A Bivariate Load Sharing Model

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Abstract

The motivation of this work came from a data set obtained from an experiment performed on diabetic patients, with diabetic retinopathy disorder. The aim of this experiment is to test whether there is any significant difference between two different treatments which are being used for this disease. The two eyes can be considered as a two component load sharing system. In a two component load sharing system after the failure of one component, the surviving component has to shoulder extra load. Hence, it is prone to failure at an earlier time than what is expected under the original model. It may also happen sometimes that the failure of one component may release extra resources to the survivor, thus delaying the failure. In most of the existing literature it has been assumed that at the beginning the lifetime distributions of the two components are independently distributed, which may not be very reasonable in this case. In this paper we have introduced a new bivariate load sharing model where the independence assumptions of the lifetime distributions of the two components at the beginning has been relaxed. In this present model they may be dependent. Further, there is a positive probability that the two components may fail simultaneously. If the two components do not fail simultaneously, it is assumed that the lifetime of the surviving component changes based on the tampered failure rate assumption. The proposed bivariate distribution has a singular component. The likelihood inference of the unknown parameters have been provided. Simulation results and the analysis of the data set have been presented to show the effectiveness of the proposed model.

Key Words and Phrases: Load sharing model; bivariate distribution; singular component; likelihood inference; Marshall-Olkin bivariate exponential

AMS Subject Classifications: 62F10, 62F03, 62H12.

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1 Introduction

In a load sharing system, it is assumed that the failure of one component changes the load on the remaining surviving components. Hence, the failure of one component decreases the expected survival time of the remaining components than what is expected under the original model. On some occasion it may also happen that the failure of one component may release extra resources to the survivors, hence, the expected remaining survival times increase. The model which takes care this stochastic aspect of the lifetimes of different components of a parallel system, is known as the load sharing model.

Daniels [5] first introduced the load sharing model while analyzing textile data. Since then an extensive amount of work has been done in this area. In a reliability analysis it has been used in electrical generator systems, central processing units (CPUs) in a multiprocessor computer systems, cables in suspension bridge, valves or pumps in a hydraulic system etc, see for example Amari et al. [1], Kvam and Pena [15] and the references cited therein. Gross et al. [10] considered the load sharing model in a two organ biological system, for example eyes or kidneys. For some of the recent work on load sharing model interested readers are referred to Wang et al. [21], Xu et al. [22], Park et al. [17], Rychlik and Spizzichino [19] and see the references cited therein.

One of the most used bivariate load sharing model was introduced by Freund [7]. In this model, it has been assumed that at the beginning the lifetime of both the components follow exponential distributions and they are independently distributed. The failure of one component changes the parameter of the lifetime distribution of the surviving to a different one. Several extensions of the above model have been suggested by several authors, see for example Asha et al. [2], Franco et al. [6] and the references cited therein in this respect. But in all these cases the basis assumption of independence of the lifetime distributions at the beginning has been maintained.

The motivation of this work came from a bivariate data (Y_1, Y_2) obtained from an experiment performed on diabetic patients with diabetic retinopathy disorder. In this experiment, for each patient with diabetic retinopathy disorder, in one eye laser treatment has been administered, where as in the other eye the traditional treatment has been provided. The main aim of this experiment is to test whether there is any significant difference between the laser treatment and the traditional treatment. The time to the onset of blindness of both the eyes have been observed. The two eyes of a human being can be treated as a two component load sharing system, it is expected that if one eye becomes blind, then it creates extra load on the other eye. Moreover, the lifetime distributions of the two eyes of a patient at the beginning of the experiment cannot be independently distributed. Further, a significant proportion of the data has tied observations, i.e. $Y_1 = Y_2$, hence they cannot be ignored. None of the existing models available so far can take care all these issues.

The main aim of this manuscript is to introduce a bivariate load sharing model where the lifetime distributions of the two components at the beginning may not be independently distributed. There is a positive probability that both the components may fail simultaneously. On the other hand if the two components do not fail simultaneously, then once one of the components fails, the load on the other component changes. It is assumed at the beginning the lifetime distributions of the two components follow bivariate proportional hazard model. If the two components do not fail simultaneously, the lifetime distribution of the surviving component changes following the tampered failure rate assumption as suggested by Bhattacharyya and Soejoeti [4]. In this model assumption the failure rate of the surviving component changes in a proportional manner. This is a very natural model in the step-stress set up, which is the case in this situation.

The proposed model is a very flexible bivariate load sharing model. It has a singular component in the sense there is a positive probability that $Y_1 = Y_2$, which is not possible in any of the existing load sharing models. We have provided the maximum likelihood estimators of the unknown parameters of the different models and also addressed a testing of hypothesis problem mainly testing whether there is any significant difference between the two treatments or not. Extensive simulations have been performed to show the effectiveness of the proposed procedures, and the analyses of the data set has been performed to show the effectiveness of the model.

Rest of the paper is organized as follows. In Section 2, we have provided some preliminaries. We have briefly discussed about the bivariate proportional hazard model and also discussed about the tampered failure rate model. In Section 3 we have discussed about the bivariate load sharing model and indicated its different properties and interpretations. The classical inference has been provided in Section 4. In Section 5 and Section 6 we have provided the simulation results and the analysis of the bivariate retinopathy data set. Finally we have concluded the paper in Section 7

2 Preliminaries

2.1 Bivariate Proportional Hazard Model

Suppose we have a baseline distribution function $F_B(t)$, and it has the support on the positive real axis, i.e. $F_B(t) = 0$, if $t < 0$. The baseline distribution function $F_B(t)$ may depend on some parameter, but we are not making it explicit at this moment. Whenever it is needed we will make it clear. Let us denote the corresponding survival function as $S_B(t) = 1 - F_B(t)$. A class of distribution function, for $\lambda > 0$

$$S_{PHC}(t; \lambda) = [S_B(t)]^{\lambda}; \quad t > 0,$$

is known as the proportional hazard class (PHC) of distribution functions. It is immediate that if $S_B(t)$ is a proper absolutely continuous survival function, then $S_{PHC}(t;\lambda)$ is also a proper absolutely continuous survival function. If the probability density function (PDF) associated with the baseline survival function $S_B(t)$ is denoted by $f_B(t)$, then the corresponding PDF associated with the survival function $S_{PHC}(t;\lambda)$ is

$$f_{PHC}(t;\lambda) = \lambda f_B(t)[S_B(t)]^{\lambda-1}. \tag{1}$$

If the hazard function of the baseline distribution is denoted by $h_B(t)$, the associated hazard function of the corresponding PHC becomes

$$h_{PHC}(t,\lambda) = \frac{\lambda f_B(t)[S_B(t)]^{\lambda-1}}{[S_B(t)]^{\lambda}} = \lambda \frac{f_B(t)}{S_B(t)} = \lambda h_B(t). \tag{2}$$

A random variable with the PDF (1) will be denoted by $\text{PH}(S_B,\lambda)$. Note that if X follows ($\sim$) $\text{PH}(S_B,\lambda)$, then the conditional PDF of X given $\{X > a\}$ is

$$f_{X|\{X>a\}}(t) = \begin{cases} \lambda f_B(t)[S_B(t)]^{\lambda-1}[S_B(a)]^{-\lambda} & \text{if} \quad t > a \\ 0 & \text{if} \quad t \le a \end{cases} \tag{3}$$

Now we will briefly mention about the bivariate proportional hazard (BPH) model. Suppose V_1, V_2 and V_3 are three independent random variables with survival functions $S_1(t;\lambda_1) = [S_B(t)]^{\lambda_1}$, $S_2(t;\lambda_2) = [S_B(t)]^{\lambda_2}$ and $S_3(t;\lambda_3) = [S_B(t)]^{\lambda_3}$, respectively, then the bivariate random variable (X,Y), where

$$X = \min\{V_1,V_3\} \quad \text{and} \quad Y = \min\{V_2,V_3\},$$

is called the bivariate proportional hazard model, and it will be denoted by $\text{BPH}(S_B,\lambda_1,\lambda_2,\lambda_3)$. If $(X,Y) \sim \text{BPH}(S_B,\lambda_1,\lambda_2,\lambda_3)$, then the joint survival function and the associated PDF for $x > 0$ and $y > 0$, can be written as

$$S_{X,Y}(x,y) = P(X > x, Y > y) = \begin{cases} [S_B(x)]^{\lambda_1+\lambda_3}[S_B(y)]^{\lambda_2} & \text{if} \quad x > y \\ [S_B(x)]^{\lambda_1}[S_B(y)]^{\lambda_2+\lambda_3} & \text{if} \quad x \le y \end{cases}$$

and

$$f_{X,Y}(x,y) = \begin{cases} \lambda_1(\lambda_2+\lambda_3) f_B(x)[S_B(x)]^{\lambda_1-1} f_B(y)[S_B(y)]^{\lambda_2+\lambda_3-1} & \text{if} \quad x<y \\ \lambda_2(\lambda_1+\lambda_3) f_B(x)[S_B(x)]^{\lambda_1+\lambda_3-1} f_B(y)[S_B(y)]^{\lambda_2-1} & \text{if} \quad x>y \\ \lambda_3 f_B(x)[S_B(x)]^{\lambda_1+\lambda_2+\lambda_3-1} & \text{if} \quad x=y, \end{cases}$$

respectively. Note that several cases can be obtained from the above class of distributions; for example the well known Marshall-Olkin bivariate exponential distribution, Marshall and Olkin [16], Marshall-Olkin bivariate Weibull distribution, Kundu and Gupta [14]. It may be mentioned that it has the same Marshall-Olkin copula, hence it has the same dependence properties and the same dependence measure as the Marshall-Olkin bivariate exponential distribution, see for example Kundu [12].

2.2 Tampered Failure Rate Model

Tampered failure rate model (TFRM) was introduced by Bhattacharyya and Soejoeti [4] in the step-stress set up. The main assumption here is that the effect of changing the stress level from s_0 to s_1 is equivalent to multiply the initial failure rate function at the stress level s_0 by an unknown factor $\alpha > 0$. Therefore, the hazard function of the lifetime of the experimental unit under the TFRM is given by

$$h_{TFRM}(t) = \begin{cases} h(t) & \text{if} \quad 0<t<\tau \\ \alpha h(t) & \text{if} \quad \tau \le t < \infty. \end{cases} \tag{4}$$

Here $h(t)$ denotes the hazard function at the stress level s_0, $\alpha h(t)$ is the hazard function at the stress level s_1 and τ is the stress changing time. It is clear from (2) that if the lifetime distribution of an experimental unit follows a proportional hazard model with a given baseline distribution at the stress level s_0, then at the stress level s_1 also it has the proportional hazard model with the same baseline distribution. For a detailed discussion on the TFRM model, one is referred to the monograph by Kundu and Ganguly [13].

3 Bivariate Load Sharing Model

In this section first we define a bivariate load sharing model with a singular component. It is assumed that there is a two component system, and the system fails when both the components fail. We denote them as component A and component B, respectively. When the system starts the two components start working simultaneously. At the beginning the lifetimes of A and B are denoted by X and Y, respectively. It is assumed that both X and Y are continuous random variables, but they may be dependent. The following three cases can occur:

Case 1: Both the components A and B fail Simultaneously and the system stops.

Case 2: The component A fails but B does not fail. Hence, system does not fail. But since A has failed, the load on B has changed. The lifetime of B has changed from Y to Y^*. Then eventually B fails, and the system stops.

Case 3: Similar to Case 2, if the component B fails first, and A does not fail, then the lifetime of A changes from X to X^*. Eventually, A also fails, and the system stops.

If Y_1 and Y_2 denote the lifetimes of A and B, respectively, at the time of the system failure, then (Y_1, Y_2) can be written as follows:

$$(Y_1, Y_2) = \begin{cases} (X, Y) & \text{if} \quad Y_1 = Y_2 \\ (X, Y^*) & \text{if} \quad Y_1 < Y_2 \\ (X^*, Y) & \text{if} \quad Y_1 > Y_2 \end{cases} \tag{5}$$

Now we make the following assumptions on X, Y, X^* and Y^* to derive the joint distribution of (Y_1, Y_2). It is assumed that $(X, Y) \sim \text{BPH}(S_B, \lambda_1, \lambda_2, \lambda_3)$. Further, based on (3) and (4), we define X^* and Y^* conditionally as follows. The conditional PDF of X^* given $\{X^* > Y = a\}$ is given by

$$f_{X^*|\{X^*>Y=a\}}(t) = \begin{cases} \lambda_4 f_B(t)[S_B(t)]^{\lambda_4 - 1}[S_B(a)]^{-\lambda_4} & \text{if} \quad t > a \\ 0 & \text{if} \quad t \le a. \end{cases} \tag{6}$$

Here $\lambda_4 = \delta(\lambda_1 + \lambda_3)$, for some $\delta > 0$. We call the parameter δ as the load factor on component one, due to the failure of the second component first. If $\delta = 1$, there is no load factor, but if $\delta > 1$ ($\delta < 1$), then it has a harsher (milder) load factor. The conditional hazard of X^* at t, given $\{X^* > Y = a\}$ for $t > a$, becomes

$$h_{X^*|\{X^*>Y=a\}}(t) = \lambda_4 h_B(t) = \delta(\lambda_1 + \lambda_3) h_B(t) = \lambda_4 h_B(t). \tag{7}$$

Note that (7) has the same form as in (4). Similarly, it is assumed that the conditional PDF of Y^* given $\{Y^* > X = a\}$ is given by

$$f_{Y^*|\{Y^*>X=a\}}(t) = \begin{cases} \lambda_5 f_B(t)[S_B(t)]^{\lambda_5 - 1}[S_B(a)]^{-\lambda_5} & \text{if} \quad t > a \\ 0 & \text{if} \quad t \leq a. \end{cases} \tag{8}$$

Here $\lambda_5 = \gamma(\lambda_2 + \lambda_3)$, for some $\gamma > 0$. Here γ is the load factor on component two, due to the failure of the component one first. Hence, based on the assumption (6) and on the PDF of Y, the PDF of X^* can be obtained as, and similarly from (8) and from the PDF of X, the PDF of Y^* also can be obtained. The details are provided in the Appendix A.

Now based on the assumptions on (X, Y) and on the conditional assumptions on X^* and Y^* we will derive the joint PDF of Y_1 and Y_2.

Case 1: If $y_1 = y_2 = y$, then

$$\begin{aligned} f_{Y_1,Y_2}(y, y)dy &= P(y \leq Y_1 \leq y + dy, y \leq Y_2 \leq y + dy) \\ &= P(y \leq X \leq y + dy, y \leq Y \leq y + dy) \\ &= \lambda_3 f_B(y)[S_B(y)]^{\lambda_1+\lambda_2+\lambda_3-1} \\ &= \lambda_3 h_B(y)[S_B(y)]^{\lambda_1+\lambda_2+\lambda_3}. \end{aligned} \tag{9}$$

Case 2: If $y_1 < y_2$, then

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2)dy_1 dy_2 &= P(y_1 \leq Y_1 \leq y_1 + dy_1, y_2 \leq Y_2 \leq y_2 + dy_2) \\ &= P(y_1 \leq X \leq y_1 + dy_1, y_2 \leq Y^* \leq y_2 + dy_2) \end{aligned}$$

$$
\begin{aligned}
&= P(y_1 \le \min\{X,Y\} \le y_1+dy_1, X<Y, y_2 \le Y^* \le y_2+dy_2) \\
&= P(y_1 \le \min\{X,Y\} \le y_1+dy_1) \times P(X<Y|y_1 \le \min\{X,Y\} \le y_1+dy_1) \times \\
&\quad P(y_2 \le Y^* \le y_2+dy_2|y_1 \le \min\{X,Y\} \le y_1+dy_1, X<Y)
\end{aligned}
$$

Now we will provide all the terms one by one.

$$
P(y_1 \le \min\{X,Y\} \le y_1+dy_1) = (\lambda_1+\lambda_2+\lambda_3) f_B(y_1;\alpha)[S_B(y_1;\alpha)]^{\lambda_1+\lambda_2+\lambda_3-1} dy_1,
$$

$$
\begin{aligned}
P(X<Y|y_1 \le \min\{X,Y\} \le y_1+dy_1) &= P(y_1 = X < Y|y_1 \le \min\{X,Y\} \le y_1+dy_1) \\
&= \frac{P(y_1 = X < Y \& y_1 \le \min\{X,Y\} \le y_1+dy_1)}{P(y_1 \le \min\{X,Y\} \le y_1+dy_1)} \\
&= \frac{\lambda_1 f_B(y_1)[S_B(y_1)]^{\lambda_1+\lambda_2+\lambda_3}}{(\lambda_1+\lambda_2+\lambda_3) f_B(y_1)[S_B(y_1)]^{\lambda_1+\lambda_2+\lambda_3}} \\
&= \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3},
\end{aligned}
$$

$$
\begin{aligned}
&P(y_2 \le Y^* \le y_2+dy_2|y_1 \le \min\{X,Y\} \le y_1+dy_1, X<Y) \\
= \quad & P(y_2 \le Y^* \le y_2+dy_2|y_1 \le \min\{X,Y\} \le y_1+dy_1, y_1 = X<Y) \\
= \quad & \lambda_5 f_B(y_2)[S_B(y_2)]^{\lambda_5-1}[S_B(y_1)]^{-\lambda_5} dy_2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= \lambda_1\lambda_5 f_B(y_1) f_B(y_2)[S_B(y_1)]^{\lambda_1+\lambda_2+\lambda_3-\lambda_5-1}[S_B(y_2)]^{\lambda_5-1} \\
&= \lambda_1\lambda_5 h_B(y_1) h_B(y_2)[S_B(y_1)]^{\lambda_1+\lambda_2+\lambda_3-\lambda_5}[S_B(y_2)]^{\lambda_5} \qquad (10)
\end{aligned}
$$

Similarly, for

Case 3: If $y_1 > y_2$, then

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= \lambda_2\lambda_4 f_B(y_1) f_B(y_2)[S_B(y_1)]^{\lambda_4}[S_B(y_2)]^{\lambda_1+\lambda_2+\lambda_3-\lambda_4-1} \\
&= \lambda_2\lambda_4 h_B(y_1) h_B(y_2)[S_B(y_2)]^{\lambda_1+\lambda_2+\lambda_3-\lambda_4}[S_B(y_1)]^{\lambda_4}. \qquad (11)
\end{aligned}
$$

It easily follows

$$P(Y_1 = Y_2) = \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}, \quad P(Y_1 < Y_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}, \quad P(Y_1 > Y_2) = \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}.$$

Now we provide the joint survival function of (Y_1, Y_2). For brevity, we use the following notations. $\lambda = \lambda_1 + \lambda_2 + \lambda_3$, $\eta = \lambda_1 + \lambda_2 + \lambda_3 - \lambda_5$ and $\xi = \lambda_1 + \lambda_2 + \lambda_3 - \lambda_4$. Then the joint survival function of (Y_1, Y_2) can be expressed as the following. The details are available in the Appendix B.

Case 1: If $y_1 < y_2$,

For $\eta \neq 0$

$$P(Y_1 > y_1, Y_2 > y_2) = [S_B(y_2)]^{\lambda} + \frac{\lambda_1}{\eta}[S_B(y_2)]^{\lambda_5}\left[[S_B(y_1)]^{\eta} - [S_B(y_2)]^{\eta}\right], \tag{12}$$

and for $\eta = 0$,

$$P(Y_1 > y_1, Y_2 > y_2) = [S_B(y_2)]^{\lambda} + \lambda_1[S_B(y_2)]^{\lambda_5}\left[\ln S_B(y_1) - \ln S_B(y_2)\right]. \tag{13}$$

Case 2: If $y_1 > y_2$,

For $\xi \neq 0$

$$P(Y_1 > y_1, Y_2 > y_2) = [S_B(y_1)]^{\lambda} + \frac{\lambda_2}{\xi}[S_B(y_1)]^{\lambda_4}\left[[S_B(y_2)]^{\xi} - [S_B(y_1)]^{\xi}\right], \tag{14}$$

and for $\xi = 0$,

$$P(Y_1 > y_1, Y_2 > y_2) = [S_B(y_1)]^{\lambda} + \lambda_2[S_B(y_1)]^{\lambda_4}\left[\ln S_B(y_2) - \ln S_B(y_1)\right]. \tag{15}$$

Case 3: If $y_1 = y_2 = y$,

$$P(Y_1 > y, Y_2 > y) = [S_B(y)]^{\lambda}. \tag{16}$$

Hence, the marginal survival functions of Y_1 and Y_2 can be obtained as

$$P(Y_1 > y_1) = [S_B(y_1)]^{\lambda} + \lambda_2[S_B(y_1)]^{\lambda_4} \times \begin{cases} \frac{1}{\xi}(1 - [S_B(y_1)]^{\xi}) & \text{if} \quad \xi \neq 0 \\ 1 - \ln S_B(y_1) & \text{if} \quad \xi = 0 \end{cases}$$

and

$$P(Y_2 > y_2) = [S_B(y_2)]^{\lambda} + \lambda_1 [S_B(y_2)]^{\lambda_5} \times \begin{cases} \frac{1}{\eta}(1 - [S_B(y_2)]^{\eta}) & \text{if} \quad \eta \neq 0 \\ 1 - \ln S_B(y_2) & \text{if} \quad \eta = 0 \end{cases}$$

respectively. If there is no load factor on both the components, i.e. if $\delta = \gamma = 1$, it implies $\lambda_4 = \lambda_1 + \lambda_3$ and $\lambda_5 = \lambda_2 + \lambda_3$, then the joint survival function of (Y_1, Y_2) becomes

$$P(Y_1 > y_1, Y_2 > y_2) = \begin{cases} [S_B(y_1)]^{\lambda_1} [S_B(y_2)]^{\lambda_2 + \lambda_3} & y_1 \leq y_2 \\ [S_B(y_1)]^{\lambda_1 + \lambda_3} [S_B(y_2)]^{\lambda_2} & y_1 > y_2. \end{cases}$$

As expected, it is same as the joint survival function of (X, Y). Now let us look at the individual hazard intensities given the current information of both the components. These expressions depend on the state either active of failed, of the other component. Let us consider the time t at which both the components are still active, i.e. $\{Y_1 \geq t, Y_2 \geq t\}$. The hazard intensity of Y_1 at the time point t, given $\{Y_1 \geq t, Y_2 \geq t\}$ is

$$h_{Y_1|Y_1 \geq t, Y_2 \geq t}(t) = \lim_{\Delta t \to 0^+} \left\{ \frac{1}{\Delta t} P(t \leq Y_1 \leq t + \Delta t | Y_1 \geq t, Y_2 \geq t) \right\} = \lambda_1 h_B(t), \tag{17}$$

and similarly, the hazard intensity of Y_2 at the time point t, given $\{Y_1 \geq t, Y_2 \geq t\}$ is

$$h_{Y_2|Y_1 \geq t, Y_2 \geq t}(t) = \lim_{\Delta t \to 0^+} \left\{ \frac{1}{\Delta t} P(t \leq Y_2 \leq t + \Delta t | Y_1 \geq t, Y_2 \geq t) \right\} = \lambda_2 h_B(t). \tag{18}$$

It may be easily seen that (17) and (18) are same as the hazard intensity of X given $\{X \geq t, Y \geq t\}$ and Y given $\{X \geq t, Y \geq t\}$,respectively. It may be easily seen that the hazard intensity of Y_1 (Y_2) at t, given $\{Y_1 \geq t, Y_2 = s\}$ ($\{Y_1 = s, Y_2 \geq t\}$), for $t > s$ is $\lambda_4 h_B(t)$ ($\lambda_5 h_B(t)$), as expected.

The following algorithm can be used to generate random sample (Y_1, Y_2) from (5) for a given baseline survival function $S_B(\cdot)$, and it is assumed that $S_B^{-1}(\cdot)$ can be obtained in explicit form.

Algorithm:

Step 1: Generate u_1, u_2, u_3 three independent samples from uniform (0,1) and set $v_1 = S_B^{-1}(u_1^{1/\lambda_1})$, $v_2 = S_B^{-1}(u_2^{1/\lambda_2})$ and $v_3 = S_B^{-1}(u_3^{1/\lambda_3})$.

Step 2: Set $x = \min\{v_1, v_3\}$ and $y = \min\{v_2, v_3\}$.

Step 3: If $x = y$, then $y_1 = y_2 = v_3$ is the desired sample.

Step 4: If $x > y$, then set $y_2 = y$. Generate u from uniform (0,1), and set $y_1 = S_B^{-1}(u^{1/\lambda_4} S_B(y_2))$. If $x < y$, then set $y_1 = x$. Generate u from uniform (0,1), and set $y_2 = S_B^{-1}(u^{1/\lambda_5} S_B(y_1))$.

4 Inference

4.1 Maximum Likelihood Estimators

In this section we would like to discuss about the maximum likelihood estimators of the unknown parameters of the proposed model based on a random sample of size n. It is assumed that we have the following data set.

$$\mathcal{D} = \{(y_{11}, y_{21}), \ldots, (y_{1n}, y_{2n})\}. \tag{19}$$

We use the following notations:

$$I_1 = \{i : y_{1i} < y_{2i}\}, \quad I_2 = \{i : y_{1i} > y_{2i}\}, \quad I_3 = \{i : y_{1i} = y_{2i} = y_i\},$$

and also the number of elements in I_1, I_2 and I_3 are n_1, n_2 and n_3, respectively. We would like to obtain the maximum likelihood estimators (MLEs) of the unknown parameters. We consider two cases namely when the baseline distribution is completely known and when it is unknown. From the joint density function derived in (9), (10) and (11), the log-likelihood function can be written as $l(\Theta, \alpha|\mathcal{D}) = l_1(\Theta, \alpha|\mathcal{D}) + l_2(\alpha|\mathcal{D})$, where $\Theta = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)^\top$, and α is the parameter which is associated with the base line distribution $S_0(\cdot)$, and it may be vector valued also or it may be absent depending on the base line distribution. Note that

$l_1(\Theta, \alpha|\mathcal{D})$ and $l_2(\alpha|\mathcal{D})$ can be written as follows.

$$
\begin{aligned}
l_1(\Theta, \alpha|\mathcal{D}) &= n_3 \ln \lambda_3 + n_1 \ln \lambda_1 + n_1 \ln \lambda_5 + n_2 \ln \lambda_2 + n_2 \ln \lambda_4 + (\lambda_1 + \lambda_2 + \lambda_3) \sum_{i \in I_3} \ln S_B(y_i; \alpha) + \\
&\quad (\lambda_1 + \lambda_2 + \lambda_3) \sum_{i \in I_1} \ln S_B(y_{1i}; \alpha) + \lambda_5 \sum_{i \in I_1} \{\ln S_B(y_{2i}; \alpha) - \ln S_B(y_{1i}; \alpha)\} + \\
&\quad (\lambda_1 + \lambda_2 + \lambda_3) \sum_{i \in I_2} \ln S_B(y_{2i}; \alpha) + \lambda_4 \sum_{i \in I_2} \{\ln S_B(y_{1i}; \alpha) - \ln S_B(y_{2i}; \alpha)\} \\
&= n_3 \ln \lambda_3 + n_1 \ln \lambda_1 + n_1 \ln \lambda_5 + n_2 \ln \lambda_2 + n_2 \ln \lambda_4 - (\lambda_1 + \lambda_2 + \lambda_3) \sum_{i \in I_3} H_B(y_i; \alpha) \\
&\quad -(\lambda_1 + \lambda_2 + \lambda_3) \sum_{i \in I_1} H_B(y_{1i}; \alpha) - \lambda_5 \sum_{i \in I_1} \{H_B(y_{2i}; \alpha) - H_B(y_{1i}; \alpha)\} \\
&\quad -(\lambda_1 + \lambda_2 + \lambda_3) \sum_{i \in I_2} H_B(y_{2i}; \alpha) - \lambda_4 \sum_{i \in I_2} \{H_B(y_{1i}; \alpha) - H_B(y_{2i}; \alpha)\} \qquad (20)
\end{aligned}
$$

and

$$
l_2(\alpha|\mathcal{D}) = \sum_{i \in I_3} \ln h_B(y_i; \alpha) + \sum_{i \in I_1 \cup I_2} (\ln h_B(y_{1i}; \alpha) + \ln h_B(y_{2i}; \alpha)). \qquad (21)
$$

Here $H_B(y; \alpha) = \int_0^y h_B(u; \alpha) du$ denotes the cumulative baseline hazard function.

4.2 Testing of Hypotheses

In this section we mainly want to test whether the two marginals are equal or not, i.e. we want to test the hypothesis

$$
H_0 : \lambda_1 = \lambda_2 \;\; \& \;\; \lambda_4 = \lambda_5 \quad vs. \quad H_1 : \text{At least one is different.} \qquad (22)
$$

To test the hypothesis (22) we propose to use the likelihood ratio test. Hence, we need to maximize the log-likelihood function under the restriction H_0. Note that under H_0, the log-likelihood function $l_0(\Theta_0, \alpha|\mathcal{D}) = l_{10}(\Theta_0, \alpha|\mathcal{D}) + l_2(\alpha|\mathcal{D})$, where $\Theta_0 = (\lambda_1, \lambda_3, \lambda_4)^\top$,

$$
\begin{aligned}
l_{10}(\Theta_0, \alpha|\mathcal{D}) &= n_3 \ln \lambda_3 + (n_1 + n_2) \ln \lambda_1 + (n_1 + n_2) \ln \lambda_4 \\
&\quad -\lambda_4 \left(\sum_{i \in I_1} \{H_B(y_{2i}; \alpha) - H_B(y_{1i}; \alpha)\} + \sum_{i \in I_2} \{H_B(y_{1i}; \alpha) - H_B(y_{2i}; \alpha)\} \right) \\
&\quad -(2\lambda_1 + \lambda_3) \left(\sum_{i \in I_3} H_B(y_i; \alpha) + \sum_{i \in I_1} H_B(y_{1i}; \alpha) + \sum_{i \in I_2} H_B(y_{2i}; \alpha) \right) \qquad (23)
\end{aligned}
$$

and $l_2(\alpha|\mathcal{D})$ is same as defined in (21). Hence, we reject the null hypothesis if $-2(l_0(\widehat{\Theta}_0, \widehat{\alpha}|\mathcal{D}) - l(\widehat{\Theta}, \widehat{\alpha}|\mathcal{D}))$ is greater than the appropriate upper percentage of the χ^2_2 value.

Now we will consider three different base line distribution functions namely (a) Exponential, (b) Weibull, (c) Piecewise constant hazard function and we will provide the maximum likelihood estimators of the unknown parameters with and without restriction in each case.

4.3 Baseline Distribution: Exponential and Weibull

In this case the baseline distribution is completely known. Here $\alpha = 1$, and $H_B(y) = y$. Hence, the MLEs of the unknown parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ can be obtained in explicit forms and they are as follows:

$$\widehat{\lambda}_1 = \frac{n_1}{\sum_{i\in I_3} y_i + \sum_{i\in I_1} y_{1i} + \sum_{i\in I_2} y_{2i}}, \tag{24}$$

$$\widehat{\lambda}_2 = \frac{n_2}{\sum_{i\in I_3} y_i + \sum_{i\in I_1} y_{1i} + \sum_{i\in I_2} y_{2i}}, \tag{25}$$

$$\widehat{\lambda}_3 = \frac{n_3}{\sum_{i\in I_3} y_i + \sum_{i\in I_1} y_{1i} + \sum_{i\in I_2} y_{2i}}, \tag{26}$$

$$\widehat{\lambda}_4 = \frac{n_2}{\sum_{i\in I_2}(y_{1i} - y_{2i})}, \tag{27}$$

$$\widehat{\lambda}_5 = \frac{n_1}{\sum_{i\in I_1}(y_{2i} - y_{1i})}. \tag{28}$$

Under H_0 as in (22), the maximum likelihood estimators of the unknown parameters are

$$\widehat{\lambda}_{10} = \frac{n_1 + n_2}{2(\sum_{i\in I_3} y_i + \sum_{i\in I_1} y_{1i} + \sum_{i\in I_2} y_{2i})}, \tag{29}$$

$$\widehat{\lambda}_{30} = \frac{n_3}{\sum_{i\in I_3} y_i + \sum_{i\in I_1} y_{1i} + \sum_{i\in I_2} y_{2i}}, \tag{30}$$

$$\widehat{\lambda}_{40} = \frac{n_1 + n_2}{\sum_{i\in I_2}(y_{1i} - y_{2i}) + \sum_{i\in I_1}(y_{2i} - y_{1i})}. \tag{31}$$

Hence, if $-2(l_0(\widehat{\lambda}_{10}, \widehat{\lambda}_{30}, \widehat{\lambda}_{40}|\mathcal{D}) - l(\widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3, \widehat{\lambda}_4, \widehat{\lambda}_5)|\mathcal{D})$ is greater than the upper γ-th percentile point of χ^2_2, we reject the null hypothesis with the level of significance γ.

When the base line distribution is Weibull, it depends on only one parameter α, and $H_B(y;\alpha) = y^\alpha$. Hence, for fixed α, the MLEs of $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ can be obtained as

$$\begin{aligned}
\widehat{\lambda}_1(\alpha) &= \frac{n_1}{\sum_{i\in I_3} y_i^\alpha + \sum_{i\in I_1} y_{1i}^\alpha + \sum_{i\in I_2} y_{2i}^\alpha}, &(32)\\
\widehat{\lambda}_2(\alpha) &= \frac{n_2}{\sum_{i\in I_3} y_i^\alpha + \sum_{i\in I_1} y_{1i}^\alpha + \sum_{i\in I_2} y_{2i}^\alpha}, &(33)\\
\widehat{\lambda}_3(\alpha) &= \frac{n_3}{\sum_{i\in I_3} y_i^\alpha + \sum_{i\in I_1} y_{1i}^\alpha + \sum_{i\in I_2} y_{2i}^\alpha}, &(34)\\
\widehat{\lambda}_4(\alpha) &= \frac{n_2}{\sum_{i\in I_2} (y_{1i}^\alpha - y_{2i}^\alpha)}, &(35)\\
\widehat{\lambda}_5(\alpha) &= \frac{n_1}{\sum_{i\in I_1} (y_{2i}^\alpha - y_{1i}^\alpha)}. &(36)
\end{aligned}$$

Since, in this case

$$l_2(\alpha|\mathcal{D}) = \alpha \sum_{i\in I_3} y_i^{\alpha-1} + \alpha \sum_{i\in I_1\cup I_2} (y_{1i}^{\alpha-1} + y_{2i}^{\alpha-1}), \tag{37}$$

the MLE of α can be obtained by maximizing $h(\alpha)$ with respect to α, where

$$h(\alpha) = n_3 \ln \widehat{\lambda}_3 + n_1 \ln \widehat{\lambda}_1 + n_1 \ln \widehat{\lambda}_5 + n_2 \ln \widehat{\lambda}_2 + n_2 \ln \widehat{\lambda}_4 + l_2(\alpha|\mathcal{D}). \tag{38}$$

Hence, the MLE of α can be obtained by solving a one dimensional optimization problem. Once $\widehat{\alpha}$, the MLE of α is obtained , the MLEs of $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ can be obtained from (32) to (36) by replacing α with $\widehat{\alpha}$.

Now under H_0, the MLEs of the unknown parameters for given α can be obtained as

$$\begin{aligned}
\widehat{\lambda}_{10}(\alpha) &= \frac{n_1 + n_2}{2(\sum_{i\in I_3} y_i^\alpha + \sum_{i\in I_1} y_{1i}^\alpha + \sum_{i\in I_2} y_{2i}^\alpha)}, &(39)\\
\widehat{\lambda}_{30}(\alpha) &= \frac{n_3}{\sum_{i\in I_3} y_i^\alpha + \sum_{i\in I_1} y_{1i}^\alpha + \sum_{i\in I_2} y_{2i}^\alpha}, &(40)\\
\widehat{\lambda}_{40}(\alpha) &= \frac{n_1 + n_2}{\sum_{i\in I_2} (y_{1i}^\alpha - y_{2i}^\alpha) + \sum_{i\in I_1} (y_{2i}^\alpha - y_{1i}^\alpha)}. &(41)
\end{aligned}$$

The MLE of α can be obtained by maximizing

$$h_0(\alpha) = n_3 \ln \widehat{\lambda}_{30}(\alpha) + (n_1 + n_2) \ln \widehat{\lambda}_{10}(\alpha) + (n_1 + n_2) \ln \widehat{\lambda}_{40}(\alpha) + l_2(\alpha|\mathcal{D}), \tag{42}$$

with respect to α. In this case also, if $-2(l_0(\widehat{\lambda}_{10}, \widehat{\lambda}_{30}, \widehat{\lambda}_{40}, \widehat{\alpha}|\mathcal{D}) - l(\widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3, \widehat{\lambda}_4, \widehat{\lambda}_5, \widehat{\alpha})|\mathcal{D})$ is greater than the upper γ-th percentile point of χ^2_2, we reject the null hypothesis with the level of significance γ.

4.4 Baseline Distribution: Piecewise Constant Hazard

In this section we do not assume any specific form of the baseline distribution function. We have considered a fairly general class of distributions. It is assumed that the baseline hazard function $h_B(t)$ is a piecewise constant, i.e. it is based on the following assumptions. The number of constants, M, and the cut points $0 = \tau_0 < \tau_1 < \ldots < \tau_{M-1} < \tau_M$, to be used. Further, it is assumed that $h_B(t;\alpha)$ has the following form

$$h_B(t;\alpha) = \sum_{j=1}^{M} c_j I_{[\tau_{j-1},\tau_j)}(t), \tag{43}$$

here c_j's are constants, and $I_{[\tau_{j-1},\tau_j)}(t)$ is the indicator function which is defined as follows:

$$I_{[\tau_{j-1},\tau_j)}(t) = \begin{cases} 1 & \text{if} \quad t \in [\tau_{j-1},\tau_j) \\ 0 & \quad\quad \text{otherwise.} \end{cases}$$

The main advantage of using the piecewise constant hazard function as the baseline distribution is that it is a very flexible distribution. It can have different shapes of the hazard function, and moreover, it is a completely data driven technique, which chooses the baseline distribution. It may be noted that under the proportional hazard model set up, $c_1, \ldots, c_M$ are identifiable only up to a multiplicative constant, and due to this without loss of generality, it is assumed that $c_M = 1$. Hence, $\alpha = (c_1, \ldots, c_{M-1})^\top$. When $M = 1$, it becomes an exponential distribution with $h_B(t) = 1$. The assumption of the piecewise hazard function is not very uncommon in the reliability or survival analysis. It brings a great flexibility in the data analysis compared to a purely parametric model. Note that for a specific parametric baseline distribution, the shape of the baseline hazard function becomes fixed. For example if we assume the baseline distribution to be exponential, it means the baseline hazard function can be only constant. Similarly if we assume Weibull as the baseline distribution, the baseline hazard function can be monotone (either increasing or decreasing) only. But if we assume the baseline hazard function as in (43), then it can take any shape depending on the τ_i's and c_i's. So it becomes a data driven model, and it can be called as the semi-parametric-parametric approach. It has been attempted by different researchers, see for example Balakrishnan et al. [3], Ibrahim et al. [11], Samanta and Kundu [20], Ganguly et al. [8] and see the references cited therein in this respect. One major issue about all these papers is that it is assumed that the number of cut points and the position of the cut points are known. Here we have addressed that issue. First it is assumed that the number of cut points and the position of the cut points are known, and then we propose a method to estimate them.

Now we will discuss how to obtain the MLEs of the unknown parameters namely Θ and $\alpha = (c_1, \ldots, c_{M-1})^\top$, from the observed data when M and $\tau_1, \ldots, \tau_M$ are known. From the above $h_B(t;\alpha)$ as given in (43), let us compute $S_B(t;\alpha)$, the base line survival function, and $f_B(t;\alpha)$, the base line PDF. The cumulative base line hazard function $H_B(t;\alpha)$ is

$$H_B(t;\alpha) = \sum_{j=1}^{M} c_j(\min\{t, \tau_j\} - \tau_{j-1}) I_{[\tau_{j-1}, \tau_M)}(t).$$

From the above $H_B(t;\alpha)$, $S_B(t;\alpha)$ and $f_B(t;\alpha)$ can be obtained as

$$S_B(t;\alpha) = e^{-H_B(t;\alpha)} \quad \text{and} \quad f_B(t;\alpha) = h_B(t;\alpha) e^{-H_B(t;\alpha)}.$$

In this section only we prefer to write y_{1i}, y_{2i} and y_i as follows: for $y_{1i}(y_{2i}) \in I_1 \cup I_2$, we write it as $y_{1uvw}(y_{2uvw})$, here $u = 1$ or 2, if $i \in I_1$ or $i \in I_2$, $v = m$, if $\tau_{m-1} < y_{1i}(y_{2i}) \leq \tau_m$, for $m = 1, \ldots, M$, and $w = 1, \ldots, n_{1um}(n_{2um})$. Here, $n_{1um}(n_{2um})$ denotes the number of $\{i; y_{1i}(y_{2i}) \in I_u, \tau_{m-1} < y_{1i}(y_{2i}) \leq \tau_m\}$. Similarly, we write each y_i as y_{vw}, and define n_{um}. Based on the above notations, we can write

$$\sum_{i \in I_1} H_B(y_{1i};\alpha) = c_1 D_{11} + c_2 D_{12} + \ldots + c_{M-1} D_{1,M-1} + D_{1M} \tag{44}$$

$$\sum_{i \in I_2} H_B(y_{1i};\alpha) = c_1 D_{21} + c_2 D_{22} + \ldots + c_{M-1} D_{2,M-1} + D_{2M} \tag{45}$$

$$\sum_{i \in I_1} H_B(y_{2i};\alpha) = c_1 E_{11} + c_2 E_{12} + \ldots + c_{M-1} E_{1,M-1} + E_{1M} \tag{46}$$

$$\sum_{i \in I_2} H_B(y_{2i};\alpha) = c_1 E_{21} + c_2 E_{22} + \ldots + c_{M-1} E_{2,M-1} + E_{2M} \tag{47}$$

$$\sum_{i \in I_3} H_B(y_{i};\alpha) = c_1 F_1 + c_2 F_2 + \ldots + c_{M-1} F_{M-1} + F_M \tag{48}$$

The explicit expressions of D_{ij}'s, E_{ij}'s and F_j's are presented in Appendix C. Now if we use the notations $U_j = D_{1j} + E_{2j} + F_j$, $V_j = D_{2j} - E_{2j}$ and $W_j = E_{1j} - D_{1j}$, for $j = 1,\ldots,M$, and $k_j = \#\{j : \tau_{j-1} < y_j \le \tau_j\} + \#\{j : \tau_{j-1} < y_{1j} \le \tau_j\} + \#\{j : \tau_{j-1} < y_{2j} \le \tau_j\}$, for $j = 1,\ldots,M$, then the MLEs of $\alpha = (c_1,\ldots,c_{M-1})^\top$ can be obtained by maximizing

$$\begin{aligned} h(c_1,\ldots,c_{M-1}) &= -n\ln(c_1U_1 + \ldots + c_{M-1}U_{M-1} + U_M) - n_2\ln(c_1V_1 + \ldots + c_{M-1}V_{M-1} + V_M) \\ &\quad -n_1\ln(c_1W_1 + \ldots + c_{M-1}W_{M-1} + W_M) + k_1\ln c_1 + \ldots + k_{M-1}\ln c_{M-1}. \end{aligned} \tag{49}$$

Note that the maximization of (49) needs to be done numerically, and it involves solving a $M-1$ dimensional optimization problem. Let the MLEs be $\widehat{\alpha} = (\widehat{c}_1,\ldots,\widehat{c}_{M-1})^\top$. Once $\widehat{\alpha}$ is obtained the MLEs of $\lambda_1,\ldots,\lambda_5$ can be obtained as

$$\begin{aligned} \widehat{\lambda}_1 &= \frac{n_1}{\widehat{c}_1U_1 + \ldots + \widehat{c}_{M-1}U_{M-1} + U_M}, \\ \widehat{\lambda}_2 &= \frac{n_2}{\widehat{c}_1U_1 + \ldots + \widehat{c}_{M-1}U_{M-1} + U_M}, \\ \widehat{\lambda}_3 &= \frac{n_3}{\widehat{c}_1U_1 + \ldots + \widehat{c}_{M-1}U_{M-1} + U_M}, \\ \widehat{\lambda}_4 &= \frac{n_2}{\widehat{c}_1V_1 + \ldots + \widehat{c}_{M-1}V_{M-1} + V_M}, \\ \widehat{\lambda}_5 &= \frac{n_1}{\widehat{c}_1W_1 + \ldots + \widehat{c}_{M-1}W_{M-1} + W_M}. \end{aligned}$$

Under H_0, the MLEs of α can be obtained by maximizing

$$\begin{aligned} h_0(c_1,\ldots,c_{M-1}) &= -n\ln(c_1U_1 + \ldots + c_{M-1}U_{M-1} + U_M) \\ &\quad -(n_1+n_2)\left(\ln(c_1(V_1+W_1) + \ldots + c_{M-1}(V_{M-1}+W_{M-1}) + V_M + W_M)\right) \end{aligned}$$

$$+k_1 \ln c_1 + \ldots + k_{M-1} \ln c_{M-1}. \tag{50}$$

Once, $\widehat{\alpha}_0 = (\widehat{c}_{10}, \ldots, \widehat{c}_{M-1,0})^\top$ is obtained, the MLEs of λ_1, λ_3 and λ_4 can be obtained as

$$\begin{aligned}
\widehat{\lambda}_{10} &= \frac{n_1+n_2}{2(\widehat{c}_{10}U_1 + \ldots + \widehat{c}_{M-1,0}U_{M-1} + U_M)}, \\
\widehat{\lambda}_{30} &= \frac{n_3}{\widehat{c}_{10}U_1 + \ldots + \widehat{c}_{M-1,0}U_{M-1} + U_M}, \\
\widehat{\lambda}_{40} &= \frac{n_1+n_2}{\widehat{c}_{10}(V_1+W_1) + \ldots + \widehat{c}_{M-1,0}(V_{M-1}+W_{M-1}) + V_M + W_M}.
\end{aligned}$$

Once the MLEs are obtained in both the cases, the likelihood ratio test can be constructed exactly the same way as before.

In practice it has been observed by several authors, see for example Balakrishnan et al. [3], Ibrahim et al. [11], Samanta and Kundu [20], Ganguly et al. [8], that with $M = 2$ or 3, the model works quite well. Hence, the MLEs of the unknown parameters can be obtained by solving either one or two dimensional optimization problem. Note that we have assumed that M and $\tau_1, \ldots, \tau_{M-1}$ are known, although in practice they are not known. In Section 6 we will show how we can estimate M and $\tau_1, \ldots, \tau_{M-1}$ for a given data set.

5 Simulations

In this section we perform some simulation experiments to see how the proposed maximum likelihood estimators work in practice for different models and for different parameter values. We have considered three different base line distribution namely (i) exponential, (ii) Weibull and (iii) piecewise constant hazard function. In each case we have considered different parameter values and different sample sizes. We have computed the maximum likelihood estimators of the unknown parameters as suggested in the previous section, and we report the average estimates (AE) and the mean squared errors (MSEs) based on 1000 replications. Note that in case of exponential distribution the maximum likelihood estimates can be obtained in explicit forms. In case of Weibull or piecewise constant hazard function one needs to solve either one or two dimensional optimization problem. We have used the optimization routine available in Press et al. [18] for this purpose.

5.1 Baseline Distribution: Exponential and Weibull

In this case we have considered the following parameter values:

Set 1: $\lambda_1 = 0.03$, $\lambda_2 = 0.051$, $\lambda_3 = 0.015$, $\lambda_4 = 0.074$, $\lambda_5 = 0.057$

Set 2: $\lambda_1 = \lambda_2 = 0.038$, $\lambda_3 = 0.015$, $\lambda_4 = \lambda_5 = 0.066$

and different sample sizes, namely 25,50,75 and 100. The results are reported in Table 1 and Table 2 for Set 1 and Set 2, respectively.

n	λ_1	λ_2	λ_3	λ_4	λ_5
25	0.031652	0.053215	0.015402	0.080099	0.067657
	(0.000136)	(0.000217)	(0.000064)	(0.000671)	(0.001047)
50	0.030636	0.052010	0.015231	0.077006	0.060082
	(0.000067)	(0.000110)	(0.000029)	(0.000240)	(0.000297)
75	0.030493	0.052129	0.015393	0.075842	0.058992
	(0.000042)	(0.000078)	(0.000020)	(0.000159)	(0.000166)
100	0.029931	0.051419	0.015349	0.075926	0.058876
	(0.000029)	(0.000049)	(0.000015)	(0.000111)	(0.000126)

Table 1: The average estimates and the associated mean squared errors (within bracket below) for different parameters in case of parameter Set 1.

In case of Weibull baseline distribution we have considered the following parameter values:

Set 3: $\lambda_1 = 0.019$, $\lambda_2 = 0.032$, $\lambda_3 = 0.0097$, $\lambda_4 = 0.046$, $\lambda_5 = 0.034$, $\alpha = 1.10$

Set 4: $\lambda_1 = \lambda_2 = 0.02$, $\lambda_3 = 0.025$, $\lambda_4 = \lambda_5 = 0.08$, $\alpha = 1.25$

and different sample sizes, namely 25,50,75 and 100.

n	λ_1	λ_2	λ_3	λ_4	λ_5
25	0.040042	0.039592	0.015399	0.073666	0.073063
	(0.000168)	(0.000155)	(0.000061)	(0.000831)	(0.000699)
50	0.038520	0.038613	0.015333	0.069703	0.069347
	(0.000073)	(0.000073)	(0.000029)	(0.000295)	(0.000280)
75	0.038669	0.038836	0.015195	0.068395	0.067580
	(0.000055)	(0.000053)	(0.000018)	(0.000164)	(0.000170)
100	0.038086	0.038326	0.015263	0.067977	0.067637
	(0.000037)	(0.000037)	(0.000015)	(0.000123)	(0.000110)

Table 2: The average estimates and the associated mean squared errors (within bracket below) for different parameters in case of parameter Set 2.

n	λ_1	λ_2	λ_3	λ_4	λ_5	α
25	0.017239	0.028488	0.086309	0.037811	0.030839	1.170954
	(0.000091)	(0.000189)	(0.000034)	(0.000539)	(0.000434)	(0.059501)
50	0.018114	0.030541	0.090968	0.040362	0.030977	1.140171
	(0.000051)	(0.000117)	(0.000168)	(0.000327)	(0.000235)	(0.013034)
75	0.018453	0.031139	0.095191	0.041239	0.031155	1.126328
	(0.000036)	(0.000079)	(0.000013)	(0.000241)	(0.000159)	(0.081219)
100	0.018264	0.031100	0.095567	0.041501	0.031361	1.120864
	(0.000025)	(0.000060)	(0.000009)	(0.000185)	(0.000125)	(0.006061)

Table 3: The average estimates and the associated mean squared errors (within bracket below) for different parameters in case of parameter Set 3.

5.2 Baseline Distribution: Piecewise Constant Hazard

In this case we have considered for $M = 2$ and $M = 3$. In case of $M = 2$, we have taken $\tau_1 = 42$ and for $M = 2$, $\tau_1 = 42$, $\tau_2 = 55$. The following parameter values for $M = 2$ and $M = 3$, respectively have been considered.

Set 5: $\lambda_1 = 0.09$, $\lambda_2 = 0.15$, $\lambda_3 = 0.045$, $\lambda_4 = 0.2$, $\lambda_5 = 0.15$, $c_1 = 0.31$

Set 6: $\lambda_1 = 0.055$, $\lambda_2 = 0.091$, $\lambda_3 = 0.027$, $\lambda_4 = 0.12$, $\lambda_5 = 0.091$, $c_1 = 0.51$ and $c_2 = 1.8$

and different sample sizes, namely 25,50,75 and 100. The results are presented in Table 5

n	λ_1	λ_2	λ_3	λ_4	λ_5	α
25	0.023492	0.023221	0.086563	0.035927	0.035597	1.168937
	(0.000155)	(0.000158)	(0.000035)	(0.000569)	(0.000511)	(0.044967)
50	0.024494	0.024697	0.091745	0.037577	0.037385	1.139756
	(0.000074)	(0.000078)	(0.000016)	(0.000289)	(0.000294)	(0.012639)
75	0.025249	0.025242	0.094387	0.037901	0.037487	1.127524
	(0.000059)	(0.000058)	(0.000013)	(0.000210)	(0.000212)	(0.082729)
100	0.025040	0.025156	0.094253	0.037776	0.037791	1.122738
	(0.000045)	(0.000043)	(0.000010)	(0.000157)	(0.000172)	(0.005989)

Table 4: The average estimates and the associated standard errors (reported within bracket below) for different parameters in case of parameter Set 4.

and Table 6.

n	λ_1	λ_2	λ_3	λ_4	λ_5	c_1
25	0.081145	0.011457	0.041657	0.146561	0.111232	0.179845
	(0.000355)	(0.000101)	(0.000016)	(0.003257)	(0.003112)	(0.006787)
50	0.084587	0.012341	0.042216	0.167656	0.122876	0.225647
	(0.000256)	(0.000071)	(0.000011)	(0.002263)	(0.002192)	(0.004809)
75	0.086798	0.014576	0.043387	0.187882	0.140115	0.288632
	(0.000206)	(0.000059)	(0.000009)	(0.001882)	(0.001824)	(0.003941)
100	0.089108	0.148785	0.044971	0.198767	0.014866	0.301468
	(0.000177)	(0.000050)	(0.000007)	(0.001613)	(0.001499)	(0.003112)

Table 5: The average estimates and the associated mean squared errors (within bracket below) for different parameters in case of parameter Set 5.

In case of exponential distribution, the maximum likelihood estimators can be obtained in explicit forms, where as in the other cases, they need to be obtained by solving optimization problem. In all the cases, the performances of the maximum likelihood estimators of the unknown parameters are along the expected line. In all these cases, it is observed that as the sample size increases, the average biases and mean squared errors of all the estimators decrease. It suggests the consistency properties of the maximum likelihood estimators.

n	λ_1	λ_2	λ_3	λ_4	λ_5	c_1	c_2
25	0.051121	0.081156	0.022176	0.091345	0.082387	0.423134	1.213457
	(0.000016)	(0.000345)	(0.000010)	(0.000519)	(0.000401)	(0.011145)	(0.032451)
50	0.052321	0.084545	0.024465	0.099987	0.085121	0.467891	1.412453
	(0.000011)	(0.000243)	(0.000007)	(0.000365)	(0.000286)	(0.007857)	(0.022812)
75	0.054011	0.087897	0.025719	0.111564	0.088898	0.496127	1.789161
	(0.000009)	(0.000210)	(0.000006)	(0.000299)	(0.000234)	(0.006372)	(0.018824)
100	0.054891	0.090879	0.026765	0.011961	0.090011	0.500954	1.889921
	(0.000008)	(0.000170)	(0.000005)	(0.000245)	(0.000201)	(0.005115)	(0.015987)

Table 6: The average estimates and the associated mean squared errors (reported within bracket below) for different parameters in case of parameter Set 6.

6 Choice of Cut Points and Data Analysis

6.1 Choice of Cut Points

So far it has been assumed that the number of cut points and the position of the cut points are known. But definitely for a given data set, they are not known. It has not been addressed properly also in the literature. In this section we will address this issue. Let us denote the maximum time point as τ, and that we will take as $\tau = \max\{y_i, y_{1i}, y_{2i}; i = 1, \ldots, n\}$. Note that it is not possible to estimate the hazard function beyond this point without any further parametric assumption. It is assumed that the position of the cut points are modeled as the arrival times of the events in $[0, \tau]$, from a non-homogeneous Poisson process $\{N(t); t \geq 0\}$ with intensity function $u(t) = \delta t^{\delta-1}$, for $\delta > 0$. But the arrival times of the events from $N(t)$ are not observed, hence they are being treated as missing values. For the given number of cut points $M - 1$ in $[0, \tau]$, the likelihood function of the observation is treated as a function of the unknown parameters $\theta_1, \ldots, \theta_M, \delta$, when the arrival times of the events from $N(t)$, which are missing, are replaced by their corresponding expected values. For a given M we compute the maximum likelihood estimators of the unknown parameters, and then based on some information theoretic criteria, like Akaike Information Criterion (AIC) we estimate M

also.

Now we provide the joint PDF of the arrival times of a non-homogeneous Poisson process. Suppose $\{N(t); t \geq 0\}$ is a non-homogeneous Poisson process with intensity function $u(t) = \delta t^{\delta-1}$, for $\delta > 0$. Let $T_1, T_2, \ldots$ denote the arrival times of the first, second etc. arrival times of the events from $N(t)$. Then the joint PDF of $T_1, \ldots, T_n$, given that $N(\tau) = n$, can be written as

$$f_{T_1,\ldots,T_n|N(\tau)=n}(t_1,\ldots,t_n) = \frac{n!\delta^n}{\tau^{n\delta}} t_1^{\delta-1} \ldots t_n^{\delta-1}; \qquad 0 < t_1 < t_2 \ldots < t_n < \tau, \tag{51}$$

and zero, otherwise. Hence, for $j = 1, \ldots, n$,

$$\begin{aligned}
E(T_j|N(\tau)=n) &= \frac{n!\delta^n}{\tau^{n\delta}} \int_0^{t_2}\int_0^{t_3} \cdots \int_0^{t_n}\int_0^{\tau} t_1^{\delta-1} \ldots t_{j-1}^{\delta-1} t_j^{\delta} t_{j+1}^{\delta-1} \ldots t_n^{\delta-1} dt_1 dt_2 \ldots dt_{n-1} dt_n \\
&= \frac{n!\delta^n}{\tau^{n\delta}} \times \frac{\tau^{n\delta+1}}{\alpha(2\delta)\ldots((j-1)\delta)(j\delta+1)((j+1)\delta+1)\ldots(n\delta+1)} \\
&= \frac{n!}{(j-1)!} \frac{\delta^{n-j+1}\tau}{(j\delta+1)((j+1)\delta+1)\ldots(n\delta+1)} \\
&= \frac{\tau}{(1+\frac{1}{j\delta})\ldots(1+\frac{1}{n\delta})}
\end{aligned}$$

Therefore, for homogeneous Poisson process i.e. when $\delta = 1$,

$$E(T_j|N(\tau)=n) = \frac{j\tau}{n+1}.$$

In this case the cut points are equidistant.

Hence, in practice we can implement the above method as follows. For fixed δ and M, we can write the the likelihood function in terms of the unknown parameters. Then for a given δ and M we can obtain the maximum likelihood estimators of the other parameters. We treat this as profile likelihood of δ and then obtain the maximum likelihood estimator of δ also for a given M. Then we can find M by using AIC.

6.2 Data Analysis

In this section we analyze a bivariate diabetic retinopathy data set to show how the proposed model can be used to analyze this data set. Extensive research has been done during the last two decades for diabetic retinopathy disorder particularly for early detection and for innovative treatment. One of the treatments which has been invented is the laser treatment. The main goal of this experiment is to test whether the laser treatment has any effect in delaying the onset of blindness compared to the traditional treatment. The experiment has been conducted as follows: For each patient one eye has been chosen at random, and the laser treatment has been given to that eye, where as the other eye has been given the traditional treatment. For each patient the time to the onset of blindness of both the eyes has been recorded. The data are presented in the Appendix D. We have considered the uncensored observations only. Here, X denotes the time to vision loss of the laser treated eye, and Y denotes the same for the other eye. For this data set $n = 38$, and 12, 20 and 6 observations for which $X < Y$, $X > Y$ and $X = Y$, respectively. Based on the observed data, we want to test whether the laser treatment has any significant effect in delaying the onset of blindness or not, further we also want to test if any eye becomes blind, whether it has any effect with respect to the onset of blindness on the other eye or not. Before progressing further, we have plotted the survival functions of the time to blindness of both the eyes in Figure 1. Now we have fitted the proposed bivariate load sharing model with different baseline distributions.

6.3 Baseline Distribution: Exponential and Weibull

We have fitted the proposed load sharing model with the baseline distribution as the exponential distribution. In this case the maximum likelihood estimators can be obtained in explicit forms. The maximum likelihood estimates and the associated standard errors (within brackets) are as follows: $\widehat{\lambda}_1$ = 2.8331E-02 ($\mp$ 1.2122E-03), $\widehat{\lambda}_2$ = 4.7219E-02 ($\mp$ 1.9876E-

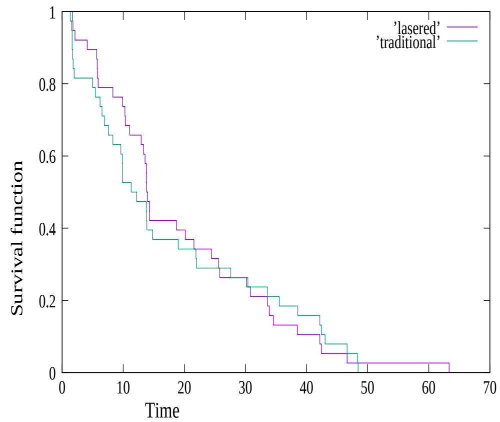

Figure 1: Survival functions of the time to blindness of the two eyes.

03), $\widehat{\lambda}_3$ = 1.4166E-02 ($\mp$ 5.1765E-04), $\widehat{\lambda}_4$ = 7.3962E-02 (2.4767E-03) and $\widehat{\lambda}_5$ = 5.6847E-02 (1.9956E-03). The corresponding log-likelihood value is -285.8590. The maximum likelihood estimators under H_0 becomes $\widehat{\lambda}_{10}$ = 3.7775E-02 ($\mp$ 1.8156E-03), $\widehat{\lambda}_{30}$ = 1.4166E-02 ($\mp$ 5.1765E-04), $\widehat{\lambda}_{40}$ = 6.6459E-02 (2.1034E-03) and the corresponding log-likelihood value is -287.1345. The test statistic becomes 2.5510 and the associated p value is 0.2794. Hence, based on the above data set, we cannot reject H_0.

We have also fitted the same model when the baseline distribution is Weibull. In this case the maximum likelihood estimators cannot be obtained explicitly. First we obtain the maximum likelihood estimate of α by maximizing the profile log-likelihood function $h(\alpha)$ as defined in (38) and once we obtain the maximum likelihood estimate of α, the maximum likelihood estimates of the other parameters can be obtained in explicit forms.

The maximum likelihood estimates and the associated standard errors of all the parameters are as as follows: $\widehat{\alpha}$ = 1.1312 ($\mp$ 0.1124), $\widehat{\lambda}_1$ = 1.9411E-02 ($\mp$ 1.0457E-03), $\widehat{\lambda}_2$ = 3.2351E-02 ($\mp$ 1.9945E-03), $\widehat{\lambda}_3$ = 9.7054E-03 ($\mp$ 5.2011E-04), $\widehat{\lambda}_4$ = 4.5584E-02 (2.5001E-03) and $\widehat{\lambda}_5$ = 3.3984E-02 (2.1123E-03). The associated log-likelihood value is -285.2415. Under H_0, the MLEs become $\widehat{\alpha}_0$ = 1.1219 ($\mp$ 0.1157), $\widehat{\lambda}_{10}$ = 2.6601E-02 ($\mp$ 1.5745E-03),

$\widehat{\lambda}_3 = 9.7054\text{E-}03$ ($\mp$ 5.1865E-04), $\widehat{\lambda}_{04} = 4.1869\text{E-}02$ ($\mp$ 2.6679E-03) and the associated log-likelihood value is -286.5804. In this case the test statistic is 2.6778, and the corresponding p value is 0.2621. Hence, we do not reject H_0.

6.4 Baseline Distribution: Piecewise Constant Hazard

In this case we have fitted the model with $M = 2$ and 3. Note that when $M = 1$, it becomes the exponential model. Since the change points are unknown, we have estimated the change points as suggested in Section 6.1. For both $M = 2$ and 3, we have taken $\tau = \max\{X, Y\} = 63.33$. When $M = 2$, the results are as follows: $\widehat{\delta} = 1.9924(\mp 0.2125)$, $\widehat{\tau}_1 = 42.17(\mp 1.3217)$, $\widehat{\lambda}_1 = 9.0089\text{E-}02$($\mp$ 1.2314E-03), $\widehat{\lambda}_2 = 0.1501$ ($\mp$ 1.2732E-02),$\widehat{\lambda}_3 =$ 4.5044E-02 ($\mp$ 9.2156E-04), $\widehat{\lambda}_4 = 0.2011$ ($\mp$ 0.6451E-03),$\widehat{\lambda}_5 = 0.1533$ ($\mp$ 1.1989E-02), $\widehat{c}_1 = 0.3140$ ($\mp$ 4.1163E-02), and the associated log-likelihood value is -282.9532. Under H_0, the results are as follows:$\widehat{\delta}_0 = 1.9924(\mp 0.2145)$, $\widehat{\tau}_{10} = 42.17(\mp 1.3645)$, $\widehat{\lambda}_{10} = 0.1201$($\mp$ 1.7983E-03), $\widehat{\lambda}_{30} = 4.5044\text{E-}02$ ($\mp$ 9.2218E-04), $\widehat{\lambda}_{40} = 0.1801$ ($\mp$ 0.5449E-03), $\widehat{c}_{10} = 0.3140$ ($\mp$ 4.2211E-02), and the associated log-likelihood value is -284.2457. In this case the test statistic is 2.5850, and the corresponding p value is 0.2745. Hence, we do not reject H_0.

When $M = 3$, the results are as follows: $\widehat{\delta} = 3.1701(\mp 0.3156)$, $\widehat{\tau}_1 = 42.02(\mp 1.1221)$, $\widehat{\tau}_2 = 55.2806(\mp 2.6512)$, $\widehat{\lambda}_1 = 5.4837\text{E-}02$($\mp$ 9.1241E-04), $\widehat{\lambda}_2 = 9.1359\text{E-}02$ ($\mp$ 1.1976E-02),$\widehat{\lambda}_3 = 2.7419\text{E-}02$ ($\mp$ 3.1178E-04), $\widehat{\lambda}_4 = 0.1251$ ($\mp$ 0.3351E-03),$\widehat{\lambda}_5 = 9.1428\text{E-}02$ ($\mp$ 1.0113E-02), $\widehat{c}_1 = 0.5149$ ($\mp$ 6.2216E-02), $\widehat{c}_2 = 1.7750$ ($\mp$ 7.2287E-02) and the associated log-likelihood value is -282.9130. Under H_0, the results are as follows:$\widehat{\delta}_0 = 3.1701(\mp 0.3047)$, $\widehat{\tau}_{10} = 42.02(\mp 1.1176)$, $\widehat{\tau}_{20} = 55.2806(\mp 2.6124)$, $\widehat{\lambda}_{10} = 7.3116\text{E-}02$($\mp$ 9.5952E-04), $\widehat{\lambda}_{30} =$ 2.7419E-02 ($\mp$ 3.2006E-04), $\widehat{\lambda}_{40} = 0.1099$ ($\mp$ 0.4456E-03), $\widehat{c}_{10} = 0.5149$ ($\mp$ 6.0116E-02), $\widehat{c}_{20} = 1.7750$ ($\mp$ 7.1114E-02) and the associated log-likelihood value is -284.2008. In this case the test statistic is 2.5756, and the corresponding p value is 0.2758. Hence, H_0 can not be rejected.

Now all the models suggest that there is no significant difference between the two treatments. Between these four models, to choose the best one we use the Akaike Information Criterion (AIC). The AIC values for the exponential, Weibull, PCH-1 and PCH-2 are 584.269, 585.161, 582.491 and 584.402, respectively. Hence, based on the AIC values, we conclude that PCH-1 provides the best fit to the data set.

7 Conclusions

In this paper we have proposed a very flexible bivariate load sharing model. One advantage of the proposed model is that it can have a very flexible data driven baseline distribution. One does not need to assume any specific form of the baseline hazard function, say for example increasing, decreasing or any other specific shape, the data can detect the shape of the baseline hazard function. Hence, it behaves like a semi-parametric model. The implementation of the proposed model is also quite easy in practice. Further, the model can be applied where there are ties in the data set, which none of the existing load sharing model can accommodate. In this paper we have mainly developed the classical inference of the unknown parameters. It will be interesting to explore the Bayesian inference also. More work is needed along that direction.

Acknowledgements:

The author would like to thank two unknown reviewers and the associate editor for their constructive comments which have helped to improve the earlier version of the manuscript significantly.

Appendix A

The PDF of X^* can be obtained as

$$f_{X^*}(t) = \int_0^t f_{X^*|X^*>Y=u}(t) f_Y(u) du = \lambda_4(\lambda_2+\lambda_3) f_B(t)[S_B(t)]^{\lambda_4-1} \int_0^t f_B(u)[S_B(u)]^{\lambda_2+\lambda_3-\lambda_4-1} du.$$

Hence, if $\lambda_2 + \lambda_3 = \lambda_4$, then

$$f_{X^*}(t) = \lambda_4^2 f_B(t)[S_B(t)]^{\lambda_4-1}(-\ln S_B(t)),$$

and if $\lambda_2 + \lambda_3 \neq \lambda_4$, then

$$f_{X^*}(t) = \frac{\lambda_4(\lambda_2+\lambda_3)}{(\lambda_2+\lambda_3-\lambda_4)} f_B(t)[S_B(t)]^{\lambda_4-1}(1-[S_B(t)]^{\lambda_2+\lambda_3-\lambda_4}).$$

Similarly, the PDF of Y^* can be obtained from the above expressions by replacing λ_2 and λ_4 with λ_1 and λ_5, respectively.

Appendix B

In this Appendix, we will show how to obtain the joint survival function of (Y_1, Y_2). We will show only Case 1, Case 2 and Case 3 can be obtained along the same line. Since $a < b$, therefore,

$$\begin{aligned} P(Y_1 > y_1, Y_2 > y_2) &= P(Y_1 = Y_2 > y_2) + P(y_1 \leq Y_1 \leq y_2, Y_2 \geq y_2) \\ &\quad + P(Y_1 > y_2, Y_2 > Y_1) + P(Y_2 \geq y_2, Y_1 > Y_2) \end{aligned}$$

Now

$$\begin{aligned} P(Y_1 = Y_2 > y_2) &= \int_{y_2}^{\infty} \lambda_3 f_B(x)[S_B(x)]^{\lambda-1} dx = \frac{\lambda_3}{\lambda}[S_B(y_2)]^{\lambda} \\ P(y_1 < Y_1 \leq y_2, Y_2 > y_2) &= \lambda_1\lambda_5 \int_{y_1}^{y_2}\int_{y_2}^{\infty} f_B(x) f_B(y)[S_B(x)]^{\eta-1}[S_B(y)]^{\lambda_5-1} dy dx \end{aligned}$$

$$
\begin{aligned}
&= \lambda_1 [S_B(y_2)]^{\lambda_5} \int_{y_1}^{y_2} f_B(x)[S_B(x)]^{\eta-1} dx \\
&= \lambda_1 [S_B(y_2)]^{\lambda_5} \times \begin{cases} \ln S_B(y_1) - \ln S_B(y_2) & \text{if} \quad \eta = 0 \\ \frac{1}{\eta}([S_B(y_1)]^{\eta} - [(S_B(y_2))]^{\eta}) & \text{if} \quad \eta \neq 0 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
P(Y_1 > y_2, Y_1 > Y_2) &= \lambda_1 \lambda_5 \int_{y_2}^{\infty} \int_{x}^{\infty} f_B(x) f_B(y) [S_B(x)]^{\eta-1} [S_B(y)]^{\lambda_5 - 1} dy dx \\
&= \lambda_1 \int_{y_2}^{\infty} f_B(x)[S_B(x)]^{\lambda-1} dx = \frac{\lambda_1}{\lambda} [S_B(b)]^{\lambda} \\
P(Y_2 > y_2, Y_1 > Y_2) &= \lambda_2 \lambda_4 \int_{y_2}^{\infty} \int_{y}^{\infty} f_B(x) f_B(y) [S_B(x)]^{\lambda_4 - 1} [S_B(y)]^{\xi-1} dx dy \\
&= \lambda_2 \int_{y_2}^{\infty} f_B(y)[S_B(y)]^{\lambda-1} dx = \frac{\lambda_2}{\lambda} [S_B(y_2)]^{\lambda}
\end{aligned}
$$

Appendix C

In this Appendix, we provide the explicit expressions of $D'_{ij}s$, $E'_{ij}s$ and $F'_{j}s$ as defined in (44) to (48).

$$
\begin{aligned}
D_{11} &= \sum_{w=1}^{n_{111}} y_{111w} + \tau_1 \sum_{m=2}^{M} n_{11m} \\
D_{12} &= \sum_{w=1}^{n_{112}} (y_{112w} - \tau_1) + (\tau_2 - \tau_1) \sum_{m=3}^{M} n_{11m} \\
\vdots \quad & \quad \vdots \\
D_{1(M-1)} &= \sum_{w=1}^{n_{11(M-1)}} (y_{11(M-1)w} - \tau_{M-2}) + (\tau_{M-1} - \tau_{M-2}) n_{11M} \\
D_{1M} &= \sum_{w=1}^{n_{11M}} (y_{11Mw} - \tau_{M-1})
\end{aligned}
$$

$$
\begin{aligned}
D_{21} &= \sum_{w=1}^{n_{121}} y_{121w} + \tau_1 \sum_{m=2}^{M} n_{12m} \\
D_{22} &= \sum_{w=1}^{n_{122}} (y_{122w} - \tau_1) + (\tau_2 - \tau_1) \sum_{m=3}^{M} n_{12m} \\
\vdots \quad & \quad \vdots
\end{aligned}
$$

$$
\begin{aligned}
D_{2(M-1)} &= \sum_{w=1}^{n_{12(M-1)}} (y_{12(M-1)w} - \tau_{M-2}) + (\tau_{M-1} - \tau_{M-2}) n_{12M} \\
D_{2M} &= \sum_{w=1}^{n_{12M}} (y_{12Mw} - \tau_{M-1})
\end{aligned}
$$

$$
\begin{aligned}
E_{11} &= \sum_{w=1}^{n_{211}} y_{211w} + \tau_1 \sum_{m=2}^{M} n_{21m} \\
E_{12} &= \sum_{w=1}^{n_{212}} (y_{212w} - \tau_1) + (\tau_2 - \tau_1) \sum_{m=3}^{M} n_{21m} \\
\vdots & \quad \vdots \\
E_{1(M-1)} &= \sum_{w=1}^{n_{21(M-1)}} (y_{21(M-1)w} - \tau_{M-2}) + (\tau_{M-1} - \tau_{M-2}) n_{21M} \\
E_{1M} &= \sum_{w=1}^{n_{21M}} (y_{21Mw} - \tau_{M-1})
\end{aligned}
$$

$$
\begin{aligned}
E_{21} &= \sum_{w=1}^{n_{221}} y_{221w} + \tau_1 \sum_{m=2}^{M} n_{22m} \\
E_{22} &= \sum_{w=1}^{n_{222}} (y_{222w} - \tau_1) + (\tau_2 - \tau_1) \sum_{m=3}^{M} n_{22m} \\
\vdots & \quad \vdots \\
E_{2(M-1)} &= \sum_{w=1}^{n_{22(M-1)}} (y_{22(M-1)w} - \tau_{M-2}) + (\tau_{M-1} - \tau_{M-2}) n_{22M} \\
E_{2M} &= \sum_{w=1}^{n_{22M}} (y_{22Mw} - \tau_{M-1})
\end{aligned}
$$

$$
\begin{aligned}
F_1 &= \sum_{w=1}^{n_{11}} y_{1w} + \tau_1 \sum_{m=2}^{M} n_{1m} \\
F_2 &= \sum_{w=1}^{n_{12}} (y_{2w} - \tau_1) + (\tau_2 - \tau_1) \sum_{m=3}^{M} n_{1m} \\
\vdots & \quad \vdots
\end{aligned}
$$

$$
\begin{aligned}
F_{M-1} &= \sum_{w=1}^{n_{1(M-1)}} (y_{(M-1)w} - \tau_{M-2}) + (\tau_{M-1} - \tau_{M-2}) n_{1M} \\
F_M &= \sum_{w=1}^{n_{1M}} (y_{Mw} - \tau_{M-1})
\end{aligned}
$$

Appendix D: Data Set 1

Table 7: Bivariate Retinopathy Data Set. Here Y_1 denotes the time to blindness of the lasered treated eye and Y_2 denotes the same for the other eye.

Sl. No.	Y_1	Y_2	Sl. No.	Y_1	Y_2	Sl. No.	Y_1	Y_2	Sl. No.	Y_1	Y_2
1.	20.17	6.90	2.	10.27	1.63	3.	5.67	13.83	4.	5.77	1.33
5.	5.90	35.53	6.	25.63	21.90	7.	33.90	14.80	8.	1.73	6.20
9.	30.20	22.00	10.	25.80	13.87	11.	5.73	48.30	12.	9.90	9.90
13.	1.73	1.73	14.	1.77	43.03	15.	8.30	8.30	16.	18.70	6.53
17.	42.17	42.17	18.	14.30	48.43	19.	13.33	9.60	20.	14.27	7.60
21.	34.57	1.80	22.	4.10	12.20	23.	21.57	9.90	24.	13.77	13.77
25.	33.63	33.63	26.	63.33	27.60	27.	38.47	1.63	28.	10.33	0.83
29.	13.83	1.57	30.	11.07	1.97	31.	2.10	11.30	32.	12.93	4.97
33.	24.43	9.87	34.	13.97	30.40	35.	6.30	56.97	36.	13.80	19.00
37.	13.57	5.43	38.	42.77	42.77	39.	42.43	46.63	40.	2.70	2.70

Conflict of Interest Statement: The author does not have any conflict of interest in publishing this article.

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