Comments on "Specifying Prior Distributions in Reliability Analysis" by Qinglong Tian, Colin Lewis-Beck, Jarad B. Niemi and William W. Meeker

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First of all I would like to thank Professors Tian, Lewis-Beck, Niemi and Meeker for their timely, comprehensive and lucidly written review article on this very important topic about specifying prior distributions mainly for lifetime probability models, and also to Professor Ruggeri for inviting me to write comments on this very interesting paper. Bayesian inference has received a significant amount of attention since Bayesian computation became easier. One can easily use different forms of the Markov Chain Monte Carlo (MCMC) techniques even in a complicated higher dimensional problem without much difficulty, where implementation of the classical method becomes very difficult. Another advantage of the Bayesian inference is that its interpretation is quite intuitive and it can be used even for small sample sizes. But one major issue in the Bayesian analysis is the choice of priors on the unknown parameters. It is a long standing issue. One standard approach was to choose the conjugate prior, but it has a very limited application. Except few cases it is difficult to find the conjugate priors. On the other hand the final answer will depend very much on the choice of the priors. A general discussions on the choice of priors can be found in different places in the literature, and one such place is the 5th chapter of the book by Ghosh, Delampady

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and Samanta [4]. They have discussed different general methods which are available till the beginning of 2000.

In this paper Tian, Lewis-Beck, Niemi and Meeker mainly considered Weibull and log-normal distributions, the two most important distributions which have been used in analyzing lifetime data. Their main emphasis is on the two-parameter Weibull distribution and analogous results have been also presented for two parameter log-normal distribution. Both distributions belong to the log-location-scale family of distributions, and due to this they enjoy certain properties. Although the maximum likelihood estimators have been studied quite extensively in case of two-parameter Weibull distribution, they have several issues if the majority of the data are censored. In this case the Bayesian analysis seems to be the natural choice. The authors have discussed the choice of several non-informative priors which usually do not need any hyper-parameters and also compared Bayes estimators based on weekly informative priors and the maximum likelihood estimators.

1 Three-Parameter Weibull Distribution

Since Tian, Lewis-Beck, Niemi and Meeker did not mention about the three-parameter Weibull distribution, I would like to make some comments about the choice of priors for this lifetime distribution. I will be using the following form of the three-parameter Weibull distribution. The random variable T is said to have three-parameter Weibull distribution with parameters α, λ and μ, if

$$P(T \le t; \alpha, \lambda, \mu) = F(t; \alpha, \lambda, \mu) = 1 - \exp\left[-\lambda \left(t - \mu\right)^{\alpha}\right], \quad t > \mu. \tag{1}$$

The corresponding density function becomes

$$f_T(t; \alpha, \lambda, \mu) = \alpha\lambda(t - \mu)^{\alpha} \exp[-\lambda(t - \mu)^{\alpha}], \quad t > \mu. \tag{2}$$

In this case $\alpha > 0$ is the shape parameter, $\mu > 0$ is the location parameter and $\lambda^{-1/\alpha}$ is the scale parameter. Note that in general for a three-parameter Weibull distribution $\mu \in (-\infty, \infty)$, but since we are mainly dealing with lifetime distribution, we restrict to $\mu > 0$. It is immediate from (1) and (2) that if the location parameter μ is known, it can be converted to a two-parameter Weibull distribution. Hence, here it is assumed that all the three parameters are unknown. We will be discussing about the point and interval estimation of the unknown parameters based on a random sample of size n, $\mathcal{D} = \{t_1, \ldots, t_n\}$. We have considered the complete sample case, since samples obtained from different censoring schemes also can be handled along the same line. Without loss of generality, it is assumed $0 < t_1 < t_2 \ldots < t_n < \infty$, and let us denote $\Theta = (\alpha, \lambda, \mu)$.

The main issue in the three-parameter Weibull distribution is the non-existence of the maximum likelihood estimators. It can be easily shown that if all the three parameters are unknown, then there exists a path in the parameter space, $(0, \infty) \times (0, \infty) \times [0, \infty)$ along which the likelihood function diverges to infinity, see for example Smith and Naylor [12]. The global maximum is $+\infty$ achieved at $\mu = t_1$. There are good reasons for ignoring this. Since all data are really discrete, the singularity disappears on taking this into account, see Barnard [1]. Therefore, in practice one looks for the local maxima of the likelihood function, and the asymptotic results for maximum likelihood estimators hold provided $\alpha > 2$. Another important point should be mentioned that even when $\alpha > 2$, but if it is close to 2, then the observed Fisher information matrix becomes quite unstable, therefore the construction of the confidence intervals is problematic. Further, due to the presence of the location parameter μ, which is also the threshold parameter in this case, the bootstrap method is not very effective.

Hence, in case of three-parameter Weibull distribution, the Bayesian inference seems to be a reasonable alternative. Interestingly, an extensive amount of work has been done on the Bayesian analysis of two-parameter Weibull distribution. Apart from the different non-informative priors as mentioned by Tian, Lewis-Beck, Niemi and Meeker in the article, several informative priors have also been suggested in the literature, see for example Papadopoulos and Tsokos [8], Singpurwalla and Song [11], Kaminskiy and Krivtsov [6], Kundu [7] and the references cited there in.

2 Priors and Importance Sampling Technique

Compared to the two-parameter Weibull distribution, there has been very little discussion on Bayesian inference of the three-parameter Weibull distribution. Reilly [9] first considered this problem, and discussed one method using a rather crude numerical algorithm. Smith and Naylor [12] considered three different informative priors and compared the Bayes estimates with the maximum likelihood estimates. They have computed the maximum likelihood estimates in the sense which we have discussed above. They have computed the Bayes estimates based on numerical integrations. Green et al. [5] proposed Jeffreys's prior on the shape and scale parameters, and uniform prior on the location parameter. They have used Gibbs sampling technique to generate samples from the full conditionals, and provided the Bayes estimates and the associated credible intervals. There are certain problems they had indicated in generating samples from the full conditionals.

In this discussion we suggest a proper set of priors on the unknown parameters and propose a simple importance sampling technique to implement it in practice. They can be used as weekly informative priors also with the proper choice of the hyper-parameters. We use the following notations now. A random variable X has a gamma distribution with the parameter $a > 0$ and $b > 0$, if it has the density function

$$f_X(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}; \qquad x > 0,$$

and zero, otherwise. It will be denoted by Gamma(a, b). We choose the following priors on

α, λ and μ. It is assumed α is distributed as Gamma(a, b), λ is distributed as Gamma(c, d) and μ is distributed as Uniform in $[0, M]$, for some $M > t_1 > 0$. They are all independently distributed. Based on the above priors, the posterior density function can be written as

$$\pi(\alpha, \lambda, \mu|\mathcal{D}) \propto f_1(\alpha|\mu, \mathcal{D}) f_2(\lambda|\alpha, \mu, \mathcal{D}) f_3(\mu|\mathcal{D}) e^{-h(\mu,\mathcal{D})}, \tag{3}$$

here

$$f_1(\alpha|\mu, \mathcal{D}) \sim \text{ Gamma}(n + a, b - \sum_{i=1}^{n} \ln(t_i - \mu)) \tag{4}$$

$$f_2(\lambda|\alpha, \mu, \mathcal{D}) \sim \text{ Gamma}(n + c, d + \sum_{i=1}^{n} (t_i - \mu)^{\alpha}) \tag{5}$$

$$f_3(\mu|\mathcal{D}) = \frac{K}{t_n - \mu}; \quad K = \frac{1}{\ln t_n - \ln(t_n - t_1)}, \quad 0 < \mu < t_1, \tag{6}$$

$$h(\mu, \mathcal{D}) = (n + c) \ln \left(d + \sum_{i=1}^{n} (t_i - \mu)^{\alpha} \right) + (n + a) \ln \left(b - \sum_{i=1}^{n} \ln(t_i - \mu) \right) + \sum_{i=1}^{n-1} \ln(t_i - \mu).$$

Note that in (4) we scale the data properly so that $b - \sum_{i=1}^{n} \ln(t_i - \mu) > 0$. Since, the generation from (6), (4) and (5) is quite straight forward, the importance sampling technique can be used quite effectively to compute the Bayes estimates and the associated highest posterior density credible interval of any function of α, λ and μ following the method proposed by Chen and Shao [2].

3 Illustrative Example

We have provided an illustrative data example here. The data represents the strength measureed in Gigapascal (GPa) for single carbon fibers and impregnated 1000-carbon fiber tows. Here the sample size is 69. The minimum, maximum, mean and variance of the scaled data are 0.525,1.434,0.981 and 0.039, respectively. The histogram has been provided in Figure 1. We have used weekly informative prior as suggested by Congdon [3] as $a = b = c = d = 0.01$. Based on the importance sampling technique with 10000 generated

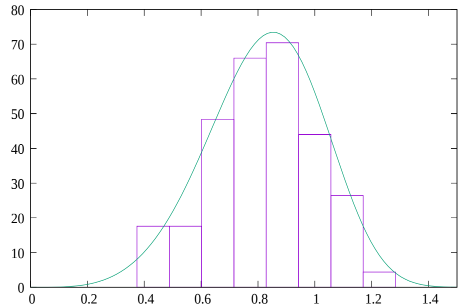

Figure 1: The histogram of the scaled data and the fitted three-parameter probability density function

samples, we obtain the posterior mean (Bayes estimates with respect to squared error loss function) of α, λ and μ as 4.6104, 1.6245 and 0.1503, respectively. The associated 95% highest posterior density credible intervals are (3.7216,4.9786), (1.4231,1.8765) and (0.1313,0.1867), respectively. The fitted probability density function is also provided in Figure 1. The Kolmogorov-Smirnov distance between the empirical cumulative distribution function and the fitted distribution function is 0.0565 with the associate p-value 0.9804. It indicates a very good fit. The above weekly informative priors and importance sampling technique may be used in case of three-parameter log-normal and several other lifetime distribution also.

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