.8500	94.4000	94.0500
		λ	0.7770	0.7717	0.7584	0.0273	0.0286	0.0219	0.6298	0.6696	0.6388	0.6216	0.5968	95.3500	93.7000	94.3000	93.7500	96.0000
		β	2.0815	2.0424	2.0478	0.0739	0.0702	0.0645	0.9620	1.0121	0.9447	0.9325	0.9234	94.1000	89.1000	92.7500	92.7000	93.0000
	65	α	1.0174	1.0221	1.0165	0.0312	0.0316	0.0287	0.6686	0.7050	0.6785	0.6665	0.6534	94.9000	94.0000	93.7000	94.7000	95.0500
		λ	0.7625	0.7732	0.7502	0.0218	0.0223	0.0185	0.5730	0.6010	0.5857	0.5754	0.5485	94.6000	93.9500	92.7500	95.0000	95.3000
		β	2.0574	2.0311	2.0341	0.0507	0.0494	0.0464	0.8373	0.8751	0.8236	0.8167	0.8066	95.0000	91.2000	94.2000	92.4500	94.1000
50,60	85	α	1.0253	1.0164	1.0233	0.0293	0.0274	0.0271	0.6272	0.6526	0.6297	0.6168	0.6141	94.9000	93.0500	94.4000	94.4500	94.9000
		λ	0.7669	0.7648	0.7579	0.0149	0.0146	0.0133	0.4581	0.4733	0.4618	0.4532	0.4439	94.9000	93.9000	93.8500	94.4000	95.0000
		β	2.0444	2.0215	2.0241	0.0397	0.0384	0.0381	0.7537	0.7774	0.7429	0.7266	0.7219	95.6000	93.0000	94.5000	92.5000	93.4000
	95	α	1.0164	1.0193	1.0158	0.0252	0.0250	0.0236	0.5977	0.6209	0.6027	0.5951	0.5863	94.5000	94.1000	93.6000	94.7500	95.2500
		λ	0.7575	0.7667	0.7510	0.0127	0.0133	0.0116	0.4326	0.4434	0.4372	0.4332	0.4203	94.5500	94.2500	93.7000	95.1500	95.0500
		β	2.0399	2.0216	2.0236	0.0355	0.0326	0.0340	0.6904	0.7094	0.6809	0.6637	0.6602	94.2000	91.5000	93.8000	93.3000	91.9000

7 Real Data Example

In this section, we illustrate the theoretical findings on a real data example. We used progressively censored samples generated from the breaking strengths of jute fiber at different gauge lengths 20mm, 15mm for X and Y respectively which is given by Xia et al [26]. Recently, Çetinkaya [8] fitted this datasets to Weibull distributions and used for the stress-strength problem under generalized progressive hybrid censoring scheme. We scale both samples by dividing to 100 for better fitting to the re-parameterized Weibull model. This data-set of failure times are denoted by X (20mm) and Y (15mm), respectively. The complete (non-censored) failure times of each group with sizes $n_1 = n_2 = 30$ is given in the following

$$X \Rightarrow \begin{array}{l} 0.7146\ 4.1902\ 2.8464\ 5.8557\ 4.5660\ 1.1385\ 1.8785\ 6.8816\ 6.6266\ 0.4558 \\ 5.7862\ 7.5670\ 5.9429\ 1.6649\ 0.9972\ 7.0736\ 7.6514\ 1.8713\ 1.4596\ 3.5070 \\ 5.4744\ 1.1699\ 3.7581\ 5.8160\ 1.1986\ 0.4801\ 2.0016\ 0.3675\ 2.4453\ 0.8355 \end{array}$$

$$Y \Rightarrow \begin{array}{l} 5.9440\ 2.0275\ 1.6837\ 5.7486\ 2.2565\ 0.7638\ 1.5667\ 1.2781\ 8.1387\ 5.6239 \\ 4.6847\ 1.3509\ 0.7224\ 4.9794\ 3.5556\ 5.6907\ 6.4048\ 2.0076\ 5.5042\ 7.4875 \\ 4.8966\ 6.7806\ 4.5771\ 1.06737.1630\ 0.4266\ 0.8040\ 3.3922\ 0.7009\ 1.9342 \end{array}$$

These data sets can also be fitted by the exponential distributions as a constant failure rate distribution. We compare the fitting model selection measurements and goodness of fit statistics between the exponential and Weibull distributions. Thus, we can decide the better model for these data sets. We report the comparison results in Table 7-8. It is clearly seen that the Weibull model have higher log-likelihood with lower Akaike information criterion (AIC) and Bayesian information criterion (BIC) values. In the both cases, data sets fit the Weibull and exponential distributions since the p-value based on the Kolmogorov-Smirnov (KS), Cramér–von Mises (CvM) and Anderson Darling (AD) test statistics are obtained as larger than 0.05. Despite both distributions fitting these data sets well, model selection measurements (log-likelihood, AIC, BIC) indicate better fitting with the Weibull distributions.

Then, we consider two different adaptive progressive censoring scheme as providing $T < W_m$ as given in the following

- CS-I: $m = 44$, $T = 5$, $R = (0_{(43)}, 16_{(1)})$

Table 7: Model-selection measurements and goodness of fit test statistics with their corresponding p-values (in brackets) for the breaking strengths of jute fiber data at different gauge lengths 20mm (X).

Distribution	Log-Likelihood	AIC	BIC	K-S	CvM	AD
WE$(0.1670, 1.3606)$	-64.895	133.791	136.593	0.149 (0.473)	0.122 (0.490)	0.760 (0.501)
Exp(0.294)	-66.779	135.557	136.958	0.133 (0.618)	0.122 (0.492)	0.903 (0.412)

Table 8: Model-selection measurements and goodness of fit test statistics with their corresponding p-values (in brackets) for the breaking strengths of jute fiber data at different gauge lengths 15mm (Y).

Distribution	Log-Likelihood	AIC	BIC	K-S	CvM	AD
WE$(0.1230, 1.498)$	-65.675	135.349	138.151	0.169 (0.323)	0.168 (0.341)	0.945 (0.387)
Exp(0.2748)	-68.749	139.498	140.899	0.182 (0.240)	0.206 (0.257)	1.346 (0.218)

- CS-II: $m = 40$, $T = 2$, $R = (0_{(10)}, 1_{(20)}, 0_{(10)})$

where $R = (0_{(43)}, 16_{(1)})$ denotes the removals in progressive censoring schemes. For example, $(0_{(43)}, 16_{(1)})$ means never withdraws in first 43 failures and 16 withdraws in $m-$th failure. Based on the proposed censoring schemes, joint censored datasets, as $(w; z; s) = (w; z; R - q)$, are given in the following

CS-I

$(0.3675; 1; 0), (0.4266; 0; 0), (0.4558; 1; 0), (0.4801; 1; 0), (0.7009; 0; 0), (0.7146; 1; 0)$
$(0.7224; 0; 0), (0.7638; 0; 0), (0.8040; 0; 0), (0.8355; 1; 0), (0.9972; 1; 0), (1.0673; 0; 0)$
$(1.1385; 1; 0), (1.1699; 1; 0), (1.1986; 1; 0), (1.2781; 0; 0), (1.3509; 0; 0), (1.4596; 1; 0)$
$(1.5667; 0; 0), (1.6649; 1; 0), (1.6837; 0; 0), (1.8713; 1; 0), (1.8785; 1; 0), (1.9342; 0; 0)$
$(2.0016; 1; 0), (2.0076; 0; 0), (2.0275; 0; 0), (2.2565; 0; 0), (2.4453; 1; 0), (2.8464; 1; 0)$
$(3.5556; 0; 0), (3.7581; 1; 0), (4.1902; 1; 0), (4.5660; 1; 0), (3.3922; 0; 0), (3.5070; 1; 0)$
$(4.5771; 0; 0), (4.6847; 0; 0), (4.8966; 0; 0), (4.9794; 0; 0), (5.4744; 1; 0), (5.5042; 0; 0)$
$(5.6239; 0; 0), (5.6907; 0; 9)$

CS-II

$(0.3675; 1; 0), (0.4266; 0; 0), (0.4558; 1; 0), (0.4801; 1; 0), (0.7009; 0; 0), (0.7146; 1; 0)$
$(0.7224; 0; 0), (0.7638; 0; 0), (0.8040; 0; 0), (0.8355; 1; 0), (0.9972; 1; 0), (1.0673; 0; 0)$
$(1.1385; 1; 0), (1.1699; 1; 1), (1.1986; 1; 1), (1.2781; 0; 0), (1.4596; 1; 0), (1.5667; 0; 0)$
$(1.6649; 1; 0), (1.6837; 0; 0), (1.8785; 1; 0), (2.0016; 1; 0), (2.0076; 0; 0), (2.2565; 0; 1)$
$(2.4453; 1; 0), (2.8464; 1; 0), (3.3922; 0; 0), (3.5070; 1; 0), (3.5556; 0; 0), (4.1902; 1; 0)$
$(4.5771; 0; 0), (4.8966; 0; 0), (5.4744; 1; 0), (5.5042; 0; 0), (5.6239; 0; 0), (5.6907; 0; 0)$
$(5.8557; 1; 0), (5.9429; 1; 0), (5.9440; 0; 0), (6.6266; 1; 2)$

It is known that $(w; z; q) = (w; z; R - s)$. In complete cases, parameter estimations are obtained as 0.167 for α, 0.1230 for λ, 1.3606 and 1.4977 for β. We considered shape parameters as equal and rounded them to 1.400. We run the bootstrap loop for 10 000 replications and we use 100 000 iterations in MCMC for Bayesian estimations. In Bayesian estimations, we use MLEs of the parameters as their initial values so we do not use burn-in period. We take every 10th variate in the thinning procedure and we obtain uncorrelated Markov Chains. Based on the adaptive progressive censoring schemes which are given above, we obtain the estimates with their standard deviations and approximate confidence intervals with their lengths. All results are reported in Tables 9-10.

Table 9: Estimations and their standard deviations (second rows) for the real data example. (The columns of the Bayesian estimations are the results based on the non-informative and informative priors, respectively.)

CS	$\hat{\alpha}$	$\hat{\alpha}^B$		$\hat{\lambda}$	$\hat{\lambda}^B$		$\hat{\beta}$	$\hat{\beta}^B$	
I	0.1777	0.1883	0.1865	0.1800	0.1903	0.1885	1.1609	1.1357	1.1425
	0.0544	0.0570	0.0555	0.0545	0.0572	0.0556	0.1503	0.1467	0.1443
II	0.2121	0.2243	0.2217	0.1381	0.1464	0.1449	1.2671	1.2399	1.2448
	0.0658	0.0691	0.0672	0.0441	0.0464	0.0453	0.1571	0.1547	0.1517

Real data results in Table 9-10 clearly show that we obtain consistent results. Since our sample sizes are not reasonably small and the MLE and

Table 10: Confidence intervals with lengths (second rows) for the real data example. (The columns of the HPD intervals are the results based on the non-informative and informative priors, respectively.)

θ	CS	ACI	Boot-p	Boot-t	HPD	
	I	0.0710;0.2844	0.0833;0.3007	0.0158;0.2624	0.0977;0.3181	0.0953;0.3127
		0.2134	0.2174	0.2466	0.2204	0.2174
α	II	0.0831;0.3410	0.0962; 0.3365	0.0000;0.3055	0.1136;0.3811	0.1140; 0.3730
		0.2579	0.2403	0.3055	0.2675	0.2590
	I	0.0730;0.2869	0.0853; 0.3014	0.0196;0.2637	0.0990;0.3191	0.0988;0.3161
		0.2138	0.2160	0.2441	0.2201	0.2173
λ	II	0.0517;0.2245	0.0665;0.2714	0.0199;0.2145	0.0729;0.2506	0.0732;0.2475
		0.1728	0.2050	0.1945	0.1776	0.1743
	I	0.8663;1.4555	0.9296;1.5757	0.8794;1.4603	0.8568;1.4353	0.8671;1.4399
		0.5892	0.6461	0.5809	0.5785	0.5727
β	II	0.9592;1.5750	1.0214;1.7244	0.9697;1.5784	0.9416;1.5566	0.9524;1.5547
		0.6158	0.7030	0.6086	0.6149	0.6022

Bayesian estimation methods provide almost equal performances. In terms of their credible intervals, we observe similar performances. Despite all removals being at the last failure in CS-I and as messy in the middle failures in CS-II, the estimation performances are not affected by this difference. These results prove that we can use our findings confidently in all joint progressive and joint adaptive progressive censoring plans.

We also check the convergence of the Markov Chains. For this purpose, we use the Markov Chains generated for the real data example. Firstly, we draw trace plots that show the values of the parameters against the iteration number at each iteration. The trace plots for the parameters which are given in Fig. 4 show that Markov Chains for all parameters fluctuate around their centers with similar variations. Further, we see from the Fig. 5, the density plots seem in symmetric and unimodal shapes. The unimodal peaks in these plots determine the mode of the posterior distributions of the parameters and they are the values with the most support from the data and the chosen priors. Additionally, we use the running mean (ergodic average) plots as another diagnostic tool. An ergodic average plot draws the mean of sampled values up to iteration t, and the precision of an ergodic average depends on the autocorrelations of the chain. We see from the Fig. 6 they stabilize with increasing iteration number which means that the chains have achieved stationarity. All plots to check the convergence in a Markov Chain can be drawn using library MCMC plots [11] in R [24] software.

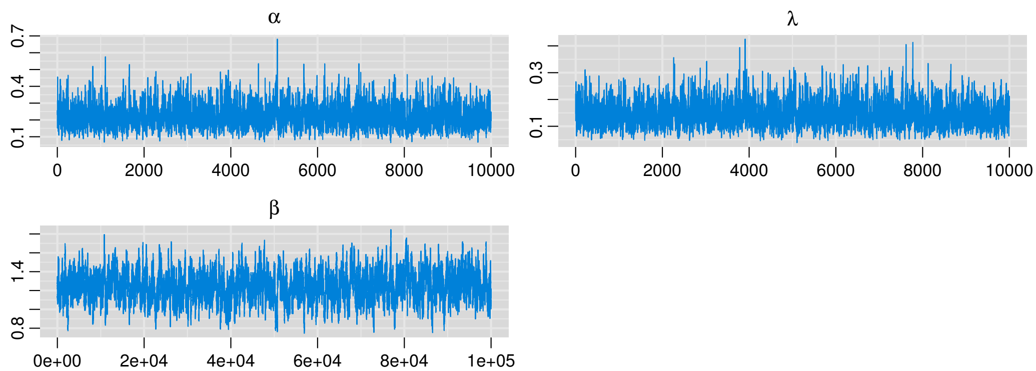

Figure 4: Trace plots of the posterior distribution the parameters.

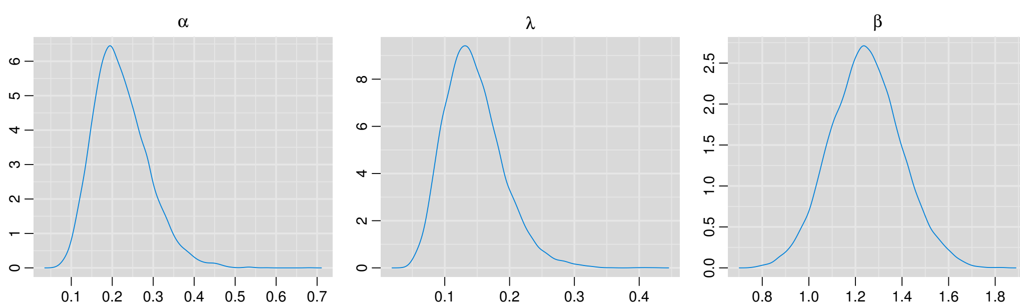

Figure 5: Density plots of the posterior distribution the parameters.

8 Balanced Joint Adaptive Progressive Censoring

The balanced joint adaptive progressive censoring (JAPC) scheme proposed by Sultana et. al [23] is an useful alternate to usual balanced progressive censoring (BJPC) scheme studied by Mondal and Kundu [17]. Balanced JAPC scheme has the advantage of getting same number of failures as in

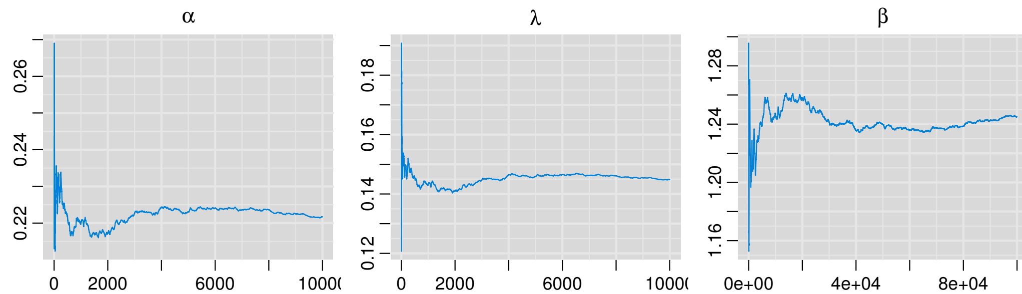

Figure 6: Running mean (ergodic average) plots of the posterior distribution the parameters.

BJPC scheme within a shorter experimental time. First we briefly describe the balanced JAPC scheme proposed by Sultana et. al [23]. Let us simultaneously place two independent samples of size n on the experiment. Before starting the life testing experiment, we prefixed an integer m, a time point T, and $(R_1, R_2, \cdots, R_m)$ is set of pre-fixed non-negative integers such that $\sum_{i=1}^{m} R_i = n - m$. Suppose the first failure occurs from Sample-I, then R_1 surviving units are randomly removed from the remaining $(n-1)$ surviving units of Sample-I and at the same time, $(R_1 + 1)$ units are randomly chosen and removed from the Sample-II. Further, if the second failure occurs from Sample-II, we remove R_2 units randomly from the remaining $(n - R_1 - 2)$ surviving units of Sample-II. At the same time $(R_2 + 1)$ units are randomly removed from the surviving $(n - R_1 - 1)$ units of Sample-I. We continue the process until either the m^{th} failure occurs or the time point T is reached. Suppose, m^{th} failure occurs at $W_{(m)}(< T)$, then we remove all the remaining surviving units from both the samples and stop the experiment. However, if only $J(< m)$ failures occurred before time point T, we follow the same procedure as before until $W_{(J)}$. But when the $(J+1)^{th}$ failure occurs at $W_{(J+1)}$, we will not remove any surviving units from the experiment until the m^{th} failure takes place from one of the samples. Thus, the experiment terminates by removing all the remaining items from it. In this scheme, we always observe a pre-fixed number of failure times and both the samples have the same number of surviving units at any time of the experiment. Also, notice that, if $T \to \infty$ then we get the usual balanced joint progressive Type-II censoring scheme proposed by Mandal and Kundu [17] and if $T = 0$ then we have $R_1 = R_2 = \cdots = R_{m-1} = 0$ and $R_m = n - m$, which corresponds to the joint Type-II censoring scheme proposed by Balakrishnan and Rasouli [5].

The joint censoring scheme proposed by Sultana et. al [23] as described in the above, we can formulate our modeling in the following form. Suppose we pre-fixed two independent random samples, say, Sample-I and Sample-II with the same number of sample points i.e., $n_1 = n_2 = n$. Further we fixed the number of removals $R_1, R_2, \cdots, R_m$ such that $\sum_{i=1}^{m} R_i = n-m$ and a pre-determined time point T. Now, like the previous discussion, the failures may occur either from Sample-I or Sample-II. Suppose, when first failure occurred from Sample-I (say), then we will remove R_1 surviving units from Sample-I, i.e., $s_1 = R_1$ and (R_1+1) units from Sample-II i.e., $q_1 = (R_1+1)$, to maintain the same number of samples in the test. Further, when a second failure occurs from Sample-II (say), then $R_2(= s_2)$ surviving units are randomly removed from Sample-II and $(R_2 + 1) = q_2$ surviving units removed from Sample-I. We continue the test until either m^{th} failure occurs or pre-fixed time point have been reached. The stopping criteria in this scenario is the same as the previous one. In this case, we will get the same number of data as in the previous case, and we denote it $(W_{(1)}, , \cdots, W_{(m)})$.

Let $d_1 = \displaystyle\sum_{i=1}^{m} z_i$ and $d_2 = \displaystyle\sum_{i=1}^{m}(1 - z_i)$ denotes the number of failures in the experiment from Sample-I and Sample-II, respectively. Now, as in the previous case Z_i, $i = 1, \cdots, m$, are the random variables which take the values either 0 or 1 depending on whether failures come from Sample-I or Sample-II, respectively. Therefore, the likelihood function can be written as

$$L(\mathbf{w},\mathbf{z}) \propto \prod_{i=1}^{m} f(w_i)^{z_i} g(w_i)^{1-z_i} [1 - F(w_i)]^{R_i^*+1-z_i} [1 - G(w_i)]^{R_i^*+z_i},$$

where R_i^* can be define as (see Sultana et. al [23])

$$R_i^* = \begin{cases} R_i, & \text{if } i = 1, \cdots, m \text{ and } w_m \leq T \text{ or } i = 1, \cdots, d \text{ and } T < w_m, \\ 0, & \text{if } i = d+1, \cdots, m-1 \text{ and } T < w_m, \\ n - d - \sum_{i=1}^{d} R_i, & \text{if } i = m \text{ and } T < w_m. \end{cases}$$

Hence, the likelihood function can be simplified as

$$L(\mathbf{w},\mathbf{z}) \propto \alpha^{d_1} \lambda^{d_2} \beta^m \, e^{(\beta-1)\sum_{i=1}^{m} \log w_i} \, e^{-(\alpha+\lambda)\sum_{i=1}^{m} w_i^{\beta}(1+R_i^*)}.$$

Now, taking the logarithm and solving the log-likelihood equation with respect to the unknown parameters we can obtain the MLEs of the model parameters. We can also construct the confidence intervals and the Bayesian analysis also can be done in the similar manner discussed in the previous sections.

9 Concluding Remarks

In this manuscript we have considered the joint adaptive progressive censoring scheme when the lifetime of the experimental units follow Weibull distributions. We have developed both the classical and Bayesian infernce of the unknown parameters. In this study we have assumed that the shape parameters of the two Weibull populations are same. This assumption is mainly due to the analytical tractability of the problem. But it may be mentioned that the assumption of the equal shape parameters for two Weibull populations is not very uncommon in the statistical literature. However, one can consider different shape and scale parameters for both the populations which will be more challenging problem. More work is needed in that direction.

It should be mentioned that all the computations are done using the software **R**. If anyone is interested, the codes are available upon request to the corresponding author.

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Appendix

It can be shown that the MLEs of α and λ maximize the log-likelihood function $\ell(\mathbf{w},\mathbf{z},\alpha,\lambda,\beta)$ for given β. For this purpose, let assume that $\phi(\theta) = \phi(\alpha,\lambda)$ be the Hessian matrix of $\ell(\mathbf{w},\mathbf{z},\alpha,\lambda,\beta)$ at $(\hat{\alpha},\hat{\lambda})$. Thus,

$$\phi_{ii}(\theta) = \frac{\partial^2 \ell}{\partial \theta_i^2} = -\frac{d_i}{\theta_i^2}, \quad i = 1,2 \quad \text{and} \quad \phi_{12}(\theta) = \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = 0$$

then, the determinant of the Hessian matrix is obtained as

$$\det[\phi(\hat{\theta})] = \phi_{11}(\hat{\alpha},\hat{\lambda})\phi_{22}(\hat{\alpha},\hat{\lambda}) - \big[\phi_{12}(\hat{\alpha},\hat{\lambda})\big]^2 = \frac{d_1 d_2}{\hat{\alpha}^2\hat{\lambda}^2} > 0.$$

It is clearly seen that $(\hat{\alpha},\hat{\lambda})$ is the local maximum of $\ell(\mathbf{w},\mathbf{z},\alpha,\lambda,\beta)$ for given β. Since there is no singular point of $\ell(\mathbf{w},\mathbf{z},\alpha,\lambda,\beta)$ and it has a single critical point, $\hat{\alpha}$ and $\hat{\lambda}$ are the absolute maximum of log-likelihood function.
On the other hand, we can express the equation (3.6) as

$$\frac{m}{\beta} = \hat{\alpha}\xi_1^{'}(\beta) + \hat{\lambda}\xi_2^{'}(\beta) - \sum_{i=1}^{m}\ln(w_i) \tag{9.10}$$

Let denote the left and the right hand sides of equation (9.10) by $\psi_1(\beta,\mathbf{w},\mathbf{z})$ and $\psi_2(\beta,\mathbf{w},\mathbf{z})$ respectively as given in the following

$$\psi_1(\beta,\mathbf{w},\mathbf{z}) = \frac{m}{\beta} \quad \text{and} \quad \psi_2(\beta,\mathbf{w},\mathbf{z}) = \hat{\alpha}\xi_1^{'}(\beta) + \hat{\lambda}\xi_2^{'}(\beta) - \sum_{i=1}^{m}\ln(w_i)$$

For a given sample of $\mathbf{w}$, it can be shown that $\psi_1(\beta,\mathbf{w},\mathbf{z})$ is a monotone increasing function of β with a finite and positive limit as $\beta \to \infty$. The plots of $\psi_1(\beta,\mathbf{w},\mathbf{z})$ and $\psi_2(\beta,\mathbf{w},\mathbf{z})$ would intersect exactly once at $\hat{\beta}$ since $\frac{m}{\beta}$ is strictly decreasing with right limit ∞ at 0. For proof, it can be shown that $\frac{\partial\psi_2(\beta,\mathbf{w},\mathbf{z})}{\partial\beta} \geq 0$. It is clearly seen that

$$\begin{aligned}
\frac{\partial\psi_2(\beta,\mathbf{w},\mathbf{z})}{\partial\beta} &= \hat{\alpha}\xi_1^{''}(\beta) + \hat{\lambda}\xi_2^{''}(\beta) \\
&= \hat{\alpha}\big[\sum_{i=1}^{m} z_i w_i^{\beta}\ln^2(w_i) + \sum_{i=1}^{J} s_i w_i^{\beta}\ln^2(w_i) + s^* w_m^{\beta}\ln^2(w_m)\big] \\
&+ \hat{\lambda}\big[\sum_{i=1}^{m}(1-z_i) w_i^{\beta}\ln^2(w_i) + \sum_{i=1}^{J} q_i w_i^{\beta}\ln^2(w_i) + q^* w_m^{\beta}\ln^2(w_m)\big] \\
&\geq 0
\end{aligned}$$

It shows that $\psi_2(\beta,\mathbf{w},\mathbf{z})$ is a monotone increasing function of β. Furthermore, $\lim_{\beta\to 0}\psi_2(\beta,\mathbf{w},\mathbf{z}) > 0$ and $\lim_{\beta\to\infty}\psi_2(\beta,\mathbf{w},\mathbf{z}) \to \infty$ indicates that the curves of $\psi_1(\beta,\mathbf{w},\mathbf{z})$ and $\psi_2(\beta,\mathbf{w},\mathbf{z})$ have a unique intersection point. Therefore, $\hat{\beta}$, as the root of equation $\psi_1(\beta,\mathbf{w},\mathbf{z}) = \psi_2(\beta,\mathbf{w},\mathbf{z})$ exist and unique.

Following the obtaining unique MLE of β, the MLE of α and λ can be obtained uniquely as $\hat{\alpha} = \hat{\alpha}(\hat{\beta})$ and $\hat{\lambda} = \hat{\lambda}(\hat{\beta})$, respectively on the condition that $m > \max(n_1, n_2)$. Thus, the existence and uniqueness of the MLEs are proved.

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