Bayesian Inference for Weibull distribution Under The Balanced Joint Type-II Progressive Censoring Scheme

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Abstract

Progressive censoring schemes have received considerable attention in recent times. All these development are mainly based on a single population. In the last few years the joint progressive censoring scheme has been introduced in the literature and it has received some attention. Recently Mondal and Kundu [15] introduced the balanced joint progressive censoring scheme (BJPC) and study the exact inference for two exponential populations. In this article we implement the BJPC scheme on two Weibull populations with the common shape parameter. The treatment here is purely Bayesian in nature. Under the Bayesian set up we assume a Beta Gamma prior of the scale parameters and a Gamma prior for the common shape parameter. Under this prior assumption closed form of the Bayes estimators can not be obtained and we rely on importance sampling technique to derive the Bayes estimators and associated credible intervals. When the order restriction between the scale parameters is apriori we propose ordered Beta Gamma prior and perform Bayes estimation. We develop one precision criteria based on the expected volume of joint credible set of model parameters to find out optimum censoring scheme. We perform rigorous simulation work to study the performance of the estimators and finally analyze one real data set for illustrative purpose.

Key Words and Phrases: Type-I censoring scheme; type-II censoring scheme; progressive censoring scheme; joint progressive censoring scheme; Bayes estimator; credible interval, optimum censoring scheme.

AMS Subject Classifications: 62N01, 62N02, 62F10.

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1 Introduction

In any life testing experiment implementation of some censoring scheme is essential mainly to control cost and time. In practice type-I, type-II censoring schemes are the two most basic censoring schemes. Rcently an extensive amount of work has been done on different hybrid censoring schemes. Interested readers may refer to the review article by Balakrishnan and Kundu [2]. But in all these censoring schemes one is not allowed to remove any item unless it fails. In a practical scenario, sometimes it is necessary to withdraw some experimental units before they fail. Different progressive censoring schemes allow removal of experimental units at different stages of the experiment.

In a life testing experiment, due to the cost constraint, along with the estimation of the reliability of the products, there is an attempt to study other factors not directly related to the corresponding failure mechanism. To study these auxiliary variables, sometimes experimental units are subject to measurements which are destructive in nature and needed to be removed from the main experiment set up before termination of the main experiment. In engineering study, system efficiency are subject to human operator. To study the human interaction on the efficiency of the system, it is required to withdraw few systems from main experimental set up before termination of the main experiment. In clinical study, doctors come across situations when some individual patients drop out before completion of the study.

All these multistage censoring are studied under different progressive censoring schemes. Herd [8] was the first to introduce the progressive censoring schemes and referred it as the "multi-censored samples". Cohen [5], [6] described the importance of the progressive censoring scheme as well as presented situations where progressively censored data are coming naturally. Mann [14] and Lemon [13] provided estimation procedure for Weibull parameters when data coming from progressive censoring scheme. Viveros and Balakrishnan [25] studied interval estimation on underlined distribution under progressive censoring scheme. Ng et al. [16] provided optimal progressive censoring plan based on Weibull distribution whereas Kundu [9] studied Bayesian inference of Weibull population under progressive censoring scheme. Wang et al. [26] provided inference of certain life time distributions under progressive type-II right censored scheme. An exhaustive collection of works on different progressive censoring scheme has been captured in Balakrishnan and Cramer [1].

Here we briefly describe the progressive type-II censoring scheme as follows. Suppose n units are put on test and k be the number of failures to be observed. Let $R_1, \ldots, R_k$ be the non-negative integers satisfying $\sum_{i=1}^{k}(R_i + 1) = n$. In progressive type-II censoring scheme at the time of first failure, R_1 units are removed from the $n-1$ surviving units. Similarly at second failure time point, R_2 units are removed from rest of the $n - R_1 - 2$ surviving units. The process continues until kth failure occurs. At kth failure remaining R_k surviving units are removed from the test.

When comparative study among more then one population is the interest, we can rely on different joint censoring schemes. Recently Rasouli and Balakrishnan [24] developed joint type-II progressive censoring scheme (JPC) for life testing experiment on two populations. The JPC scheme is briefly described here. Suppose we have a sample of size m from product line-A, and another sample of size n from product line-B. Let $R_1, \ldots, R_k$ be non-negative integers such that $\sum_{i=1}^{k}(R_i + 1) = m + n$. Under the JPC scheme we combine both the samples and put all the $m+n$ units on test. At the time of first failure, whether it is coming from line-A or B, we remove R_1 units from the $m + n - 1$ surviving units. These R_1 units consist of S_1 units from line-A and T_1 units from line-B where S_1 and T_1 are random and $S_1 + T_1 = R_1$. Similarly at second failure time point we remove R_2 units which consist of S_2 units from line-A and T_2 units from line-B. The process will be continued till k th failure when all the remaining R_k units will be removed.

Under the JPC scheme, Rasouli and Balakrishnan [24] studied the likelihood inference of exponential distributions. Parsi and Ganjali [19] applied the JPC scheme on two Weibull populations. Parsi and Bairamov [18] determined the expected number of failures in life testing experiment under the JPC scheme. Doostparast and Ahmadi et al. [7] provided the Bayesian estimation of the parameters from underlined populations under the JPC scheme. Balakrishnan and Su [3] extended the JPC scheme for more than two exponential populations and provided the likelihood and Bayesian inference. Under the JPC scheme since both S_i's and T_i's are random, it is observed that analytically it is quite difficult to handle in general.

Mondal and Kundu [15] recently introduced a Balanced joint type-II progressive censoring scheme (BJPC). The BJPC scheme is analytically easier to handle than the JPC scheme, hence the properties of the estimators can be stated more explicitly. Also it has some advantages over JPC scheme. Mondal and Kundu [15] provided likelihood inference of two exponential populations under the BJPC scheme. The main aim of this paper is to extend the results for two Weibull populations.

This article is developed under the BJPC scheme when two populations are two-parameter Weibull distributions. Assuming that two Weibull populations have common shape parameter, we study the Bayesian inference of unknown model parameters. As the motivation of the JPC or the BJPC scheme is to study the relative merits of similar kind of product under similar condition, assumption of common shape parameter is quite expected.

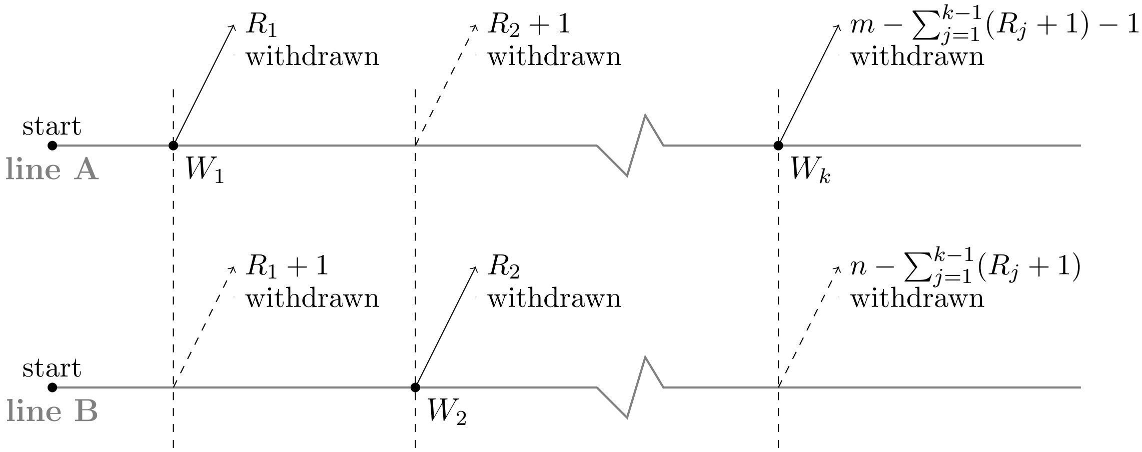
Following the idea of Pena and Gupta [22], Kundu and pradhan [12], joint prior of two scale parameters is assumed to be Beta-Gamma distribution. As gamma distribution has log-concave density function on $(0, \infty)$, and Jeffery's prior is a special case of gamma prior, the common shape parameter is assumed to follow gamma distribution. As under these prior assumptions the Bayes estimator out of squared error loss and the credible intervals can not be obtained in closed form we rely on importance sampling technique.

In this article we also develop Bayes inference of the unknown parameters, when it is known apriori, that there is an order restrictions on the two scale parameters. For instance in an accelerated life test if one sample is put under higher stress keeping the other one in normal stress, it is quiet expected that expected life time of units put under higher stress will be smaller than that of units kept under normal stress. In inference study we incorporate this information imposing order restrictions between scale parameters. Under order restriction joint prior of scale parameters is considered as ordered Beta-Gamma distribution. Further, we propose a precision criteria to compare two different sampling schemes and eventually to find out the optimum censoring scheme out of some given censoring schemes under Bayesian set-up for Weibull populations. When two equal sized samples are drawn from two Weibull populations under BJPC scheme, a joint credible set of model parameters can be obtained. The precision criteria derived here is based on the expected volume of this joint credible set. An exhaustive simulation work is performed to study the performance of the estimators. It is important to note that both the point and interval estimators are performing better in informative priors than the non-informative prior for without order restricted and with order restricted cases. Finally we analyze two data sets for illustrative purpose.

Rest of the paper is organized as follows. In section 2 we describe the BJPC scheme with a schematic diagram. Notations, model assumption and likelihood function are given in section 3. Section 4 consists of prior assumptions. We study posterior analysis in section 5. We propose precision criteria in section 6. Simulation study and data analysis are performed in section 7. Finally we conclude this paper in section 8.

2 Model Description

The balanced joint progressive type-II censoring scheme (BJPC) introduced by Mondal and Kundu [15] is described here with a schematic diagram below. Suppose reliability of two similar kind of products from two different lines has to be studied. A sample of m units is drawn from one product line (say line A) and another sample of n units is drawn from the other product line (say line B). Let k be the total number of failures to be observed in the experiment and $R_1, \ldots, R_{k-1}$ be non-negative integers satisfying $\sum_{i=1}^{k-1}(R_i+1) < min(m,n)$. Under the BJPC scheme, units from two samples are put on test simultaneously. Among these two samples, suppose the first failure comes from the sample of product line A, with failure time point W_1. At W_1 time point R_1 units from the remaining $m-1$ surviving units of the sample from product line A and R_1+1 units from n units of the sample from product line B are randomly removed from the experiment. Next, if the second failure comes from the sample of line B with failure time point W_2, R_2+1 units from $m-R_1-1$ remaining surviving units of the sample from product line A and R_2 units from $n-R_1-2$ remaining surviving units of the sample from product line B are randomly removed at W_2 time point. In general, at ith ($i = 1, \ldots, k-1$), failure time point W_i, we remove R_i units from the sample where ith failure occurs and remove R_i+1 units from the other sample. The experiment is continued until kth failure occurs. Under the BJPC scheme along with $W_1, \ldots, W_k$ another set of random variables, $Z_1, \ldots, Z_k$ are introduced where $Z_i = 1$ or 0 if ith failure comes from the sample of product line A or line B respectively. Let $K_1 = \sum_{i=1}^{k} Z_i$ and $K_2 = \sum_{i=1}^{k}(1-Z_i)$ denote total the number of failures observed from the sample of product line A and line B respectively. Under the BJPC scheme data consists of $((W_1, Z_1), \ldots (W_k, Z_k))$.



3 Notations, Model Assumption and Likelihood

3.1 Notations

PDF :	Probability density function.
i.i.d. :	Independent and identically distributed.
$Beta(a,b)$:	Beta distribution with PDF $\frac{\Gamma(a+b)}{\Gamma a \Gamma b} x^{a-1}(1-x)^{b-1}; 0<x<1$.
$GA(\beta,\lambda)$:	Gamma distribution with PDF $\frac{\lambda^\beta}{\Gamma\beta} x^{\beta-1}e^{-\lambda x}; x>0$.
$WE(\beta,\lambda)$:	Weibull distribution with PDF $\beta\lambda x^{\beta-1}e^{-\lambda x^\beta}; x>0$.

3.2 Model Assumption and Likelihood

It is assumed that m units of line A, say $X_1,\ldots X_m$ are i.i.d random variables from $WE(\beta,\lambda_1)$ and n units of line B, say $Y_1,\ldots,Y_n$ are i.i.d. random variables from $WE(\beta,\lambda_2)$.

For given $m,n,k,(R_1,\ldots,R_{k-1})$ the data can be generated as $((w_1,z_1),\ldots,(w_k,z_k))$. The contribution of $(w_i, z_i=1)$ in likelihood equation is

$$\left(\beta\lambda_1 w_i^{\beta-1}e^{-\lambda_1 w_i^\beta}\right)\times\left(e^{-\lambda_1 w_i^\beta}\right)^{R_i}\times\left(e^{-\lambda_2 w_i^\beta}\right)^{(R_i+1)}=\beta\lambda_1 w_i^{\beta-1}e^{-\lambda_1(R_i+1)w_i^\beta}e^{-\lambda_2(R_i+1)w_i^\beta}.$$

Similarly contribution of $(w_i,z_i=0)$ in likelihood equation is

$$\left(\beta\lambda_2 w_i^{\beta-1}e^{-\lambda_2 w_i^\beta}\right)\times\left(e^{-\lambda_1 w_i^\beta}\right)^{(R_1+1)}\times\left(e^{-\lambda_2 w_i^\beta}\right)^{R_i}=\beta\lambda_2 w_i^{\beta-1}e^{-\lambda_1(R_i+1)w_i^\beta}e^{-\lambda_2(R_i+1)w_i^\beta}.$$

Hence the contribution of general (w_i, z_i) is obtained as

$$\beta \lambda_1^{z_i} \lambda_2^{1-z_i} w_i^{\beta-1} e^{-\lambda_1 (R_1+1) w_i^{\beta}} e^{-\lambda_2 (R_1+1) w_i^{\beta}}.$$

The likelihood function can be obtained as

$$L(\beta, \lambda_1, \lambda_2 | (w_1, z_1), \ldots, (w_k, z_k)) \propto \beta^k {\lambda_1}^{k_1} {\lambda_2}^{k_2} \prod_{i=1}^{k} {w_i}^{\beta-1} e^{-\lambda_1 A_1(\beta)} e^{-\lambda_2 A_2(\beta)} \tag{1}$$

where $k_1 = \sum_{i=1}^{k} z_i$, $k_2 = \sum_{i=1}^{k} (1 - z_i) = k - k_1$, $A_1(\beta) = \sum_{i=1}^{k-1} (R_i + 1) w_i^{\beta} + (m - \sum_{i=1}^{k-1} (R_i + 1)) w_k^{\beta}$, and $A_2(\beta) = \sum_{i=1}^{k-1} (R_i + 1) w_i^{\beta} + (n - \sum_{i=1}^{k-1} (R_i + 1)) w_k^{\beta}$.

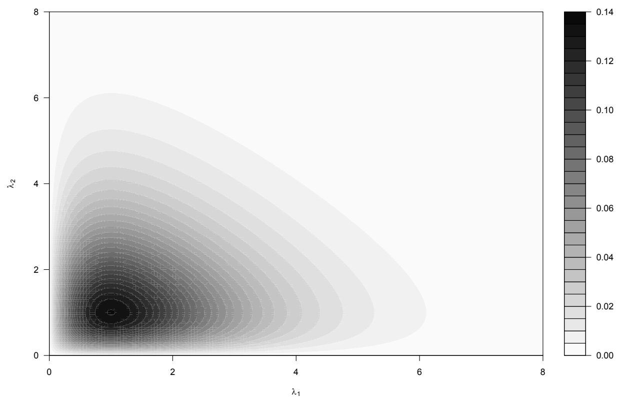
4 Prior Assumption

4.1 Prior Assumption Without Order Restriction

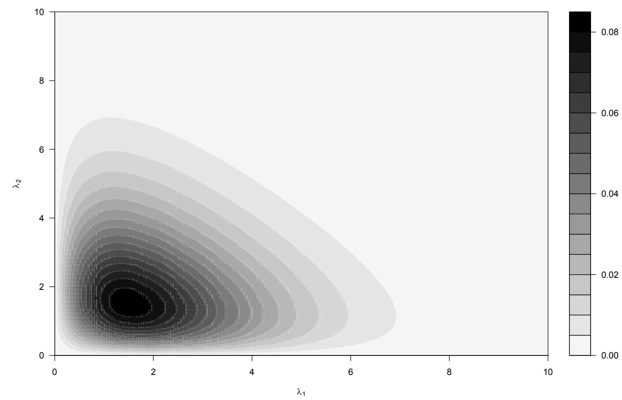
Under the given assumptions, β, λ_1, λ_2 are three unknown model parameters. In absence of order restriction between λ_1 and λ_2, it is assumed that $\lambda_1 + \lambda_2 \sim GA(a_0, b_0)$ and $\frac{\lambda_1}{\lambda_1+\lambda_2} \sim Beta(a_1, a_2)$ independently where $a_0 > 0, b_0 > 0, a_1 > 0, a_2 > 0$. Similar approach is done in Kundu and pradhan [12] under competing risk setup, which is a special case from Pena and Gupta [22]. Applying simple transformation the joint PDF of (λ_1, λ_2) is given by

$$\begin{aligned} \pi(\lambda_1, \lambda_2 | a_0, b_0, a_1, a_2) &= \frac{\Gamma(a_1 + a_2)}{\Gamma a_0 \Gamma a_1 \Gamma a_2} b_0^{a_0} \lambda_1^{a_1-1} \lambda_2^{a_2-1} (\lambda_1 + \lambda_2)^{a_0-a_1-a_2} \\ & \quad e^{-b_0(\lambda_1+\lambda_2)}; \quad \text{where} \quad 0 < \lambda_1, \lambda_2 < \infty. \end{aligned} \tag{2}$$

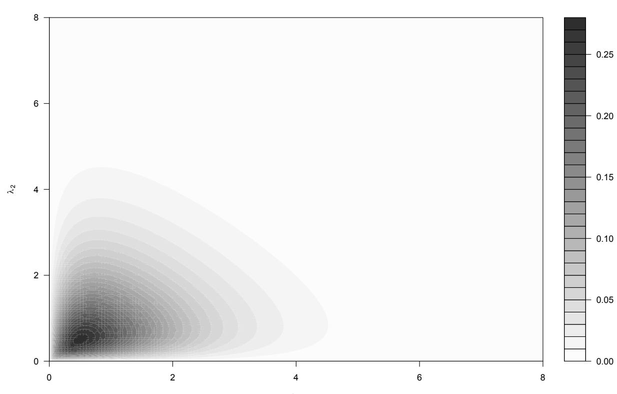
This is the PDF of Beta-Gamma distribution (BG$(\cdot)$). Under BJPC scheme, as similar kinds of products are tested, it is quite expected to assume scale parameters have certain association. Apart from incorporating dependency structure in scale parameters , different set of hyper-parameters in Beta-Gamma prior provide a varied rage of joint density functions of the scale parameters. The correlation between λ_1 and λ_2 can be determined by the hyper parameter a_0, a_1, a_2. λ_1 and λ_2 are independent when $a_1 + a_2 = a_0$, positively correlated if $a_1 + a_2 > a_0$ and they are negatively correlated if $a_1 + a_2 < a_0$. Contour plots of the prior are provided in Figure (1) for different values of hyper-parameters. In Figure (1a) $a_0 = a_1 + a_2$ indicates λ_1, λ_2 are independently distributed where as in Figure (1b) and Figure (1c) the values of hyper-parameters indicate λ_1 and λ_2 are negatively and positively correlated respectively.



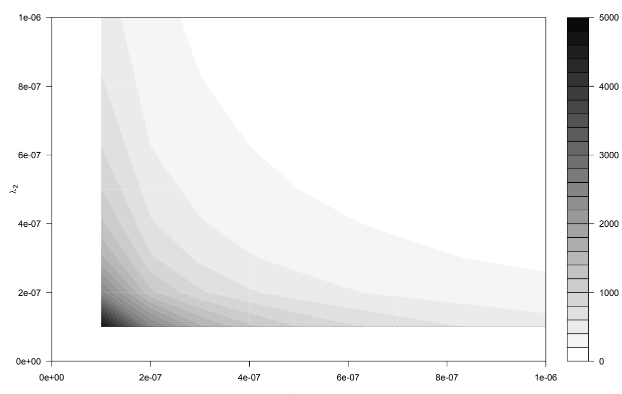
(a) $b_0 = 1, a_0 = 4, a_1 = 2, a_2 = 2$



(b) $b_0 = 1, a_0 = 5, a_1 = 2, a_2 = 2$



(c) $b_0 = 1, a_0 = 3, a_1 = 2, a_2 = 2$



(d) $b_0 = a_0 = a_1 = a_2 = 10^{-5}$

Figure 1: Contour plot of $\pi(\lambda_1, \lambda_2)$ for different values of hyper-parameters

RESULT 1: If $(\lambda_1, \lambda_2) \sim BG(a_0, b_0, a_1 a_2)$,

$$\begin{aligned}
E(\lambda_i) &= \frac{a_0}{b_0}\frac{a_i}{(a_1+a_2)} \quad \text{for} \quad i = 1,2; \\
E(\lambda_i^2) &= \frac{a_0(a_0+1)}{b_0^2}\frac{a_i(a_i+1)}{(a_1+a_2)(a_1+a_2+1)} \quad \text{for} \quad i = 1,2; \\
E(\lambda_1\lambda_2) &= \frac{a_0(a_0+1)}{b_0^2}\frac{a_1 a_2}{(a_1+a_2)(a_1+a_2+1)}.
\end{aligned}$$

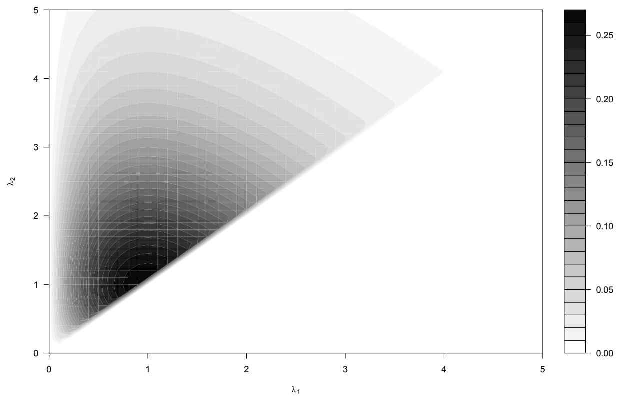
Due to the flexibility of gamma distribution, it is assumed that the common shape parameter $\beta \sim \pi(\beta) \equiv GA(a,b)$ where $a > 0, b > 0$. Gamma distribution has log-concave PDF and Jeffrey's prior (non-informative) is a special case of gamma prior. Prior distribution of shape parameter β is independent of prior distribution of (λ_1, λ_2).

4.2 Prior Assumption With Order Restriction

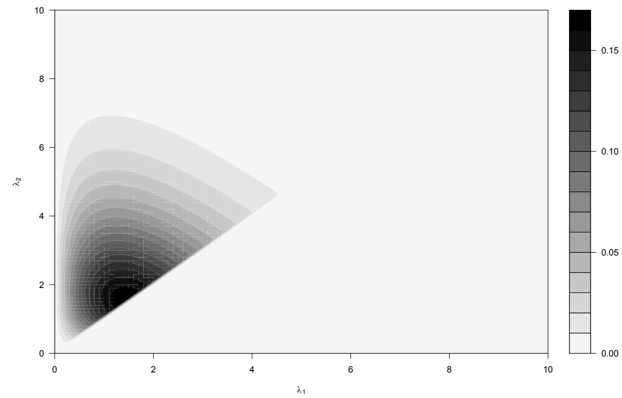
When the order restriction of the shape parameters λ_1 and λ_2 is considered, i.e. $\lambda_1 < \lambda_2$, the joint prior of (λ_1, λ_2) is given as

$$\begin{aligned}
\pi(\lambda_1, \lambda_2 | a_0, b_0, a_1, a_2) &= \frac{\Gamma(a_1+a_2)}{\Gamma a_0 \Gamma a_1 \Gamma a_2} b_0^{a_0} (\lambda_1^{a_1-1}\lambda_2^{a_2-1} + \lambda_1^{a_2-1}\lambda_2^{a_1-1}) \\
&\quad (\lambda_1+\lambda_2)^{a_0-a_1-a_2} e^{-b_0(\lambda_1+\lambda_2)}; \quad \text{where} \quad 0 < \lambda_1 < \lambda_2 < \infty. \quad (3)
\end{aligned}$$

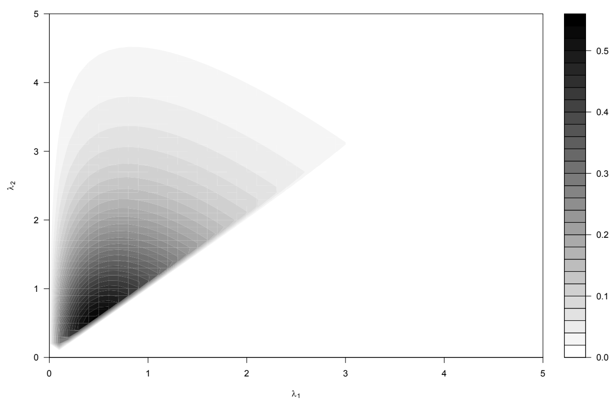
From the form of the joint prior above, it is clear that, it is the PDF of ordered random variables $(\lambda_{(1)}, \lambda_{(2)})$ where $(\lambda_{(1)}, \lambda_{(2)}) = (\lambda_1, \lambda_2)$ if $\lambda_1 < \lambda_2$ and $(\lambda_{(1)}, \lambda_{(2)}) = (\lambda_2, \lambda_1)$ if $\lambda_1 > \lambda_2$ and $(\lambda_1, \lambda_2) \sim BG(a_0, b_0, a_1, a_2)$. The prior in (3) is denoted as OBG(a_0, b_0, a_1, a_2). In Figure 2 we provide contour plot of the density of ordered beta-gamma distribution for different sets of hyper-parameters. The common shape parameter β has the same prior $\pi(\beta)$ independent of (λ_1, λ_2) as before.



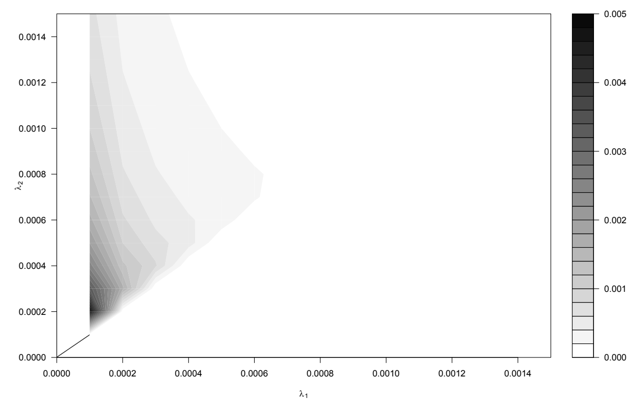
(a) $b_0 = 1, a_0 = 4, a_1 = 2, a_2 = 2$



(b) $b_0 = 1, a_0 = 5, a_1 = 2, a_2 = 2$



(c) $b_0 = 1, a_0 = 3, a_1 = 2, a_2 = 2$



(d) $b_0 = a_0 = a_1 = a_2 = 10^{-5}$

Figure 2: Contour plot of $\pi(\lambda_1, \lambda_2)$ for different values of hyper-parameters

5 Posterior Analysis

In this section, based on the prior assumption we derive the Bayes estimators based on the squared error loss function. We compute Bayes estimators and associated credible intervals when shape parameter is known and when it is unknown. All these estimators are derived both for without order restriction and with order restriction on scale parameters.

5.1 Posterior Analysis Without Order Restriction

5.1.1 shape parameter known

When shape parameter β is known joint posterior density function of λ_1 and λ_2 can be obtained as

$$\pi(\lambda_1, \lambda_2|\beta, data) \quad \propto \quad \lambda_1^{a_1+k_1-1}\lambda_2^{a_2+k_2-1}(\lambda_1+\lambda_2)^{a_0-a_1-a_2}e^{-b_0(\lambda_1+\lambda_2)}e^{-\lambda_1 A_1(\beta)}e^{-\lambda_2 A_2(\beta)} \quad (4)$$

When $m = n$, $A_1(\beta) = A_2(\beta) = A(\beta)$ (say), posterior distribution of λ_1, λ_2 is $BG(a_0 + k, b_0 + A(\beta), a_1 + k_1, a_2 + k_2)$. For known shape parameter and two equal sample sizes, Beta Gamma prior is conjugate one. The Bayes estimate of λ_i, $i = 1, 2$ can be obtained as $\frac{a_0+k}{b_0+A(\beta)}\frac{a_i+k_i}{(a_1+a_2+k)}$ for $\quad i = 1, 2$.

For the development of credible intervals the following theorem will be useful.

Theorem 1: $(\lambda_1, \lambda_2) \sim \mathrm{BG}(a_0, b_0, a_1, a_2)$ if nad only if $\lambda_1 + \lambda_2 \sim \mathrm{GA}(a_0, b_0)$ and $\frac{\lambda_1}{\lambda_1+\lambda_2} \sim \mathrm{Beta}(a_1, a_2)$ independently.

Bsed on THEOREM 1 a $100(1-\gamma)\%$ credible set of (λ_1, λ_2) can be constructed as
$C_{1-\gamma}(\lambda_1, \lambda_2|\beta) = \{(\lambda_1, \lambda_2) : \lambda_1 > 0, \lambda_2 > 0, \quad B_1 \leq \lambda_1 + \lambda_2 \leq B_2, C_1 \leq \frac{\lambda_1}{\lambda_1+\lambda_2} \leq C_2\}$.
Here B_1, B_2, C_1, C_2, are such that

$$P(B_1 \leq \lambda_1 + \lambda_2 \leq B_2) \times P(C_1 \leq \frac{\lambda_1}{\lambda_1 + \lambda_2} \leq C_2) = 1 - \gamma,$$

Note that $C_{1-\gamma}(\lambda_1, \lambda_2|\beta)$ is a trapezoid with area $\frac{(B_2^2-B_1^2)(C_2-C_1)}{2}$. The HPD credible set of (λ_1, λ_2) can be determined choosing B_1, B_2, C_1, C_2 which will minimize this area.

When two samples are not of equal sized, given prior is not the conjugate one and we can not obtain the Bayesian estimators in closed form. We discuss the estimation procedure using importance sampling technique in the Appendix A.1.

5.1.2 Shape parameter not known

When shape parameter β is unknown, the joint posterior density function of $\beta, \lambda_1, \lambda_2$ can be written as

$$\begin{aligned}\pi(\beta, \lambda_1, \lambda_2|data) \;\propto\;& \lambda_1^{a_1+k_1-1}\lambda_2^{a_2+k_2-1}(\lambda_1+\lambda_2)^{a_0-a_1-a_2}e^{-b_0(\lambda_1+\lambda_2)}e^{-\lambda_1 A_1(\beta)}e^{-\lambda_2 A_2(\beta)}\\ & \beta^{a+k-1}e^{-\beta(b-\sum_{i=1}^{k}\ln w_i)} \qquad (5)\end{aligned}$$

When $m = n$, $A_1(\beta) = A_2(\beta) = A(\beta)$ (say) and $\pi(\beta, \lambda_1, \lambda_2|data)$ can be decomposed as

$$\pi(\beta, \lambda_1, \lambda_2|data) \propto \pi_1^*(\lambda_1, \lambda_2|\beta, data) \times \pi_2^*(\beta|data) \qquad (6)$$

where $\pi_1^*(\lambda_1, \lambda_2|\beta, data)$ is the PDF of BG$(a_0+k, b_0+A(\beta), a_1+k_1, a_2+k_2)$, and $\pi_2^*(\beta|data) \propto \frac{\beta^{a+k-1}e^{-\beta(b-\sum_{i=1}^{k}\ln w_i)}}{(b_0+A(\beta))^{a_0+k}}$.

The Bayes estimate of a function $g(\beta, \lambda_1, \lambda_2)$ is obtained by

$$\begin{aligned}E(g(\beta, \lambda_1, \lambda_2)|data) &= \int_0^\infty\int_0^\infty\int_0^\infty g(\beta, \lambda_1, \lambda_2)\pi(\beta, \lambda_1, \lambda_2|data)d\beta d\lambda_1 d\lambda_2\\ &= \frac{\int_0^\infty\int_0^\infty\int_0^\infty g(\beta, \lambda_1, \lambda_2)\pi_1^*(\lambda_1, \lambda_2|\beta, data) \times \pi_2^*(\beta|data)d\beta d\lambda_1 d\lambda_2}{\int_0^\infty\int_0^\infty\int_0^\infty \pi_1^*(\lambda_1, \lambda_2|\beta, data) \times \pi_2^*(\beta|data)d\beta d\lambda_1 d\lambda_2},\end{aligned}$$

provided it exists. The following algorithm can be used to approximate Bayes estimator and credible interval of a function $g(\beta, \lambda_1, \lambda_2)$.

Algorithm 1

Step 1: Given data, generate β from $\pi_2^*(\beta|data)$.

Step 2: Given a generated β, generate λ_1, λ_2 from $\pi_1^*(\lambda_1, \lambda_2|\beta, data)$.

Step 3: Repeat the process say N times to generate $\big((\beta_1, \lambda_{11}, \lambda_{21}) \ldots (\beta_N, \lambda_{1N}, \lambda_{2N})\big)$.

Step 4: To obtain Bayes estimate of $g(\beta, \lambda_1, \lambda_2)$, compute $(g_1, \ldots, g_N)$, where $g_i = g(\beta_i, \lambda_{1i}, \lambda_{2i})$.

Step 5: Bayes estimate of $g(\beta, \lambda_1, \lambda_2)$ can be approximated as $\frac{\sum_{i=1}^{N} g_i}{N}$.

Step 6: To compute $100(1-\gamma)\%$ CRI of $g(\beta, \lambda_1, \lambda_2)$, arrange g_i in ascending order to obtain $(g_{(1)}, \ldots g_{(N)})$. A $100(1-\gamma)\%$ CRI can be obtained as $(g_{(j)}, g_{(j+[N(1-\gamma)])})$ where $j = 1, 2, \ldots, [N\gamma]$. The $100(1-\gamma)\%$ highest posterior density (HPD) CRI can be obtained as $(g_{(j^*)}, g_{(j^*+[N(1-\gamma)])})$, such that $g_{(j^*+[N(1-\gamma)])} - g_{(j^*)} \leq g_{(j)} - g_{(j+[N(1-\gamma)])}$ for $j = 1, 2, \ldots, [N\gamma]$.

For $m \neq n$ case, the estimation procedure is discussed in the Appendix.

5.2 Posterior Analysis With Order Restriction

5.2.1 Shape Parameter Known

When the order restriction $\lambda_1 < \lambda_2$ is considered, the joint posterior density function of λ_1, λ_2 can be written as

$$\begin{aligned} \pi(\lambda_1, \lambda_2|\beta, data) \;\propto\; & (\lambda_1^{a_1+k_1-1}\lambda_2^{a_2+k_2-1} + \lambda_1^{a_2+k_1-1}\lambda_2^{a_1+k_2-1})(\lambda_1+\lambda_2)^{a_0-a_1-a_2}e^{-b_0(\lambda_1+\lambda_2)} \\ & e^{-\lambda_1 A_1(\beta)}e^{-\lambda_2 A_2(\beta)}. \end{aligned} \tag{7}$$

(7) can be re-written as

$$\begin{aligned} \pi(\lambda_1, \lambda_2|\beta, data) \;\propto\; & (\lambda_1^{a_1+J-1}\lambda_2^{a_2+J-1} + \lambda_1^{a_2+J-1}\lambda_2^{a_1+J-1})(\lambda_1+\lambda_2)^{a_0-a_1-a_2}e^{-(\lambda_1+\lambda_2)(b_0+U(\beta))} \\ & \lambda_1^{k_1-J}\lambda_2^{k_2-J}e^{-\lambda_1(A_1(\beta)-U(\beta))}e^{-\lambda_2(A_2(\beta)-U(\beta))} \end{aligned} \tag{8}$$

here $J = min(k_1, k_2)$, $U(\beta) = min(A_1(\beta), A_2(\beta))$.

From (8) it is clear that, the joint posterior density of λ_1, λ_2 can be decomposed as

$$\pi(\lambda_1, \lambda_2|\beta, data) \propto \pi_1^*(\lambda_1, \lambda_2|\beta, data) \times h(\lambda_1, \lambda_2). \tag{9}$$

where $\pi_1^*(\lambda_1, \lambda_2|\beta, data)$ is the PDF of OBG$(a_0+2J, b_0+U(\beta), a_1+J, a_2+J)$, and $h(\lambda_1, \lambda_2) = \lambda_1^{k_1-J}\lambda_2^{k_2-J}e^{-\lambda_1(A_1(\beta)-U(\beta))}e^{-\lambda_2(A_2(\beta)-U(\beta))}$. The Bayes estimator and CRI of any function of λ_1, λ_2 can be derived using the Algorithm 2 (see Appendix A.1).

5.2.2 Shape parameter not Known

Under the order restriction between scale parameters when shape parameter β is not known, the joint posterior density function of $\beta, \lambda_1, \lambda_2$ can be written as

$$\begin{aligned}\pi(\beta, \lambda_1, \lambda_2|data) \;\propto\; & (\lambda_1^{a_1+k_1-1}\lambda_2^{a_2+k_2-1} + \lambda_1^{a_2+k_1-1}\lambda_2^{a_1+k_2-1})(\lambda_1+\lambda_2)^{a_0-a_1-a_2}e^{-b_0(\lambda_1+\lambda_2)} \\ & e^{-\lambda_1 A_1(\beta)}e^{-\lambda_2 A_2(\beta)}\beta^{a+k-1}e^{-\beta(b-\sum_{i=1}^{k}\ln w_i)}. \end{aligned} \tag{10}$$

(10) can be re-written as

$$\begin{aligned}\pi(\beta, \lambda_1, \lambda_2|data) \;\propto\; & (\lambda_1^{a_1+J-1}\lambda_2^{a_2+J-1} + \lambda_1^{a_2+J-1}\lambda_2^{a_1+J-1})(\lambda_1+\lambda_2)^{a_0-a_1-a_2}e^{-(\lambda_1+\lambda_2)(b_0+U(\beta))} \\ & \lambda_1^{k_1-J}\lambda_2^{k_2-J}e^{-\lambda_1(A_1(\beta)-U(\beta))}e^{-\lambda_2(A_2(\beta)-U(\beta))}\beta^{a+k-1}e^{-\beta(b-\sum_{i=1}^{k}\ln w_i)} \end{aligned} \tag{11}$$

here $J = min(k_1, k_2)$, $U(\beta) = min(A_1(\beta), A_2(\beta))$.

From (11) it is clear that, the joint posterior density of $\beta, \lambda_1, \lambda_2$ can be decomposed as

$$\pi(\beta, \lambda_1, \lambda_2|data) \propto \pi_1^*(\lambda_1, \lambda_2|\beta, data) \times \pi_2^*(\beta|data) \times h(\beta, \lambda_1, \lambda_2) \tag{12}$$

where $\pi_1^*(\lambda_1, \lambda_2|\beta, data)$ is the PDF of OBG$(a_0+2J, b_0+U(\beta), a_1+J, a_2+J)$, $\pi_2^*(\beta|data) \propto \frac{\beta^{a+k-1}e^{-(b-\sum_{i=1}^{k}\ln w_i)}}{(b_0+U(\beta))^{a_0+2J}}$ and $h(\beta, \lambda_1, \lambda_2) = \lambda_1^{k_1-J}\lambda_2^{k_2-J}e^{-\lambda_1(A_1(\beta)-U(\beta))}e^{-\lambda_2(A_2(\beta)-U(\beta))}$.

The Bayes estimator and CRI of any function of $\beta, \lambda_1, \lambda_2$ can be derived using the Algorithm 3 (see Appendix A.2).

6 Optimum Censoring Scheme

So far we discussed Bayesian inference of unknown model parameters $\beta, \lambda_1, \lambda_2$ assuming that sample sizes m and n, effective sample size k and censoring scheme $(R_1, \ldots, R_{k-1})$ are known. In practical situation to conduct a life testing experiment, it is advisable to choose the optimum censoring scheme (OCS) out of all possible censoring schemes, i.e. the censoring scheme which will provide maximum *information* about the unknown parameters based on some *scientific criteria*. Burkschat et al. [4] provided optimum censoring scheme based on minimizing variance of best linear unbiased estimator. Ng et al. [16] obtained optimum censoring scheme in terms of minimizing variance of maximum likelihood estimators from Weibull distribution. Pradhan and Kundu [21] chosen optimum scheme based on variance of quantile estimator in frequentest set up. In Bayesian frame-work several works have been done in finding the optimum censoring scheme. Kundu [9], Pareek and Kundu et al. [20], Kundu and Pradhan [11], Kundu and Pradhan [12] studied in finding out OCS based on posterior variance of quantile of underlined distributions.

In our model when both the sample sizes are equal, i.e. $m = n = m_0 \quad (say)$ and m_0 and effective sample size k are fixed, the set of all possible censoring schemes consists of $R_i's, i = 1, \ldots, k-1$, such that $\sum_{i=1}^{k-1}(R_i + 1) < m_0$. Following the idea of Kundu [10] we propose a precision criteria based on the volume of joint credible set of unknown model parameters to find out the OCS.

6.1 Precision Criteria

When $m = n$, and order restriction between λ_1, λ_2 is not considered, as described section 5.1.1 we can obtain a $100\big((1-\gamma_1)(1-\gamma_2)\%$ joint credible set of λ_1, λ_2 for a given β as

$C_{(1-\gamma_1)(1-\gamma_2)}(\lambda_1, \lambda_2|\beta)$,

$$C_{(1-\gamma_1)(1-\gamma_2)}(\lambda_1, \lambda_2|\beta) = \{(\lambda_1, \lambda_2) : \lambda_1 > 0, \lambda_2 > 0, \quad B_1 \leq \lambda_1 + \lambda_2 \leq B_2, C_1 \leq \frac{\lambda_1}{\lambda_1 + \lambda_2} \leq C_2\}.$$

Here $B_1,\ B_2,\ C_1,\ C_2$ are such that

$$P(B_1 \leq \lambda_1 + \lambda_2 \leq B_2) = 1 - \gamma_1,\ P(C_1 \leq \frac{\lambda_1}{\lambda_1 + \lambda_2} \leq C_2) = 1 - \gamma_2.$$

Note that $C_{(1-\gamma_1)(1-\gamma_2)}(\lambda_1, \lambda_2|\beta)$ is a trapezoid whose area say $Area(C_{(1-\gamma_1)(1-\gamma_2)}(\lambda_1, \lambda_2|\beta))$ is $\frac{(B_2^2 - B_1^2)(C_2 - C_1)}{2}$.

When β is known precision criteria to find out OCS can be defined as $E_{data}(Area(C_{(1-\gamma_1)(1-\gamma_2)}(\lambda_1, \lambda_2|\beta))$.

Further when β is not known a $100(1-\gamma_3)\%$ credible interval of β is obtained as

$$\{(L_1, L_2) : \int_{L_1}^{L_2} \pi_2^*(\beta|data) d\beta = 1 - \gamma_3\}.$$

where $\pi_2^*(\beta|data) \propto \frac{\beta^{a+k-1} e^{-\beta(b - \sum_{i=1}^{k} \ln w_i)}}{(b_0 + A(\beta))^{a_0+k}}$.

Let $(1-\gamma_1)(1-\gamma_2)(1-\gamma_3) = (1-\gamma)$. A $100(1-\gamma)\%$ joint credible set of $\beta, \lambda_1, \lambda_2$ can be obtained as

$$\{(\beta, \lambda_1, \lambda_2) : L_1 \leq \beta \leq L_2, (\lambda_1, \lambda_2) \in C_{(1-\gamma_1)(1-\gamma_2)}(\lambda_1, \lambda_2|\beta)\} = D_\gamma \quad \text{(say)}. \tag{13}$$

The volume of D_γ, say $Vol(D_\gamma)$ is obtained as

$$Vol(D_\gamma) = \int_{L_1}^{L_2} \frac{(B_2^2 - B_1^2)(C_2 - C_1)}{2} d\beta. \tag{14}$$

Based on (14) the precision criteria to find out the OCS is defined as $E_{data}(Vol(D_\gamma))$. Between two censoring schemes $R_1 = (R_{1,1}, \ldots, R_{k-1,1})$ and $R_2 = (R_{1,2}, \ldots, R_{k-1,2})$, if R_1 provides smaller value of $E_{data}(Vol(D_\gamma))$ than R_2, R_1 is considered to be better scheme than R_2.

7 Simulation and Data Analysis

7.1 Simulation

In this section we perform extensive simulation work to study the performance of the estimators. To compute Bayes estimates and CRIs we consider $m = 27, n = 25$, $\beta = 2$, $\lambda_1 = .7$, and $\lambda_2 = 1$. Simulation work is done based on one informative prior (IP) where the hyper-parameters are taken as $b_0 = 1$, $a_0 = (\lambda_1 + \lambda_2)b_0$, $a_1 = 2$, $a_2 = \frac{\lambda_2}{\lambda_1}a_1$, $b = 2$, $a = \beta b$ where β, λ_1 and λ_2 are as considered above. In case of non-informative prior (NIP) it is assumed that $b_0 = a_0 = a_1 = a_2 = a = b = 10^{-5}$ which is close to zero. We could not take a_1, a_2, a, b as zero as this might lead to improper posterior density. These values of hyper-parameters make the prior distribution very diffuse and hence non-informative. Based on the algorithm in section 5 we compute Bayes estimates of the parameters for 1000 samples. For different censoring censoring scheme in Table (1) we compute average of the Bayes estimates (BE) and corresponding mean squared errors (MSE) for both informative and non-informative priors without considering order restrictions between scale parameters. We record the corresponding results in Table (2) considering order restriction between scale parameters. In Table (3) and Table (4) we compute the average length (AL) and coverage percentage (CP) of 90% symmetric CRIs of the parameters with and without order restriction for different censoring schemes. The AL and CP of 90% HPD CRIs are reported in Table (5) and Table (6). In all the cases AL and CP are computed based on 1000 replications. For fixed sample sizes m_0 and fixed effective sample size k, number of all possible censoring scheme is $\binom{m_0-1}{k-1}$ which is large even for small m_0, k. Hence we have reported the results for some corner point of the lattice of all possible censoring schemes. In Table (7) we compute $E_{data}(Vol(D_\gamma)$ taking $\gamma_1 = \gamma_2 = \gamma_3 = 0.025$ for certain censoring schemes along with the average total time for the test (ETOT). These computation are based on 1000 generations Here $R = (10, 0_{(13)})$ indicates $R_1 = 10$ and $R_2 = R_3 = \ldots = R_{14} = 0$.

Table 1: $m = 27, n = 25, \beta = 2, \lambda_1 = 0.7, \lambda_2 = 1$ (Without Order Restriction)

censoring scheme	Parameter	BAYES IP		BAYES NIP	
		BE	MSE	BE	MSE
k=15,R=$(10,0_{(13)})$	β	2.109	0.145	2.172	0.227
	λ_1	0.756	0.068	0.791	0.122
	λ_2	1.078	0.123	1.156	0.289
k=15,R=$(0_{(5)},10,0_{(8)})$	β	2.078	0.127	2.132	0.196
	λ_1	0.751	0.063	0.785	0.110
	λ_2	1.083	0.116	1.118	0.218
k=15,R=$(0_{(10)},10,0_{(3)})$	β	2.026	0.127	2.087	0.219
	λ_1	0.737	0.056	0.790	0.128
	λ_2	1.057	0.098	1.128	0.241
k=15,R=$(0_{(14)})$	β	1.960	0.126	2.021	0.196
	λ_1	0.715	0.050	0.756	0.096
	λ_2	1.026	0.082	1.096	0.171
k=20,R=$(5,0_{(18)})$	β	2.100	0.124	2.120	0.166
	λ_1	0.739	0.050	0.747	0.071
	λ_2	1.066	0.097	1.077	0.141
k=20,R=$(0_{(5)}\ 5,0_{(13)})$	β	2.086	0.116	2.108	0.159
	λ_1	0.744	0.051	0.751	0.075
	λ_2	1.064	0.097	1.079	0.139
k=20,R=$(0_{(10)}\ 5,0_{(8)})$	β	2.078	0.120	2.120	0.153
	λ_1	0.743	0.050	0.756	0.075
	λ_2	1.070	0.096	1.080	0.137
k=20,R=$(0_{(15)}\ 5,0_{(3)})$	β	2.068	0.123	2.131	0.188
	λ_1	0.740	0.048	0.752	0.076
	λ_2	1.060	0.086	1.110	0.177
k=20,R=$(0_{(19)})$	β	2.054	0.122	2.100	0.191
	λ_1	0.738	0.048	0.749	0.068
	λ_2	1.057	0.081	1.084	0.135

Table 2: $m = 27, n = 25, \beta = 2, \lambda_1 = 0.7, \lambda_2 = 1$ (With Order Restriction)

censoring scheme	Parameter	BAYES IP		BAYES NIP	
		BE	MSE	BE	MSE
k=15,R=(10,$0_{(13)}$)	β	2.120	0.155	2.206	0.267
	λ_1	0.703	0.043	0.677	0.074
	λ_2	1.179	0.143	1.284	0.301
k=15,R=($0_{(5)}$,10,$0_{(8)}$)	β	2.109	0.127	2.160	0.239
	λ_1	0.703	0.042	0.680	0.062
	λ_2	1.191	0.155	1.271	0.259
k=15,R=($0_{(10)}$,10,$0_{(3)}$)	β	2.115	0.168	2.140	0.251
	λ_1	0.699	0.045	0.698	0.070
	λ_2	1.183	0.147	1.301	0.309
k=15,R=($0_{(14)}$)	β	2.081	0.149	2.077	0.241
	λ_1	0.685	0.040	0.670	0.085
	λ_2	1.164	0.133	1.268	0.354
k=20,R=(5,$0_{(18)}$)	β	2.114	0.130	2.154	0.191
	λ_1	0.683	0.034	0.662	0.041
	λ_2	1.143	0.109	1.196	0.165
k=20,R=($0_{(5)}$ 5,$0_{(13)}$)	β	2.114	0.130	2.139	0.172
	λ_1	0.688	0.033	0.657	0.042
	λ_2	1.138	0.099	1.182	0.139
k=20,R=($0_{(10)}$ 5,$0_{(8)}$)	β	2.121	0.125	2.135	0.166
	λ_1	0.684	0.033	0.672	0.046
	λ_2	1.147	0.103	1.202	0.156
k=20,R=($0_{(15)}$ 5,$0_{(3)}$)	β	2.096	0.128	2.159	0.207
	λ_1	0.698	0.035	0.669	0.045
	λ_2	1.157	0.109	1.202	0.166
k=20,R=($0_{(19)}$)	β	2.097	0.140	2.124	0.194
	λ_1	0.694	0.034	0.669	0.043
	λ_2	1.143	0.102	1.174	0.133

Table 3: $m = 27, n = 25, \beta = 2, \lambda_1 = 0.7, \lambda_2 = 1$; 90% Symmetric CRI:(Without Order Restriction)

censoring scheme	Parameter	BAYES IP		BAYES NIP	
		AL	CP	AL	CP
k=20,R=$(5,0_{(18)})$	β	1.071	89.8%	1.122	83.4%
	λ_1	0.734	92.3%	0.798	87.0%
	λ_2	1.016	90.4%	1.101	88.3%
k=20,R=$(0_{(10)},5,0_{(8)})$	β	0.987	85.7%	1.016	79.6%
	λ_1	0.741	91.5%	0.795	87.8%
	λ_2	1.011	91.9%	1.107	89.3%
k=20,R=$(0_{(15)},5,0_{(3)})$	β	0.985	83.4%	1.016	78.4%
	λ_1	0.741	94.0%	0.815	90.2%
	λ_2	1.013	93.0%	1.100	89.6%
k=20,R=$(0_{(19)})$	β	1.005	85.0%	1.037	80.5%
	λ_1	0.764	92.3%	0.820	91.4%
	λ_2	1.013	92.6%	1.084	91.5%

Table 4: $m = 27, n = 25, \beta = 2, \lambda_1 = 0.7, \lambda_2 = 1$; 90% Symmetric CRI (With Order Restriction)

censoring scheme	Parameter	BAYES IP		BAYES NIP	
		AL	CP	AL	CP
k=20,R=$(5,0_{(18)})$	β	1.099	89.6%	1.159	88.5%
	λ_1	0.630	91.1%	0.661	86.8%
	λ_2	0.920	91.4%	1.022	89.2%
k=20,R=$(0_{(10)},5,0_{(8)})$	β	1.024	87.3%	1.088	85.6%
	λ_1	0.639	93.6%	0.667	87.7%
	λ_2	0.930	93.1%	1.022	92.2%
k=20,R=$(0_{(15)},5,0_{(3)})$	β	1.051	88.5%	1.078	84.2%
	λ_1	0.638	93.0%	0.656	89.5%
	λ_2	0.927	92.3%	1.021	90.6%
k=20,R=$(0_{(19)})$	β	1.088	87.3%	1.135	82.7%
	λ_1	0.645	91.9%	0.683	88.4%
	λ_2	0.906	94.4%	0.988	91.8%

Table 5: $m = 27, n = 25, \beta = 2, \lambda_1 = 0.7, \lambda_2 = 1$; 90% HPD CRI: (Without Order Restriction)

censoring scheme	Parameter	BAYES IP		BAYES NIP	
		AL	CP	AL	CP
k=20,R=$(5,0_{(18)})$	β	1.047	88.2%	1.102	86.0%
	λ_1	0.694	87.9%	0.761	85.7%
	λ_2	0.968	90.7%	1.063	89.0%
k=20,R=$(0_{(10)},5,0_{(8)})$	β	0.955	86.8%	0.976	80.7%
	λ_1	0.695	88.1%	0.756	86.5%
	λ_2	0.979	91.6%	1.077	87.8%
k=20,R=$(0_{(15)},5,0_{(3)})$	β	0.966	82.8%	0.968	79.8%
	λ_1	0.711	88.7%	0.757	88.7%
	λ_2	0.966	91.5%	1.050	89.2%
k=20,R=$(0_{(19)})$	β	1.000	85.1%	1.016	77.9%
	λ_1	0.719	90.3%	0.779	89.5%
	λ_2	0.960	91.9%	1.036	90.0%

Table 6: $m = 27, n = 25, \beta = 2, \lambda_1 = 0.7, \lambda_2 = 1$; 90% HPD CRI:(With Order Restriction)

censoring scheme	Parameter	BAYES IP		BAYES NIP	
		AL	CP	AL	CP
k=20,R=$(5,0_{(18)})$	β	1.065	90.7%	1.107	86.0%
	λ_1	0.589	88.0%	0.624	84.9%
	λ_2	0.845	90.7%	0.930	87.2%
k=20,R=$(0_{(10)},5,0_{(8)})$	β	1.007	88.0%	1.045	84.9%
	λ_1	0.604	89.0%	0.628	84.0%
	λ_2	0.858	91.5%	0.949	88.5%
k=20,R=$(0_{(15)},5,0_{(3)})$	β	1.008	85.8%	1.051	81.5%
	λ_1	0.605	87.6%	0.626	84.5%
	λ_2	0.842	90.7%	0.919	89.3%
k=20,R=$(0_{(19)})$	β	1.032	85.8%	1.079	80.0%
	λ_1	0.608	87.9%	0.636	83.8%
	λ_2	0.814	88.9%	0.873	89.0%

Table 7: $m = 30, n = 30, \beta = 2, \lambda_1 = 0.7, \lambda_2 = 1$

censoring scheme	Vol (IP)	Vol (NIP)	ETOT)
k=25,R=(5,$0_{(23)}$)	1.056	1.275	1.477
k=25,R=($0_{(3)}$,5,$0_{(20)}$)	1.041	1.251	1.473
k=25,R=($0_{(5)}$,5,$0_{(18)}$)	1.036	1.238	1.469
k=25,R=($0_{(8)}$,5,$0_{(15)}$)	1.044	1.232	1.463
k=25,R=($0_{(12)}$,5,$0_{(11)}$)	1.055	1.258	1.440
k=25,R=($0_{(15)}$,5,$0_{(8)}$)	1.056	1.256	1.432
k=25,R=($0_{(18)}$,5,$0_{(5)}$)	1.064	1.281	1.394
k=25,R=($0_{(20)}$,5,$0_{(3)}$)	1.077	1.286	1.344
k=25,R=($0_{(23)}$,5)	1.096	1.309	1.203
k=25,R=($0_{(24)}$)	1.109	1.319	1.002

From Table (1) ,Table (2) it is evident that MSEs of Bayes estimators are decreasing as sample size increases as well as Bayes estimators are performing better in the informative prior than the non-informative prior for both the cases, without order restriction and with order restriction between scale parameters. Bayes estimates of λ_1 are performing better when order restriction is considered than without order restriction.

From Table (3)-(6) it is observed that both the credible intervals perform quite good in terms of coverage percentage. In most of the cases coverage percentages are close to nominal level. Also Symmetric and HPD credible intervals are performing better for informative priors over non-informative prior. For scale parameters these two credible intervals are slightly better for order restriction case than without order restriction in terms of average length.

From Table (7) it is observed that among these investigated censoring schemes, early stage censoring schemes are slightly better than the late stage censoring schemes for both informative and non-informative prior based on the given precision criteria. Also the expected total time on test is decreasing as units are censored late stages which is quite expected.

On a precise note, we can conclude that when we have information about the prior hyperparameter we must opt for informative priors. Also order restriction improves the estimation in case it is known apriori.

7.2 Data Analysis

For illustrative purpose we perform data analysis on simulated and real data sets in this section.

Example 1: We generate two samples of size 25 from $WE(2, 0.7)$ and $WE(2, 1)$. On these two samples we apply a BJPC scheme with $k = 15$ and $R = (9, 0_{(13)})$. Applying the censoring scheme on the given data sets we generate a balanced joint progressive censored sample which consist of the form $((w_i, z_i)\ i = 1, \ldots, k)$. The data are obtained as $((0.103, 1), (0.160, 0), (0.350, 1), (0.380, 0), (0.383, 0), (0.430, 0), (0.445, 1), (0.611, 0), (0.632, 0),$ $(0.802, 0), (0.822, 1), (0.830, 1), (0.884, 0), (0.889, 1), (1.376, 0))$.

In Table 8 we compute Bayesian estimator and 90% credible intervals of the parameters and in Table 9 the corresponding order restricted estimates are recorded. We compute all the estimates based on non-informative prior. As two samples are taken as equal size we can apply Algorithm 1 to generate Bayes estimate and CRIs. In data analysis we approximate $\pi_2^*(\beta|data)$ by a gamma distribution whose parameters are determined by equating mean and variance of $\pi_2^*(\beta|data)$ by those of the gamma distribution. In Figure (3) we construct contour plot of $\pi_1^*(\lambda_1, \lambda_2|\beta, data)$ for true value of β. In Figure (4) we plot both the $\pi_2^*(\beta|data)$ and its approximated gamma PDF based on the generated data which depicts the approximation is quite good for this data set.

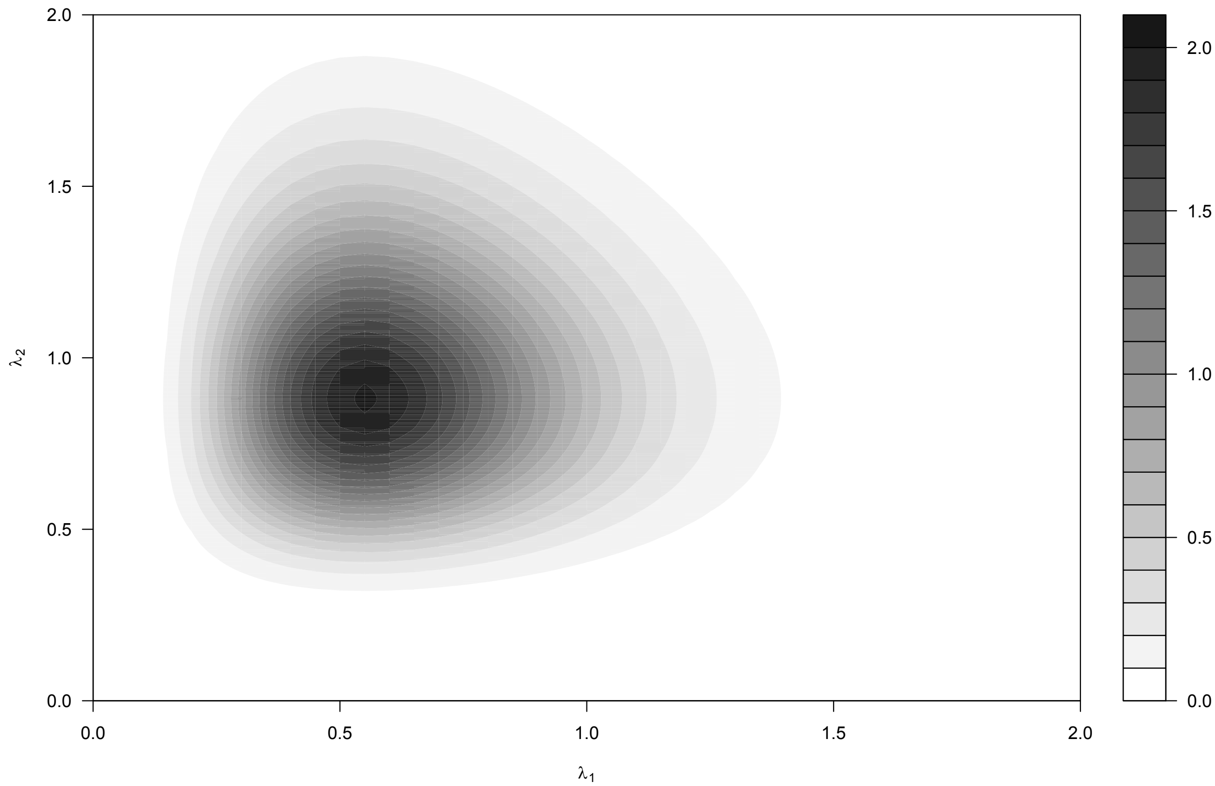


Figure 3: plot of $\pi_1^*(\lambda_1, \lambda_2|\beta, data)$
for true value of $\beta = 2$

Table 8: Bayes estimator and credible intervals for simulated data set

Parameter	Bayes estimate	Symmetric CRI (90%)		HPD CRI (90%)	
		LL	UL	LL	UL
β	1.811	1.288	2.425	1.246	2.356
λ_1	0.639	0.273	1.125	0.212	1.017
λ_2	0.959	0.493	1.538	0.445	1.460

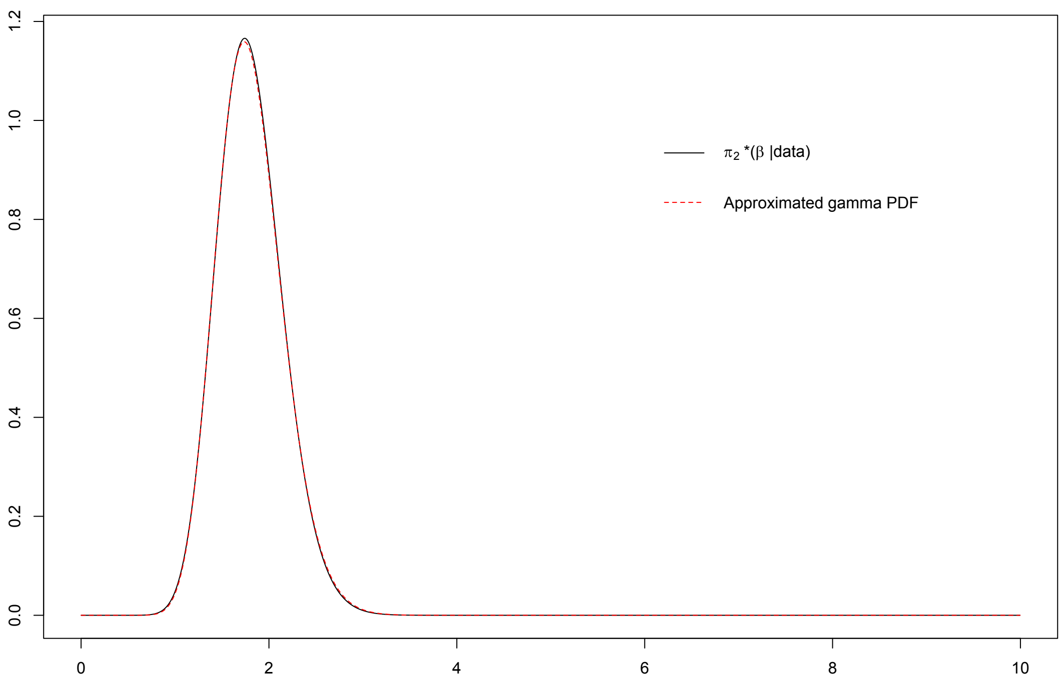


Figure 4: Gamma approximation of $\pi_2^*(\beta|data)$ for simulated data

Table 9: Bayes estimator and credible intervals for simulated data set with order restriction

Parameter	Bayes estimate	Symmetric CRI (90%)		HPD CRI (90%)	
		LL	UL	LL	UL
β	1.808	1.293	2.378	1.192	2.364
λ_1	0.567	0.272	0.906	0.245	0.868
λ_2	1.037	0.580	1.374	0.608	1.501

Example 2: The data indicate the failure times (in hour) of air-conditioning system of plane "7913" and "7914" which are originally taken from Proschan [23]. It is assumed that two data sets are independent as well as failure times in each data set are also independent. The data are presented below.

Data set 1 (Plane 7914): 3, 5, 5, 13, 14, 15, 22, 22, 23, 30, 36, 39, 44, 46, 50, 72, 79, 88, 97, 102, 139, 188, 197, 210.

Data set 2 (Plane 7913): 1, 4, 11, 16, 18, 18, 18, 24, 31, 39, 46, 51, 54, 63, 68, 77, 80, 82, 97, 106, 111, 141, 142, 163, 191, 206, 216.

Here $m = 24$ and $n = 27$.

We divide both the data sets by 100 and fit Weibull distributions. Maximum likelihood estimators (MLEs) and Kolmogorov -Simnov (K-S) distance between empirical distribution functions and fitted distributions along with p values are provided in Table (10) for both the data sets. Also we perform likelihood ratio test for equality of shape parameters between these two data sets and p value is obtained 0.682. Based on complete and combined data sets maximum likelihood estimate of shape parameter is obtained as 1.072 and maximum likelihood estimates of scale parameters are 1.559 and 1.293 for data set 1 and data set 2 respectively. For further development we assume that after dividing by 100 data set 1 follows $WE(\beta, \lambda_1)$ as well as data set 2 follows $WE(\beta, \lambda_2)$.

Table 10: MLEs and K-S distance

Data set	MLE from complete sample shape parameter	scale parameter	K-S distance	p value
Data set1	$\beta = 1.024$	$\lambda = 1.560$	0.089	0.991
Data set 2	$\beta = 1.123$	$\lambda = 1.286$	0.120	0.831

We consider the same censoring plans with $k = 15$ and $R_1 = 9$ and $R_i = 0$ for $i = 2, \ldots, 14$. The generated sample is given as $((1,0),(4,0),(5,1),(5,1),(18,0),(22,1),(30,1),(39,1),$ $(39,0),(44,1),(50,1),(51,0),(54,0),(63,0),(82,0))$.

In Table (11) we compute Bayes estimates and 90% credible intervals (CRI) based on non-informative prior. Corresponding results based on order restriction is given in Table (12).

Table 11: Bayes estimator and credible intervals for real data set

Parameter	Bayes estimate	Symmetric CRI(901%) LL	UL	HPD CRI(90%) LL	UL
β	1.146	0.809	1.371	0.834	1.567
λ_1	1.533	0.645	2.509	0.671	3.137
λ_2	1.143	0.537	1.992	0.448	1.953

Table 12: Bayes estimator and credible intervals for real data set with order restriction

Parameter	Bayes estimate	Symmetric CRI (90%) LL	UL	HPD CRI(90%) LL	UL
β	1.133	0.821	1.512	0.798	1.420
λ_1	1.728	0.910	2.808	0.735	2.616
λ_2	1.008	0.521	1.675	0.429	1.551

Note that, for both the data analysis order restriction reduces the length of credible intervals.

8 Conclusion

In this article we study the Bayesian inference of two Weibull populations under the newly proposed the BJPC scheme. We assume Beta Gamma prior for the scale parameters and Gamma prior for common shape parameter. These prior assumption is fairly general in some sense. The Bayes estimates out of squared error loss and credible intervals are obtained using importance sampling technique. We also study the corresponding results when order restriction between two scale parameters is apriori. Further we set a precision criteria based on expected volume of joint credible set of model parameters and compare certain censoring schemes based on this criteria. Note that all the results here are based on two populations. Further development can be done for more than two populations also.

Appendix

A.1 Known shape parameter: $(m \neq n)$

When shape parameter β is known and $m \neq n$, posterior density of λ_1, λ_2 can be obtained as $\pi(\lambda_1, \lambda_2|\beta, data) \propto \lambda_1^{a_1+k_1-1}\lambda_2^{a_2+k_2-1}(\lambda_1+\lambda_2)^{a_0-a_1-a_2}e^{(\lambda_1+\lambda_2)U(\beta)}e^{-\lambda_1(A_1(\beta)-U(\beta))}e^{-\lambda_2(A_2(\beta)-U(\beta))}$, where $U(\beta) = min(A_1(\beta), A_2(\beta))$.

The joint posterior density of λ_1, λ_2 can be decomposed as $\pi(\lambda_1, \lambda_2|\beta, data) \propto \pi_1^*(\lambda_1, \lambda_2|\beta) \times h(\lambda_1, \lambda_2)$ where $\pi_1^*(\lambda_1, \lambda_2|\beta)$ is the PDF of $\text{BG}(a_0 + k, b_0 + U(\beta), a_1 + k_1, a_2 + k_2)$ and $h(\lambda_1, \lambda_2) = e^{-\lambda_1(A_1(\beta)-U(\beta))}e^{-\lambda_2(A_2(\beta)-U(\beta))}$. The Bayes estimate of a function $g(\lambda_1, \lambda_2)$ is given by

$E(g(\lambda_1,\lambda_2)|\beta,data)$, provided it exists, where

$$\begin{aligned} E(g(\lambda_1,\lambda_2)|\beta,data) &= \int_0^\infty \int_0^\infty \int_0^\infty g(\lambda_1,\lambda_2)\pi(\lambda_1,\lambda_2|\beta,data)d\lambda_1 d\lambda_2 \\ &= \frac{\int_0^\infty \int_0^\infty \int_0^\infty g(\lambda_1,\lambda_2)\pi_1^*(\lambda_1,\lambda_2|\beta,data)\times h(\lambda_1\lambda_2)d\lambda_1 d\lambda_2}{\int_0^\infty \int_0^\infty \int_0^\infty \pi_1^*(\lambda_1,\lambda_2|\beta,data)\times h(\lambda_1\lambda_2)d\lambda_1 d\lambda_2}. \end{aligned}$$

As $E(g(\lambda_1,\lambda_2)|\beta,data)$ can not be obtained in closed form, we apply importance sampling technique to derive approximate Bayes estimates as well as credible intervals (CRI). Algorithm 2 can be used for approximation purpose.

Algorithm 2:

Step 1: Generate λ_1,λ_2 from $\pi_1^*(\lambda_1,\lambda_2|\beta,data)$.

Step 2: Repeat the process say N times to generate $\big((\lambda_{11},\lambda_{21})\ldots(\lambda_{1N},\lambda_{2N})\big)$.

Step3: To obtain Bayes estimate of $g(\lambda_1,\lambda_2)$, compute $(g_1,\ldots,g_N)$, where $g_i = g(\lambda_{1i},\lambda_{2i})$ as well as compute $(h_1,\ldots,h_N)$, where $h_i = h(\lambda_{1i},\lambda_{2i})$.

Step 4: Bayes estimate of $g(\lambda_1,\lambda_2)$ can be approximated as $\frac{\sum_{i=1}^N h_i g_i}{\sum_{j=1}^N h_j} = \sum_{i=1}^N v_i h_i$ where $v_i = \frac{h_i}{\sum_{j=1}^N h_j}$.

Step 5: To compute $100(1-\gamma)\%$ CRI of $g(\lambda_1,\lambda_2)$, arrange g_i in ascending order to obtain $(g_{(1)},\ldots g_{(N)})$ and record the corresponding v_i as $(v_{(1)},\ldots,v_{(N)})$. A $100(1-\gamma)\%$ CRI can be obtained as $(g_{(j_1)},g_{(j_2)})$ where j_1,j_2 such that

$$j_1 < j_2, \quad j_1,j_2 \in \{1,\ldots,N\} \quad \text{and} \quad \sum_{i=j_1}^{j_2} v_i \leq 1-\gamma < \sum_{i=j_1}^{j_2+1} v_i \tag{15}$$

The $100(1-\gamma)\%$ highest posterior density (HPD) CRI can be obtained as $(g_{(j_1^*)},g_{(j_2^*)})$, such that $g_{(j_2^*)} - g_{(j_1^*)} \leq g_{(j_2)} - g_{(j_1)}$ and j_1^*,j_2^* satisfying (15) for all j_1,j_2 satisfying (15).

A.2 Shape parameter unknown: $(m \neq n)$

For unknown shape parameter β when $m \neq n$, $\pi(\beta, \lambda_1, \lambda_2)$ can be written as follow

$$\begin{aligned}\pi(\beta, \lambda_1, \lambda_2|data) \;\; \propto \;\; & \lambda_1^{a_1+k_1-1}\lambda_2^{a_2+k_2-1}(\lambda_1+\lambda_2)^{a_0-a_1-a_2}e^{-(\lambda_1+\lambda_2)(b_0+U(\beta))}e^{-\lambda_1(A_1(\beta)-U(\beta))}e^{-\lambda_2(A_2(\beta)-U(\beta))} \\ & \beta^{a+k-1}e^{-\beta(b-\sum_{i=1}^{k}\ln w_i)} \qquad (16)\end{aligned}$$

Here $U(\beta) = min(A_1(\beta), A_2(\beta))$.

The joint posterior density of $\beta, \lambda_1, \lambda_2$ can be decomposed as

$$\pi(\beta, \lambda_1, \lambda_2|data) \propto \pi_1^*(\lambda_1, \lambda_2|\beta, data) \times \pi_2^*(\beta|data) \times h(\beta, \lambda_1, \lambda_2) \qquad (17)$$

Here $\pi_1^*(\lambda_1, \lambda_2|\beta, data)$ is the PDF of BG$(a_0+k, b_0+U(\beta), a_1+k_1, a_2+k_2)$,

$\pi_2^*(\beta|data) \propto \frac{\beta^{a+k-1}e^{-(b-\sum_{i=1}^{k}\ln w_i)}}{(b_0+U(\beta))^{a_0+k}}$ and $h(\beta, \lambda_1, \lambda_2) = e^{-\lambda_1(A_1(\beta)-U(\beta))}e^{-(A_2(\beta)-U(\beta))}$.

The Bayes estimate of a function $g(\beta, \lambda_1, \lambda_2)$ is given by

$$\begin{aligned} E(g(\beta, \lambda_1, \lambda_2)|data) &= \int_0^\infty \int_0^\infty \int_0^\infty g(\beta, \lambda_1, \lambda_2)\pi(\beta, \lambda_1, \lambda_2|data)d\beta d\lambda_1 d\lambda_2 \\ &= \frac{\int_0^\infty \int_0^\infty \int_0^\infty g(\beta, \lambda_1, \lambda_2)\pi_1^*(\lambda_1, \lambda_2|\beta, data) \times \pi_2^*(\beta|data) \times h(\beta, \lambda_1\lambda_2)d\beta d\lambda_1 d\lambda_2}{\int_0^\infty \int_0^\infty \int_0^\infty \pi_1^*(\lambda_1, \lambda_2|\beta, data) \times \pi_2^*(\beta|data) \times h(\beta, \lambda_1\lambda_2)d\beta d\lambda_1 d\lambda_2} \end{aligned}$$

provided it exists.

As $E(g(\beta, \lambda_1, \lambda_2)|data)$ can not be obtained in closed form, we apply importance sampling technique to derive approximate Bayes estimates as well as credible intervals (CRI). Algorithm 3 can be used for approximation purpose.

Algorithm 3:

Step 1: Given data, generate β from $\pi_2^*(\beta|data)$.

Step 2: Given a generated β, generate λ_1, λ_2 from $\pi_1^*(\lambda_1, \lambda_2|\beta, data)$.

Step 3: Repeat the process say N times to generate $\big((\beta_1, \lambda_{11}, \lambda_{21}) \ldots (\beta_N, \lambda_{1N}, \lambda_{2N})\big)$.

Step 4: To obtain Bayes estimate of $g(\beta, \lambda_1, \lambda_2)$, compute $(g_1, \ldots, g_N)$, where $g_i = g(\beta_i, \lambda_{1i}, \lambda_{2i})$ as well as compute $(h_1, \ldots, h_N)$, where $h_i = h(\beta_i, \lambda_{1i}, \lambda_{2i})$.

Step 5: Bayes estimate of $g(\beta, \lambda_1, \lambda_2)$ can be approximated as $\frac{\sum_{i=1}^N h_i g_i}{\sum_{j=1}^N h_j} = \sum_{i=1}^N v_i h_i$ where $v_i = \frac{h_i}{\sum_{j=1}^N h_j}$.

Step 6: To compute $100(1-\gamma)\%$ CRI of $g(\beta, \lambda_1, \lambda_2)$, arrange g_i in ascending order to obtain $(g_{(1)}, \ldots g_{(N)})$ and record the corresponding v_i as $(v_{(1)}, \ldots, v_{(N)})$. A $100(1-\gamma)\%$ CRI can be obtained as $(g_{(j_1)}, g_{(j_2)})$ where j_1, j_2 such that

$$j_1 < j_2, \quad j_1, j_2 \in \{1, \ldots, N\} \quad \text{and} \quad \sum_{i=j_1}^{j_2} v_i \le 1-\gamma < \sum_{i=j_1}^{j_2+1} v_i \tag{18}$$

The $100(1-\gamma)\%$ highest posterior density (HPD) CRI can be obtained as $(g_{(j_1^*)}, g_{(j_2^*)})$, such that $g_{(j_2^*)} - g_{(j_1^*)} \le g_{(j_2)} - g_{(j_1)}$ and j_1^*, j_2^* satisfying (18) for all j_1, j_2 satisfying (18).

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