Bayesian Inference for Sequential k-out-of-n Systems

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Abstract

Consider a system of n components and the component fails one after another. The system fails when the first $(n-k+1)$ components fail. It is known as k-out-of-n system. It is assumed that the failure of a component alters the lifetimes of the remaining components. The lifetime of the first $(n-k+1)$ failures are observed. Based on the sequential order statistics of k-out-of-n system one needs to draw inference of the lifetime of the components. It is assumed that the lifetime distributions of the components are from a proportional hazard class of distributions. The baseline distribution of the proportional hazard class can be either exponential, Weibull or a very flexible piecewise constant hazard functions. The Bayes estimators of the unknown parameters under a very general set of priors have been provided. We have further obtained the order restricted Bayesian inference also based on the order restricted priors. Extensive simulations results have been presented to show the effectiveness of the proposed estimators, and finally one data set has been analyzed for illustrative purposes.

Key Words and Phrases: Weibull distribution, exponential distribution; piecewise constant hazard function; Dirichlet gamma prior, posterior density function.

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1 Introduction

In this chapter, we consider a system consisting of n components that fail sequentially over time. The system continues to function as long as at least k $(1 \leq k \leq n)$ components remain operational. Such a system is known in the reliability literature as a k-out-of-n system. Note that the special cases $k = 1$ and $k = n$ correspond to series and parallel system, respectively. A substantial body of work has studied k-out-of-n systems under the assumption that component lifetimes are independent and identically distributed (i.i.d.) random variables: see for example Barlow and Proschan [2], Meeker and Escobar [14], Miziula and Rychlik [16], Navarro and Ryechlik [18], Rychlik and Szymkowiak [20] and see the references cited there in. Under this framework, it is implicitly assumed that the failure of one component does not affect the lifetimes of the remaining active components. However, this assumption may be unrealistic in many practical situations.

For instance, consider a vehicle equipped with a V8 engine. The car may continue to operate if only four cylinders are functioning, but it becomes inoperable if fewer than four cylinders fire. This system can be modeled as a 4-out-of-8 system. In such a setting, it is unreasonable to assume that the lifetimes of the remaining cylinders remain unchanged after a cylinder fails.

Motivated by such considerations, it is more realistic to assume that the failure of a component alters the stress levels and hence the lifetimes of the remaining active components To model such systems, sequential order statistics (SOS) were introduced, allowing the underlying lifetime distributions of surviving components to change after each failure; see for example Kamps [8, 9]. Since their introduction, SOS have been studied extensively. For general theoretical properties and applications, see Cramer and Kamps [6].

In this chapter, we adopt a formulation similar to that of Balakrishnan et al. [1], Beutner [3], and Mies and Bedbur [15]. Specifically, we assume that component lifetimes follow a proportional hazard class (PHC) of distributions with a common baseline cumulative distribution function (CDF) F, and are characterized by positive parameters $\alpha_1, \ldots, \alpha_n$. Balakrishnan et al. [1] considered the case when the baseline CDF F is known or belongs to a location-scale family of distributions. Beutner [3] and Mies and Bedbur [15] studied nonparametric and semiparametric estimation of F, respectively.

The contributions of this chapter are multifold. We consider several choices for the baseline distribution, including the exponential, Weibull, and a highly flexible piecewise constant hazard function. The exponential baseline corresponds to a constant hazard rate, while the Weibull baseline allows for either increasing or decreasing hazards depending on the shape parameter. The piecewise constant hazard model is particularly flexible, as it can approximate a wide range of hazard shapes without requiring a priori specification. Since the shape of the hazard function is largely determined by the data, we treat this model as semiparametric.

For the exponential baseline, the hazard function is fully specified as $h_0(x) = 1$. In the Weibull case, the baseline hazard takes the form $h_0(x) = \beta x^{\beta-1}$, for $\beta > 0$ is an unknown shape parameter. For piecewise constant hazard function, the baseline hazard function is of the form (9), with $\theta_1 + \ldots + \theta_M = 1$. Hence, in this case the unknown parameters are $\theta_1, \ldots, \theta_M$, with the above restriction. Note that in case of exponential distribution only $\alpha_1, \ldots, \alpha_{n-k+1}$ are estimable. Let us denote $r = n - k + 1$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$. In case of Weibull $\boldsymbol{\alpha}$ and β are estimable and similarly, in case of piecewise constant hazard function $\boldsymbol{\alpha}$ and $\theta_1, \ldots, \theta_M$, such that $\theta_1 + \ldots + \theta_M = 1$, are estimable.

In this chapter we have provided the Bayesian inference of the unknown parameters based on a very flexible set of priors. In case of exponential distribution it has been assumed that $(\alpha_1, \ldots, \alpha_r)$ has a Dirichlet-Gamma prior as in Pena and Gupta [19]. Based on this prior, the

Bayes estimators cannot be obtained in explicit forms. We can use very effective importance sampling technique to compute the Bayes estimates and the associated credible intervals of the unknown parameters.

In case of Weibull baseline distribution, it is assumed that $(\alpha_1, \ldots, \alpha_r)$ has the same prior as above and β has an independent gamma prior. In this case the Bayes estimators cannot be obtained in explicit forms, we have proposed importance sampling technique to compute the Bayes estimators and the associated credible intervals. Similarly, for piecewise constant baseline distribution, it is assumed that $(\alpha_1, \ldots, \alpha_r)$ has the same prior as above and $\theta_1, \ldots, \theta_M$ has an independent Dirichlet prior. In this case also we have proposed importance sampling technique to compute the Bayes estimators and the associated credible intervals of the unknown parameters.

In a k-out-of-n system it is very natural that the parameters α_j's are ordered, i.e. $\alpha_1 \geq \ldots \geq \alpha_r$, see for example Balakrishnan et al. [1]. In developing Bayesian inference it has been assumed that $(\alpha_1, \ldots, \alpha_r)$ has a ordered Dirichlet-Gamma distribution as in Mondal and Kundu [17]. We have developed the Bayesian inference of the unknown parameters based on importance sampling technique. Extensive simulation studies demonstrate the effectiveness of the proposed estimators. One data set has been analyzed for illustrative purposes.

The main contributions of this chapter are as follows:

(i) Bayesian inference for sequential k-out-of-n systems under exponential, Weibull, and piecewise constant baseline hazards;

(ii) the use of flexible Dirichlet–Gamma and ordered Dirichlet–Gamma priors;

(iii) development of efficient importance sampling algorithms for posterior inference; and

(iv) a comprehensive simulation study and real-data illustration.

Rest of the chapter has been organized as follows. In Section 2 we have provided some preliminaries. The likelihood function and the prior distributions have been discussed in Section 3. The Bayesian inference has been presented in Section 4. In Section 5 the simulation results have been presented. One illustrative data analysis results have been presented in Section 6. Finally in Section 7 we have concluded the manuscript.

2 Preliminaries

2.1 Proportional Hazard Class

A class of distribution functions $\{F(x;\lambda); \lambda > 0\}$ is said to form a proportional hazard class (PHC) of distribution functions if there exists a distribution function $F_0(x)$, such that

$$S(x;\lambda) = (S_0(x))^{\lambda}, \tag{1}$$

for all $x > 0$ and $\lambda > 0$. Here $S(x;\lambda) = 1 - F(x;\lambda)$, the survival function of a member of the above PHC and $S_0(x) = 1 - F_0(x)$ is the baseline survival function. From now on the class which is defined through (1) will be denoted by PHC(S_0, λ). Further, it is assumed here that $F_0(x)$ is absolutely continuous with the probability density function (PDF) $f_0(x)$. Hence, $F(x;\lambda)$ is also an absolutely continuous distribution function with the corresponding PDF as $f(x;\lambda) = \lambda(S_0(x))^{\lambda-1}f_0(x)$. It may be mentioned that the baseline distribution function $F_0(x)$ may have some parameters, and we will make it explicit whenever it is required. Note that exponential, Weibull, Pareto etc. are some examples of the PHC of distributions.

It can be easily seen that if the baseline distribution function $F_0(x)$ has the hazard function $h_0(x)$, then $h(x;\lambda)$, the hazard function of $F(x,\lambda)$ can be written as $h(x;\lambda) = \lambda h_0(x)$. This is the reason the above class of distributions is known as the PHC of distributions. Since the inception of the model by Cox [4] an extensive amount of work has been done related to Cox's PHMs. Most of the standard statistical books on survival analysis discuss this model in details, see for example Cox and Oakes [5], Therneau and Grambsch [21]. The PHC of distributions have been extended to two dimensional case also, see for example Kundu [11, 12] and the references cited therein.

2.2 Sequential Order Statistics

The definition of sequential order statistics as given by Cramer and Kamps [7] in terms of inverse distribution functions is very useful, and it is as follows.

Definition 1: Let $F_1, \ldots, F_n$ be distribution functions with $F_1^{-1}(1) \leq \cdots \leq F_n^{-1}(1)$, and let $V_1, \ldots, V_n$ be independent random variables with $V_r \sim \text{Beta}(n-r+1,1)$, $1 \leq r \leq n$. Then, the random variables

$$X_r^* = F_r^{-1}(1 - V_r R_r(X_{r-1}^*)), \quad 1 \leq r \leq n, \quad \text{with} \quad X_0^* = -\infty,$$

are called sequential order statistics (based on $F_1, \ldots, F_n$), where $R_r(\cdot) = 1 - F_r(\cdot)$ denotes the reliability function.

Assumption 1: It is assumed that $F_1, \ldots, F_n$, belong to the PHC class of distributions with baseline distribution function $F_0(x)$, i.e.

$$F_i(x) = 1 - (1 - F_0(x))^{\alpha_i}, \tag{2}$$

for a vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i > 0$ for $i = 1, \ldots, n$. Here, $\alpha_1, \ldots, \alpha_n$ are parameters. The baseline distribution function $F_0(x)$ is absolutely continuous with probability density function (PDF) $f_0(x)$, and it may be completely known or it may contain some unknown parameter(s).

Based on Assumption 1, it can be seen that the hazard function $\lambda_i(x)$ of $F_i(x)$ is of the form

$$\lambda_i(x) = \alpha_i \lambda_0(x), \quad i = 1, \ldots, n,$$

where $\lambda_0(x)$ is the hazard function of $F_0(x)$. Note that in this case the parameters α_i plays the adjustment role of the failure rate of the remaining survival components. The load on each component, when $(n-i+1)$ components are still working is α_i times the hazard function of the underlying distribution function of $F_0(x)$. It is expected that the load on the remaining components increases after each failure. Hence, there exists a natural order restriction on the parameters as $\alpha_1 \leq \ldots \leq \alpha_n$. In this manuscript we provide the MLEs of the unknown parameters in both the cases namely (i) without any order restriction and (ii) with order restriction. The joint density of the first r $(1 \leq r \leq n)$ sequential order statistics, $X_1, \ldots, X_r$ is

$$\begin{aligned} f_{X_1,\ldots,X_r}(x_1,\ldots,x_r) &= \frac{n!}{(n-r)!}\left(\prod_{j=1}^{r}\alpha_j\right)\left(\prod_{j=1}^{r-1}(1-F_0(x_j))^{m_j} f_0(x_j)\right) \\ &\quad \times(1-F_0(x_r))^{(n-r+1)\alpha_r+1} f_0(x_r); \quad x_1 \leq \ldots \leq x_r, \end{aligned} \tag{3}$$

where $m_j = (n-j+1)\alpha_j - (n-j)\alpha_{j+1} - 1$, $1 \leq j \leq r-1$. Alternatively, (3) can be written as

$$\begin{aligned} f_{X_1,\ldots,X_r}(x_1,\ldots,x_r) &= \frac{n!}{(n-r)!}\left(\prod_{j=1}^{r}\alpha_j\right)\left(\prod_{j=1}^{r-1}(S_0(x_j))^{m_j+1} h_0(x_j)\right) \\ &\quad \times(S_0(x_r))^{(n-r+1)\alpha_r} h_0(x_r); \quad x_1 \leq \ldots \leq x_r, \end{aligned} \tag{4}$$

2.3 Dirichlet-Gamma Distribution

Suppose $(U_1, \ldots, U_r)$ has the Dirichlet distribution with parameters $\boldsymbol{c} = (c_1, \ldots, c_r)$, $c_1 > 0, \ldots, c_r > 0$ and having the following PDF

$$\pi_1(u_1,\ldots,u_r|\boldsymbol{c}) = \frac{\Gamma\left(\sum_{i=1}^{r} c_i\right)}{\prod_{i=1}^{r}\Gamma(c_i)}\prod_{i=1}^{r} u_i^{c_i-1}; \tag{5}$$

for $0 < u_1, \ldots, u_r < 1, \sum_{i=1}^{r} u_i = 1$. From now on a r-variate Dirichlet distribution with the parameter vector $\boldsymbol{c}$ will be denoted by $\text{Dirichlet}_r(\boldsymbol{c})$. Suppose the random variable G has a gamma distribution with parameters $a > 0$ and $b > 0$, and it has the PDF

$$\pi_2(x|a,b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}; \qquad x > 0. \tag{6}$$

It is assumed that $(U_1, \ldots, U_r)$ and G are independent. Then the random variables $W_1 = U_1 G, \ldots, W_r = U_r G$ have the following joint PDF

$$\pi_0(w_1, \ldots, w_r|a, b, \boldsymbol{c}) = \frac{\Gamma(\overline{c})}{\Gamma(a)} (b\overline{w})^{a-\overline{c}} \times \prod_{i=1}^{r} \frac{b^{c_i}}{\Gamma(c_i)} w_i^{c_i-1} e^{-bw_i}, \tag{7}$$

where $\overline{c} = c_1 + \ldots + c_r$ and $\overline{w} = w_1 + \ldots + w_r$, is said to have r-variate Dirichlet-Gamma (DG) distribution and it will be denoted by $\mathrm{DG}_r(a, b, \boldsymbol{c})$.

This is a very flexible multivariate distribution and it was originally introduced by Pena and Gupta [19] as a conjugate prior for the Marshall-Olkin bivariate exponential distribution. Here depending on the parameters, the random variables can be independent, positively or negatively correlated. Kundu and Gupta [13] provided a very effective method of generating random samples from a Dirichlet-Gamma distribution.

2.4 Ordered Dirichlet-Gamma Distribution

Suppose $(W_1, \ldots, W_r) \sim \mathrm{DG}_r(a, b, \boldsymbol{c})$, then the ordered W_i's, say $(W_{(1)}, \ldots, W_{(r)})$, where $W_{(1)} \geq \ldots \geq W_{(r)}$ is said to have r-variate ordered Dirichlet-Gamma distribution. The joint PDF of $W_{(1)} \geq \ldots \geq W_{(r)}$ is

$$\pi_3(w_1, \ldots, w_r|a, b, \boldsymbol{c}) = \frac{b^a \Gamma(\overline{c})}{\Gamma(a) \prod_{i=1}^{r} \Gamma(c_i)} (\overline{w})^{a-\overline{c}} e^{-b\overline{w}} \times \sum_{i_1, \ldots, i_r \in \mathcal{P}} \left(w_1^{c_{i_1}-1} \ldots w_r^{c_{i_r}-1} \right), \tag{8}$$

for $0 < w_r \leq \ldots \leq w_1 < \infty$, and zero, elsewhere. Here $\mathcal{P}$ denotes the class of all permutations of $\{1, \ldots, r\}$ and it will be denoted by $\mathrm{ODG}_r(a, b, \boldsymbol{c})$.

Ordered Dirichlet-Gamma distribution was introduced by Mondal and Kundu [17] as a very flexible prior for ordered parameters. The authors have also provided a very effective method of generating random samples from a ordered Dirichlet-Gamma distribution and it has been used quite efficiently to compute Bayes estimates and associated credible intervals.

2.5 Semiparametric Model

It is assumed that the baseline hazard function $h_0(t)$ is piecewise constant, and it is based on the following: the number of constants, M and the cut points $\tau_0 = 0 < \tau_1 < \ldots \tau_{M-1} < \tau_M = \tau$ are assumed to be known. Later we will discuss how to estimate these constants. Further, it is assumed that $h_0(t)$ has the following form:

$$h_0(t) = \sum_{j=1}^{M} \theta_j I_{[\tau_{j-1}\tau_j)}(t), \tag{9}$$

where the θ_j's are unknown positive constants and $I_{[\tau_{j-1}\tau_j)}(t)$ is the indicator function, i.e.

$$I_{[\tau_{j-1}\tau_j)}(t) = \begin{cases} 1 & t \in [\tau_{j-1}, \tau_j) \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that under this set up $\theta_1, \ldots, \theta_M$ are identifiable only up to a multiplicative constant. Hence, without loss of generality, it is assumed that $\theta_1 + \cdots + \theta_M = 1$. Notice that when $M = 1$, $h_0(t)$ becomes an exponential baseline hazard. Now let us compute $S_0(t)$ and $f_0(t)$ which will be be needed later. The cumulative baseline hazard function $H_0(t)$ can be written as

$$H_0(t) = \sum_{j=1}^{M} \theta_j (\min\{t, \tau_j\} - \tau_{j-1}) I_{[\tau_{j-1}, \tau_M)}(t).$$

From now on we will use the notation $A_j = [\tau_{j-1}, \tau_j)$, for $j = 1, \ldots, M$. The cumulative hazard function can be written as

$$H_0(t) = \begin{cases} \theta_1 t & \text{if} \quad t \in A_1 \\ \theta_1\tau_1 + \theta_2(t - \tau_1) & \text{if} \quad t \in A_2 \\ \vdots & \vdots \quad \vdots \\ \sum_{j=1}^{k-1} \theta_j(\tau_j - \tau_{j-1}) + \theta_k(t - \tau_{k-1}) & \text{if} \quad t \in A_k \\ \vdots & \vdots \quad \vdots \\ \sum_{j=1}^{M-1} \theta_j(\tau_j - \tau_{j-1}) + \theta_M(t - \tau_{M-1}) & \text{if} \quad t \in A_M. \end{cases} \tag{10}$$

Once $H_0(t)$ is obtained, $S_0(t)$ and $f_0(t)$ can be obtained as

$$S_0(t) = e^{-H_0(t)} \quad \text{and} \quad f_0(t) = h_0(t)e^{-H_0(t)}.$$

3 Likelihood Functions and Prior Distributions

3.1 Likelihood Function

In this section we provide the likelihood function of the observed data for three different baseline distribution functions; namely when the baseline distribution function is (a) exponential, (b) Weibull and (c) having a piecewise constant hazard function. In case of the piecewise constant hazard function, it is assumed in this section that M and $\tau_1, \ldots, \tau_M$ are known. Note that under this setup $c_1, \ldots, c_M$ are identifiable only up to a normalizing constant. It is assumed, without loss of generality $c_1 + \ldots + c_M = 1$. In case of exponential distribution the unknown parameter vector $\boldsymbol{\theta} = (\alpha_1, \ldots, \alpha_r)$, for Weibull it is $\boldsymbol{\theta} = (\beta, \alpha_1, \ldots, \alpha_r)$ and for piecewise constant hazard function $\boldsymbol{\theta} = (c_1, \ldots, c_M, \alpha_1, \ldots, \alpha_r)$.

It is assumed that we have $\mathcal{D}$, $m \geq 1$ independent and identically distributed observations of some sequential $(n-r+1)$-out-of-n system as follows:

$$\mathcal{D} = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1r} \\ x_{21} & x_{22} & \ldots & x_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \ldots & x_{mr} \end{bmatrix}. \tag{11}$$

The corresponding sequential order statistics are denoted by

$$\begin{bmatrix} X_{11} & X_{12} & \ldots & X_{1r} \\ X_{21} & X_{22} & \ldots & X_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ X_{m1} & X_{m2} & \ldots & X_{mr} \end{bmatrix}. \tag{12}$$

Here $\boldsymbol{Y}_j = \{X_{j1}, \ldots, X_{jr}\}$ has the joint PDF (3) for $j = 1, \ldots, m$ and $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m$ are independently distributed. The following notations have been used; $x_{i0}=0$, $r_{i1}+\ldots+r_{iM}=r$

$x_{i1}<\ldots<x_{ir_{i1}}<\tau_1<x_{i(r_{i1}+1)}<\ldots,x_{i(r_{i1}+r_{i2})}<\tau_2<\ldots<x_{i(r_{i1}+\ldots+r_{iM-1})}<\ldots<x_{ir}<\tau_M$,

for $i=1,\ldots,m$.

Based on the observations $\mathcal{D}$, if we assume $x_{i0}=0$ for $i=1,\ldots,m$ and $S_0(0)=1$, then the log-likelihood function without the additive constant can be written as

$$\begin{aligned}
l(\boldsymbol{\theta}|\mathcal{D}) &= m\sum_{j=1}^{r}\ln(\alpha_j)+\sum_{j=1}^{r-1}((n-j+1)\alpha_j-(n-j)\alpha_{j+1})\sum_{i=1}^{m}\ln(S_0(x_{ij}))+\sum_{i=1}^{m}\sum_{j=1}^{r-1}\ln(h_0(x_{ij})) \\
&\quad +((n-r+1)\alpha_r)\sum_{i=1}^{m}\ln(S_0(x_{ir}))+\sum_{i=1}^{m}\ln(h_0(x_{ir})) \\
&= m\sum_{j=1}^{r}\ln(\alpha_j)+\sum_{j=1}^{r}(n-j+1)\alpha_j\sum_{i=1}^{m}(\ln(S_0(x_{ij}))-\ln(S_0(x_{i(j-1)}))+\sum_{i=1}^{m}\sum_{j=1}^{r}\ln(h_0(x_{ij})) \\
&= m\sum_{j=1}^{r}\ln(\alpha_j)-\sum_{j=1}^{r}(n-j+1)\alpha_j\sum_{i=1}^{m}(H_0(x_{ij})-H_0(x_{i(j-1)})+\sum_{i=1}^{m}\sum_{j=1}^{r}\ln(h_0(x_{ij})).
\end{aligned}$$

Now we consider three different cases separately.

Exponential Distribution

In this case for $x>0$, $h_0(x)=1$, hence, $H_0(x)=x$. Therefore, the log-likelihood function without additive constant becomes

$$l(\boldsymbol{\theta}|\mathcal{D})=m\sum_{j=1}^{r}\ln(\alpha_j)-\sum_{j=1}^{r}(n-j+1)\alpha_j\sum_{i=1}^{m}(x_{ij}-x_{i(j-1)}). \tag{13}$$

Weibull Distribution

In this case for $x>0$, $h_0(x)=\beta x^{\beta-1}$, hence $H_0(x)=x^{\beta}$. Therefore, the log-likelihood function without the additive constant becomes

$$l(\boldsymbol{\theta}|\mathcal{D})=m\sum_{j=1}^{r}\ln(\alpha_j)-\sum_{j=1}^{r}(n-j+1)\alpha_j\sum_{i=1}^{m}(x_{ij}^{\beta}-x_{i(j-1)}^{\beta})+mr\ln\beta+\beta\sum_{i=1}^{m}\sum_{j=1}^{r}\ln(x_{ij}).$$

Piecewise Constant Hazard Function Distribution

In this case

$$\begin{aligned} h_0(x_{ij}) &= \sum_{k=1}^{M} \theta_k I_{A_k}(x_{ij}) \\ H_0(x_{ij}) &= \sum_{k=1}^{M} \theta_k \{\min(x_{ij}, \tau_k) - \tau_{k-1}\} I_{A_k}(x_{ij}). \end{aligned}$$

We will write for $j = 1, \ldots, r$,

$$\sum_{i=1}^{m} H_0(x_{ij}) = \theta_1 U_{1j} + \ldots + \theta_M U_{Mj},$$

$$\sum_{i=1}^{m} (H_0(x_{ij}) - H_0(x_{i(j-1)}) = \theta_1 V_{1j} + \ldots + \theta_M V_{Mj},$$

and

$$\sum_{i=1}^{m} \sum_{j=1}^{r} \ln(h_0(x_{ij})) = n_1 \ln \theta_1 + \ldots + n_M \ln \theta_M.$$

The exact expressions of U_{ij}'s, V_{ij}'s and n_j's are provided in the Appendix A. Therefore, the log-likelihood function without the additive constant becomes

$$l(\boldsymbol{\theta}|\mathcal{D}) = m \sum_{j=1}^{r} \ln(\alpha_j) - \sum_{j=1}^{r} (n - j + 1)\alpha_j \sum_{k=1}^{M} \theta_k V_{kj} + \sum_{k=1}^{M} n_k \ln \theta_k.$$

3.2 Prior Distribution

In case of exponential distribution for the unordered case it is assumed that $\boldsymbol{\alpha} \sim \text{DG}_r(a_1, b_1, \boldsymbol{c})$ and for the ordered case $\boldsymbol{\alpha} \sim \text{ODG}_r(a_1, b_1, \boldsymbol{c})$, where $a_1 > 0, b_1 > 0, c_1 > 0, \ldots, c_r > 0$ and a_1, b_1 and $\boldsymbol{c} = (c_1, \ldots, c_r)$, are hyper-parameters. In case of Weibull distribution it is assumed that $\boldsymbol{\alpha} \sim \text{DG}_r(a_1, b_1, \boldsymbol{c})$, $\beta \sim \text{Gamma}(a_2, b_2)$; $a_2 > 0, b_2 > 0$ and they are independently distributed. For the ordered case it is assumed that $\boldsymbol{\alpha} \sim \text{ODG}_r(a_1, b_1, \boldsymbol{c})$, $\beta \sim \text{Gamma}(a_2, b_2)$ and they are independently distributed. In case of piecewise constant hazard function, for the unordered case, $\boldsymbol{\alpha} \sim \text{DG}_r(a_1, b_1, \boldsymbol{c})$ as before, and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_M) \sim \text{Dirichlet}_M(\boldsymbol{d})$, where $d_1 > 0, \ldots, d_M > 0$, $\boldsymbol{d} = (d_1, \ldots, d_M)$ and they are independently distributed. For the ordered case, $\boldsymbol{\alpha} \sim \text{ODG}_r(a_1, b_1, \boldsymbol{c})$ as before, and $\boldsymbol{\theta} \sim \text{Dirichlet}_M(\boldsymbol{d})$ and they are independently distributed.

4 Bayes Estimators

4.1 Exponential Distribution

In this case based on the prior on $\boldsymbol{\alpha}$ for the unordered case as given in Section 3.2, the joint posterior distribution of $\boldsymbol{\alpha}$ given the observation $\mathcal{D}$ becomes

$$\pi_{EXU}(\boldsymbol{\alpha}|\mathcal{D}) \propto \overline{\alpha}^{\{a_1-\overline{c}\}} \left\{ \prod_{j=1}^{r} \alpha_j^{m+c_j-1} e^{-(b_1+D_j)\alpha_j} \right\}, \tag{14}$$

here $\overline{\alpha} = \alpha_1 + \ldots + \alpha_r$, $\overline{c} = c_1 + \ldots + c_r$ and $D_j = (n-j+1)\sum_{i=1}^{m}(x_{ij} - x_{i(j-1)})$. A very convenient importance sampling technique can be used to compute the Bayes estimates and the associated credible intervals of the unknown parameters.

In case of the ordered case, based on the prior as given in Section 3.2, the joint posterior distribution of $\boldsymbol{\alpha}$ given the observation $\mathcal{D}$ becomes

$$\pi_{EXO}(\boldsymbol{\alpha}|\mathcal{D}) \propto \overline{\alpha}^{\{a_1-\overline{c}\}} \sum_{i_1,\ldots,i_r \in \mathcal{P}} \left(\alpha_1^{c_{i_1}-1} \ldots \alpha_r^{c_{i_r}-1}\right) \left\{ \prod_{j=1}^{r} \alpha_j^{m} e^{-(b_1+D_j)\alpha_j} \right\}. \tag{15}$$

Here, $\overline{\alpha}$, $\overline{c}$ and D_j are same as defined above. In this case also, a very convenient importance sampling technique can be used to compute the Bayes estimates and the associated credible intervals of the unknown parameters. The explicit algorithms have been provided in the Appendix B.

4.2 Weibull Distribution

In this case based on the priors on $\boldsymbol{\alpha}$ and β given in Section 3.2, the joint posterior distribution of $\boldsymbol{\alpha}$ and β for the unordered case, given the observation $\mathcal{D}$ becomes

$$\pi_{WEU}(\boldsymbol{\alpha}, \beta|\mathcal{D}) \propto \overline{\alpha}^{\{a_1-\overline{c}\}} \left\{ \prod_{j=1}^{r} \alpha_j^{m+c_j-1} e^{-(b_1+D_j(\beta))\alpha_j} \right\} \beta^{mr+a_2-1} e^{-\beta(b-\sum_{i=1}^{m}\sum_{j=1}^{r} \ln x_{ij})}, \quad (16)$$

here $\overline{\alpha} = \alpha_1 + \ldots + \alpha_r$, $\overline{c} = c_1 + \ldots + c_r$ and $D_j(\beta) = (n-j+1)\sum_{i=1}^{m}(x_{ij}^{\beta} - x_{i(j-1)}^{\beta})$. The joint posterior distribution of $\boldsymbol{\alpha}$ and β for the ordered case based on the priors as given in Section 3.2, becomes

$$\begin{aligned} \pi_{WEO}(\boldsymbol{\alpha}, \beta|\mathcal{D}) \;\; \propto \;\; & \overline{\alpha}^{\{a_1-\overline{c}\}} \sum_{i_1,\ldots,i_r \in \mathcal{P}} \left(\alpha_1^{c_{i_1}-1} \ldots \alpha_r^{c_{i_r}-1}\right) \\ & \times \left\{ \prod_{j=1}^{r} \alpha_j^{m} e^{-(b_1+D_j(\beta))\alpha_j} \right\} \beta^{mr+a_2-1} e^{-\beta(b-\sum_{i=1}^{m}\sum_{j=1}^{r} \ln x_{ij})}. \end{aligned} \quad (17)$$

In this case also, a very convenient importance sampling technique can be used to compute the Bayes estimates and the associated credible intervals of the unknown parameters. The explicit algorithms have been provided in the Appendix C.

4.3 Piecewise Constant Hazard Function

In this case based on the priors on $\boldsymbol{\alpha}$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_M)$ in Section 3.2, the joint posterior distribution of $\boldsymbol{\alpha}$ and $\boldsymbol{\theta}$ for the unordered case, given the observation $\mathcal{D}$ becomes

$$\pi_{PCU}(\boldsymbol{\alpha}, \boldsymbol{\theta}|\mathcal{D}) \propto \overline{\alpha}^{\{a_1-\overline{c}\}} \left\{ \prod_{j=1}^{r} \alpha_j^{m+c_j-1} e^{-(b_1+W_j(\boldsymbol{\theta}))\alpha_j} \right\} \prod_{k=1}^{M} \theta_k^{n_k+d_k-1}, \quad (18)$$

here $\overline{\alpha} = \alpha_1 + \ldots + \alpha_r$, $\overline{c} = c_1 + \ldots + c_r$ and $W_j(\boldsymbol{\theta}) = (n-j+1)\sum_{k=1}^{M}\theta_k V_{kj}$. The joint posterior distribution of $\boldsymbol{\alpha}$ and $\boldsymbol{\theta}$ for the ordered case based on the priors as given in Section 3.2, becomes

$$\pi_{PCO}(\boldsymbol{\alpha}, \boldsymbol{\theta}|\mathcal{D}) \;\; \propto \;\; \overline{\alpha}^{\{a_1-\overline{c}\}} \sum_{i_1,\ldots,i_r \in \mathcal{P}} \left(\alpha_1^{c_{i_1}-1} \ldots \alpha_r^{c_{i_r}-1}\right) \times$$

$$\left\{\prod_{j=1}^{r} \alpha_j^m e^{-(b_1+W_j(\boldsymbol{\theta}))\alpha_j}\right\} \prod_{k=1}^{M} \theta_k^{n_k+d_k-1}. \tag{19}$$

A very convenient importance sampling technique can be used to compute the Bayes estimates and the associated credible intervals of the unknown parameters. The explicit algorithms have been provided in the Appendix D.

5 Simulations

The simulation study is designed to evaluate the performance of the proposed Bayes estimators under different baseline distributions and system configurations. A comprehensive study is carried out for three baseline distributions namely: exponential, Weibull and piecewise constant hazard (PCH), under (i) 3-out-of-5 system and (ii) 4-out-of-8 system structures. For each scenario, the Bayes estimates and their corresponding credible intervals are computed using 1,000 replications, with 10,000 importance samples generated per replication. The analysis is conducted for sample sizes $m = 15, 25$, and 30. The true parameter values are taken as $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, and $\alpha_3 = 1$ for the 3-out-of-5 system, and $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$, and $\alpha_4 = 1$ for the 4-out-of-8 system.

For the exponential baseline distribution, the hyper-parameters are chosen as $a_1 = 0.001$, $b_1 = 15$, and $\mathbf{c} = (0.5, 0.75, 1)$ for the 3-out-of-5 system, while $\mathbf{c} = (0.5, 0.6, 0.8, 1)$ is used for the 4-out-of-8 system. For the Weibull distribution, the shape parameter is fixed at $\beta = 1.5$. The same hyper-parameters a_1, b_1, and $\mathbf{c}$ as in the exponential case are adopted, along with additional hyper-parameters $a_2 = 30$ and $b_2 = 2$ corresponding to the shape parameter. For the PCH model with $M = 2$, the change point is set at $\tau_1 = 0.7$ for the 3-out-of-5 system and at $\tau_1 = 0.5$ for the 4-out-of-8 system. The baseline parameters are specified as $\boldsymbol{\theta} = (0.7, 0.3)$ for the 3-out-of-5 system and $\boldsymbol{\theta} = (0.3, 0.7)$ for the 4-out-of-8 system. The hyper-parameters are fixed at $a_1 = 0.001$, $b_1 = 15$, $\boldsymbol{d} = (1, 1)$, and the same $\boldsymbol{c}$ vectors used as in the exponential case. For the PCH model with $M = 3$, the hyper-parameters structure remains the same except that $\boldsymbol{d} = (1, 1, 1)$. The change points are taken as $(\tau_1, \tau_2) = (0.7, 2)$ for the 3-out-of-5 system and $(\tau_1, \tau_2) = (0.5, 1)$ for the 4-out-of-8 system.

For all three baseline distributions, the Bayes estimators do not have closed-form expressions and are therefore computed numerically. In every case, the mean squared error decreases as the sample size increases. A similar pattern is observed for the credible intervals: the average lengths of both the HPD and symmetric credible intervals become shorter with larger sample sizes, and the HPD intervals are consistently shorter than their symmetric counterparts. The coverage probabilities remain close to the nominal levels across all scenarios. Furthermore, the mean squared errors for the ordered case are smaller than those for the unordered case, and the corresponding interval lengths are also shorter in the ordered setting.

Table 1: The average estimates (the associated mean squared error) for exponential baseline based on m independent observations from a 3 out of 5 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.75$ and $\alpha_3 = 1$

m	Unordered			Ordered		
	α_1	α_2	α_3	α_1	α_2	α_3
15	0.6062	0.7148	0.8689	0.5351	0.7109	0.9400
	(0.0207)	(0.0099)	(0.0257)	(0.0044)	(0.0048)	(0.0089)
25	0.6300	0.7541	0.9150	0.5874	0.7505	0.9585
	(0.0200)	(0.0063)	(0.0147)	(0.0040)	(0.0026)	(0.0065)
30	0.6336	0.7515	0.9177	0.5966	0.7517	0.9522
	(0.0127)	(0.0054)	(0.0128)	(0.0026)	(0.0022)	(0.0064)

Table 2: The AL (CP) of 95% Symmetric intervals for exponential baseline based on m independent observations from a 3 out of 5 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.75$ and $\alpha_3 = 1$

m	Unordered			Ordered		
	α_1	α_2	α_3	α_1	α_2	α_3
15	0.6047	0.7088	0.8576	0.4495	0.5185	0.7528
	(0.97)	(0.97)	(0.98)	(0.98)	(0.98)	(0.98)
25	0.4895	0.5834	0.7065	0.3899	0.4379	0.6202
	(0.97)	(0.96)	(0.97)	(0.97)	(0.98)	(0.98)
30	0.4500	0.5325	0.6482	0.3636	0.4059	0.5723
	(0.98)	(0.96)	(0.98)	(0.96)	(0.98)	(0.97)

Table 3: The AL (CP) of 95% HPD intervals for exponential baseline based on m independent observations from a 3 out of 5 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.75$ and $\alpha_3 = 1$

m	Unordered			Ordered		
	α_1	α_2	α_3	α_1	α_2	α_3
15	0.5937	0.6951	0.8414	0.4460	0.5112	0.7319
	(0.98)	(0.97)	(0.98)	(0.97)	(0.97)	(0.98)
25	0.4836	0.5764	0.6979	0.3878	0.4335	0.6070
	(0.98)	(0.98)	(0.97)	(0.98)	(0.99)	(0.98)
30	0.4450	0.5267	0.6416	0.3618	0.4022	0.5613
	(0.97)	(0.98)	(0.98)	(0.98)	(0.99)	(0.99)

Table 4: The average estimates (the associated mean squared error) for exponential baseline based on m independent observations from a 4 out of 8 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$ and $\alpha_4 = 1$

m	Unordered				Ordered			
	α_1	α_2	α_3	α_4	α_1	α_2	α_3	α_4
15	0.6134	0.6907	0.8076	0.8753	0.5191	0.6602	0.8011	1.0029
	(0.0238)	(0.0268)	(0.0087)	(0.0257)	(0.0072)	(0.0064)	(0.0029)	(0.0049)
25	0.6337	0.7274	0.8464	0.9247	0.5731	0.7067	0.8354	1.0134
	(0.0218)	(0.0226)	(0.0083)	(0.0222)	(0.0052)	(0.0059)	(0.0029)	(0.0031)
30	0.5957	0.6922	0.7982	0.8663	0.5509	0.6717	0.7869	0.9401
	(0.0138)	(0.0129)	(0.0045)	(0.0217)	(0.0043)	(0.0044)	(0.0014)	(0.0033)

Table 5: The AL (CP) for 95% Symmetric intervals for exponential baseline based on m independent observations from a 4 out of 8 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$ and $\alpha_4 = 1$

m	Unordered				Ordered			
	α_1	α_2	α_3	α_4	α_1	α_2	α_3	α_4
15	0.6106	0.6861	0.7983	0.8603	0.4168	0.4372	0.5205	0.7402
	(0.97)	(0.98)	(0.97)	(0.97)	(0.96)	(0.98)	(0.98)	(0.97)
25	0.4919	0.5632	0.6540	0.7131	0.3656	0.3750	0.4363	0.6074
	(0.98)	(0.98)	(0.98)	(0.98)	(0.98)	(0.96)	(0.97)	(0.98)
30	0.4229	0.4901	0.5640	0.6112	0.3234	0.3292	0.3788	0.5174
	(0.98)	(0.96)	(0.97)	(0.97)	(0.98)	(0.98)	(0.97)	(0.98)

Table 6: The AL (CP) for 95% HPD intervals for exponential baseline based on m independent observations from a 4 out of 8 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$ and $\alpha_4 = 1$

m	Unordered				Ordered			
	α_1	α_2	α_3	α_4	α_1	α_2	α_3	α_4
15	0.5993	0.6735	0.7841	0.8443	0.4141	0.4328	0.5120	0.7186
	(0.98)	(0.96)	(0.98)	(0.98)	(0.97)	(0.96)	(0.98)	(0.98)
25	0.4860	0.5565	0.6458	0.7044	0.3640	0.3721	0.4309	0.5933
	(0.97)	(0.98)	(0.98)	(0.98)	(0.98)	(0.97)	(0.97)	(0.97)
30	0.4185	0.4850	0.5583	0.6044	0.3221	0.3270	0.3747	0.5066
	(0.97)	(0.97)	(0.98)	(0.98)	(0.97)	(0.98)	(0.97)	(0.98)

Table 7: The average estimates (the associated mean squared error) for Weibull baseline based on m independent observations from a 3 out of 5 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, $\alpha_3 = 1$ and $\beta = 1.5$

m	Unordered				Ordered			
	α_1	α_2	α_3	β	α_1	α_2	α_3	β
15	0.5678	0.6554	0.7730	1.5829	0.4964	0.6502	0.8451	1.5784
	(0.0269)	(0.0143)	(0.0573)	(0.0534)	(0.0107)	(0.0110)	(0.0267)	(0.0527)
25	0.6221	0.7382	0.9249	1.4893	0.5785	0.7423	0.9602	1.4865
	(0.0254)	(0.0051)	(0.0172)	(0.0270)	(0.0070)	(0.0025)	(0.0079)	(0.0272)
30	0.6337	0.7559	0.9131	1.5333	0.5962	0.7536	0.9497	1.5309
	(0.0202)	(0.0045)	(0.0135)	(0.0260)	(0.0052)	(0.0017)	(0.0065)	(0.0158)

Table 8: The AL (CP) for 95% Symmetric intervals for Weibull baseline based on m independent observations from a 3 out of 5 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, $\alpha_3 = 1$ and $\beta = 1.5$

m	Unordered				Ordered			
	α_1	α_2	α_3	β	α_1	α_2	α_3	β
15	0.5941	0.6510	0.7635	0.7533	0.4232	0.4735	0.6667	0.7530
	(0.98)	(0.97)	(0.95)	(0.91)	(0.97)	(0.97)	(0.97)	(0.92)
25	0.5163	0.5722	0.7150	0.5692	0.4011	0.4432	0.6334	0.5690
	(0.97)	(0.96)	(0.97)	(0.92)	(0.98)	(0.96)	(0.98)	(0.92)
30	0.4792	0.5371	0.6464	0.5270	0.3802	0.4111	0.5692	0.5263
	(0.96)	(0.97)	(0.96)	(0.91)	(0.97)	(0.96)	(0.98)	(0.92)

Table 9: The AL (CP) for 95% HPD intervals for Weibull baseline based on m independent observations from a 3 out of 5 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, $\alpha_3 = 1$ and $\beta = 1.5$

m	Unordered				Ordered			
	α_1	α_2	α_3	β	α_1	α_2	α_3	β
15	0.5813	0.6390	0.7495	0.7482	0.4192	0.4671	0.6483	0.7472
	(0.98)	(0.98)	(0.92)	(0.91)	(0.98)	(0.96)	(0.96)	(0.93)
25	0.5082	0.5653	0.7062	0.5664	0.3990	0.4391	0.6192	0.5664
	(0.98)	(0.98)	(0.95)	(0.91)	(0.98)	(0.97)	(0.97)	(0.91)
30	0.4734	0.5312	0.6391	0.5243	0.3783	0.4081	0.5582	0.5230
	(0.97)	(0.96)	(0.95)	(0.92)	(0.96)	(0.97)	(0.98)	(0.92)

Table 10: The average estimates (the associated mean squared error) for Weibull baseline based on m independent observations from a 4 out of 8 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$, $\alpha_4 = 1$ and $\beta = 1.5$

m	Unordered					Ordered				
	α_1	α_2	α_3	α_4	β	α_1	α_2	α_3	α_4	β
15	0.5897	0.6695	0.7689	0.9305	1.4771	0.5104	0.6509	0.7901	0.9928	1.4772
	(0.0109)	(0.0113)	(0.0184)	(0.0145)	(0.0252)	(0.0005)	(0.0072)	(0.0074)	(0.0038)	(0.0252)
25	0.5772	0.6301	0.7095	0.8079	1.5703	0.5156	0.6203	0.7208	0.8622	1.5702
	(0.0085)	(0.0097)	(0.0118)	(0.0108)	(0.0220)	(0.0005)	(0.0070)	(0.0069)	(0.0025)	(0.0221)
30	0.5813	0.6766	0.7869	0.8890	1.4702	0.5435	0.6655	0.7820	0.9357	1.4702
	(0.0079)	(0.0089)	(0.0050)	(0.0095)	(0.0145)	(0.0002)	(0.0048)	(0.0014)	(0.0023)	(0.0145)

Table 11: The AL (CP) for 95% Symmetric intervals for Weibull baseline based on m independent observations from a 4 out of 8 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$, $\alpha_4 = 1$ and $\beta = 1.5$

m	Unordered					Ordered				
	α_1	α_2	α_3	α_4	β	α_1	α_2	α_3	α_4	β
15	0.6341	0.6760	0.7673	0.9071	0.5782	0.4282	0.4463	0.5211	0.7414	0.5790
	(0.97)	(0.98)	(0.98)	(0.97)	(0.95)	(0.98)	(0.97)	(0.97)	(0.98)	(0.94)
25	0.4751	0.4960	0.5514	0.6192	0.4653	0.3293	0.3272	0.3691	0.5052	0.4629
	(0.98)	(0.98)	(0.97)	(0.96)	(0.91)	(0.97)	(0.97)	(0.98)	(0.97)	(0.92)
30	0.4455	0.4862	0.5609	0.6248	0.4133	0.3382	0.3375	0.3823	0.5181	0.4129
	(0.98)	(0.98)	(0.96)	(0.98)	(0.92)	(0.98)	(0.98)	(0.96)	(0.98)	(0.92)

Table 12: The AL (CP) for 95% HPD intervals for Weibull baseline based on m independent observations from a 4 out of 8 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$, $\alpha_4 = 1$ and $\beta = 1.5$

m	Unordered					Ordered				
	α_1	α_2	α_3	α_4	β	α_1	α_2	α_3	α_4	β
15	0.6192	0.6631	0.7523	0.8911	0.5754	0.4251	0.4413	0.5144	0.7203	0.5750
	(0.98)	(0.97)	(0.98)	(0.98)	(0.94)	(0.97)	(0.97)	(0.98)	(0.98)	(0.95)
25	0.4631	0.4843	0.5582	0.6671	0.4573	0.3444	0.3512	0.3903	0.5884	0.4560
	(0.98)	(0.98)	(0.96)	(0.97)	(0.92)	(0.98)	(0.97)	(0.97)	(0.98)	(0.93)
30	0.4386	0.4811	0.5330	0.6186	0.4113	0.3362	0.3359	0.3783	0.5084	0.4107
	(0.98)	(0.98)	(0.98)	(0.97)	(0.91)	(0.97)	(0.98)	(0.98)	(0.98)	(0.91)

Table 13: The average estimates (the associated mean squared error) for PCH($M = 2$) baseline based on m independent observations from a 3 out of 5 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, $\alpha_3 = 1$ and $\theta_1 = 0.7$

m	Unordered				Ordered			
	α_1	α_2	α_3	θ_1	α_1	α_2	α_3	θ_1
15	0.8233	0.8387	0.8715	0.7040	0.6554	0.8204	1.0176	0.7040
	(0.1058)	(0.0090)	(0.0175)	(0.0067)	(0.0244)	(0.0053)	(0.0108)	(0.0066)
25	0.7088	0.7212	0.7382	0.7060	0.5971	0.7099	0.8392	0.7060
	(0.0539)	(0.0011)	(0.0118)	(0.0044)	(0.0195)	(0.0017)	(0.0060)	(0.0044)
30	0.7256	0.7390	0.7566	0.7026	0.6227	0.7293	0.8502	0.7024
	(0.0512)	(0.0005)	(0.0095)	(0.0036)	(0.0151)	(0.0005)	(0.0026)	(0.0036)

Table 14: The AL (CP) for 95% Symmetric intervals for PCH($M = 2$) baseline based on m independent observations from a 3 out of 5 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, $\alpha_3 = 1$ and $\theta_1 = 0.7$

m	Unordered				Ordered			
	α_1	α_2	α_3	θ_1	α_1	α_2	α_3	θ_1
15	0.8233	0.8319	0.8579	0.2522	0.5300	0.5637	0.7377	0.2525
	(0.95)	(0.98)	(0.98)	(0.91)	(0.98)	(0.98)	(0.98)	(0.91)
25	0.5526	0.5574	0.5688	0.1991	0.3683	0.3776	0.4743	0.1989
	(0.96)	(0.98)	(0.98)	(0.90)	(0.98)	(0.98)	(0.98)	(0.90)
30	0.5162	0.5236	0.5340	0.1833	0.3501	0.3525	0.4409	0.1836
	(0.97)	(0.97)	(0.98)	(0.91)	(0.97)	(0.98)	(0.98)	(0.91)

Table 15: The AL (CP) for 95% HPD intervals for PCH($M = 2$) baseline based on m independent observations from a 3 out of 5 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, $\alpha_3 = 1$ and $\theta_1 = 0.7$

m	Unordered				Ordered			
	α_1	α_2	α_3	θ_1	α_1	α_2	α_3	θ_1
15	0.7995	0.8081	0.8330	0.2479	0.5189	0.5484	0.7092	0.2481
	(0.97)	(0.98)	(0.98)	(0.90)	(0.98)	(0.98)	(0.98)	(0.91)
25	0.5406	0.5459	0.5564	0.1962	0.3622	0.3692	0.4604	0.1957
	(0.96)	(0.97)	(0.98)	(0.91)	(0.98)	(0.98)	(0.98)	(0.91)
30	0.5060	0.5138	0.5233	0.1807	0.3446	0.3458	0.4291	0.1809
	(0.97)	(0.98)	(0.97)	(0.91)	(0.98)	(0.97)	(0.98)	(0.92)

Table 16: The average estimates (the associated mean squared error) for PCH($M = 2$) baseline based on m independent observations from a 4 out of 8 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$, $\alpha_4 = 1$ and $\theta_1 = 0.3$

m	Unordered					Ordered				
	α_1	α_2	α_3	α_4	θ_1	α_1	α_2	α_3	α_4	θ_1
15	0.6769	0.6838	0.7012	0.7115	0.3184	0.5106	0.6212	0.7243	0.8680	0.3178
	(0.0615)	(0.0272)	(0.0099)	(0.0834)	(0.0058)	(0.0100)	(0.0105)	(0.0057)	(0.0175)	(0.0058)
25	0.7383	0.7462	0.7585	0.7653	0.3165	0.5973	0.6948	0.7825	0.9019	0.3166
	(0.0569)	(0.0215)	(0.0088)	(0.0752)	(0.0031)	(0.0094)	(0.0090)	(0.0043)	(0.0096)	(0.0031)
30	0.7273	0.7336	0.7463	0.7500	0.3158	0.5999	0.6888	0.7677	0.8744	0.3158
	(0.0518)	(0.0179)	(0.0029)	(0.0626)	(0.0023)	(0.0080)	(0.0079)	(0.0010)	(0.0058)	(0.0023)

Table 17: The AL (CP) for 95% Symmetric intervals for PCH($M = 2$) baseline based on m independent observations from a 4 out of 8 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$, $\alpha_4 = 1$ and $\theta_1 = 0.3$

m	Unordered					Ordered				
	α_1	α_2	α_3	α_4	θ_1	α_1	α_2	α_3	α_4	θ_1
15	0.6703	0.6806	0.6912	0.6970	0.2257	0.3914	0.3904	0.4353	0.5760	0.2273
	(0.98)	(0.97)	(0.98)	(0.98)	(0.91)	(0.98)	(0.97)	(0.97)	(0.98)	(0.91)
25	0.5730	0.5764	0.5850	0.5896	0.1781	0.3493	0.3340	0.3649	0.4715	0.1782
	(0.98)	(0.98)	(0.97)	(0.97)	(0.91)	(0.97)	(0.97)	(0.98)	(0.98)	(0.91)
30	0.5147	0.5189	0.5278	0.5279	0.1635	0.3187	0.3032	0.3272	0.4195	0.1629
	(0.97)	(0.98)	(0.98)	(0.98)	(0.92)	(0.98)	(0.98)	(0.97)	(0.97)	(0.92)

Table 18: The AL (CP) for 95% HPD intervals for PCH($M = 2$) baseline based on m independent observations from a 4 out of 8 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$, $\alpha_4 = 1$ and $\theta_1 = 0.3$

m	Unordered					Ordered				
	α_1	α_2	α_3	α_4	θ_1	α_1	α_2	α_3	α_4	θ_1
15	0.6514	0.6617	0.6716	0.6787	0.2227	0.3836	0.3804	0.4215	0.5531	0.2233
	(0.97)	(0.98)	(0.98)	(0.97)	(0.92)	(0.97)	(0.97)	(0.98)	(0.98)	(0.91)
25	0.5611	0.5647	0.5716	0.5777	0.1758	0.3435	0.3274	0.3559	0.4566	0.1759
	(0.98)	(0.97)	(0.97)	(0.97)	(0.93)	(0.97)	(0.98)	(0.97)	(0.97)	(0.92)
30	0.5051	0.5087	0.5169	0.5177	0.1616	0.3139	0.2968	0.3199	0.4065	0.1610
	(0.98)	(0.98)	(0.98)	(0.98)	(0.92)	(0.98)	(0.97)	(0.98)	(0.98)	(0.92)

Table 19: The average estimates (the associated mean squared error) for PCH ($M = 3$) based on m independent observations from a 3 out of 5 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, $\alpha_3 = 1$ and $\theta_1 = 0.7$, $\theta_2 = 0.2$

m	Unordered					Ordered				
	α_1	α_2	α_3	θ_1	θ_2	α_1	α_2	α_3	θ_1	θ_2
15	0.8032	0.8180	0.8500	0.6911	0.1810	0.6393	0.8000	0.9912	0.6911	0.1809
	(0.0932)	(0.0056)	(0.0533)	(0.0071)	(0.0037)	(0.0196)	(0.0028)	(0.0205)	(0.0071)	(0.0037)
25	0.7311	0.7431	0.7616	0.7015	0.1761	0.6160	0.7325	0.8660	0.7017	0.1760
	(0.0538)	(0.0040)	(0.0471)	(0.0036)	(0.0029)	(0.0135)	(0.0014)	(0.0181)	(0.0036)	(0.0029)
30	0.7157	0.7305	0.7459	0.6994	0.1758	0.6148	0.7205	0.8397	0.6993	0.1760
	(0.0468)	(0.0027)	(0.0448)	(0.0035)	(0.0023)	(0.0132)	(0.0009)	(0.0158)	(0.0035)	(0.0023)

Table 20: The AL (CP) of 95% Symmetric intervals for PCH ($M = 3$) based on m independent observations from a 3 out of 5 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, $\alpha_3 = 1$ and $\theta_1 = 0.7$, $\theta_2 = 0.2$

m	Unordered					Ordered				
	α_1	α_2	α_3	θ_1	θ_2	α_1	α_2	α_3	θ_1	θ_2
15	0.8018	0.8125	0.8318	0.2528	0.2092	0.5165	0.5482	0.7169	0.2521	0.2085
	(0.93)	(0.94)	(0.94)	(0.92)	(0.92)	(0.97)	(0.97)	(0.98)	(0.92)	(0.91)
25	0.5691	0.5736	0.5859	0.1992	0.1643	0.3803	0.3871	0.4890	0.1983	0.1644
	(0.94)	(0.93)	(0.95)	(0.93)	(0.92)	(0.98)	(0.97)	(0.97)	(0.92)	(0.92)
30	0.5071	0.5177	0.5256	0.1835	0.1510	0.3453	0.3485	0.4364	0.1829	0.1517
	(0.94)	(0.95)	(0.94)	(0.92)	(0.93)	(0.97)	(0.98)	(0.98)	(0.93)	(0.93)

Table 21: The AL (CP) of 95% HPD intervals for PCH ($M = 3$) based on m independent observations from a 3 out of 5 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, $\alpha_3 = 1$ and $\theta_1 = 0.7$, $\theta_2 = 0.2$

m	Unordered					Ordered				
	α_1	α_2	α_3	θ_1	θ_2	α_1	α_2	α_3	θ_1	θ_2
15	0.7791	0.7903	0.8092	0.2486	0.2042	0.5054	0.5320	0.6907	0.2473	0.2029
	(0.94)	(0.94)	(0.95)	(0.93)	(0.92)	(0.98)	(0.97)	(0.97)	(0.92)	(0.92)
25	0.5571	0.5617	0.5742	0.1964	0.1613	0.3741	0.3787	0.4743	0.1956	0.1612
	(0.94)	(0.95)	(0.96)	(0.92)	(0.92)	(0.97)	(0.97)	(0.98)	(0.93)	(0.93)
30	0.4968	0.5074	0.5154	0.1811	0.1483	0.3399	0.3418	0.4244	0.1805	0.1488
	(0.95)	(0.95)	(0.95)	(0.92)	(0.93)	(0.98)	(0.98)	(0.98)	(0.92)	(0.93)

Table 22: The average estimates (the associated mean squared error) for PCH($M = 3$) baseline based on m independent observations from a 4 out of 8 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$, $\alpha_4 = 1$, $\theta_1 = 0.3$ and $\theta_2 = 0.3$

m	Unordered						Ordered					
	α_1	α_2	α_3	α_4	θ_1	θ_2	α_1	α_2	α_3	α_4	θ_1	θ_2
15	0.8309	0.8420	0.8612	0.8723	0.3239	0.2970	0.6273	0.7634	0.8894	1.0662	0.3240	0.2969
	(0.1100)	(0.0590)	(0.0141)	(0.0566)	(0.0063)	(0.0035)	(0.0123)	(0.0268)	(0.0081)	(0.0046)	(0.0063)	(0.0035)
25	0.7501	0.7567	0.7711	0.7775	0.3158	0.2915	0.6068	0.7058	0.7946	0.9157	0.3161	0.2914
	(0.0628)	(0.0448)	(0.0110)	(0.0496)	(0.0035)	(0.0022)	(0.0104)	(0.0192)	(0.0071)	(0.0032)	(0.0035)	(0.0022)
30	0.7684	0.7773	0.7910	0.7967	0.3111	0.2924	0.6360	0.7299	0.8134	0.9265	0.3112	0.2925
	(0.0522)	(0.0316)	(0.0103)	(0.0415)	(0.0027)	(0.0021)	(0.095)	(0.0169)	(0.0002)	(0.0025)	(0.0026)	(0.0021)

Table 23: The AL (CP) for 95% Symmetric intervals for PCH($M = 2$) baseline based on m independent observations from a 4 out of 8 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$, $\alpha_4 = 1$, $\theta_1 = 0.3$ and $\theta_2 = 0.3$

m	Unordered						Ordered					
	α_1	α_2	α_3	α_4	θ_1	θ_2	α_1	α_2	α_3	α_4	θ_1	θ_2
15	0.8259	0.8352	0.8517	0.8539	0.2246	0.2209	0.4805	0.4778	0.5352	0.7113	0.2246	0.2206
	(0.96)	(0.98)	(0.96)	(0.97)	(0.95)	(0.94)	(0.97)	(0.98)	(0.97)	(0.96)	(0.95)	(0.95)
25	0.5851	0.5881	0.5934	0.5967	0.1761	0.1733	0.3532	0.3393	0.3705	0.4785	0.1767	0.1739
	(0.97)	(0.97)	(0.98)	(0.97)	(0.95)	(0.95)	(0.96)	(0.98)	(0.98)	(0.97)	(0.95)	(0.95)
30	0.5467	0.5534	0.5570	0.5599	0.1613	0.1586	0.3362	0.3199	0.3483	0.4436	0.1616	0.1593
	(0.96)	(0.97)	(0.97)	(0.96)	(0.96)	(0.94)	(0.98)	(0.97)	(0.98)	(0.97)	(0.95)	(0.95)

Table 24: The AL (CP) for 95% HPD intervals for PCH($M = 2$) baseline based on m independent observations from a 4 out of 8 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$, $\alpha_4 = 1$, $\theta_1 = 0.3$ and $\theta_2 = 0.3$

m	Unordered						Ordered					
	α_1	α_2	α_3	α_4	θ_1	θ_2	α_1	α_2	α_3	α_4	θ_1	θ_2
15	0.8040	0.8103	0.8266	0.8301	0.2214	0.2175	0.4706	0.4659	0.5183	0.6829	0.2214	0.2173
	(0.97)	(0.97)	(0.97)	(0.98)	(0.95)	(0.95)	(0.98)	(0.98)	(0.96)	(0.97)	(0.94)	(0.95)
25	0.5725	0.5745	0.5814	0.5834	0.1741	0.1712	0.3476	0.3323	0.3621	0.4636	0.1743	0.1717
	(0.96)	(0.98)	(0.96)	(0.98)	(0.94)	(0.94)	(0.97)	(0.97)	(0.98)	(0.97)	(0.94)	(0.95)
30	0.5360	0.5417	0.5459	0.5489	0.1595	0.1568	0.3308	0.3138	0.3403	0.4311	0.1597	0.1573
	(0.97)	(0.97)	(0.97)	(0.97)	(0.95)	(0.95)	(0.98)	(0.98)	(0.96)	(0.96)	(0.95)	(0.95)

5.1 Model Misspecification Analysis

In this section, we investigate the effect of model misspecification on the performance of the proposed Bayes estimators. Specifically, data are generated from the exponential baseline distribution but are incorrectly fitted using the Weibull, PCH($M = 2$), and PCH($M = 3$) baseline models to estimate the parameters $\alpha_1, \alpha_2, \ldots, \alpha_r$ within a Bayesian framework. Similarly, data generated from the Weibull and PCH baseline distributions are misfitted using the remaining baseline families. For each misspecification scenario, the mean squared errors (MSEs) of the estimators of $\alpha_1, \alpha_2, \ldots, \alpha_r$ are computed for different sample sizes.

Data generated from the exponential distribution We consider the case where the data are generated from an exponential distribution but are incorrectly fitted the Weibull and PCH distributions to estimate the model parameters through their respective posterior distributions. For this scenario, the hyperparameters are fixed as follows: $a_1 = 0.001$, $b_1 = 4$, and $\boldsymbol{c} = (0.5, 0.75, 1)$. For the Weibull model, the hyperparameters associated with the scale parameter β are chosen as $a_2 = 1$ and $b_2 = 1$. For the PCH($M = 2$) and PCH($M = 3$) baseline models, the hyperparameters for $\boldsymbol{\theta}$ are taken as $\boldsymbol{d} = (1, 1)$ and $\boldsymbol{d} = (1, 1, 1)$, respectively. It is observed that varying these hyperparameters does not have a noticeable effect on either the Bayes estimates or the corresponding MSEs.

For the 3-out-of-5 system, the estimated Weibull shape parameter β is obtained as 0.7157, 0.7053, and 0.7382 for sample sizes $m = 15, 25$, and 30, respectively. Under the PCH($M = 2$) baseline, the corresponding estimates of θ_1 are 0.8363, 0.8394, and 0.8398. For the PCH($M = 3$) specification, the estimates of θ_1 are 0.8367, 0.8391, and 0.8396, while the corresponding estimates of θ_2 are 0.1467, 0.1459, and 0.1450. The corresponding MSEs of the Bayes estimators for α_1, α_2 and α_3 are reported in Table 25.

For the 4-out-of-8 system, the estimated Weibull shape parameter β is 0.8637, 0.8190, and 0.7768 for sample sizes $m = 15, 25$, and 30, respectively. Under the PCH($M = 2$) baseline, the estimates of θ_1 are 0.8265, 0.8333, and 0.8362. Under the PCH($M = 3$) specification, the corresponding estimates of θ_1 are 0.8264, 0.8333, and 0.8360, while the estimates of θ_2 are 0.1506, 0.1496, and 0.1484 for the same sample sizes. The corresponding MSEs of the Bayes estimators for $\alpha_1, \ldots \alpha_4$ are reported in Table 26.

Data generated from the Weibull distribution. We next consider the case in which the data are generated from a Weibull distribution with shape parameter $\beta = 1.5$, but are incorrectly fitted using the exponential and PCH baseline models to estimate the model parameters within a Bayesian framework. For this scenario, the hyperparameters are fixed as follows: $a_1 = 0.001$, $b_1 = 4$, and $\boldsymbol{c} = (0.5, 0.75, 1)$. For the Weibull model, the hyperparameters associated with the shape parameter β are chosen as $a_2 = 30$ and $b_2 = 2$. For the PCH($M = 2$) and PCH($M = 3$) baseline models, the hyperparameters for $\boldsymbol{\theta}$ are taken as $\boldsymbol{d} = (1, 1)$ and $\boldsymbol{d} = (1, 1, 1)$, respectively. Similar to the exponential-generated data case, it is observed that varying the hyperparameters for does not have any effect on either the Bayes estimates or the corresponding MSEs of $\alpha_1, \ldots, \alpha_r$. However, since the true value of the shape parameter is $\beta = 1.5$, changes in the hyperparameters associated with β lead to changes in its estimator.

For the 3-out-of-5 system, the estimated values of the Weibull shape parameter β are 1.5433, 1.5525, and 1.5088 for sample sizes $m = 15, 25$, and 30, respectively. Under the misspecified PCH($M = 2$) baseline, the estimates of θ_1 are 0.4240, 0.4808, and 0.6086. For the PCH($M = 3$) specification, the estimates of θ_1 are 0.4199, 0.4820, and 0.6084, while the corresponding estimates of θ_2 are 0.4763, 0.4440, and 0.3702. The corresponding MSEs of the Bayes estimators for α_1, α_2 and α_3 are reported in Table 27.

For the 4-out-of-8 system, the estimated Weibull shape parameters are 1.5419, 1.5309, and 1.5413 for sample sizes $m = 15, 25$, and 30, respectively. The corresponding estimates of θ_1 under the PCH($M = 2$) baseline are 0.6796, 0.6813, and 0.6828. Under the PCH($M = 3$) specification, the estimates of θ_1 are 0.6794, 0.6812, and 0.6829, while the corresponding estimates of θ_2 are 0.2985, 0.3020, and 0.3017. The corresponding MSEs of the Bayes estimators for $\alpha_1, \ldots, \alpha_4$ are reported in Table 28. .

Data generated from the PCH($M = 2$) distribution. We next consider the case in which data are generated from the PCH($M = 2$) baseline distribution. For the 3-out-of-5 system, the data are generated with change point $\tau_1 = 0.7$ and hazard rates $\boldsymbol{\theta} = (0.7, 0.3)$. For this scenario, the hyperparameters are chosen as follows: $a_1 = 0.001$, $b_1 = 15$, and $\boldsymbol{c} = (0.5, 0.75, 1)$. The hyperparameters associated with the Weibull shape parameter β are chosen as $a_2 = 1$ and $b_2 = 1$. For the PCH($M = 2$) and PCH($M = 3$), the hyperparameters for $\boldsymbol{\theta}$ are taken as $\boldsymbol{d} = (1, 1)$ and $\boldsymbol{d} = (1, 1, 1)$, respectively. Similar to earlier cases, changing the hyperparameters has no noticeable impact on the Bayes estimates or MSEs of $\alpha_1, \ldots, \alpha_r$. The MSEs of α_1,..., α_r presented in Tables 29 and 30.

For the 3-out-of-5 system, the estimated Weibull shape parameters are 1.7078, 1.5997, and 1.1021 for sample sizes $m = 15, 25$, and 30, respectively. Under the correctly specified PCH($M = 2$) model, the corresponding estimates of θ_1 are 0.6960, 0.7058, and 0.7024. Under the misspecified PCH($M = 3$) assumption, the estimates of θ_1 are 0.5380, 0.5492, and 0.5455, while the corresponding estimates of θ_2 are 0.2348, 0.2333, and 0.2341.

For the 4-out-of-8 system, data are generated from the PCH($M = 2$) baseline distribution with $\tau_1 = 0.5$ and hazard rates $\boldsymbol{\theta} = (0.3, 0.7)$. In this case, the hyperparameters are taken to be the same as those used for the 3-out-of-5 system. The estimated Weibull shape parameters are 2.4495, 1.4490, and 1.5449 for sample sizes $m = 15, 25$, and 30, respectively. Under the PCH($M = 2$) model, the estimates of θ_1 are 0.3243, 0.3208, and 0.3209. Under the misspecified PCH($M = 3$) model, the corresponding estimates of θ_1 are 0.3194, 0.3178, and 0.3185, while the estimates of θ_2 are 0.5195, 0.5223, and 0.5255.

Data generated from the PCH($M = 2$) distribution. Finally, we consider the case in which the data are generated from the PCH($M = 3$) distribution. The hyperparameter choices for this scenario are taken to be the same as those used for the PCH($M = 2$) case. The results corresponding to this misspecification setting are reported in Tables 31 and 32.

For 3-out-of-5 systems, the change points and hazard rates have been taken as $(\tau_1, \tau_2) = (0.7, 2)$ and $\boldsymbol{\theta} = (0.7, 0.2, 0.1)$ respectively. The estimated Weibull shape parameters are 2.1595, 1.9869, and 1.7481 for $m = 15, 25$, and 30. For the PCH($M = 2$) model, the estimates of θ_1 are 0.5517, 0.5539, and 0.5498. Under the PCH($M = 3$) model, the estimates of θ_1 are 0.6854, 0.6963, and 0.6950, while the estimates of θ_2 are 0.1812, 0.1765, and 0.1791.

For the 4-out-of-8 system, with data generated from the PCH($M = 3$) baseline having $(\tau_1, \tau_2) = (0.5, 1)$ and hazard rates $\boldsymbol{\theta} = (0.3, 0.5, 0.2)$, the estimated Weibull shape parameters are 1.8288, 1.5276, and 1.7726. The estimates of θ_1 under the PCH($M = 2$) model are 0.3177, 0.3147, and 0.3163, while for the PCH($M = 3$) model, the estimates of θ_1 are 0.3128, 0.3117, and 0.3137, and the estimates of θ_2 are 0.2912, 0.2897, and 0.2901.

From Tables 25–32, it is evident that the mean squared errors under misspecified baseline distributions are consistently higher than those obtained under the correctly specified models. This highlights that model misspecification leads to a clear loss of estimation accuracy. Such findings are important because, in practical applications, the true baseline distribution is rarely known in advance. The results highlight the importance of selecting an appropriate baseline model and show that using more flexible baseline families, such as the PCH models, helps reduce the adverse effects of model misspecification.

Table 25: The mean squared error based on m independent observations generated from exponential from a 3 out of 5 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.75$ and $\alpha_3 = 1$

m	distribution	Unordered			Ordered		
		α_1	α_2	α_3	α_1	α_2	α_3
15	EXP	0.1349	0.1179	0.1693	0.0668	0.0910	0.2232
	WEI	0.2060	0.1343	0.1789	0.0877	0.1092	0.2327
	PCH2	3.9741	1.7267	0.8564	3.8010	1.6323	0.7962
	PCH3	3.7593	1.4605	0.4897	3.6016	1.3788	0.4497
25	EXP	0.1743	0.1663	0.3473	0.1223	0.1562	0.3804
	WEI	0.2432	0.1866	0.3540	0.1624	0.1728	0.3947
	PCH2	8.5909	3.5501	1.8913	8.3821	3.4426	1.8206
	PCH3	8.1399	2.9206	1.0696	7.9479	2.8349	1.0235
30	EXP	0.1905	0.2006	0.4207	0.1443	0.1896	0.4518
	WEI	0.3260	0.2815	0.4360	0.2013	0.1938	0.4600
	PCH2	10.7927	4.2733	2.2991	10.5869	4.1708	2.2242
	PCH3	10.1925	3.5225	1.2993	9.9893	3.4315	1.2537

Table 26: The mean squared error based on m independent observations generated from exponential distribution from a 4 out of 8 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$ and $\alpha_4 = 1$

m	distribution	Unordered				Ordered			
		α_1	α_2	α_3	α_4	α_1	α_2	α_3	α_4
15	EXP	0.0062	0.0700	0.1547	0.1885	0.0041	0.0397	0.0915	0.2680
	WEI	0.0423	0.1152	0.1672	0.1950	0.0165	0.0714	0.1071	0.2952
	PCH2	4.8092	2.6164	1.3371	0.6529	4.6697	2.5426	1.2956	0.6048
	PCH3	4.7648	2.5324	1.3030	0.6104	4.6172	2.4654	1.2321	0.5835
25	EXP	0.0052	0.1089	0.1985	0.4559	0.0041	0.0770	0.1147	0.2958
	WEI	0.0479	0.1345	0.1877	0.1851	0.0213	0.0812	0.1249	0.5331
	PCH2	10.9008	5.1738	2.6599	1.3487	10.6445	5.0191	2.5491	1.3379
	PCH3	10.8725	5.1171	2.5347	1.2947	10.6965	4.9852	2.4969	1.2282
30	EXP	0.0045	0.1220	0.2216	0.4576	0.0039	0.1027	0.2398	0.5209
	WEI	0.0126	0.1458	0.3197	0.5042	0.0102	0.1201	0.2563	0.5767
	PCH2	13.9197	6.2649	3.1778	1.6606	13.6614	6.1607	3.1019	1.596
	PCH3	13.8681	6.2236	3.0828	1.5639	13.5791	6.0395	2.9940	1.5211

6 Illustrative Data Analysis

6.1 Choice of cut points

Earlier analyses assumed that both the number of cut points and their locations were known. In practical settings, however, this information is usually unavailable, and only limited lit-

Table 27: The mean squared error based on m independent observations generated from Weibull distribution from a 3 out of 5 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.75$ and $\alpha_3 = 1$

m	distribution	Unordered			Ordered		
		α_1	α_2	α_3	α_1	α_2	α_3
15	WEI	0.0126	0.0754	0.1606	0.0088	0.0494	0.2219
	EXP	0.1881	0.1128	0.2405	0.0813	0.0987	0.2575
	PCH2	3.2589	1.5057	0.7980	3.1085	1.4160	0.7399
	PCH3	3.0837	1.2692	0.4719	2.9492	1.1958	0.4322
25	WEI	0.0137	0.1428	0.3648	0.0132	0.1226	0.3969
	EXP	0.1284	0.1727	0.5318	0.0953	0.1507	0.5381
	PCH2	6.6102	3.0421	1.7597	6.4370	2.9393	1.6910
	PCH3	6.2688	2.5526	1.0643	6.1079	2.4690	1.0181
30	WEI	0.0154	0.1651	0.4391	0.0153	0.1444	0.4620
	EXP	0.1167	0.1825	0.6306	0.0939	0.1658	0.6330
	PCH2	8.1802	3.6876	2.1709	7.9956	3.5880	2.1067
	PCH3	7.8080	3.0912	1.3040	7.6397	3.0094	1.2570

erature addresses this issue. To overcome this limitation, we develop a Bayesian approach that enables the estimation of cut points when they are unknown.

Let τ denote the maximum observed time. Beyond this value, the hazard function cannot be inferred without additional modelling assumptions. To provide a flexible framework, we treat the cut points as ordered arrival times of a non-homogeneous Poisson process (NHPP) $\{N(t), t \geq 0\}$ defined on $[0, \tau]$, with intensity function $u(t) = \delta t^{\delta-1}$ for $\delta > 0$. As the arrival times are not observed, they are treated as latent variables in the model. Conditional on δ and a specified value of $M-1$, the joint density of the ordered arrival times $T_1, \ldots, T_{M-1}$, given that $N(\tau) = M-1$, is

$$f_{T_1,\ldots,T_{M-1}|N(\tau)=M-1}(t_1, \ldots, t_{M-1}) = \frac{(M-1)!\delta^{M-1}}{\tau^{(M-1)\delta}} t_1^{\delta-1} \ldots t_{M-1}^{\delta-1};\ 0 < t_1 < \ldots < t_{M-1} < \tau \tag{20}$$

and zero otherwise. Under the Bayesian framework, prior distributions are assigned to all unknown quantities, including the NHPP parameter δ and the cut points. Specifically, we consider the following priors:

Table 28: The mean squared error based on m independent observations generated from Weibull distribution from a 4 out of 8 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$ and $\alpha_4 = 1$

m	distribution	Unordered				Ordered			
		α_1	α_2	α_3	α_4	α_1	α_2	α_3	α_4
15	WEI	0.0222	0.1409	0.2960	0.4133	0.0168	0.1103	0.2194	0.2979
	EXP	0.1390	0.1781	0.3168	0.4268	0.0550	0.1270	0.2628	0.3358
	PCH2	3.4770	1.9187	1.0269	0.5224	3.3102	1.806	0.9552	0.4747
	PCH3	3.4399	1.8869	0.9905	0.4683	3.2709	1.7839	0.9233	0.4237
25	WEI	0.0172	0.1381	0.2984	0.3024	0.0123	0.1081	0.2229	0.4912
	EXP	0.1848	0.2616	0.3493	0.3493	0.1129	0.2194	0.2860	0.5261
	PCH2	7.1882	3.6635	2.0439	1.1598	6.9836	3.5409	1.9641	1.1036
	PCH3	7.1321	3.6148	1.9704	1.0446	6.9379	3.4966	1.8936	0.9929
30	WEI	0.0158	0.1452	0.3538	0.4063	0.0132	0.1221	0.2729	0.5488
	EXP	0.1936	0.2828	0.4133	0.4133	0.1317	0.2565	0.3422	0.5876
	PCH2	8.9115	4.3896	2.4344	1.409	8.7037	4.2725	2.3602	1.3560
	PCH3	8.8492	4.3337	2.3521	1.2692	8.6463	4.2208	2.2783	1.2202

Table 29: The mean squared error based on m independent observations generated from PCH (M=2) distribution from a 3 out of 5 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.75$ and $\alpha_3 = 1$

m	distribution	Unordered			Ordered		
		α_1	α_2	α_3	α_1	α_2	α_3
15	PCH2	0.0030	0.1105	0.3632	0.0039	0.1145	0.3696
	EXP	0.0504	0.2121	0.4634	0.0741	0.2108	0.3985
	WEI	0.0459	0.2698	0.6024	0.1066	0.2620	0.4408
	PCH3	0.0039	0.1133	0.3867	0.0047	0.1180	0.3946
25	PCH2	0.0126	0.0146	0.1793	0.0102	0.0159	0.1825
	EXP	0.0610	0.1362	0.3305	0.0355	0.1330	0.2773
	WEI	0.0676	0.1808	0.4526	0.0573	0.1698	0.3114
	PCH3	0.0166	0.0170	0.2130	0.0136	0.0185	0.2180
30	PCH2	0.0500	0.0013	0.1239	0.0512	0.0018	0.1264
	EXP	0.1530	0.1161	0.2908	0.1261	0.1125	0.2452
	WEI	0.1206	0.1598	0.4132	0.1451	0.1483	0.2819
	PCH3	0.0534	0.0031	0.1611	0.0806	0.0035	0.1653

NHPP shape parameter:

$$\pi(\delta) = \delta \sim \text{Gamma}(a_\delta, b_\delta), \quad a_\delta > 0, b_\delta > 0.$$

Table 30: The mean squared error based on m independent observations generated from PCH (M=2) distribution from a 4 out of 8 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$ and $\alpha_4 = 1$

m	distribution	Unordered				Ordered			
		α_1	α_2	α_3	α_4	α_1	α_2	α_3	α_4
15	PCH2	0.1023	0.1783	0.3882	0.6785	0.1044	0.1810	0.3922	0.6837
	EXP	0.1418	0.2102	0.4125	0.6951	0.1547	0.2184	0.4132	0.6533
	WEI	0.1253	0.2105	0.4266	0.7238	0.1532	0.2184	0.4157	0.6607
	PCH3	0.1022	0.1780	0.3875	0.6777	0.1044	0.1808	0.3914	0.6829
25	PCH2	0.0410	0.0968	0.2672	0.5215	0.0424	0.0990	0.2705	0.526
	EXP	0.1101	0.1564	0.3170	0.5567	0.1169	0.1628	0.3232	0.5227
	WEI	0.0863	0.1571	0.3387	0.6011	0.1100	0.1618	0.3310	0.5446
	PCH3	0.0409	0.0962	0.2656	0.5196	0.0423	0.0981	0.2688	0.5240
30	PCH2	0.0211	0.0665	0.2177	0.4555	0.0221	0.0681	0.2207	0.4596
	EXP	0.0990	0.1373	0.2799	0.4996	0.1038	0.1423	0.2876	0.4700
	WEI	0.0730	0.1373	0.3036	0.5491	0.0937	0.1403	0.2977	0.4984
	PCH3	0.0210	0.0657	0.2158	0.4528	0.0220	0.0673	0.2186	0.4568

Table 31: The mean squared error based on m independent observations generated from PCH (M=3) distribution from a 3 out of 5 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.75$ and $\alpha_3 = 1$

m	distribution	Unordered			Ordered		
		α_1	α_2	α_3	α_1	α_2	α_3
15	PCH3	0.0605	0.2533	0.5758	0.0626	0.2573	0.5820
	EXP	0.1021	0.3324	0.6784	0.1323	0.3314	0.6069
	WEI	0.1160	0.4162	0.8372	0.1874	0.4166	0.6763
	PCH2	0.0604	0.2581	0.5960	0.0625	0.2622	0.6018
25	PCH3	0.0078	0.1319	0.4000	0.0085	0.1346	0.4047
	EXP	0.0636	0.2671	0.5863	0.0955	0.2635	0.5032
	WEI	0.0761	0.3561	0.7596	0.1548	0.3539	0.5652
	PCH2	0.0077	0.1393	0.4371	0.0084	0.1422	0.4416
30	PCH3	0.0003	0.0899	0.3330	0.0004	0.0921	0.3371
	EXP	0.0526	0.2471	0.5531	0.0840	0.2414	0.4693
	WEI	0.0649	0.3413	0.7362	0.1458	0.3367	0.5298
	PCH2	0.0003	0.0986	0.3782	0.0004	0.1010	0.3823

Cut points condition on δ:

$$\pi(\tau_1, \ldots, \tau_{M-1}|\delta) = \frac{(M-1)!\delta^{M-1}}{\tau^{(M-1)\delta}} \prod_{j=1}^{M-1} \tau_j^{\delta-1}.$$

Table 32: The mean squared error based on m independent observations generated from PCH (M=3) distribution from a 4 out of 8 system with true parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$ and $\alpha_4 = 1$

m	distribution	Unordered				Ordered			
		α_1	α_2	α_3	α_4	α_1	α_2	α_3	α_4
15	PCH3	0.0578	0.1195	0.301	0.5651	0.0602	0.1231	0.3063	0.5711
	EXP	0.1256	0.1904	0.3743	0.6392	0.1409	0.1971	0.3763	0.5931
	WEI	0.1193	0.2398	0.4911	0.8303	0.1863	0.2583	0.4634	0.6846
	PCH2	0.0595	0.1249	0.3144	0.5891	0.0616	0.1285	0.3189	0.5953
25	PCH3	0.0056	0.0375	0.1647	0.3814	0.0063	0.0391	0.1680	0.3870
	EXP	0.1012	0.1505	0.2955	0.5197	0.1106	0.1532	0.3027	0.4859
	WEI	0.0941	0.2178	0.4564	0.7901	0.1693	0.2350	0.4291	0.6171
	PCH2	0.0071	0.0459	0.1901	0.4307	0.0079	0.0476	0.1934	0.4353
30	PCH3	0.0004	0.0144	0.1145	0.3082	0.0001	0.0155	0.1174	0.3129
	EXP	0.0914	0.1352	0.2697	0.4691	0.0991	0.1367	0.2748	0.4423
	WEI	0.0788	0.1991	0.4325	0.7502	0.1526	0.2150	0.3993	0.5821
	PCH2	0.0004	0.0218	0.1431	0.3659	0.0004	0.0230	0.1461	0.3702

So the posterior distribution will be

$$\Pi(\delta, \boldsymbol{\tau}, \theta) = \pi(\delta) \times \pi(\tau_1, \ldots, \tau_{M-1}|\delta) \times \pi(\theta) \times l(\theta|D), \tag{21}$$

where $l(\theta|D)$ denotes likelihood based on the data.

Since this posterior distribution does not have a closed-form expression, estimation is carried out using an importance sampling approach. Candidate values for δ and $\boldsymbol{\tau}$ are drawn from convenient proposal distributions—for instance, a Gamma proposal for δ and Dirichlet-based transformations for the cut points. Importance weights are then computed and normalized to approximate the posterior distribution. This procedure yields posterior summaries such as posterior means, standard deviations, and credible intervals for δ and the cut points.

6.2 Data Analysis

To illustrate the analysis for a 3-out-of-5 system, lifetime data were generated for 20 independent systems. Each system contains five components, and the lifetime of each component follows a Weibull distribution with shape parameter 1.5 and scale parameter 2. The lifetimes for a given system are denoted by X_1, X_2, X_3, X_4, and X_5. Since the system has a 3-out-of-5 structure, it fails upon the failure of the third component. Accordingly, for each system, the five component lifetimes were arranged in increasing order and the first three order statistics (X_1^*, X_2^*, X_3^*) were recorded, where X_3^* represents the system failure time.

The complete component failure times for all systems along with their corresponding ordered failure times are presented in Appendix E. These data will be used in subsequent analyses, including parameter estimation and the assessment of the system's survival behavior.

First, we consider the exponential baseline distribution. The hyper-parameters are taken as $a_1 = 0.001$, $b_1 = 1$, and $\boldsymbol{c} = (1, 1, 1)$. The Bayes estimates and their associated standard errors (given in parentheses) are obtained as follows:

$$\hat{\alpha}_1 = 0.2936\,(\pm 0.0663), \quad \hat{\alpha}_2 = 0.5985\,(\pm 0.1378), \quad \hat{\alpha}_3 = 0.7517\,(\pm 0.1766)$$

for the unordered case, and

$$\hat{\alpha}_1 = 0.2928\,(\pm 0.0656), \quad \hat{\alpha}_2 = 0.5727\,(\pm 0.1181), \quad \hat{\alpha}_3 = 0.7837\,(\pm 0.1616)$$

for the ordered case. The corresponding 95% HPD and symmetric credible intervals for all parameters are reported in Table 33.

Next, for the Weibull baseline distribution, the hyper-parameters are specified as $a_1 = 0.001$, $b_1 = 1$, $\boldsymbol{c} = (1, 1, 1)$, $a_2 = 1$, and $b_2 = 40$. The Bayes estimates and their associated

Table 33: 95% HPD and Symmetric intervals for the exponential baseline

		Unordered			Ordered		
		α_1	α_2	α_3	α_1	α_2	α_3
HPD	LL	0.1671	0.3459	0.4515	0.1686	0.3654	0.5109
	UL	0.4191	0.8540	1.1002	0.4159	0.7959	1.0930
SYM	LL	0.1797	0.3713	0.4599	0.1804	0.3713	0.5290
	UL	0.4351	0.8886	1.1160	0.4314	0.8069	1.1290

standard errors (given in parentheses) are obtained as follows:

$$\hat{\alpha}_1 = 0.3533\,(\pm 0.0831), \quad \hat{\alpha}_2 = 0.4080\,(\pm 0.1063), \quad \hat{\alpha}_3 = 0.4121\,(\pm 0.1299), \quad \hat{\beta} = 1.6174\,(\pm 0.2010)$$

for the unordered case, and

$$\hat{\alpha}_1 = 0.3064\,(\pm 0.0608), \quad \hat{\alpha}_2 = 0.3888\,(\pm 0.0787), \quad \hat{\alpha}_3 = 0.4785\,(\pm 0.1093), \quad \hat{\beta} = 1.6174\,(\pm 0.2010)$$

for the ordered case. The corresponding 95% HPD and symmetric credible intervals for all parameters are presented in Table 34.

Table 34: 95% HPD and Symmetric intervals for the Weibull baseline

		Unordered				Ordered			
		α_1	α_2	α_3	β	α_1	α_2	α_3	β
HPD	LL	0.2092	0.2303	0.2006	1.2224	0.1942	0.2547	0.3048	1.2202
	UL	0.5190	0.6051	0.6423	2.0154	0.4180	0.5251	0.6651	2.0175
SYM	LL	0.2161	0.2417	0.2178	1.2435	0.1999	0.2686	0.3303	1.2419
	UL	0.5305	0.6261	0.6796	2.0413	0.4257	0.5486	0.7034	2.04166

Next, we consider the PCH model with $M = 2$. The change point is estimated as described in Section 6.1, and the hyper-parameters for the change-point parameter are taken as $a_\delta = 0.001$ and $b_\delta = 0.001$. For the remaining parameters, the hyper-parameters are specified as $a_1 = 0.001$, $b_1 = 1$, $\boldsymbol{c} = (1, 1, 1)$, and $\boldsymbol{d} = (1, 1)$. The Bayes estimates and standard errors of the parameters δ and τ_1 are obtained as $1.0208\,(\pm 0.0279)$ and $0.6468\,(\pm 0.0315)$, respectively.

The Bayes estimates and their associated standard errors (in parentheses) for the model parameters are as follows:

$$\hat{\alpha}_1 = 0.8117\,(\pm 0.3515), \quad \hat{\alpha}_2 = 0.9908\,(\pm 0.2801), \quad \hat{\alpha}_3 = 1.4242\,(\pm 0.2664), \quad \hat{\theta}_1 = 0.1676\,(\pm 0.0448)$$

for the unordered case, and

$$\hat{\alpha}_1 = 1.1461\,(\pm 0.3477), \quad \hat{\alpha}_2 = 1.2887\,(\pm 0.2739), \quad \hat{\alpha}_3 = 2.1333\,(\pm 0.2652), \quad \hat{\theta}_1 = 0.1598\,(\pm 0.0443)$$

for the ordered case. The corresponding 95% HPD and symmetric credible intervals for all the parameters are presented in Table 35.

Table 35: 95% HPD and Symmetric intervals for the PCH (M=2) baseline

		Unordered				Ordered			
		α_1	α_2	α_3	θ_1	α_1	α_2	α_3	θ_1
HPD	LL	0.6770	0.7829	1.3809	0.1427	0.5530	0.7306	1.4524	0.0630
	UL	1.0769	1.2242	1.7517	0.1977	1.9911	1.9061	2.6733	0.2370
SYM	LL	0.7587	0.7797	1.3125	0.1274	0.5530	0.7807	1.4667	0.0686
	UL	1.1745	1.3765	1.7352	0.1942	2.2889	2.2654	2.6886	0.2478

Finally, we consider the PCH model with $M = 3$. The change points are estimated as described in Section 6.1, and the hyper-parameters for the change-point parameter are taken as $a_\delta = 0.001$ and $b_\delta = 0.001$. For the remaining parameters, the hyper-parameters are specified as $a_1 = 0.001$, $b_1 = 1$, $\boldsymbol{c} = (1, 1, 1)$, and $\boldsymbol{d} = (1, 1, 1)$. The Bayes estimates and standard errors of δ, τ_1, and τ_2 are obtained as 1.0538 (±0.0145), 0.6370 (±0.0141), and 1.9537 (±0.0115), respectively.

The Bayes estimates and their associated standard errors (in parentheses) for the remaining parameters are as follows:

$$\hat{\alpha}_1 = 0.8750\,(\pm 0.3637), \quad \hat{\alpha}_2 = 1.1486\,(\pm 0.2622), \quad \hat{\alpha}_3 = 1.5677\,(\pm 0.1999),$$

$$\hat{\theta}_1 = 0.0890\,(\pm 0.0299), \quad \hat{\theta}_2 = 0.6821\,(\pm 0.0484)$$

for the unordered case, and

$$\hat{\alpha}_1 = 0.8723\,(\pm 0.3581), \quad \hat{\alpha}_2 = 1.1396\,(\pm 0.2558), \quad \hat{\alpha}_3 = 1.5647\,(\pm 0.1700),$$

$$\hat{\theta}_1 = 0.0891\,(\pm 0.0297), \quad \hat{\theta}_2 = 0.6825\,(\pm 0.0480)$$

for the ordered case. The corresponding 95% HPD and symmetric credible intervals for all parameters are presented in Table 35.

Table 36: 95% HPD and Symmetric intervals for the PCH (M=2) baseline

		Unordered					Ordered				
		α_1	α_2	α_3	θ_1	θ_2	α_1	α_2	α_3	θ_1	θ_2
HPD	LL	0.5113	0.6898	0.9032	0.0368	0.5839	0.5031	0.6869	0.9439	0.0359	0.5903
	UL	1.2464	1.6879	2.2548	0.1510	0.7770	1.2495	1.6550	2.2875	0.1486	0.7759
SYM	LL	0.5365	0.7036	0.9529	0.0392	0.5818	0.5319	0.7074	0.9578	0.0397	0.5855
	UL	1.2939	1.7169	2.3387	0.1561	0.7757	1.2911	1.6854	2.3150	0.1547	0.7721

7 Conclusions

In this chapter, we developed a unified Bayesian framework for analyzing sequential order statistics from k-out-of-n systems under a proportional hazards structure. The methodology accommodates exponential, Weibull, and flexible piecewise constant hazard baselines, and incorporates both unrestricted and order-restricted priors. Since the posterior distributions do not admit closed-form expressions, importance sampling was used for efficient computation. Simulation studies demonstrate the accuracy and robustness of the proposed estimators, highlighting the advantages of order-restricted inference and the sensitivity to model misspecification. The analysis of a real data set further illustrates the practical utility of the approach. Overall, the proposed framework provides a flexible and effective tool for Bayesian reliability analysis in systems with sequential component failures. Future work may explore extensions to hierarchical system structures, optimal design strategies, and alternative non-parametric or semiparametric baseline hazard formulations.

Appendix A

We define a new set of variables $x_{ij1}, \ldots, x_{ijM}$, for $j = 1, \ldots, r$ as follows: if $x_{ij} \in A_p$, for $1 \leq p \leq M$, then

$$x_{ijk} = \begin{cases} x_{ij} & \text{if} \quad k = p \\ 0 & \text{if} \quad k \neq p \end{cases}$$

Further let $n_{jk} = \#\{x_{ij} : x_{ij} \in A_k\}$, for $j = 1, \ldots, r$ and $k = 1, \ldots, M$. Then

$$\begin{aligned} \sum_{i=1}^{m} H_0(x_{ij}) \quad = \quad & \theta_1 \left(\sum_{i=1}^{m} \sum_{k=1}^{M} x_{ijk} + n_{12} + \ldots + n_{1M} \right) + \\ & \theta_2 \left(\sum_{i=1}^{m} \sum_{k=1}^{M} (x_{ijk} - \tau_1)^+ + n_{13} + \ldots + n_{1M} \right) + \\ & \qquad \vdots \\ & \theta_{M-1} \left(\sum_{i=1}^{m} \sum_{k=1}^{M} (x_{ijk} - \tau_{M-2})^+ + n_{1M} \right) + \\ & \theta_M \left(\sum_{i=1}^{m} \sum_{k=1}^{M} (x_{ijk} - \tau_{M-1})^+ \right). \end{aligned}$$

Here $(a-b)^+ = (a-b)$, if $a > b$ and it is 0, otherwise. Also, $n_k = \sum_{j=1}^{r} n_{jk}$ for $k = 1, \ldots, M$. Let use define,

$$\begin{aligned} U_{1j} \quad &= \quad \left(\sum_{i=1}^{m} \sum_{k=1}^{M} x_{ijk} + n_{12} + \ldots + n_{1M} \right) \\ U_{2j} \quad &= \quad \left(\sum_{i=1}^{m} \sum_{k=1}^{M} (x_{ijk} - \tau_1)^+ + n_{13} + \ldots + n_{1M} \right) \\ \vdots \\ U_{Mj} \quad &= \quad \left(\sum_{i=1}^{m} \sum_{k=1}^{M} (x_{ijk} - \tau_{M-1})^+ \right) \end{aligned} \tag{22}$$

and $V_{kj} = U_{kj} - U_{k(j-1)}; k = 1, \ldots M$.

Appendix B

For the exponential baseline, we use the importance sampling procedure to compute the Bayes estimate of α_j and also to construct the corresponding credible intervals of α_j, $j = 1, \ldots, r$. We use the following procedure:

Step 1: Generate

$$\alpha_j \sim \text{Gamma}(m + c_j, D_j + b_1), \quad j = 1, \ldots, r$$

for unordered case and

$$\alpha_j \sim \text{Gamma}(m + 1, D_j + b_1), \quad j = 1, \ldots, r$$

for ordered case.

Step 2: Repeat this N times to obtain $\{\alpha_{ji}; j = 1, \ldots, r; i = 1, \ldots, N\}$.

Step 3: For squared error loss, the Bayes estimate of α_j can be obtained as

$$\hat{\alpha_j} = \frac{\displaystyle\sum_{i=1}^{N} \alpha_{ji} h(\alpha_{ji})}{\displaystyle\sum_{i=1}^{N} h(\alpha_{ji})}, \tag{23}$$

where $h(\alpha_{ji}) = \overline{\alpha}^{(a_1 - \overline{c})}$ for unordered case and $h(\alpha_{ji}) = \overline{\alpha}^{\{a_1 - \overline{c}\}} \sum_{i_1, \ldots, i_r \in \mathcal{P}} \left(\alpha_1^{c_{i_1} - 1} \ldots \alpha_r^{c_{i_r} - 1}\right)$ for ordered case.

Now to construct a $(1 - \gamma)100\%$ HPD credible interval of α_{ji} :

- First compute

$$w_i = \frac{h(\alpha_{ji})}{\displaystyle\sum_{i=1}^{N} h(\alpha_{ji})}, \tag{24}$$

- Sort the pairs $\{(\alpha_{j1}, w_1), \ldots, (\alpha_{j2}, w_N)\}$ as $\{(\alpha_{(j1)}, w_{(1)}), \ldots, (\alpha_{(jN)}, w_{(N)})\}$, where $\alpha_{(j1)} < \ldots < \alpha_{(jN)}$ and $w'_{(i)}$s are associated with $\alpha_{(ji)}$.

- For any $p \in (0, 1)$,

$$\sum_{i=1}^{N_p} w_{(i)} \leq p \leq \sum_{i=1}^{N_p+1} w_{(i)} \text{ and } \hat{\alpha}_{jp} = \alpha_{(jN_p)} \tag{25}$$

- A $(1-\gamma)100\%$ credible interval of $\hat{\alpha}_j$ can be obtained as $(\hat{\alpha}_{j\delta}, \hat{\alpha}_{j(\delta+1-\gamma)})$, for $\gamma = w_{(1)}, w_{(1)} + w_{(2)}, \ldots, \sum_{i=1}^{N_\gamma} w_{(i)}$. Hence, a HPD $(1-\gamma)100\%$ credible interval of $g(\Theta)$ can be obtained as $(\hat{\alpha}_{j\delta*}, \hat{\alpha}_{j(\delta*+1-\gamma)})$, where

$$\hat{\alpha}_{j(\delta*+1-\gamma)} - \hat{\alpha}_{j\delta*} \leq \hat{\alpha}_{j(\delta+1-\gamma)} - \hat{\alpha}_{j\delta}, \quad \text{for all } \delta. \tag{26}$$

Appendix C

The algorithm for the Weibull baseline is as follows:

Step 1: Generate $\beta \sim \text{Gamma}(mr + a_2, b - \sum_{i=1}^{m} \sum_{j=1}^{r} \ln x_{ij})$.

Step 2: Generate

$$\alpha_j \sim \text{Gamma}(m + c_j, D_j(\beta) + b_1), \quad j = 1, \ldots, r$$

for unordered and

$$\alpha_j \sim \text{Gamma}(m + 1, D_j(\beta) + b_1), \quad j = 1, \ldots, r$$

for ordered case.

Step 3: Repeat the steps 1 and 2 N times to obtain $\{\beta_i, \alpha_{ji}; j = 1, \ldots, r; i = 1, \ldots, N\}$.

Step 4: For squared error loss, the Bayes estimate of $g = g(\beta, \boldsymbol{\alpha})$ can be obtained as

$$\hat{g}(\beta, \boldsymbol{\alpha}) = \frac{\sum_{i=1}^{N} g_i h(\alpha_{ji})}{\sum_{i=1}^{N} h(\alpha_{ji})}, \tag{27}$$

where $h(\alpha_{ji}) = \overline{\alpha}^{(a_1 - \overline{c})}$ for unordered case and $h(\alpha_{ji}) = \overline{\alpha}^{\{a_1 - \overline{c}\}} \sum_{i_1, \ldots, i_r \in \mathcal{P}} \left(\alpha_1^{c_{i_1}-1} \ldots \alpha_r^{c_{i_r}-1} \right)$ for ordered case.

Now to construct a $(1-\gamma)100\%$ HPD credible interval of $g(\beta, \boldsymbol{\alpha})$:

- First compute

$$w_i = \frac{h(\alpha_{(ji)})}{\sum_{i=1}^{N} h(\alpha_{(ji)})}, \tag{28}$$

- Sort the pairs $\{(g_1, w_1), \ldots, (g_N, w_N)\}$ as $\{(g_{(1)}, w_{(1)}), \ldots, (g_{(N)}, w_{(N)})\}$, where $g_{(1)} < \ldots < g_{(N)}$ and $w'_{(i)}$s are associated with $g_{(i)}$.

- For any $p \in (0, 1)$,

$$\sum_{i=1}^{N_p} w_{(i)} \leq p \leq \sum_{i=1}^{N_p+1} w_{(i)} \text{ and } \hat{g}_p = g_{(N_p)} \tag{29}$$

- A $(1-\gamma)100\%$ credible interval of $g(\beta, \boldsymbol{\alpha})$ can be obtained as $(\hat{g}_\delta, \hat{g}_{\delta+1-\gamma})$, for $\gamma = w_{(1)}, w_{(1)} + w_{(2)}, \ldots, \sum_{i=1}^{N_\gamma} w_{(i)}$. Hence, a HPD $(1-\gamma)100\%$ credible interval of $g(\beta, \boldsymbol{\alpha})$ can be obtained as $(\hat{g}_\delta *, \hat{g}_{\delta*+1-\gamma})$, where

$$\hat{g}_{\delta*+1-\gamma} - \hat{g}_{\delta*} \leq \hat{g}_{\delta+1-\gamma} - \hat{g}_\delta, \quad \text{for all } \delta. \tag{30}$$

Appendix D

The algorithm for the PCH baseline is as follows:

Step 1: Generate $\theta_k \sim \text{Dirichlet}_M(n_k + d_k)$; $k = 1, \ldots, k$.

Step 2: Generate

$$\alpha_j \sim \text{Gamma}(m + c_j, W_j(\boldsymbol{\theta}) + b_1), \quad j = 1, \ldots, r$$

for unordered and

$$\alpha_j \sim \text{Gamma}(m + 1, W_j(\boldsymbol{\theta}) + b_1), \quad j = 1, \ldots, r$$

for ordered case.

Step 3: Repeat the steps 1 and 2 N times to obtain $\{\theta_{ki}, \alpha_{ji}; j = 1, \ldots, r; i = 1, \ldots, N\}$.

Step 4: For squared error loss, the Bayes estimate of $g = g(\boldsymbol{\theta}, \boldsymbol{\alpha})$ can be obtained as

$$\hat{g}(\boldsymbol{\theta}, \boldsymbol{\alpha}) = \frac{\displaystyle\sum_{i=1}^{N} g_i h(\alpha_{ji})}{\displaystyle\sum_{i=1}^{N} h(\alpha_{ji})}, \tag{31}$$

where $h(\alpha_{ji}) = \overline{\alpha}^{(a_1 - \bar{c})}$ for unordered case and $h(\alpha_{ji}) = \overline{\alpha}^{\{a_1 - \bar{c}\}} \sum_{i_1, \ldots, i_r \in \mathcal{P}} \left(\alpha_1^{c_{i_1} - 1} \ldots \alpha_r^{c_{i_r} - 1}\right)$ for ordered case.

The credible intervals are constructed by applying the same algorithmic steps employed for the Weibull baseline case.

Appendix E

Table 37: Complete component lifetimes and ordered failure times for the 20 systems.

System	X_1	X_2	X_3	X_4	X_5	X_1^*	X_2^*	X_3^*
1	1.84	0.72	3.11	1.29	0.96	0.72	0.96	1.29
2	0.55	2.43	1.06	1.88	3.52	0.55	1.06	1.88
3	2.97	1.14	0.68	2.21	1.41	0.68	1.14	1.41
4	0.92	0.47	1.33	3.44	2.18	0.47	0.92	1.33
5	2.62	1.73	0.81	1.09	3.07	0.81	1.09	1.73
6	1.46	0.38	0.91	2.68	3.15	0.38	0.91	1.46
7	0.77	1.56	2.04	0.59	1.12	0.59	0.77	1.12
8	3.51	1.24	0.83	1.96	2.67	0.83	1.24	1.96
9	1.17	2.88	0.64	1.49	3.72	0.64	1.17	1.49
10	0.42	1.01	1.88	3.11	0.93	0.42	0.93	1.01
11	2.54	1.71	3.22	0.97	1.33	0.97	1.33	1.71
12	0.66	2.14	1.28	3.59	0.88	0.66	0.88	1.28
13	1.95	2.63	0.57	1.12	1.47	0.57	1.12	1.47
14	3.18	1.38	0.92	2.07	1.69	0.92	1.38	1.69
15	1.24	0.51	0.88	2.92	3.06	0.51	0.88	1.24
16	1.66	2.73	1.05	1.91	0.79	0.79	1.05	1.66
17	2.82	0.74	1.43	1.09	2.11	0.74	1.09	1.43
18	1.51	2.45	1.92	0.68	3.28	0.68	1.51	1.92
19	0.83	1.31	2.76	1.07	1.53	0.83	1.07	1.31
20	2.34	1.29	0.95	3.41	1.76	0.95	1.29	1.76

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