On efficient parameter estimation of elementary chirp model

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***Abstract*—Elementary chirp signals can be found in various fields of science and engineering. We propose two computationally efficient algorithms based on the choice of two different initial estimators to estimate the parameters of the elementary chirp model. It is observed that the proposed efficient estimators are consistent; they have the identical asymptotic distribution as that of the least squares estimators and they are also less computationally intensive. We also propose sequential efficient procedures to estimate the parameters of the multi-component elementary chirp model. The asymptotic properties of the sequential efficient estimators coincide with the least squares estimators. The important point about the efficient and sequential efficient algorithms is that these algorithms produce efficient frequency rate estimators in a fixed number of iterations. Another important point is that the under normal error assumption the theoretical variances of the proposed estimators achieve the Cramér-Rao lower bounds asymptotically. Simulation experiments are performed to see the performance of the proposed estimators, and it is observed that they are computationally efficient, take less time in computation than the other existing methods and perform well when two frequency rates are close to each other upto a reasonably low degree of separation. On an EEG dataset, we demonstrate the performance of the proposed algorithm.**

***Index Terms*—Elementary chirp, efficient algorithm, least squares, sequential efficient algorithm, frequency rate, consistency, asymptotic normality.**

I. Introduction

In this paper we consider the multi-component elementary chirp model embedded in noise and it can be expressed as follows:

$$y(t) = \sum_{k=1}^{p} A_k^0 e^{i\beta_k^0 t^2} + \epsilon(t); \quad t = 1, \ldots, N. \tag{1}$$

Here A_k^0s are unknown complex-valued non-zero amplitude parameters, β_k^0s are unknown frequency rates which are distinct and strictly lie between 0 and 2π, and $\epsilon(t)$ is an additive error present in the observed signal $y(t)$, satisfying assumption 1. The number of components, i.e., p is assumed to be known. Based on the observed data of size N, the problem is to estimate the unknown parameters A_k^0s and β_k^0s.

The elementary chirp signal can be found in many areas of statistical signal processing. This model has applications in radar [9], sonar [1], micro-Doppler signal [3], acoustic signal [2], see also Mboup and Adali [14], Casazza and Fickus [7], Mittal et al. [15] and the references cited therein. Estimating the chirp rate, i.e., the frequency rate, is crucial here. Some estimation techniques focus primarily on estimating the instantaneous frequency rate (IFR), which is twice the chirp rate. To name a few, the cubic phase function (CPF) [18] method and other methods inspired by CPF, for example, integrated CPF (ICPF) [20] and product CPF (PCPF) [22] are among them. Estimation methods for chirp signals based on fractional auto-correlation and fractional Fourier transform [17], fractional Fourier transform and the Wigner distribution [13], time-frequency rate distribution [19], and simplified linear canonical transform [12] have been discussed in the literature. In the literature, there is also a discussion of the Wigner-Hough transform-based approach for chirp signal detection [21]. Recently, least squares estimators (LSEs) [15] have been developed to estimate the frequency rates and the amplitude parameters of the elementary chirp model. It has been derived that the LSEs are consistent and follow normal distribution asymptotically.

The LSEs provide optimal rates of convergence for the amplitude parameters and the frequency rates, that are $O_p\left(N^{-\frac{1}{2}}\right)$ and $O_p\left(N^{-\frac{5}{2}}\right)$, respectively. Here, $a_N = O_p\left(N^{-\delta}\right)$ means that $N^{\delta} a_N$ is bounded in probability. Finding the LSEs is a computationally challenging problem due to the extremely non-linear nature of the function that needs to be optimized in terms of their parameters. We use Nelder-Mead method to obtain these estimators. However, obtaining them is a computationally intensive problem.

The objective of this paper is to obtain efficient estimators of the amplitude parameters and frequency rates that have the same optimal rates of convergence as the LSEs. We put forward two efficient algorithms based on choosing two different initial estimators of frequency rates to find efficient frequency rate estimates. It is observed that if we start with the initial estimate of

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β^0 with convergence rate $O_p\left(N^{-2}\right)$, then the proposed iterative procedure produces efficient frequency rate estimator in three steps, and if we start with the initial estimate of β^0 with convergence rate $O_p\left(N^{-\frac{3}{2}}\right)$, then the proposed iterative procedure produces efficient frequency rate estimator in five steps. In these efficient algorithms, we do not use the whole sample size at every step. At the initial steps, we use some fractions of the whole sample size, and at the final step, we use the whole sample size. Also, our iterative processes stop after a fixed number of steps. Further theoretical variances of the proposed estimators achieve their corresponding Cramér-Rao lower bound (CRLB) when errors follow normal distribution.

We do some numerical studies to assess the performance of the proposed finite step iterative algorithms for varying sample size and error variance. It is observed that the efficient algorithms work quite satisfactorily. The mean squared errors (MSEs) of the efficient estimators match well with the MSEs of the LSEs and the MSEs of all these methods are close to their theoretical asymptotic variance. The proposed efficient algorithms take significantly less time in computing the frequency rate estimates than the LSEs. Similar observations are recorded for the sequential efficient estimators in the case of multi-component model. We also observe that the sequential efficient estimators perform well when two frequency rates are close to each other to a certain degree of separation. Thus, for practical implementation, we can go with the sequential efficient estimators, which are computationally efficient.

The rest of the paper is structured as follows. We propose efficient algorithm for the one-component elementary chirp model and discuss the statistical properties of the efficient estimators in Section II. In Section III, we study statistical properties of the sequential efficient estimators for the multi-component elementary chirp model (1). We present simulation results in Section IV. Real data analysis is presented in Section V. In Section VI we conclude the paper. Proofs of all the results are given in the appendices.

II. One-Component Model

Consider the one-component elementary chirp model, mathematically expressed as follows:

$$y\left(t\right)=A^0e^{i\beta^0t^2}+\epsilon\left(t\right); \quad t=1,\ldots,N. \tag{2}$$

The aim is to estimate the unknown model parameters, i.e. the amplitude parameter and the frequency rate parameter, under the assumption on the error random variables $\epsilon\left(t\right)$s stated below.

Assumption 1: $\epsilon\left(t\right)$s are *i.i.d.* complex-valued random variables with mean 0 and variance $\frac{\sigma^2}{2}$ for both real and imaginary parts. Real and imaginary parts of $\epsilon\left(t\right)$ are assumed to be independent. Also, fourth order moment of $\epsilon\left(t\right)$ exists.

The following is the assumption on the unknown parameters:

Assumption 2: Let $\boldsymbol{\theta}^0$ be an interior point of the parameter space $\boldsymbol{\Theta}=[-S,S]\times[-S,S]\times[0,2\pi]$, and $\left|A^0\right|>0$.

We will use the following notations in this paper. A_R and A_I denote the real and the imaginary part of A; respectively, and $\epsilon_R\left(t\right)$ and $\epsilon_I\left(t\right)$ denote the real and the imaginary part of $\epsilon\left(t\right)$; respectively. $\boldsymbol{\theta}=\left(A_R,A_I,\beta\right)$, the parameter vector, $\boldsymbol{\theta}^0=\left(A_R^0,A_I^0,\beta^0\right)$, the true parameter vector, $\hat{\boldsymbol{\theta}}=\left(\hat{A}_R,\hat{A}_I,\hat{\beta}\right)$, the LSE of $\boldsymbol{\theta}^0$, and $\breve{\boldsymbol{\theta}}=\left(\breve{A}_R,\breve{A}_I,\breve{\beta}\right)$, the efficient estimators of $\boldsymbol{\theta}^0$.

LSEs are the most natural estimators to estimate the unknown parameters of the non-linear model. The LSEs of the parameters of the model (2) are determined by minimizing the following error sum of squares, say;

$$R\left(A,\beta\right)=\sum_{t=1}^{N}\left|y\left(t\right)-Ae^{i\beta t^2}\right|^2 \tag{3}$$

with respect to A and β simultaneously. Here, we do one-dimensional non-linear optimization to obtain the LSEs. For details, please see [15].

It has been proved that the LSEs of $\boldsymbol{\theta}^0$ are strongly consistent and follow normal distribution asymptotically [15] as follows:

$$(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}^0)\boldsymbol{D}^{-1}\xrightarrow{d}\mathcal{N}_3\left(0,\sigma^2\boldsymbol{\Sigma}^{-1}\right) \quad as \quad N \rightarrow\infty, \tag{4}$$

where $\boldsymbol{D}=diag\left(\frac{1}{\sqrt{N}},\frac{1}{\sqrt{N}},\frac{1}{N^2\sqrt{N}}\right)$ and

$$\boldsymbol{\Sigma}^{-1}=\begin{bmatrix}\frac{1}{2}+\frac{5A_I^{0^2}}{8|A^0|^2} & \frac{-5A_R^0A_I^0}{8|A^0|^2} & \frac{15A_I^0}{8|A^0|^2}\\ \frac{-5A_R^0A_I^0}{8|A^0|^2} & \frac{1}{2}+\frac{5A_R^{0^2}}{8|A^0|^2} & \frac{-15A_R^0}{8|A^0|^2}\\ \frac{15A_I^0}{8|A^0|^2} & \frac{-15A_R^0}{8|A^0|^2} & \frac{45}{8|A^0|^2}\end{bmatrix}. \tag{5}$$

In the next subsection, we propose two finite step iterative procedures based on the choice of initial estimators similar to the procedure of Bai et al. [4]. These methods give frequency rate estimators that converge at the same rate as the LSEs and identical in distribution as the LSEs.

A. Efficient Algorithm

We now introduce an iterative procedure that produces a frequency rate estimator with the same rate of convergence and asymptotic variance as the LSEs. First, we provide the algorithm, and then the theoretical justification is provided in Theorem 1.

If $\tilde{\beta}$ is a consistent estimator of β^0, such that $\tilde{\beta}-\beta^0=O_p\left(N^{-2-\delta}\right)$, for some $\delta>0$, then improved estimator of β^0 can be obtained as follows:

$$\breve{\beta}=\tilde{\beta}+\frac{11.25}{N^4}\operatorname{Im}\left(\frac{P_N^{\tilde{\beta}}}{Q_N^{\tilde{\beta}}}\right) \tag{6}$$

where,

$$P_N^{\tilde{\beta}}=\sum_{t=1}^{N} y(t)\left(t^2-\frac{N^2}{3}\right)e^{-i\tilde{\beta}t^2} \tag{7}$$

and

$$Q_N^{\tilde{\beta}}=\sum_{t=1}^{N} y(t)\,e^{-i\tilde{\beta}t^2}. \tag{8}$$

Here, $\operatorname{Im}(.)$ denotes the imaginary part of a complex number. We start with a consistent estimator $\tilde{\beta}$ as the initial estimator and then improve it using (6). The following result motivates the proposed algorithms.

Theorem 1: If $\tilde{\beta}-\beta^0=O_p\left(N^{-2-\delta}\right)$, where $\delta\in\left(0,\frac{1}{2}\right]$, then (a) $\breve{\beta}-\beta^0=O_p\left(N^{-2-2\delta}\right)$ if $\delta\le\frac{1}{4}$. (b) $N^{\frac{5}{2}}\left(\breve{\beta}-\beta^0\right)\xrightarrow{d}\mathcal{N}\left(0,\frac{45\sigma^2}{8|A^0|^2}\right)$ if $\delta>\frac{1}{4}$.

Proof: See Appendix A. ■

Now, we propose an efficient algorithm for two cases. First, when we use the estimator obtained using the dechirping method as the initial estimator. Second, when we use a periodogram-type estimator as the initial estimator.

1) Method 1: In this subsection, we propose an efficient algorithm based on the dechirping method. First, we present the approach to compute the estimator using dechirping method [6]. Here we obtain data from $y(t)\overline{y(t+1)}$ which can be written in the following expression:

$$z(t)=y(t)\overline{y(t+1)}=B^0e^{-i2\beta^0t}+e(t);\quad t=1,\ldots,N-1; \tag{9}$$

where, $B^0=|A^0|^2e^{-i\beta^0}$ and $e(t)=A^0e^{i\beta^0t^2}\overline{\epsilon(t+1)}+\overline{A^0}e^{-i\beta^0(t+1)^2}\epsilon(t)+\epsilon(t)\overline{\epsilon(t+1)}$. Here, $\overline{H}$ represents conjugate of a complex number H.

From equation (9), it is clear that this is a sinusoidal model with the frequency parameter $-2\beta^0$ and amplitude parameter B^0. Then the LSEs of B^0 and β^0, say, $\tilde{B}$ and $\tilde{\beta}$, respectively, are determined by minimizing the following error sum of squares:

$$R(B,\beta)=\sum_{t=1}^{N-1}\left|z(t)-Be^{-i2\beta t}\right|^2, \tag{10}$$

with respect to B and β, simultaneously. Note that we have to minimize (10) when $\beta^0\in(0,\pi)$, because (10) is a error sum of squares of a sinusoidal model with frequency $-2\beta^0$. Thus, we can only use the dechirping method for elementary chirp model when $\beta^0\in(0,\pi)$. However, the range of β^0 is $(0,2\pi)$ in general. We denote by B_R and B_I as the real and the imaginary parts of the B; respectively, and the real and the imaginary parts of the $e(t)$ as $e_R(t)$ and $e_I(t)$; respectively. Also, we denote $\tilde{\boldsymbol{v}}=\left(\tilde{B}_R,\tilde{B}_I,\tilde{\beta}\right)$ as the LSE of $\boldsymbol{v}^0=\left(B_R^0,B_I^0,\beta^0\right)$.

It can be shown that the dechirping method provides an estimator of β^0 with the convergence rate $O_p\left(N^{-\frac{3}{2}}\right)$, i.e., $\tilde{\beta}-\beta^0=O_p\left(N^{-\frac{3}{2}}\right)$.

Now, we propose the finite step algorithm when initial estimator is the estimator obtained using dechirping method. We start with the LSE of β^0 obtained using dechirping method and improve it at each step using the following finite step iterative algorithm. The m^{th} step estimator $\breve{\beta}^{(m)}$ is calculated from the $(m-1)^{th}$ step estimator $\breve{\beta}^{(m-1)}$, by the formula

$$\breve{\beta}^{(m)}=\breve{\beta}^{(m-1)}+\frac{11.25}{N_m^4}\operatorname{Im}\left[\frac{P_{N_m}^{\breve{\beta}}}{Q_{N_m}^{\breve{\beta}}}\right], \tag{11}$$

where $P_{N_m}^{\breve{\beta}}$ and $Q_{N_m}^{\breve{\beta}}$ can be determined from $P_N^{\tilde{\beta}}$ and $Q_N^{\tilde{\beta}}$ by replacing N and $\tilde{\beta}$ with N_m and $\breve{\beta}^{(m-1)}$, respectively. Here, N_m is the sample size at the m^{th} iterative step. In the following steps, $\lceil x\rceil$ denotes the least integer function, i.e., the least integer greater than or equal to x. We use formula given in (11) repeatedly by choosing N_m at every step as follows:

Step 1 Choose $N_1=\lceil N^{\frac{2}{3}}\rceil$ for $m=1$ and $\breve{\beta}^{(0)}=\tilde{\beta}$. Note that $\tilde{\beta}-\beta^0=O_p\left(N^{-\frac{3}{2}}\right)=O_p\left(N_1^{-2-\frac{1}{4}}\right)$. Then applying part (a) of theorem 1, we have;
$$\breve{\beta}^{(1)}-\beta^0=O_p\left(N_1^{-2-\frac{1}{2}}\right)=O_p\left(N^{-\frac{5}{3}}\right).$$

Step 2 Choose $N_2=\lceil N^{\frac{3}{4}}\rceil$ for $m=2$ and calculate $\breve{\beta}^{(2)}$ from $\breve{\beta}^{(1)}$ using (11). Since $\breve{\beta}^{(1)}-\beta^0=O_p\left(N^{-\frac{5}{3}}\right)=O_p\left(N_2^{-2-\frac{2}{9}}\right)$, thus using part (a) of theorem 1, we have;
$$\breve{\beta}^{(2)}-\beta^0=O_p\left(N_2^{-2-\frac{4}{9}}\right)=O_p\left(N^{-\frac{11}{6}}\right).$$

Step 3 Choose $N_3=\lceil N^{\frac{22}{27}}\rceil$ for $m=3$ and calculate $\breve{\beta}^{(3)}$ from $\breve{\beta}^{(2)}$ using (11). Since $\breve{\beta}^{(2)}-\beta^0=O_p\left(N^{-\frac{11}{6}}\right)=O_p\left(N_3^{-2-\frac{1}{4}}\right)$, therefore using part (a) of theorem 1, we have;
$$\breve{\beta}^{(3)}-\beta^0=O_p\left(N_3^{-2-\frac{1}{2}}\right)=O_p\left(N^{-\frac{55}{27}}\right).$$

Step 4 Choose $N_4=\lceil N^{\frac{220}{243}}\rceil$ for $m=4$ and calculate $\breve{\beta}^{(4)}$ from $\breve{\beta}^{(3)}$ using (11). Since $\breve{\beta}^{(3)}-\beta^0=O_p\left(N^{-\frac{55}{27}}\right)=O_p\left(N_4^{-2-\frac{1}{4}}\right)$, therefore using part (a) of theorem 1, we have;
$$\breve{\beta}^{(4)}-\beta^0=O_p\left(N_4^{-2-\frac{1}{2}}\right)=O_p\left(N^{-\frac{550}{243}}\right).$$

Step 5 Choose $N_5=N$ for $m=5$ and calculate

$\breve{\beta}^{(5)}$ from $\breve{\beta}^{(4)}$ using (11). Since $\breve{\beta}^{(4)} - \beta^0 = O_p\left(N^{-\frac{550}{243}}\right) = O_p\left(N_5^{-2-\frac{64}{243}}\right)$ and $\frac{64}{243} > \frac{1}{4}$, therefore using part (b) of theorem 1, we have;

$$N^{\frac{5}{2}}\left(\breve{\beta} - \beta^0\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{45\sigma^2}{8\left|A^0\right|^2}\right).$$

2) Method 2: In this subsection, we propose an efficient algorithm based on the periodogram-type estimator. First, we see how to obtain the periodogram-type estimator. The periodogram-type function for the elementary chirp model [15] is expressed as follows:

$$I(\beta) = \frac{1}{N}\left|\sum_{t=1}^{N} y(t)\, e^{-i\beta t^2}\right|^2. \quad (12)$$

The periodogram-type estimator is obtained by maximising the periodogram-type function (12) over the grid of the type $\frac{2\pi k}{N^2}, k = 1, \ldots, N^2 - 1$ which has the convergence rate $O_p\left(N^{-2}\right)$.

Now, we propose the finite step algorithm when initial estimator is the periodogram-type estimator. We start with the periodoram-type estimator of β^0 and improve it at each step using the following finite step iterative algorithm. We apply formula given in (11) repeatedly by choosing N_m at every step as follows:

Step 1 Choose $N_1 = \lceil N^{\frac{8}{9}} \rceil$ for $m = 1$ and $\breve{\beta}^{(0)} = \tilde{\beta}$, the periodogram-type estimator. Note that $\tilde{\beta} - \beta^0 = O_p\left(N^{-2}\right) = O_p\left(N_1^{-2-\frac{1}{4}}\right)$. Then applying part (a) of theorem 1, we have;

$$\breve{\beta}^{(1)} - \beta^0 = O_p\left(N_1^{-2-\frac{1}{2}}\right) = O_p\left(N^{-\frac{20}{9}}\right).$$

Step 2 Choose $N_2 = \lceil N^{\frac{80}{81}} \rceil$ for $m = 2$ and calculate $\breve{\beta}^{(2)}$ from $\breve{\beta}^{(1)}$ using (11). Since $\breve{\beta}^{(1)} - \beta^0 = O_p\left(N^{-\frac{20}{9}}\right) = O_p\left(N_2^{-2-\frac{1}{4}}\right)$, therefore applying part (a) of theorem 1, we have;

$$\breve{\beta}^{(2)} - \beta^0 = O_p\left(N_2^{-2-\frac{1}{2}}\right) = O_p\left(N^{-\frac{200}{81}}\right).$$

Step 3 Choose $N_3 = N$ for $m = 3$ and calculate $\breve{\beta}^{(3)}$ from $\breve{\beta}^{(2)}$ using (11). Since $\breve{\beta}^{(2)} - \beta^0 = O_p\left(N^{-\frac{200}{81}}\right) = O_p\left(N_3^{-2-\frac{38}{81}}\right)$ and $\frac{38}{81} > \frac{1}{4}$, therefore, using part (b) of theorem 1, we have:

$$N^{\frac{5}{2}}\left(\breve{\beta} - \beta^0\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{45\sigma^2}{8\left|A^0\right|^2}\right).$$

Therefore, if we have an estimator of order $O_p\left(N^{-2-\delta}\right)$ at any step, then using the above discussed algorithms, one can obtain an estimator with improved rate of convergence $O_p\left(N^{-2-2\delta}\right)$ if $\delta \leq \frac{1}{4}$, otherwise it provides an efficient estimator. Here, initial estimators are obtained using the dechirping method and the periodogram-type estimator, which are of orders $O_p\left(N^{-\frac{3}{2}}\right)$ and $O_p\left(N^{-2}\right)$, respectively. Since, an estimator of order $O_p\left(N^{-2-\delta}\right)$ is required to implement these algorithms, we choose a fraction of the full sample size at the first step of these algorithms, and then we get the initial estimator of order $O_p\left(N^{-2-\delta}\right)$. Gradually, we increase the sample size in the subsequent steps of these algorithms to get an efficient estimator of the frequency rate.

The key advantage of the proposed efficient algorithms is that they provide an efficient estimator of frequency rate with optimal rate of convergence in a fixed number of iterations. Once $\breve{\beta}$ is obtained, the amplitude parameters can be easily estimated using simple linear regression. Explicitly, we can write $\breve{A}$ as $\breve{A} = \breve{A}_R + i\breve{A}_I$, where,

$$\breve{A}_R = \frac{1}{N}\sum_{t=1}^{N}\left(y_R(t)\cos\left(\breve{\beta}t^2\right) + y_I(t)\sin\left(\breve{\beta}t^2\right)\right), \quad (13)$$

$$\breve{A}_I = \frac{1}{N}\sum_{t=1}^{N}\left(y_I(t)\cos\left(\breve{\beta}t^2\right) - y_R(t)\sin\left(\breve{\beta}t^2\right)\right). \quad (14)$$

Here, $y_R(t)$ and $y_I(t)$ are the real and imaginary parts of the data $y(t)$, (2), and expressed as follows:

$$y_R(t) = A_R^0 \cos\left(\beta^0 t^2\right) - A_I^0 \sin\left(\beta^0 t^2\right) + \epsilon_R(t) \quad (15)$$

$$y_I(t) = A_R^0 \sin\left(\beta^0 t^2\right) + A_I^0 \cos\left(\beta^0 t^2\right) + \epsilon_I(t) \quad (16)$$

The following theorem provides the asymptotic distribution of the efficient amplitude estimator.

Theorem 2: If assumptions 1 and 2 hold true, asymptotic distributions of $\breve{A}_R$ and $\breve{A}_I$ are given as follows:

$$N^{\frac{1}{2}}\left(\breve{A}_R - A_R^0\right) \xrightarrow{d} \mathcal{N}\left(0, \left(\frac{1}{2} + \frac{5A_I^{0^2}}{8|A^0|^2}\right)\sigma^2\right);$$

$$N^{\frac{1}{2}}\left(\breve{A}_I - A_I^0\right) \xrightarrow{d} \mathcal{N}\left(0, \left(\frac{1}{2} + \frac{5A_R^{0^2}}{8|A^0|^2}\right)\sigma^2\right).$$

Proof: See Appendix A. ∎

Therefore, we have efficient estimators of the unknown parameters of model (2) that are consistent, have the optimal rate of convergence, and have the same asymptotic distribution as that of the LSEs.

Remark: Note that the values of the N_m that we choose here are not unique. There are several other combinations of N_ms which can be chosen such that the proposed iterative algorithms converge in five steps as discussed in subsection II-A1 and in three steps as discussed in subsection II-A2, respectively. For example, $N_1 = \lceil N^{\frac{67}{100}} \rceil$, $N_2 = \lceil N^{\frac{74}{100}} \rceil$, $N_3 = \lceil N^{\frac{82}{100}} \rceil$, $N_4 = \lceil N^{\frac{91}{100}} \rceil$ and $N_5 = N$ is another set of such N_ms for which the algorithm converges in five steps and $N_1 = \lceil N^{\frac{90}{100}} \rceil$, $N_2 = \lceil N^{\frac{96}{100}} \rceil$ and $N_3 = N$ is another set of N_ms for which the algorithm converges in three steps. Therefore, we can choose one of the sets of the N_ms among several such sets. Also, it has been observed that results are more or less the same for different sets of N_ms.

III. Multi-Component Model

In this section, we consider the multi-component elementary chirp model (1). We propose a sequential efficient algorithm under the error assumption 1. Let us denote by $\boldsymbol{\mu}$ the parameter vector for the model (1), $\boldsymbol{\mu} = (A_{R1}, A_{I1}, \beta_1, \ldots, A_{Rp}, A_{Ip}, \beta_p)$. Also, $\boldsymbol{\mu}^0$ denote the true parameter vector.

The underlying assumptions of the unknown parameters are as follows:

Assumption 3: $\boldsymbol{\mu}^0$ is an interior point of the parameter space $\boldsymbol{\Theta}^{(p)}$; $\boldsymbol{\Theta} = [-S, S] \times [-S, S] \times [0, 2\pi]$ and the frequency rates β_k^0s are distinct for $k = 1, \cdots, p$.

Assumption 4: The amplitude parameters; A_k^0s satisfy the following relationship:

$$2S^2 > \left|A_1^0\right|^2 > \left|A_2^0\right|^2 > \cdots > \left|A_p^0\right|^2 > 0.$$

A. Sequential Efficient Algorithm

We now propose an iterative procedure that produces estimators of frequency rates with the same rate of convergence and asymptotic variance as the LSEs. First, we provide the algorithm, and then the theoretical justification is provided in Theorem 3. If $\tilde{\beta}_k$ is a consistent estimator of β_k^0, for $k = 1, \cdots, p$, such that $\tilde{\beta}_k - \beta_k^0 = O_p\left(N^{-2-\delta}\right)$, for some $\delta > 0$, then improved estimator of β_k^0 can be obtained as follows:

$$\breve{\beta}_k = \tilde{\beta}_k + \frac{11.25}{N^4} \operatorname{Im}\left(\frac{P_N^{\tilde{\beta}_k}}{Q_N^{\tilde{\beta}_k}}\right) \tag{17}$$

where,

$$P_N^{\tilde{\beta}_k} = \sum_{t=1}^{N} y_k(t)\left(t^2 - \frac{N^2}{3}\right) e^{-i\tilde{\beta}_k t^2} \tag{18}$$

and

$$Q_N^{\tilde{\beta}_k} = \sum_{t=1}^{N} y_k(t)\, e^{-i\tilde{\beta}_k t^2}. \tag{19}$$

Here, $y_k(t)$ is the data at the kth step of the sequential algorithm. We start with any consistent estimator $\tilde{\beta}_k$ as the initial estimator and then improve it using (17). The following result motivates the proposed algorithm:

Theorem 3: If $\tilde{\beta}_k - \beta_k^0 = O_p\left(N^{-2-\delta}\right)$, for $k = 1, \cdots, p$, where $\delta \in \left(0, \frac{1}{2}\right]$, then

(a) $\breve{\beta}_k - \beta_k^0 = O_p\left(N^{-2-2\delta}\right)$ if $\delta \leq \frac{1}{4}$.

(b) $N^{\frac{5}{2}}\left(\breve{\beta}_k - \beta_k^0\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{45\sigma^2}{8\left|A_k^0\right|^2}\right)$ if $\delta > \frac{1}{4}$.

Proof: See Appendix B. ■

Now, we propose a sequential efficient algorithm for two cases. First, when we use LSEs obtained through the dechirping method as the initial estimators; second, when we use periodogram-type estimators as the initial estimators.

1) Method 1: In this subsection, we propose a sequential efficient algorithm based on the dechirping method. In the sequential procedure, components of the model (1) are estimated in a sequential manner [15]. We estimate the first chirp component as discussed in subsection II-A1 and then remove its effect from the original data. Then, we use the adjusted data to estimate the second chirp component, and continue to do so until all p-chirp components have been estimated. The data at the kth step of the sequential algorithm is given as follows:

$$y_k(t) = y_{k-1}(t) - \breve{A}_{k-1} e^{i\breve{\beta}_{k-1} t^2};\ t = 1, \ldots, N. \tag{20}$$

To apply this algorithm, we need that $\tilde{\beta}_k - \beta_k^0 = O_p\left(N^{-\frac{3}{2}}\right)$; $\forall k = 1, \ldots, p$, which can be proved.

Remark: For the usual multi-component chirp model where we also have the frequency term the dechirping method does not work. Because in that case we have global maxima that occur far from the true parameter value due to the cross product term (see [5] for details).

2) Method 2: In this subsection, we propose a sequential efficient algorithm based on the periodogram-type estimator. In this procedure, we estimate the first chirp component as discussed in subsection II-A2 and then remove its effect from the original data. Then, we estimate the second chirp component using the adjusted data, and we keep doing this until we have estimated all the p-chirp components.

Through the following theorem, we provide the asymptotic distribution of the amplitude estimators at every step.

Theorem 4: If assumptions 1, 3 and 4 hold true, asymptotic distributions of $\breve{A}_{Rk}$ and $\breve{A}_{Ik}$ $\forall k = 1, \ldots, p$, are given as follows: $N^{\frac{1}{2}}\left(\breve{A}_{Rk} - A_{Rk}^0\right) \xrightarrow{d} \mathcal{N}\left(0, \left(\frac{1}{2} + \frac{5A_{Ik}^{0^2}}{8\left|A_k^0\right|^2}\right)\sigma^2\right)$ and

$$N^{\frac{1}{2}}\left(\breve{A}_{Ik} - A_{Ik}^0\right) \xrightarrow{d} \mathcal{N}\left(0, \left(\frac{1}{2} + \frac{5A_{Rk}^{0^2}}{8\left|A_k^0\right|^2}\right)\sigma^2\right).$$

Here, $\breve{A}_{Rk}$ and $\breve{A}_{Ik}$ denote the real and imaginary parts of $\breve{A}_k$ at k^{th} step and they can be obtained along the similar lines as $\breve{A}_R$ (13) and $\breve{A}_I$ (14).

Proof: See Appendix B. ■

Next, we provide the results for the consistency of the proposed sequential efficient estimators, when p are unknown.Therefore, we take the following situations: (a) when the fitted model's number of components is less than or equal to the true number of components, and (b) when the fitted model's number of components is more than the true number of components.

Theorem 5: If assumptions 1, 3 and 4 hold true, $\breve{\boldsymbol{\theta}}_k$ is a consistent estimator of $\boldsymbol{\theta}_k^0$, i.e.,

$$\breve{\boldsymbol{\theta}}_k \xrightarrow{p} \boldsymbol{\theta}_k^0 \text{ as } N \to \infty, \forall k = 1, \cdots, p.$$

Proof: The result follows from theorems 3 and 4. ■

Theorem 6: If assumptions 1, 3 and 4 hold true, the following is true:

$$\breve{A}_{R(p+k)} \xrightarrow{p} 0,\ \breve{A}_{I(p+k)} \xrightarrow{p} 0 \text{ as } N \to \infty,\ \forall k = 1, 2, \cdots.$$

Proof: See Appendix B. ∎

The results indicate that the sequential efficient estimators are consistent and have the same rate of convergence and the identical asymptotic distribution as the LSEs. They can also be computed with less computational complexity. Thus, we can use sequential efficient estimators to estimate the unknown parameters, which can be obtained in finite number of iterations.

IV. Simulation Results

In this section, we provide results for various numerical studies for different choices of sample size N, signal-to-noise ratio (SNR) and error variance σ^2, to assess the performance of the proposed efficient algorithms. The SNR is defined as $SNR = 10\log_{10}\left(\frac{\sum_{k=1}^{p}|A_k^0|^2}{\sigma^2}\right)$. We also compare the performance of the proposed algorithms with the existing methods. We present results for a one-component and for a multi-component elementary chirp model in the following subsections.

The bias is almost negligible in the simulations presented in this paper. That is the reason that the MSEs are equivalent to the asymptotic theoretical variances. Hence, we present the MSEs of the proposed efficient estimators and compare them with their asymptotic theoretical variances.

A. One-Component Model

Consider the following one-component elementary chirp model:

$$y(t) = e^{i1.5t^2} + \epsilon(t); \quad t = 1, \ldots, N. \tag{21}$$

Here, $\epsilon(t)$s are *i.i.d.* complex-valued normal random variables that satisfy assumption 1. We take $N = 101, 201, 301, 401, 501$, and $\sigma^2 = 0.1, 0.25, 0.5, 0.75, 1$. We generate the data 1000 times for each sample size and error variance. We estimate the frequency rate using efficient algorithms discussed in subsections II-A1 and II-A2, LSE [15] and CPF method [18]. We calculate MSEs of frequency rate estimates for all these methods. The theoretical asymptotic variance for all these methods has also been reported to validate the accuracy of the estimates. We also compute the time involved in estimating the frequency rate using these methods. In all the figures, "EFFDCHP" represents MSE of the efficient estimator obtained using efficient algorithm discussed in subsection II-A1, "EFFPTE" represents MSE of the efficient estimator obtained using efficient algorithm discussed in subsection II-A2, "AVARTH" represents the theoretical asymptotic variance of the LSE and "CPFTH" represents the theoretical asymptotic variance of the estimate obtained through CPF method.

We also provide the CP function for model (2) as follows for reference:

$$CPF(t, \Omega) = \sum_{m=0}^{\frac{(N-1)}{2}} y(t+m)\, y(t-m)\, e^{-i\Omega m^2}. \tag{22}$$

Here, Ω is IFR and its estimator $\hat{\hat{\Omega}}$ at time t can be obtained as

$$\hat{\hat{\Omega}} = \arg\max_{\Omega} |CPF(t, \Omega)|. \tag{23}$$

As in [18], we have chosen the time point $t = \frac{N+1}{2}$, i.e., the centre of the observed data, in the present case. Once $\hat{\hat{\Omega}}$ is obtained, the estimate of frequency rate, $\hat{\hat{\beta}}$, is obtained as $\hat{\hat{\beta}} = \frac{\hat{\hat{\Omega}}}{2}$.

To compute the efficient frequency rate estimate based on dechirping method discussed in subsection II-A1, we obtain an initial value for β to compute the initial estimate by minimising the objective function (10) over the Fourier grid, $\frac{\pi k}{N-1}, k = 1, 2, \ldots, N-2$. To compute the efficient frequency rate estimate discussed in subsection II-A2 and the LSE, the periodogram-type estimate has been used as the initial estimate. The initial value in the CPF method has been obtained by maximising the CPF over the grid points $\frac{\pi k}{N^2}, k = 1, \ldots, N^2 - 1$. We have used the Nelder-Mead optimization method to optimize the corresponding objective function in all these estimation methods. Here, the grid search is restricted among 10 number of points around the true value in all the methods to save computational time. Once we have estimate of the frequency rate, we can estimate the amplitude parameters using simple linear regression.

Fig. 1 shows the plot of MSEs of different frequency rate estimates versus sample size for different error variances. From this figure, we observe that the MSEs of the proposed estimators decrease as the sample size increases, which validates the consistency of the proposed estimators. Similar behaviour is observed for the LSE and estimator based on CPF. Furthermore, the MSEs of the discussed estimators match well with the corresponding theoretical variances.

Fig. 2 illustrates MSEs and theoretical asymptotic variances of different frequency rate estimates versus SNR for varying sample sizes. This figure shows that as the SNR increases, MSEs decrease for all the methods. From this figure, it is clear that the SNR threshold is -6 for the efficient estimator obtained using periodogram-type estimator discussed in subsection II-A2 and LSE when the sample size is 301. For lower SNR values, the

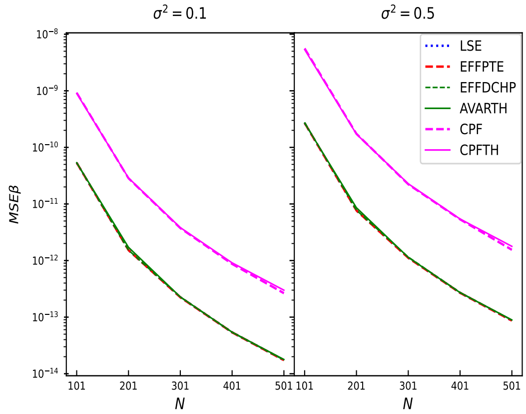

Fig. 1: MSEs and theoretical asymptotic variances of different estimates of frequency rate of the model (21) with $A^0 = 1$ and $\beta^0 = 1.5$ for different error variances versus sample size.

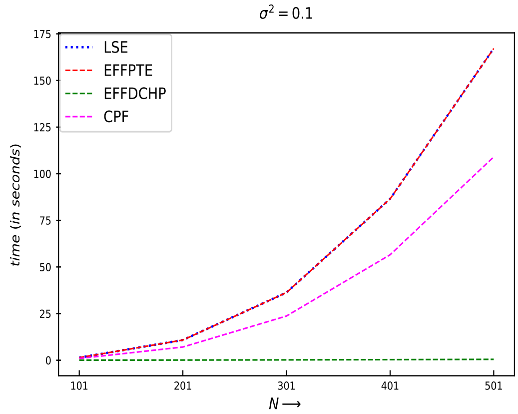

Fig. 3: Computational time in estimating the frequency rate using different estimation methods for the model (21) with $A^0 = 1$ and $\beta^0 = 1.5$ versus sample size.

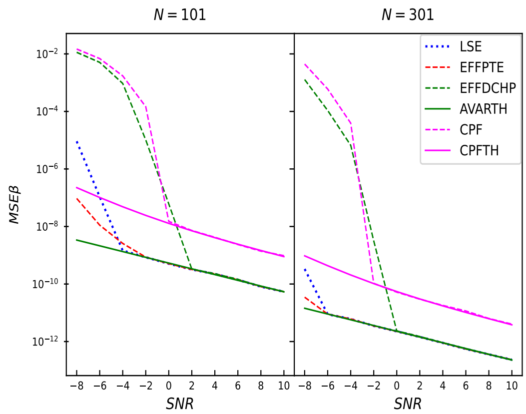

Fig. 2: MSEs and theoretical asymptotic variances of different estimates of frequency rate of the model (21) with $A^0 = 1$ and $\beta^0 = 1.5$ for different sample sizes versus SNR.

MSE of the efficient estimator based on periodogram-type estimator is closer to the theoretical variance than those of the LSE. The SNR threshold is 0 for the efficient estimator based on dechirping method discussed in subsection II-A1, when the sample size is 301. This is because, for the lower SNR, the LSE obtained using dechirping method is not that good (near to the true value). When the sample size is 301, the SNR threshold for CPF method is -2 .

The plot of the time in computing the frequency rate estimates using the different estimation methods versus sample size is shown in Fig. 3. From this figure, it is evident that the efficient algorithm based on dechirping method discussed in subsection II-A1 takes the least time for estimating the frequency rate as compared with the efficient algorithm based on periodogram-type estimator discussed in subsection II-A2, LSE and the CPF method. The reason behind this is that in efficient algorithm based on dechirping method, we have to do grid search among N grid points, whereas, in other estimation methods, we have to do grid search among N^2 number of points. The efficient algorithm based on periodogram-type estimator takes a bit less time in computing than the LSE. Moreover, the time in computing the estimates increases as the sample size increases. The proposed algorithms provide estimators with the optimal efficiency. Therefore, from Fig. 2 and Fig. 3 , we refer to the efficient algorithm based on periodogram-type estimator for estimation when SNR is very low; otherwise, we may use the efficient algorithm based on dechirping method.

B. *Multi-Component Model*

Consider the following two-component elementary chirp model:

$$y(t) = 2e^{i2.5t^2} + e^{i1.5t^2} + \epsilon(t); \quad t = 1, \ldots, N. \quad (24)$$

The simulation study has been done for the above model (24) for the varying sample sizes, and error variances same as those used for the simulation study of the one-component model (21). The error random variables follow the same distribution as used for one-component model (21). We estimate frequency rates using sequential efficient algorithms discussed in subsections III-A1 and III-A2, sequential LSEs [15] and PCPF method [22].

The following is the PCP function for the model (1) for the given L different time points:

$$PCPF(\Omega) = \prod_{l=1}^{L} CPF(t_l, \Omega). \quad (25)$$

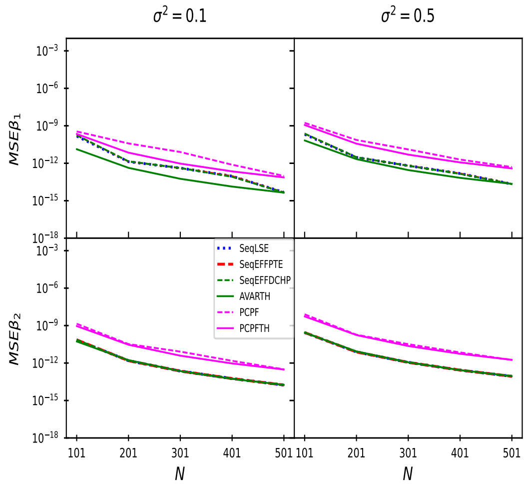

Fig. 4: MSEs and theoretical asymptotic variances of different estimates of frequency rates of the model (24) with $A_1^0 = 2, \beta_1^0 = 2.5, A_2^0 = 1$ and $\beta_2^0 = 1.5$ for different error variances versus sample size.

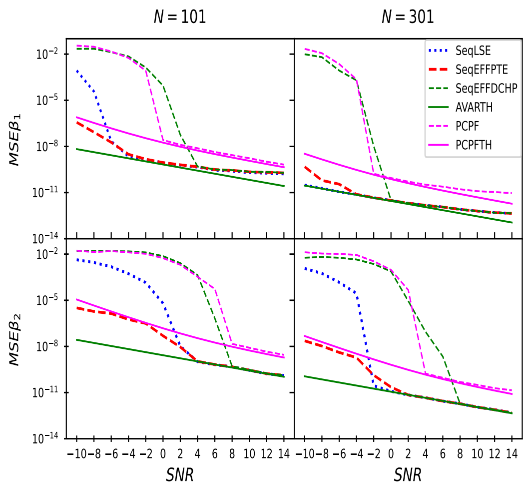

Fig. 5: MSEs and theoretical asymptotic variances of different estimates of frequency rates of the model (24) with $A_1^0 = 2, \beta_1^0 = 2.5, A_2^0 = 1$ and $\beta_2^0 = 1.5$ for different sample sizes versus SNR.

In this case, we have selected two different time points, $t_1 = \frac{N+1}{2}$ and $t_2 = 0.4N$ as in [22]. To optimize the associated objective function in each of these estimating techniques, we employed the Nelder-Mead simplex algorithm. We employed the grid search discussed in the previous subsection IV-A for all the methods to find the initial values.

We report MSEs of the frequency rate estimates based on 1000 replications. We also report the theoretical asymptotic variances of all the methods to compare with their corresponding MSEs. In all the figures, "PCPFTH" represents the theoretical asymptotic variance of the estimates obtained through the PCPF method. The MSEs and the theoretical asymptotic variances of frequency rate estimates for various error variances versus sample size are shown in Fig. 4. From this figure, it is observed that the MSEs decrease as N increases for all the methods, which shows that the frequency rate estimates get closer to the true parameter values as sample size increases. In most cases, MSEs of the frequency rate estimates are very close to the corresponding theoretical variances. Further, the MSEs of proposed sequential efficient estimators are at par with the sequential LSEs.

Fig. 5 depicts the plot of the MSEs and the theoretical variances versus the SNR. From here, it is visible that the SNR threshold is 4 for the sequential efficient estimators based on periodogram-type estimators discussed in subsection III-A2 and sequential LSEs when the sample size is 101. The SNR threshold is 8 for the efficient estimators based on dechirping method discussed in subsection III-A1. The SNR threshold is 8 and 4 for the PCPF method when the sample size is 101 and 301, respectively. For lower SNR values, MSEs of the sequential efficient estimators obtained using periodogram-type estimators are nearer to their corresponding theoretical variances than those of the sequential LSEs. We also observe that the MSEs decrease as the SNR increases.

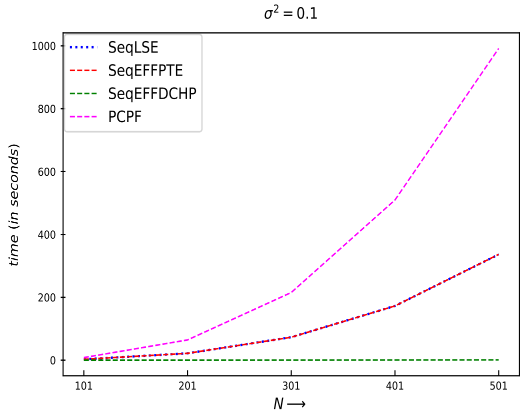

Fig. 6: Computational time in estimating the frequency rates using different estimation methods for the model (24) with $A_1^0 = 2, \beta_1^0 = 2.5, A_2^0 = 1$ and $\beta_2^0 = 1.5$ versus sample size.

The computational time to evaluate the estimates of the frequency rates using the different estimation methods versus sample size is shown in Fig. 6. Here, we report the time involved in computing both the frequency rates of the model (24). From this figure, it is clear that the sequential efficient algorithm based on dechirping method takes the least time in comparison with the sequential efficient algorithm based on periodogram-type estimators, sequential LSEs and the PCPF method. The PCPF method takes the maximum time for computation. The sequential efficient algorithm based on periodogram-type estimators takes a bit less time in computing than the sequential LSE. The proposed efficient algorithms provide the estimators with the optimal rates of convergence and also take less time in computation. Therefore, the proposed algorithms are suggested to study the real life data.

C. *Case of low degree of separation of frequency rates*

In this subsection, we study the performance of the proposed sequential efficient estimators when two frequency rates are close to each other in a two-component elementary chirp model. Here, we consider the following model:

$$y(t) = 3e^{i\beta_1^0 t^2} + e^{i\beta_2^0 t^2} + \epsilon(t); \quad t = 1, \ldots, 300, \quad (26)$$

where we vary β_1^0 by bringing it closer to $\beta_2^0 = 1.5$ while keeping all other parameters fixed. Here, $\epsilon(t)$s satisfy assumption 1 and follow complex-valued normal distribution with $\sigma^2 = 0.1$. The SNR value is 20. The values of β_1^0 are taken as $1.50001, 1.500055, 1.5001, 1.501$ and 1.51. We replicate the experiment 1000 times and calculate the MSEs and theoretical asymptotic variances of the frequency rate estimates. Fig. 7 shows the plot of the MSEs versus the gap between two frequency rates. The gap between two frequency rates is represented by δ in this figure, i.e., $\delta = \beta_1^0 - \beta_2^0$. From this figure we can see that the MSEs are near to their corresponding theoretical variances when δ is 10^{-3} and 10^{-2}. MSEs do not match well with the theoretical variances when two frequency rates get further close to each other. We observe the breakdown point is at δ equal to 10^{-5} in this particular case. When we increase the sample size or the SNR value, this breakdown point moves to a lower value, and vice versa.

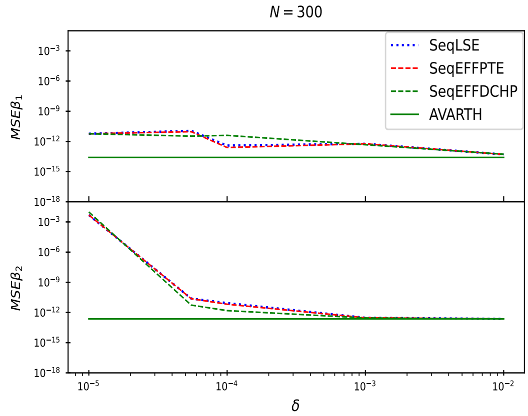

Fig. 7: MSEs and theoretical asymptotic variances of different estimates of frequency rates versus the gap between two frequency rates with $A_1^0 = 3, A_2^0 = 5, \beta_2^0 = 1.5, \sigma^2 = 0.1, SNR = 20$ and $\delta = 10^{-5}, 5.5\times 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}$.

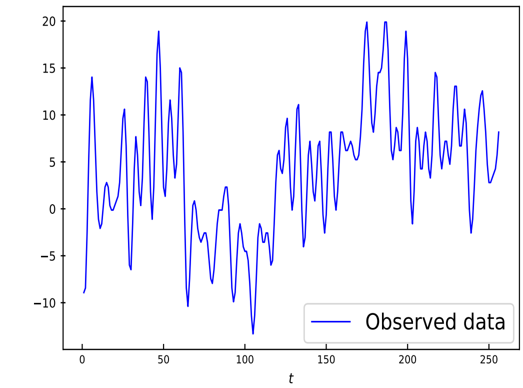

Fig. 8: Plot of the EEG data [23].

V. Real Data Analysis

In this section, we present a real data analysis to demonstrate the applicability of the elementary chirp model using the proposed method in real world. Here, we consider an EEG data [23]. The original signal can be seen in Fig. 8, which has a data size of 256. We fit an elementary chirp model to this data using the proposed sequential efficient algorithm III-A2.We take a maximum of 85 elementary chirp components to fit the data, and choose that p where the residuals first satisfy the error assumption of the model. To test the error assumption of the model, we use the test for the i.i.d. property of a time series by Dalla et al. [8]. At first, the error assumption is satisfied at the 41st component. We also show the plot of the residual sum of squares (RSS) and the plot of the absolute difference of RSS in Figs. 9 and 10, respectively. These plots show that the RSS does not decrease much in comparison with the previous step RSS after the 41st component. Thus, we fit the data at the 41st component. The plot of residuals at the 41st component can be seen in Fig. 12. We plot the fitted data using 41 elementary chirp components along with the original data in Fig. 11. The figure shows that the fitting of the data using the proposed sequential efficient algorithm III-A2 looks good.

VI. Conclusion

In this paper, we propose two efficient algorithms for estimating the elementary chirp signals. These algorithms are based on selecting two distinct ini-

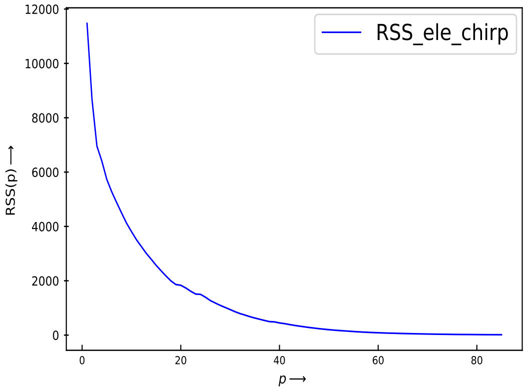

Fig. 9: Residuals sum of squares for EEG data using sequential efficient algorithm III-A2.

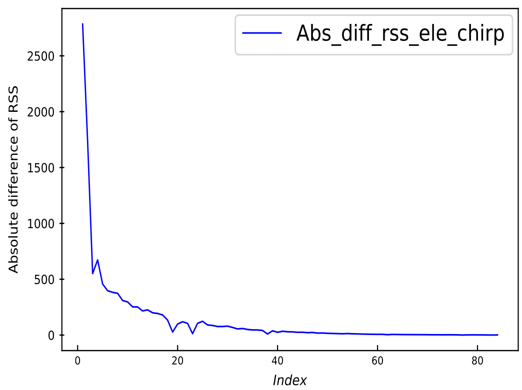

Fig. 10: Absolute difference of RSS for EEG data using sequential efficient algorithm III-A2.

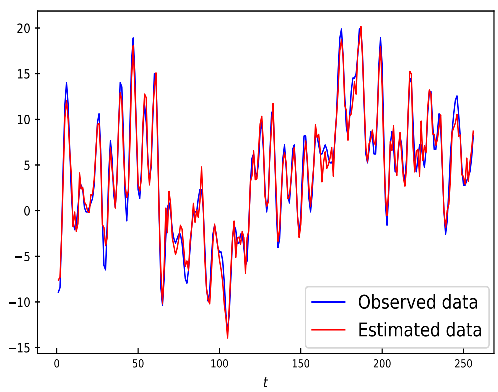

Fig. 11: Observed EEG signal and estimated EEG signal using elementary chirp model.

tial estimators of frequency rates. These estimators

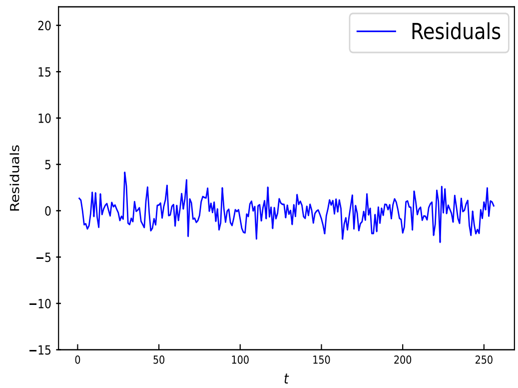

Fig. 12: Residuals at 41st elementary chirp component.

are periodogram-type estimators and LSEs obtained through the dechirping method. It has been observed that the efficient algorithms provide estimators of the frequency rate with the same rate of convergence as the LSEs. The proposed algorithms produce efficient frequency rate estimators in a fixed number of iterations. Also, under the normal error assumption, the proposed efficient estimators achieve the CRLBs asymptotically. They are also computationally less intensive and take less time in computation than the LSEs and other standard methods. Also, when two frequency rates are close to each other upto a certain extent, the proposed sequential efficient estimators can be used for practical implementation purposes.

There are some open problems that are of interest for future work. For instance, amplitude is assumed to be a fixed constant in this paper, but there are some problems where time-varying amplitude [16] is considered. It will be interesting to develop an efficient algorithm for the elementary chirp model in the time-varying amplitude case. Further, it will be interesting to develop an efficient algorithm for signals with a more complicated phase, i.e., a polynomial phase. Djurović et al. [10] provide a comprehensive discussion on parameter estimation of signals with a polynomial phase; also see the references cited therein.

VII. Acknowledgements

The authors would like to thank the associate editor and the anonymous reviewers for their constructive comments and suggestions which have helped to improve the manuscript significantly. We also thank Henri Begleiter at the Neurodynamics Laboratory at the State University of New York Health Center at Brooklyn for the EEG data.

Appendix A

Proof of Theorem 1: Note that,

$$Q_N^{\tilde{\beta}} = \sum_{t=1}^{N} y(t) e^{-i\tilde{\beta}t^2} = \sum_{t=1}^{N} \left[A^0 e^{i\beta^0 t^2} + \epsilon(t)\right] e^{-i\tilde{\beta}t^2} = \sum_{t=1}^{N} A^0 e^{i\left(\beta^0-\tilde{\beta}\right)t^2} + \sum_{t=1}^{N} \epsilon(t) e^{-i\tilde{\beta}t^2}, \quad (27)$$

$$P_N^{\tilde{\beta}} = \sum_{t=1}^{N} y(t)\left(t^2 - \frac{N^2}{3}\right) e^{-i\tilde{\beta}t^2} = \sum_{t=1}^{N} A^0 \left(t^2 - \frac{N^2}{3}\right) e^{i\left(\beta^0-\tilde{\beta}\right)t^2} + \sum_{t=1}^{N} \epsilon(t)\left(t^2 - \frac{N^2}{3}\right) e^{-i\tilde{\beta}t^2}. \quad (28)$$

Using Taylor series expansion and lemma 2 of [11], the $1st$ term of $Q_N^{\tilde{\beta}}$, (27), is written as follows:

$$\begin{aligned}\sum_{t=1}^{N} e^{i\left(\beta^0-\tilde{\beta}\right)t^2} &= \sum_{t=1}^{N} \left[e^{i\left(\beta^0-\beta^0\right)t^2} + i\left(\beta^0-\tilde{\beta}\right)t^2 e^{i\left(\beta^0-\beta^*\right)t^2}\right] \\ &= N - iO_p\left(N^{-2-\delta}\right)\sum_{t=1}^{N} t^2 e^{i\left(\beta^0-\beta^*\right)t^2} \\ &= N - iO_p\left(N^{-2-\delta}\right)\left[O_p\left(N^3\right) + iO_p\left(N^3\right)\right] \\ &= N - iO_p\left(N^{1-\delta}\right) + O_p\left(N^{1-\delta}\right), \text{where}, \beta^* \in \left(\beta^0, \tilde{\beta}\right).\end{aligned}$$

Now, choosing a large L such that $1 - L\delta < 0$, then using Taylor series expansion expand $2nd$ term of $Q_N^{\tilde{\beta}}$, (27). Using Lindeberg Feller CLT and lemma 4(b) of [11], we get:

$$\begin{aligned}\sum_{t=1}^{N} \epsilon(t) e^{-i\tilde{\beta}t^2} &= \sum_{t=1}^{N} \epsilon(t)\left[e^{-i\beta^0 t^2} + e^{-i\beta^0 t^2}\sum_{l=1}^{L-1}\frac{\left(-i\left(\tilde{\beta}-\beta^0\right)t^2\right)^l}{l!} + \frac{\left(-it^2\right)^L}{L!}\left(\tilde{\beta}-\beta^0\right)^L e^{-i\beta^* t^2}\right] \\ &= O_p\left(N^{\frac{1}{2}}\right) + \sum_{l=1}^{L-1}\frac{(-i)^l}{l!}O_p\left(N^{-2l-l\delta}\right)O_p\left(N^{2l+\frac{1}{2}}\right) + \frac{(-i)^L}{L!}O_p\left(N^{-2L-L\delta}\right)O_p\left(N^{2L+1}\right) \\ &= O_p\left(N^{\frac{1}{2}}\right), \text{where}, \beta^* \in \left(\beta^0, \tilde{\beta}\right).\end{aligned}$$

Thus $Q_N^{\tilde{\beta}}$, (27), becomes

$$Q_N = A^0\left[N - iO_p\left(N^{1-\delta}\right) + O_p\left(N^{1-\delta}\right)\right] + O_p\left(N^{\frac{1}{2}}\right)$$

$$Q_N = A^0\left(N + O_p\left(N^{1-\delta}\right)\right); \delta \in \left(0, \frac{1}{2}\right].$$

Now, we calculate the terms of $P_N^{\tilde{\beta}}$, (28). Using Taylor series expansion upto $2nd$ order and lemma 2 of [11], $1st$ term of $P_N^{\tilde{\beta}}$, (28), is written as follows:

$$\begin{aligned}\sum_{t=1}^{N}\left(t^2 - \frac{N^2}{3}\right) e^{i\left(\beta^0-\tilde{\beta}\right)t^2} &= \sum_{t=1}^{N}\left(t^2 - \frac{N^2}{3}\right)\left[e^{i\left(\beta^0-\beta^0\right)t^2} + e^{i\left(\beta^0-\beta^0\right)t^2} it^2\left(\beta^0-\tilde{\beta}\right) + \frac{\left(it^2\left(\beta^0-\tilde{\beta}\right)\right)^2}{2!}e^{i\left(\beta^0-\beta^*\right)t^2}\right] \text{where}, \beta^* \in \left(\beta^0, \tilde{\beta}\right), \\ &= \left(\frac{N^2}{2} + \frac{N}{6}\right) + i\left(\beta^0-\tilde{\beta}\right)\left[\frac{4N^5}{45} + O\left(N^4\right)\right] - \left(\beta^0-\tilde{\beta}\right)O_p\left(N^{-2-\delta}\right)\left[O_p\left(N^7\right) + iO_p\left(N^7\right)\right] \\ &= O\left(N^2\right) - O_p\left(N^{3-2\delta}\right) + i\left(\beta^0-\tilde{\beta}\right)\left[\frac{4N^5}{45} - O_p\left(N^{5-\delta}\right)\right] \\ &= i\left(\beta^0-\tilde{\beta}\right)\left[\frac{4N^5}{45} - O_p\left(N^{5-\delta}\right)\right].\end{aligned}$$

Now, choosing a large L such that $1 - L\delta < 0$ and then using Taylor series expansion, Lindeberg-Feller CLT and lemma 4(b) of [11], we calculate the $2nd$ term of $P_N^{\tilde{\beta}}$, (28), as follows:

$$\begin{aligned}\sum_{t=1}^{N}\epsilon(t)\left(t^2 - \frac{N^2}{3}\right) e^{-i\tilde{\beta}t^2} &= \sum_{t=1}^{N}\epsilon(t)\left(t^2 - \frac{N^2}{3}\right)\left[e^{-i\beta^0 t^2} + \sum_{l=1}^{L-1}\frac{\left(-i\left(\tilde{\beta}-\beta^0\right)t^2\right)^l}{l!}e^{-i\beta^0 t^2} + \frac{\left(-it^2\right)^L}{L!}\left(\tilde{\beta}-\beta^0\right)^L e^{-i\beta^* t^2}\right] \\ &= \sum_{t=1}^{N}\epsilon(t)\left(t^2 - \frac{N^2}{3}\right)e^{-i\beta^0 t^2} + \sum_{l=1}^{L-1}\frac{(-i)^l}{l!}O_p\left(N^{-2l-l\delta}\right)O_p\left(N^{2l+\frac{5}{2}}\right) + \frac{(-i)^L}{L!}O_p\left(N^{-2L-L\delta}\right)O_p\left(N^{2L+3}\right) \\ &= \sum_{t=1}^{N}\epsilon(t)\left(t^2 - \frac{N^2}{3}\right)e^{-i\beta^0 t^2} + \sum_{l=1}^{L-1}\frac{(-i)^l}{l!}O_p\left(N^{\frac{5}{2}-l\delta}\right) + \frac{(-i)^L}{L!}O_p\left(N^2\right), \text{where}, \beta^* \in \left(\beta^0, \tilde{\beta}\right).\end{aligned}$$

Thus $P_N^{\tilde{\beta}}$, (28), becomes

$$\begin{aligned}P_N^{\beta} &= \sum_{t=1}^{N} y(t)\left(t^2 - \frac{N^2}{3}\right)e^{-i\tilde{\beta}t^2} \\ &= \sum_{t=1}^{N}\epsilon(t)\left(t^2 - \frac{N^2}{3}\right)e^{-i\beta^0 t^2} + A^0 i\left(\beta^0-\tilde{\beta}\right)\left[\frac{4N^5}{45} - O_p\left(N^{5-\delta}\right)\right].\end{aligned}$$

Hence, we get:

$$\breve{\beta} = \tilde{\beta} + \frac{11.25}{N^4}\operatorname{Im}\left(\frac{P_N^{\tilde{\beta}}}{Q_N^{\tilde{\beta}}}\right)$$

$$\breve{\beta} = \tilde{\beta} + \frac{11.25}{N^4}\left(\beta^0 - \tilde{\beta}\right)\frac{\left[\frac{4N^5}{45} - O_p\left(N^{5-\delta}\right)\right]}{\left(N + O_p\left(N^{1-\delta}\right)\right)} +$$
$$\frac{11.25}{N^4}\operatorname{Im}\left[\frac{\sum_{t=1}^{N}\epsilon(t)\left(t^2 - \frac{N^2}{3}\right)e^{-i\beta^0 t^2}}{A^0\left(N + O_p\left(N^{1-\delta}\right)\right)}\right]$$
$$\breve{\beta} = \beta^0 + \left(\tilde{\beta} - \beta^0\right)O_p\left(N^{-\delta}\right) +$$
$$\frac{45}{4N^5}\operatorname{Im}\left[\frac{\sum_{t=1}^{N}\epsilon(t)\left(t^2 - \frac{N^2}{3}\right)e^{-i\beta^0 t^2}}{A^0}\right].$$

Using lemma 2 of [11], the variance of $G = \frac{45}{4N^{\frac{5}{2}}}\operatorname{Im}\left[\frac{\sum_{t=1}^{N}\epsilon(t)\left(t^2 - \frac{N^2}{3}\right)e^{-i\beta^0 t^2}}{A^0}\right]$ is $\frac{45\sigma^2}{8|A^0|^2}$, asymptotically. Thus, if $\tilde{\beta} - \beta^0 = O_p\left(N^{-2-\delta}\right)$, where $\delta \in \left(0, \frac{1}{2}\right]$, then $\breve{\beta} - \beta^0 = O_p\left(N^{-2-2\delta}\right)$ if $\delta \leq \frac{1}{4}$ and $N^{\frac{5}{2}}\left(\breve{\beta} - \beta^0\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{45\sigma^2}{8|A^0|^2}\right)$ if $\delta > \frac{1}{4}$ by Lindeberg-Feller CLT. Hence the result of theorem 1 follows.

***Proof of Theorem** 2*: Let us consider:

$$\breve{A}_R = \frac{1}{N}\sum_{t=1}^{N}\left(y_R(t)\cos\left(\breve{\beta}t^2\right) + y_I(t)\sin\left(\breve{\beta}t^2\right)\right). \tag{29}$$

Here, $y_R(t)$ and $y_I(t)$ are as defined in (15) and (16), respectively. Now, expanding $\cos\left(\breve{\beta}t^2\right)$ and $\sin\left(\breve{\beta}t^2\right)$ around the point β^0 using Taylor series, we can rewrite $\breve{A}_R$ as follows:

$$\breve{A}_R = \frac{1}{N}\sum_{t=1}^{N}y_R(t)\left\{\cos\left(\beta^0 t^2\right) - t^2\left(\breve{\beta} - \beta^0\right)\sin\left(\bar{\beta}t^2\right)\right\}$$
$$+ \frac{1}{N}\sum_{t=1}^{N}y_I(t)\left\{\sin\left(\beta^0 t^2\right) + t^2\left(\breve{\beta} - \beta^0\right)\sin\left(\bar{\beta}t^2\right)\right\},$$
$$\text{where, } \bar{\beta} \in \left(\beta^0, \breve{\beta}\right). \tag{30}$$

Now using the Lindeberg-Feller CLT and lemma 2 of [11], it can be shown that $N^{-\frac{1}{2}}\left(\sum_{t=1}^{N}\epsilon_R(t)\cos\left(\beta^0 t^2\right) + \epsilon_I(t)\sin\left(\beta^0 t^2\right)\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{2}\right)$. We have $N^{\frac{5}{2}}\left(\breve{\beta} - \beta^0\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{45\sigma^2}{8|A^0|^2}\right)$ from theorem 1. Using this result and the Chebyshev inequality, it can be easily proved that $N^2\left(\breve{\beta} - \beta^0\right) \xrightarrow{p} 0$ as $N \to \infty$. Using these results and lemmas 2 and 4(b) of [11], we have: $N^{\frac{1}{2}}\left(\breve{A}_R - A_R^0\right) \xrightarrow{d} \mathcal{N}\left(0, \left(\frac{1}{2} + \frac{5A_I^{0^2}}{8|A^0|^2}\right)\sigma^2\right)$. Proceeding in the similar manner asymptotic distribution of $\breve{A}_I$ can be obtained. Hence the result.

APPENDIX B

***Proof of Theorem** 3*: First we derive this theorem for $\breve{\beta}_1$. Note that,

$$Q_N^{\tilde{\beta}_1} = \sum_{t=1}^{N}y_1(t)e^{-i\tilde{\beta}_1 t^2} = \sum_{t=1}^{N}\left[\sum_{k=1}^{p}A_k^0 e^{i\beta_k^0 t^2} + \epsilon(t)\right]e^{-i\tilde{\beta}_1 t^2}$$
$$= \sum_{k=1}^{p}A_k^0\sum_{t=1}^{N}e^{i\left(\beta_k^0 - \tilde{\beta}_1\right)t^2} + \sum_{t=1}^{N}\epsilon(t)e^{-i\tilde{\beta}_1 t^2}. \tag{31}$$

Now, let us see the behaviour of $\sum_{t=1}^{N}e^{i\left(\beta_k^0 - \tilde{\beta}_1\right)t^2}$ for different k. For $k = 1$, using Taylor series expansion and lemma 2 of [11], we get,

$$\sum_{t=1}^{N}e^{i\left(\beta_1^0 - \tilde{\beta}_1\right)t^2} = \sum_{t=1}^{N}\left[e^{i\left(\beta_1^0 - \beta_1^0\right)t^2} + i\left(\beta_1^0 - \tilde{\beta}_1\right)\right.$$
$$\left.t^2 e^{i\left(\beta_1^0 - \beta_1^*\right)t^2}\right] = N - iO_p\left(N^{-2-\delta}\right)\sum_{t=1}^{N}t^2 e^{i\left(\beta_1^0 - \beta_1^*\right)t^2}$$
$$= N - iO_p\left(N^{-2-\delta}\right)\left[O_p\left(N^3\right) + iO_p\left(N^3\right)\right]$$
$$= N - iO_p\left(N^{1-\delta}\right) + O_p\left(N^{1-\delta}\right),$$
$$\text{where, } \beta_1^* \in \left(\beta_1^0, \tilde{\beta}_1\right).$$

For $k = 2, \cdots, p$, using conjecture 2 and lemma 2 of [11] and Taylor series expansion upto L-th order, choosing L large such that $1 - L\delta < 0$, we get,

$$\sum_{t=1}^{N}e^{i\left(\beta_k^0 - \tilde{\beta}_1\right)t^2} = \sum_{t=1}^{N}\left[e^{i\left(\beta_k^0 - \beta_1^0\right)t^2} + \right.$$
$$\sum_{l=1}^{L-1}\frac{\left(-i\left(\tilde{\beta}_1 - \beta_1^0\right)t^2\right)^l}{l!}e^{i\left(\beta_k^0 - \beta_1^0\right)t^2} +$$
$$\left.\frac{\left(-i\left(\tilde{\beta}_1 - \beta_1^0\right)t^2\right)^L}{L!}e^{i\left(\beta_k^0 - \beta_1^*\right)t^2}\right]$$
$$= o_p\left(N^{\frac{1}{2}}\right) + \sum_{l=1}^{L-1}\frac{(-i)^l}{l!}O_p\left(N^{-2l-l\delta}\right)o_p\left(N^{2l+\frac{1}{2}}\right) +$$
$$\frac{(-i)^L}{L!}O_p\left(N^{-2L-L\delta}\right)O_p\left(N^{2L+1}\right)$$
$$= o_p\left(N^{\frac{1}{2}}\right), \text{where, } \beta_1^* \in \left(\beta_1^0, \tilde{\beta}_1\right).$$

Now, choosing a large L such that $1 - L\delta < 0$ and then using Taylor series expansion, Lindeberg-Feller CLT and lemma 4(b) of [11], $2nd$ term of $Q_N^{\tilde{\beta}_1}$, (31), is written as follows:

$$\sum_{t=1}^{N}\epsilon(t)e^{-i\tilde{\beta}_1 t^2}=\sum_{t=1}^{N}\epsilon(t)\Big[e^{-i\beta_1^0 t^2}+\sum_{l=1}^{L-1}\frac{\left(-i\left(\tilde{\beta}_1-\beta_1^0\right)t^2\right)^l}{l!}e^{-i\beta_1^0 t^2}+\frac{\left(-it^2\right)^L}{L!}\left(\tilde{\beta}_1-\beta_1^0\right)^L e^{-i\beta_1^* t^2}\Big]$$

$$=O_p\left(N^{\frac{1}{2}}\right)+\sum_{l=1}^{L-1}\frac{(-i)^l}{l!}O_p\left(N^{-2l-l\delta}\right)O_p\left(N^{2l+\frac{1}{2}}\right)+\frac{(-i)^L}{L!}O_p\left(N^{-2L-L\delta}\right)O_p\left(N^{2L+1}\right)$$

$$=O_p\left(N^{\frac{1}{2}}\right),\text{where, }\beta_1^*\in\left(\beta_1^0,\tilde{\beta}_1\right).$$

Thus $Q_N^{\tilde{\beta}_1}$, (31), becomes

$$Q_N^{\tilde{\beta}_1}=A_1^0\left[N-iO_p\left(N^{1-\delta}\right)+O_p\left(N^{1-\delta}\right)\right]+O_p\left(N^{\frac{1}{2}}\right)+\sum_{k=2}^{p}A_k^0 o_p\left(N^{\frac{1}{2}}\right)$$

$$=A_1^0\left(N+O_p\left(N^{1-\delta}\right)\right);\delta\in\left(0,\frac{1}{2}\right].$$

Now, we calculate $P_N^{\tilde{\beta}_1}$.

$$P_N^{\tilde{\beta}_1}=\sum_{t=1}^{N}y_1(t)\left(t^2-\frac{N^2}{3}\right)e^{-i\tilde{\beta}_1 t^2}=\sum_{k=1}^{p}A_k^0\sum_{t=1}^{N}\left(t^2-\frac{N^2}{3}\right)e^{i\left(\beta_k^0-\tilde{\beta}_1\right)t^2}+\sum_{t=1}^{N}\epsilon(t)\left(t^2-\frac{N^2}{3}\right)e^{-i\tilde{\beta}_1 t^2}. \tag{32}$$

For, $k=1$, using Taylor series expansion and lemma 2 of [11], we get,

$$\sum_{t=1}^{N}\left(t^2-\frac{N^2}{3}\right)e^{i\left(\beta_1^0-\tilde{\beta}_1\right)t^2}=\sum_{t=1}^{N}\left(t^2-\frac{N^2}{3}\right)\Big[e^{i\left(\beta_1^0-\beta_1^0\right)t^2}+e^{i\left(\beta_1^0-\beta_1^0\right)t^2}it^2\left(\beta_1^0-\tilde{\beta}_1\right)+\frac{\left(it^2\left(\beta_1^0-\tilde{\beta}_1\right)\right)^2}{2!}e^{i\left(\beta_1^0-\beta_1^*\right)t^2}\Big]=\left(\frac{N^2}{2}+\frac{N}{6}\right)+$$

$$i\left(\beta_1^0-\tilde{\beta}_1\right)\left[\frac{4N^5}{45}+O\left(N^4\right)\right]-\left(\beta_1^0-\tilde{\beta}_1\right)O_p\left(N^{-2-\delta}\right)\left[O_p\left(N^7\right)+iO_p\left(N^7\right)\right]=O\left(N^2\right)-O_p\left(N^{3-2\delta}\right)+i\left(\beta_1^0-\tilde{\beta}_1\right)\left[\frac{4N^5}{45}-O_p\left(N^{5-\delta}\right)\right]$$

$$=i\left(\beta_1^0-\tilde{\beta}_1\right)\left[\frac{4N^5}{45}-O_p\left(N^{5-\delta}\right)\right],\text{where, }\beta_1^*\in\left(\beta_1^0,\tilde{\beta}_1\right).$$

For $k=2,\cdots,p$, using conjecture 2 and lemma 2 of [11] and Taylor series expansion upto L-th order, choosing L large such that $\frac{3}{2}-L\delta<0$, we get,

$$\sum_{t=1}^{N}\left(t^2-\frac{N^2}{3}\right)e^{i\left(\beta_k^0-\tilde{\beta}_1\right)t^2}=\sum_{t=1}^{N}\left(t^2-\frac{N^2}{3}\right)\Big[e^{i\left(\beta_k^0-\beta_1^0\right)t^2}+\sum_{l=1}^{L-1}\frac{\left(-i\left(\tilde{\beta}_1-\beta_1^0\right)t^2\right)^l}{l!}e^{i\left(\beta_k^0-\beta_1^0\right)t^2}+\frac{\left(-i\left(\tilde{\beta}_1-\beta_1^0\right)t^2\right)^L}{L!}e^{i\left(\beta_k^0-\beta_1^*\right)t^2}\Big]$$

$$=o_p\left(N^{\frac{5}{2}}\right)+\sum_{l=1}^{L-1}\frac{(-i)^l}{l!}O_p\left(N^{-2l-l\delta}\right)o_p\left(N^{2l+\frac{5}{2}}\right)+\frac{(-i)^L}{L!}O_p\left(N^{-2L-L\delta}\right)O_p\left(N^{2L+3}\right)$$

$$=o_p\left(N^{\frac{5}{2}}\right)+\sum_{l=1}^{L-1}\frac{(-i)^l}{l!}o_p\left(N^{\frac{5}{2}-l\delta}\right)+\frac{(-i)^L}{L!}O_P\left(N^{\frac{3}{2}}\right)$$

$$=o_p\left(N^{\frac{5}{2}}\right),\text{where, }\beta_1^*\in\left(\beta_1^0,\tilde{\beta}_1\right).$$

Now, choosing a large L such that $1-L\delta<0$ and then using Taylor series expansion, Lindeberg-Feller CLT and lemma 4(b) of [11], we calculate the $2nd$ term of $P_N^{\tilde{\beta}_1}$, (32), as follows:

$$\sum_{t=1}^{N}\epsilon(t)\left(t^2-\frac{N^2}{3}\right)e^{-i\tilde{\beta}_1 t^2}=\sum_{t=1}^{N}\epsilon(t)\left(t^2-\frac{N^2}{3}\right)\left[e^{-i\beta_1^0 t^2}+\sum_{l=1}^{L-1}\frac{\left(-i\left(\tilde{\beta}_1-\beta_1^0\right)t^2\right)^l}{l!}e^{-i\beta_1^0 t^2}+\frac{\left(-it^2\right)^L}{L!}\left(\tilde{\beta}_1-\beta_1^0\right)^L e^{-i\beta_1^* t^2}\right]=\sum_{t=1}^{N}\left[\epsilon(t)\left(t^2-\frac{N^2}{3}\right)e^{-i\beta_1^0 t^2}\right]+\sum_{l=1}^{L-1}\frac{(-i)^l}{l!}O_p\left(N^{-2l-l\delta}\right)$$

$$O_p\left(N^{2l+\frac{5}{2}}\right)+\frac{(-i)^L}{L!}O_p\left(N^{-2L-L\delta}\right)O_p\left(N^{2L+3}\right)$$

$$=\sum_{t=1}^{N}\epsilon(t)\left(t^2-\frac{N^2}{3}\right)e^{-i\beta_1^0 t^2}+\sum_{l=1}^{L-1}\frac{(-i)^l}{l!}O_p\left(N^{\frac{5}{2}-l\delta}\right)+\frac{(-i)^L}{L!}O_p\left(N^2\right),\text{where, }\beta_1^*\in\left(\beta_1^0,\tilde{\beta}_1\right).$$

Thus $P_N^{\tilde{\beta}_1}$, (32), becomes

$$P_N^{\tilde{\beta}_1}=\sum_{t=1}^{N}y_1(t)\left(t^2-\frac{N^2}{3}\right)e^{-i\tilde{\beta}_1 t^2}$$

$$=\sum_{t=1}^{N}\epsilon(t)\left(t^2-\frac{N^2}{3}\right)e^{-i\beta_1^0 t^2}+A_1^0 i\left(\beta_1^0-\tilde{\beta}_1\right)\left[\frac{4N^5}{45}-O_p\left(N^{5-\delta}\right)\right]+\sum_{k=2}^{p}A_k^0 o_p\left(N^{\frac{5}{2}}\right).$$

Further,

$$\breve{\beta}_1 = \tilde{\beta}_1 + \frac{11.25}{N^4} \operatorname{Im}\left(\frac{P_N^{\tilde{\beta}_1}}{Q_N^{\tilde{\beta}_1}}\right)$$

$$\breve{\beta}_1 = \tilde{\beta}_1 + \frac{11.25}{N^4}\left(\beta_1^0 - \tilde{\beta}_1\right) \frac{\left[\frac{4N^5}{45} - O_p\left(N^{5-\delta}\right)\right]}{\left(N + O_p\left(N^{1-\delta}\right)\right)}$$

$$+ \frac{11.25}{N^4} \operatorname{Im}\left[\frac{\sum_{t=1}^{N} \epsilon(t)\left(t^2 - \frac{N^2}{3}\right) e^{-i\beta_1^0 t^2}}{A_1^0\left(N + O_p\left(N^{1-\delta}\right)\right)}\right] +$$

$$\frac{11.25}{N^5} o_P\left(N^{\frac{5}{2}}\right)$$

$$\breve{\beta}_1 = \beta_1^0 + \left(\tilde{\beta}_1 - \beta_1^0\right) O_p\left(N^{-\delta}\right) + \frac{45}{4N^5}$$

$$\operatorname{Im}\left[\frac{\sum_{t=1}^{N} \epsilon(t)\left(t^2 - \frac{N^2}{3}\right) e^{-i\beta_1^0 t^2}}{A_1^0}\right] + o_P\left(N^{-\frac{5}{2}}\right). \tag{33}$$

Using lemma 2 of [11], variance of $G_1 = \frac{45}{4N^{\frac{5}{2}}} \operatorname{Im}\left[\frac{\sum_{t=1}^{N} \epsilon(t)\left(t^2 - \frac{N^2}{3}\right) e^{-i\beta_1^0 t^2}}{A_1^0}\right]$ is $\frac{45\sigma^2}{8\left|A_1^0\right|^2}$, asymptotically. Therefore, if $\tilde{\beta}_1 - \beta_1^0 = O_p\left(N^{-2-\delta}\right)$, where $\delta \in \left(0, \frac{1}{2}\right]$, then from (33), $\breve{\beta}_1 - \beta_1^0 = O_p\left(N^{-2-2\delta}\right)$ if $\delta \leq \frac{1}{4}$. If $\delta > \frac{1}{4}$, then from (33) and using Lindeberg-Feller CLT, we have $N^{\frac{5}{2}}\left(\breve{\beta}_1 - \beta_1^0\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{45\sigma^2}{8\left|A_1^0\right|^2}\right)$.

Now, we prove the result for $\breve{\beta}_2$. Using the above result and the Chebyshev inequality, it can be easily proved that

$$N^2\left(\breve{\beta}_1 - \beta_1^0\right) \xrightarrow{p} 0 \text{ as } N \to \infty. \tag{34}$$

Now we will prove the consistency of the amplitude parameters, i.e., $\breve{A}_{R1} \xrightarrow{p} A_{R1}^0$ as $N \to \infty$ and $\breve{A}_{I1} \xrightarrow{p} A_{I1}^0$ as $N \to \infty$. Let us consider:

$$\breve{A}_{R1} = \frac{1}{N}\sum_{t=1}^{N}\left(y_{R1}(t)\cos\left(\breve{\beta}_1 t^2\right) + y_{I1}(t)\sin\left(\breve{\beta}_1 t^2\right)\right). \tag{35}$$

Here, $y_{R1}(t)$ and $y_{I1}(t)$ are the real and imaginary parts of the data $y_1(t)$, respectively. Now, expanding $\cos\left(\breve{\beta}_1 t^2\right)$ and $\sin\left(\breve{\beta}_1 t^2\right)$ around the point β_1^0 using Taylor series, we can rewrite $\breve{A}_{R1}$ as follows:

$$\breve{A}_{R1} = \frac{1}{N}\sum_{t=1}^{N}\left\{\cos\left(\beta_1^0 t^2\right) - t^2\left(\breve{\beta}_1 - \beta_1^0\right)\sin\left(\bar{\beta}_1 t^2\right)\right\}$$

$$y_{R1}(t) + y_{I1}(t)\left\{\sin\left(\beta_1^0 t^2\right) + t^2\left(\breve{\beta}_1 - \beta_1^0\right)\sin\left(\bar{\beta}_1 t^2\right)\right\}$$

$$\text{where, } \bar{\beta}_1 \in \left(\beta_1^0, \breve{\beta}_1\right). \tag{36}$$

Now using (34) and lemmas 2 and 4(b) of [11], it can be proved that $\breve{A}_{R1} \xrightarrow{p} A_{R1}^0$ as $N \to \infty$. Similarly, we can show $\breve{A}_{I1} \xrightarrow{p} A_{I1}^0$ as $N \to \infty$. Therefore, we have: $\breve{A}_{R1} = A_{R1}^0 + o_p(1)$, $\breve{A}_{I1} = A_{I1}^0 + o_p(1)$ and $\breve{\beta}_1 = \beta_1^0 + o_p\left(N^{-2}\right)$. Consider $y_2(t) = y_1(t) - \breve{A}_1 e^{i\breve{\beta}_1 t^2}$. Now using the above result and then proceeding in the similar manner as done for $\breve{\beta}_1$, we obtain the desired result. This result can be proved for $k = 3, \ldots, p$ along the similar lines.

Proof of Theorem 4: This result can be derived along the similar lines as theorem 2. Let us consider,

$$\breve{A}_{R1} = \frac{1}{N}\sum_{t=1}^{N}\left(y_{R1}(t)\cos\left(\breve{\beta}_1 t^2\right) + y_{I1}(t)\sin\left(\breve{\beta}_1 t^2\right)\right). \tag{37}$$

Now, expanding $\cos\left(\breve{\beta}_1 t^2\right)$ and $\sin\left(\breve{\beta}_1 t^2\right)$ around the point β_1^0 using Taylor series, we can rewrite $\breve{A}_{R1}$ as follows:

$$\breve{A}_{R1} = \frac{1}{N}\sum_{t=1}^{N} y_{R1}(t)\left\{\cos\left(\beta_1^0 t^2\right) - t^2\left(\breve{\beta}_1 - \beta_1^0\right)\right.$$

$$\left.\sin\left(\bar{\beta}_1 t^2\right)\right\} + y_{I1}(t)\left\{\sin\left(\beta_1^0 t^2\right) + t^2\left(\breve{\beta}_1 - \beta_1^0\right)\right.$$

$$\left.\sin\left(\bar{\beta}_1 t^2\right)\right\}, \text{ where, } \bar{\beta}_1 \in \left(\beta_1^0, \breve{\beta}_1\right). \tag{38}$$

Now using the Lindeberg-Feller CLT and lemma 2 of [11], it can be shown that $N^{-\frac{1}{2}}\left(\sum_{t=1}^{N} \epsilon_{R1}(t)\cos\left(\beta_1^0 t^2\right) + \epsilon_{I1}(t)\sin\left(\beta_1^0 t^2\right)\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{2}\right)$. From above theorem we have $N^{\frac{5}{2}}\left(\breve{\beta}_1 - \beta_1^0\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{45\sigma^2}{8\left|A_1^0\right|^2}\right)$. Using these results, (34) and lemmas 2 and 4(b) of [11], we have $N^{\frac{1}{2}}\left(\breve{A}_{R1} - A_{R1}^0\right) \xrightarrow{d} \mathcal{N}\left(0, \left(\frac{1}{2} + \frac{5A_{I1}^{0^2}}{8\left|A_1^0\right|^2}\right)\sigma^2\right)$. Procceding in the similar manner asymptotic distribution of $\breve{A}_{I1}$ can be obtained. Now using; $\breve{A}_{R1} = A_{R1}^0 + o_p(1)$, $\breve{A}_{I1} = A_{I1}^0 + o_p(1)$ and $\breve{\beta}_1 = \beta_1^0 + o_p\left(N^{-2}\right)$, and $y_2(t) = y_1(t) - \breve{A}_1 e^{i\breve{\beta}_1 t^2}$, we can prove the result for $k = 2$. In the similar manner, we can prove the result for $k = 3, \ldots, p$. Hence we obtain the result.

Proof of Theorem 6 : Consider $\breve{A}_{R(p+1)}$ and $\breve{A}_{I(p+1)}$ the efficient estimators of the linear parameters $A_{R(p+1)}^0$ and $A_{I(p+1)}^0$, respectively:

$$\breve{A}_{R(p+1)} = \frac{1}{N}\sum_{t=1}^{N}\left(y_{R(p+1)}(t)\cos\left(\breve{\beta}_{p+1} t^2\right) + \right.$$

$$\left. y_{I(p+1)}(t)\sin\left(\breve{\beta}_{p+1} t^2\right)\right),$$

$$\breve{A}_{I(p+1)} = \frac{1}{N}\sum_{t=1}^{N}\left(y_{I(p+1)}(t)\cos\left(\breve{\beta}_{p+1} t^2\right) - \right.$$

$$\left. y_{R(p+1)}(t)\sin\left(\breve{\beta}_{p+1} t^2\right)\right),$$

where, $y_{p+1}(t)$ is the data obtained by eliminating the first p-components from the original data $y_1(t)$, and $y_{R(p+1)}(t)$ and $y_{I(p+1)}(t)$ are the real and imaginary parts of the data $y_{p+1}(t)$, respectively. It implies that:

$$y_{R(p+1)}(t) = y_{R1}(t) - \sum_{k=1}^{p} \left(\breve{A}_{Rk} \cos\left(\breve{\beta}_k t^2\right) - \breve{A}_{Ik} \sin\left(\breve{\beta}_k t^2\right)\right),$$
$$y_{I(p+1)}(t) = y_{I1}(t) - \sum_{k=1}^{p} \left(\breve{A}_{Rk} \sin\left(\breve{\beta}_k t^2\right) + \breve{A}_{Ik} \cos\left(\breve{\beta}_k t^2\right)\right). \quad (39)$$

Now using theorem 3 and 4, we have $\breve{\beta}_k \xrightarrow{p} \beta_k^0$, $\breve{A}_{Rk} \xrightarrow{p} A_{Rk}^0$ and $\breve{A}_{Ik} \xrightarrow{p} A_{Ik}^0$ as $N \to \infty$ $\forall k = 1, \ldots, p$. Thus, (39) can be rewritten as:

$$y_{R(p+1)}(t) = \epsilon_R(t) + o_p(1), y_{I(p+1)}(t) = \epsilon_I(t) + o_p(1).$$

Using these equations, we get:

$$\breve{A}_{R(p+1)} = \frac{1}{N} \sum_{t=1}^{N} \left(\epsilon_R(t) \cos\left(\breve{\beta}_{p+1} t^2\right) + \epsilon_I(t) \sin\left(\breve{\beta}_{p+1} t^2\right)\right) + o_p(1),$$
$$\breve{A}_{I(p+1)} = \frac{1}{N} \sum_{t=1}^{N} \left(\epsilon_I(t) \cos\left(\breve{\beta}_{p+1} t^2\right) - \epsilon_R(t) \sin\left(\breve{\beta}_{p+1} t^2\right)\right) + o_p(1).$$

Using lemma 4(b) of [11], we obtain: $\breve{A}_{R(p+1)} \xrightarrow{p} 0$, $\breve{A}_{I(p+1)} \xrightarrow{p} 0$ as $N \to \infty$. Hence, the result follows.

REFERENCES

[1] K. Abratkiewicz. "Sonar pulse detection using chirp rate estimation and CFAR algorithms," *Hydroacoustics*, 20, 7–12, 2017.

[2] K. Abratkiewicz, K. Czarnecki, D. Fourer, and F. Auger. "Estimation of time-frequency complex phase-based speech attributes using narrow band filter banks," *IEEE Signal Processing Symposium*, 1–6, 2017.

[3] K. Abratkiewicz, D Gromek, and P. Samczynski. "Chirp rate estimation and micro-Doppler signatures for pedestrian security radar systems," *IEEE Signal Processing Symposium*, 212–215, 2017.

[4] Z. D. Bai, C. R. Rao, M. Chow, and D. Kundu. "An efficient algorithm for estimating the parameters of superimposed exponential signals," *Journal of Statistical Planning and Inference*,vol 110, no. 1-2, 23–34, 2003.

[5] S. Barbarossa, A. Scaglione, and G. B. Giannakis. "Product high-order ambiguity function for multicomponent polynomial-phase signal modeling," *IEEE Transactions on Signal Processing*, vol 46, no. 3, 691–708, 1998.

[6] J. Bian, J. Xing, Z. Liu, L. Fu, and H. Li. "A computationally efficient iterative algorithm for estimating the parameter of chirp signal model," *Journal of Applied Mathematics*, 1–14, 2014.

[7] P. G. Casazza, and M. Fickus. "Fourier transforms of finite chirps," *Eurasip Journal on Advances in Signal Processing*, 1–7, 2006.

[8] V. Dalla, L. Giraitis, and P. Phillips. "Robust tests for white noise and cross-correlation," *Econometric Theory*, vol 38, no. 5, 913–941, 2022.

[9] I. Djurović, T. Thayaparan, and L. Stankovic. "SAR imaging of moving targets using polynomial Fourier transform," *IET Signal Processing*, vol 2, no. 3, 237–246, 2008.

[10] I. Djurović, M. Simeunović, and P. Wang. "Cubic phase function: A simple solution to polynomial phase signal analysis," *Signal Processing*, 135, 48–66, 2017.

[11] R. Grover, D. Kundu, and A. Mitra. "Asymptotic properties of least squares estimators and sequential least squares estimators of a chirp-like signal model parameters," *Circuits, Systems, and Signal Processing*, vol. 40, no. 11, 5421–5465, 2021.

[12] Y. Guo and L. Yang. " Method for parameter estimation of LFM signal and its application," *IET Signal Processing*, vol 13, no. 5, 538–543, 2019.

[13] Y. Huang, Y. Song, Y. Gu, and W. Gao. "A fast parameter estimation method for multicomponent LFM signal based on FRFT," *In 2020 International Conference on Virtual Reality and Intelligent Systems (ICVRIS)*, 393–396, 2020.

[14] M. Mboup, and T. Adalı. "A generalization of the Fourier transform and its application to spectral analysis of chirp-like signals," *Applied and Computational Harmonic Analysis*, vol 32, no. 2, 305–312, 2012.

[15] A. Mittal, R. Grover, D. Kundu, and A. Mitra. "Estimation of the Elementary Chirp Model Parameters," *IEEE Transactions on Aerospace and Electronic Systems*, 2023.

[16] S. Nandi, and D. Kundu. "Estimation of parameters in random amplitude chirp signal," *Signal Processing*, vol 168, Art. 107328, 2020.

[17] S. S. Moghadasian."A Fast and Accurate Method for Parameter Estimation of Multi-Component LFM Signals," *IEEE Signal Processing Letters*, vol 29, 1719-1723, 2022.

[18] P. O'shea. "A fast algorithm for estimating the parameters of a quadratic FM signal," *IEEE Transactions on Signal Processing*, vol 52, no. 2, 385–393, 2004.

[19] E. Swiercz, D. Janczak, and K. Konopko. "Detection of LFM Radar Signals and Chirp Rate Estimation Based on Time-Frequency Rate Distribution," *Sensors*, no. 16, 5415, 2021.

[20] P. Wang, H. Li, I. Djurović, and B. Himed. "Integrated cubic phase function for linear FM signal analysis," *IEEE Transactions on Aerospace and Electronic Systems*, vol 46, no. 3, 963-977, 2010.

[21] H. Wang, X. Fan, Y. Chen, and Y. Yang. "Wigner-Hough transform based on slice's entropy and its application to multi-LFM signal detection," *Journal of Systems Engineering and Electronics*, vol 28, no. 4, 634–642, 2017.

[22] P. Wang, and J. Yang. "Multicomponent chirp signals analysis using product cubic phase function," *Digital Signal Processing*, vol 16, no. 6, 654–669, 2006.

[23] H. Begleiter. https://archive.ics.uci.edu/ml/datasets/EEG%20Database.