Decision Theoretic Sampling Plans: A Review

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Abstract

Acceptance sampling plan within the decision theoretic frame work has received a considerable amount of attention during the last few decades. In this review article we have mainly discussed different acceptance sampling plans based on the decision theoretic approach under the Bayesian set up for different censoring schemes. We have discussed optimal sampling plans for one and two sample problems, in case of competing risks and also for interval censored data. Throughout the article, we mention several open problems and suggest possible future work for the benefit of readers interested in this area of research. Finally, we have presented some numerical results to compare the performances of the different Bayesian sampling plans.

Keywords and Phrases: Bayesian sampling plan, Bayes estimator, Bayes risk, Decision-theoretic approach, Exponential distribution, Censoring schemes, Maximum likelihood estimator.

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1 Introduction

In this modern world, a country's progress mainly depends on its ever-flourishing and sustainable industrial growth, which, in turn, is directly linked to the quality of manufactured products it produces. The improvement in quality is an essential issue of service of companies and industries. Maintaining the standards of durability and overall quality of the products is the primary concern of a manufacturer. Manufacturers (or customers) need quality assurance before releasing (or buying) a product. Thus, it is necessary to do quality inspection and reliability testing to ensure product quality and reliability. Acceptance sampling plan is one of the earliest aspects of quality assurance in the applications of quality control. A sample is randomly taken from a lot, and the decision on the lot depends on the information obtained from this sample. It is the area of statistical quality control, which is useful in analyzing the quality of a product and improve the process performance. Both manufacturers and customers need proper acceptance sampling plans to control the risk of rejecting good quality items or accepting bad quality items. Notably, acceptance sampling plan is used to accept or reject the items, not for estimating the quality of the lot.

Acceptance sampling plans are divided into two classes, viz., attribute sampling plan and variable sampling plan. An attribute sampling plan is used when a category measures the quality of an item, say, non-defective or defective, satisfactory or unsatisfactory, etc.. In contrast, a variable sampling plan is used when the quality of an object is measured on a continuous scale. A statistically framed variable sampling plan is more efficient than an attribute sampling plan as it needs less sampling to achieve an optimal result, see Krishnamoorthi [30]. Variable sampling plans provide more information than attribute plans. Since a numerical value for the characteristic is specified, the extent to which the item is conforming or not, can be obtained.

In this review, we mainly focus on the variable sampling plans. In case of a variable sampling plan, we take a random sample from the batch and compare its performance with some standard value. The batch will be accepted if the observations of a performance indicator, for example the lifetimes of the sampled units, is better (higher or lower depending on the indicator) than the standard value, and it will be rejected otherwise. Thus, a variable sampling plan typically requires a standard value of some measuring characteristic - mean, median, range, quantile, standard deviation, etc., before deciding whether to accept or reject a lot. If the quality of the item being inspected is measured by its lifetime, then after the completion of sampling inspection, we have data on lifetimes of sampled units. The variable sampling plan based on these lifetimes of sampled items is termed 'acceptance sampling plan based on life test' or 'reliability test plan.'

We have mainly discussed different decision theoretic sampling plans under the Bayesian set up which are available till today in the area of acceptance sampling plan. The standard methods used in decision theory can be found in many standard statistical text books. In the decision theoretic frame work one chooses a loss function, a decision function, a prior, and based on these, the Bayes risk is obtained. Finally, based on the Bayes risk, the optimal decision function has been obtained. From the point of view of economic consideration, the decision-theoretic approach is more scientific and realistic. Here, the sampling plan is determined by making an optimal decision driven by some economic considerations, such as maximizing the return or minimizing the loss. In the decision-theoretic approach, a decision δ is taken with respect to (w.r.t.) the quality of the lot, whether to accept or reject it, and it is invariably subject to the risks associated with two types of errors, i.e., rejecting a good lot, or accepting a bad lot. Hence we consider a loss function L, which is a function of various costs, e.g., inspection cost, acceptance cost, rejection cost, etc. Thus, to design a variable sampling plan, we have to determine some non-negative integer n and an acceptance region or set say, A; according to some specified criterion, it minimizes the loss L. To implement the sampling plan, we have to draw a sample of size n from the lot and measure the variable values for all the items drawn. We can then decide to accept or reject the lot by inspecting whether the variable values belong to the acceptance region A; according to some specified criteria.

The acceptance sampling plans within the decision-theoretic approach under the Bayesian framework are more appealing because of the economic consequences of accepting or rejecting a production lot and providing a way of combining prior information and judgment with the available data. Therefore, to obtain the optimal sampling plans under various scenarios, the standard Bayesian approach is quite popular, attractive, more precise, and scientific. In the usual Bayesian approach, the optimal decision function δ is determined by minimizing the Bayes risk of the sampling plan for a given loss function among all sampling plans. The Bayesian approach arises naturally when prior information is available for planning and estimation.

Optimal Bayesian decision-theoretic sampling plans have found significant applications in modern industrial settings where decisions must balance sampling cost, time, and the consequences of incorrect actions. By incorporating prior information and explicitly modeling loss functions, these approaches provide a principled framework for optimizing sampling strategies under uncertainty. Two important application domains are semiconductor manufacturing and pharmaceutical stability testing.

Semiconductor manufacturing involves highly complex and expensive processes, where even small defects can lead to substantial financial losses. Reliability and yield analysis are therefore critical. The Bayesian sampling plans are used to determine the number of chips to be tested under life-testing experiments, often under censoring schemes, to assess failure rates. The decision-theoretic frameworks help determine optimal burn-in times and sample sizes to minimize total expected cost, including warranty and replacement costs. The main advantages are the incorporation of historical production data through prior distributions. It reduces the testing costs by avoiding unnecessary sampling. It has the flexibility to adapt to rapidly changing manufacturing conditions.

Pharmaceutical products must meet stringent regulatory requirements regarding shelf life and stability. Stability testing involves monitoring degradation over time under various environmental conditions. The Bayesian sampling plans help determine optimal testing schedules and sample sizes to estimate product shelf life accurately. The decision-theoretic approaches are used to design experiments at elevated stress conditions to infer long-term stability. The Bayesian decision rules assist in determining whether a batch meets quality standards based on sampled data. The main advantages are the efficient use of prior scientific knowledge and historical data. It reduces the duration and cost of stability studies. Moreover, it has the ability to handle complex degradation models and censoring schemes. In general the optimal Bayesian decision-theoretic sampling plans play a crucial role in modern industries characterized by high costs of failure and limited testing resources.

The rest of the paper is organized as follows. In Section 2, we provide the necessary preliminaries. The developments of the decision-theoretic approach based on the complete sample is discussed in Section 3. We have reviewed different optimal sampling plans for classical Type-I and Type-II censoring, hybrid censoring, progressive hybrid censoring, generalized hybrid censoring and for interval censoring cases in Sections 4, 5, 6, 7 and 8, respectively. Results based on the joint censoring scheme and for competing risks set up are provided in Sections 9 and 10, respectively. Numerical comparisons of different optimal censoring schemes are provided in Section 11 and finally we conclude the paper, point out some open problems for future work, and provide some references for future reading in Section 12. Throughout the article, unless otherwise mentioned, it is assumed that n identical units are placed simultaneously on a life-testing experiment at time zero. The failed units are not replaced.

2 Preliminaries

2.1 Basic Censoring Schemes

We will be using the following notation throughout this section. $\{X_1, \ldots, X_n\}$ is a random sample of size n from a given distribution function $F(x)$. The ordered sample will be denoted by $\{X_{(1)}, \ldots, X_{(n)}\}$. Further, the observed values of $\{X_1, \ldots, X_n\}$ and $\{X_{(1)}, \ldots, X_{(n)}\}$

will be denoted by $\{x_1, \ldots, x_n\}$ and $\{x_{(1)}, \ldots, x_{(n)}\}$, respectively. In general we will denote the capital letter as the random variable and the corresponding observed value will be denoted by the small letter. It should be clear from the context.

The most commonly used censoring schemes are the time-censored and failure-censored schemes, also known as Type-I and Type-II censoring schemes, respectively. In a Type-I censoring scheme, the experiment starts with n number of units with a predefined termination time τ. The experiment terminates at time τ with $M \in \{0, 1, \ldots, n\}$, the number of failures in the experiment. Note that if $x_{(1)} > \tau$, then $M = 0$. If $x_{(1)} < \tau$, the observed data are:

$$x_{(1)} < \ldots < x_{(m)}, \tag{1}$$

where $x_{(m)} < \tau < x_{(m+1)}$ and $M = m$. In a Type-I censoring scheme the time of the experiment is fixed before hand, although the number of failure(s) is random. In a Type-II censoring scheme, we put n items on the test and stop the experiment when r, pre-fixed number of failures, is observed. The observed data set in this case is:

$$x_{(1)} < \ldots < x_{(r)}. \tag{2}$$

In a Type-II censoring scheme the time of the experiment is not fixed, although the number of failures is fixed.

2.2 Hybrid Censoring Schemes

The Type-I hybrid censoring scheme (Type-I HCS) is proposed by Epstein [21]. It is a mixture of both Type-I and Type-II censoring schemes. Here, we prefix censoring time τ and a positive integer $r \in \{1, \ldots, n\}$ before starting the experiment. We put n items on test and terminate the experiment at a time $\tau^* = min\{X_{(r)}, \tau\}$. As before, if M is the random variable denoting the number of failures in the experiment, then the observed data in the experiment for Type-I HCS is one of the following forms: if $X_{(1)} > \tau$, then we do not observe any failure till the time point τ, similar to the Type-I censoring scheme, if $X_{(1)} < \tau$

and $X_{(r)} > \tau$, then

$$x_{(1)} < x_{(2)} < \ldots < x_{(m)}, \tag{3}$$

where $m \in \{1, \ldots, r-1\}$ and if $X_{(r)} \leq \tau$, then

$$x_{(1)} < x_{(2)} < \ldots < x_{(r)}, \tag{4}$$

with $m = r$.

Childs *et al.* [16] proposed the Type-II hybrid censoring scheme(Type-II HCS). Here also we prefix a censoring time τ and the number of failures r, a positive integer, before the experiment starts. We put n items on test in the experiment and terminate the experiment at the time $\max\{X_{(r)}, \tau\}$. The observed data in the experiment for Type-II HCS is in one of the following forms: if $X_{(r)} > \tau$, then

$$x_{(1)} < x_{(2)} < \ldots < x_{(r)} \tag{5}$$

and if $X_{(r)} \leq \tau$, then

$$x_{(1)} < x_{(2)} < \ldots < x_{(m)}, \tag{6}$$

where $m \in \{r, r+1, \ldots, n\}$ is as defined earlier. Readers are referred to Balakrishnan and Kundu [3] for more information on hybrid censoring.

2.3 Generalized Hybrid Censoring Scheme

Chandrasekar *et al.* [4] proposed two generalized hybrid censoring schemes:

2.3.1 Generalized Type-I Hybrid Censoring (Type-I GHCS):

Fix integers $r, l \in \{1, 2, \ldots, n\}$ such that $l < r < n$, and time $\tau \in (0, \infty)$. If the l^{th} failure occurs before time τ, terminate the experiment at $min\{X_{(r)}, \tau\}$. If the l^{th} failure occurs after time τ, terminate the experiment at $X_{(l)}$. It ensures l number of failures before the experiment terminates. Here the experiment gets terminated at a random time,

$$\tau^* = min\{X_{(r)}, \tau\} I(X_{(l)} \leq \tau) + X_{(l)} I(X_{(l)} > \tau).$$

If M' and M are the number of failures before prefixed time τ, and during the entire experiment, respectively, then the data observed during the experiment can be categorized into three possible scenarios:

(a) If $x_{(l)} < \tau < x_{(r)}$, then $\tau^* = \tau$, the observed number of failures are equal, i.e., $m = m'$ and observations are $0 < x_{(1)} < \ldots < x_{(l)} < \ldots < x_{(m')} < \tau$;

(b) If $x_{(r)} < \tau$, then $\tau^* = x_{(r)}$, $m = r$ and observations are $0 < x_{(1)} < \ldots < x_{(r)} < \tau$;

(c) If $x_{(l)} > \tau$, then $\tau^* = x_{(l)}$, $m = l$ and observations are $0 < x_{(1)} < \ldots < x_{(m')} < \tau < x_{(m'+1)} < \ldots < x_{(l)}$.

2.3.2 Generalized Type-II Hybrid Censoring Scheme (Type-II GHCS):

For a fixed integer $r \in \{1, 2, \ldots, n\}$ and time points $\tau_1, \tau_2 \in (0, \infty)$, with $\tau_1 < \tau_2$, if the r^{th} failure occurs before the time point τ_1, terminate the experiment at τ_1. If the r^{th} failure occurs between τ_1 and τ_2, terminate the experiment at $X_{(r)}$. Finally, if the r^{th} failure occurs after τ_2, terminate the experiment at τ_2. Thus, this censoring scheme guarantees that the experiment will be completed by time τ_2. Let the experiment get terminated at a random time,

$$\tau^* = max\{X_{(r)}, \tau_1\} I(X_{(r)} \leq \tau_2) + \tau_2 I(X_{(r)} > \tau_2).$$

Let M_1 and M_2 be the number of failures among the n items before time τ_1 and τ_2, respectively and as before, M is the total number of failures in the experiment. Then the data observed during the course of the experiment can be categorized into three cases:

(a) If $x_{(r)} \leq \tau_1$, then $\tau^* = \tau_1$, $m = m_1$ and observations are $0 < x_{(1)} < \ldots < x_{(r)} < \ldots < x_{(m_1)} < \tau_1$;

(b) If $\tau_1 < x_{(r)} \leq \tau_2$, then $\tau^* = x_{(r)}$, $m = r$ and observations are $0 < x_{(1)} < \ldots < x_{(r)}$;

(c) If $x_{(r)} > \tau_2$, then $\tau^* = \tau_2$, $m = m_2$ and observations are $0 < x_{(1)} < \ldots < x_{(m_2)} < \tau_2$.

In order to save time, Chandrasekar *et al.* [4] added a second threshold time $\tau_2 > \tau_1$, which serves as an upper bound for the experimental duration. This further minimizes the possibility of no failures during the entire experiment. For more details on the generalized hybrid censoring schemes, readers can see Balakrishnan and Kundu [3]. Besides these, there are other censoring schemes too, which are popular in the literature, e.g., random censoring scheme, progressive and progressive hybrid censoring schemes, generalized progressive hybrid censoring etc. Readers are referred to Lawless [38], Miller [50] and Balakrishnan and Kundu [3] for details.

2.4 Progressive Censoring Scheme

In the classical Type-I and Type-II censoring schemes or in the ordinary hybrid censoring schemes, surviving units can only be removed at the terminal point of the experiment. However, in some lifetime experiments, it is not uncommon to remove surviving units during the experiment also, particularly when the items being tested are very expensive. In the statistical literature this is known as a progressive censoring scheme.

Suppose n identical units are placed on a life test. Let $m < n$ be the number of failures to be observed. A progressive censoring scheme is defined by a vector $(R_1, R_2, \ldots, R_m)$, such that $R_i \geq 0$ and $\sum_{i=1}^{m} R_i = n - m$. The experiment proceeds as follows:

- At the time of the first failure, R_1 surviving units are randomly removed.
- At the time of the second failure, R_2 surviving units are removed.
- ...
- At the time of the m-th failure, the remaining R_m units are removed.

In recent times an extensive amount of work has been done in this area, see for example the book by Balakrishnan and Cramer [2] for an excellent exposition of this topic.

Let us briefly introduce the progressive hybrid censoring scheme as introduced by Kundu and Joarder [31] and Childs et al. [15]. Childs et al. [15] proposed a Type-I progressive hybrid-censoring scheme (Type-I PHCS) in the context of life-testing experiment in which n identical units are placed on test with progressive censoring scheme $(R_1, \ldots, R_m)$, and the experiment is terminated at the time $\min\{X_{m:m:n}, T\}$, where $T \in (0, \infty)$

and $1 \leq m \leq n$ are fixed in advance, and $X_{1:m:n} \leq \cdots \leq X_{m:m:n}$ are the ordered failure times resulting from the experiment. Similarly, Type-II progressive censoring scheme (Type-II PHCS) is defined when the termination time is $\max\{X_{m:m:n}, T\}$, see for example Kundu and Joarder [31].

2.5 Interval Censoring Schemes

Sometimes, it may not be possible to monitor continuously the failure times of the experimental units due to different causes. In this type of situation, one usually performs a systematic investigation, in which the test items are inspected only at specific points in time. Thus, one can only record whether a test item fails in an interval instead of measuring the exact failure time. Suppose n items are put in a life testing experiment and the items are inspected at predetermined times $0 < \tau_1 < \ldots < \tau_k < \infty$. Here, τ_i's are usually taken to be equidistant and τ_k is the censoring time where the experiment stops. One observes the number of failures n_i in (τ_{i-1}, τ_i), for $i = 1, 2, \ldots, k+1$, where $\tau_0 = 0$ and $\tau_{k+1} = \infty$. This type of inspection is called interval censoring, and the corresponding sample obtained from such an experiment is called grouped data. Interval censoring scheme is very common in reliability engineering where the components are inspected at scheduled times, so failure is only known to occur between inspections, see for example the monograph by Sun [66].

In decision theoretic sampling plans, censoring schemes play a crucial role in determining the structure and efficiency of sampling plans. The choice of censoring scheme affects the risk function through sampling cost, time constraints, and information content. In case of Type-I censoring, the main advantage is that the experiment terminates at a fixed time duration, making it suitable when time is a primary constraint. It is particularly useful in industrial settings with strict deadlines. The main disadvantage is that in this case the number of failures may be too small, leading to poor inference. The associated risk function may have high variability due to randomness in the number of observed failures. On the other hand, Type-II censoring guarantees a fixed number of failures, ensuring sufficient information for inference. It often leads to more precise parameter estimation and associated risk function is typically more stable compared to Type-I censoring. But the main problem about the Type-II censoring is that the total experimental time is random and may be excessively long. It is not suitable when time constraints are critical.

In case of Type-I hybrid censoring scheme, it provides an upper bound on experimental time. It tries to balances time and information considerations. But it may still result in a small number of failures if failure rates are low. Further, the analytical expressions for the associated risk can be more complicated. Similar to Type-II censoring scheme, Type-II Hybrid Censoring guarantees a minimum number of failures, improving inferential quality. It avoids premature termination of the experiment and it provides more informative than Type-I hybrid censoring. But in this case also, the experimental duration can be quite large, leading to higher cost. It has less control over total testing time.

The generalized Type-I hybrid censoring has greater flexibility in designing sampling plans. It can be tailored to optimize decision-theoretic risk and it balances time, cost, and information more effectively. But it increases the model complexity and computational burden quite significantly. Moreover, it is difficult to implement in practice without sophisticated monitoring. The progressive censoring schemes provides flexibility in resource allocation and it reduces experimental cost while retaining information. But it involves more complex likelihood and risk structure. It requires careful design to avoid excessive information loss. Hence, it is clear that from a decision-theoretic perspective, no censoring scheme is uniformly optimal. Type-I censoring offers control over time but may lack sufficient data, whereas Type-II censoring ensures information at the cost of uncertain duration. Hybrid and generalized hybrid schemes attempt to strike a balance between these competing objectives. The optimal choice depends on the loss structure, cost of sampling, and practical constraints such as time and resource availability.

2.6 Competing Risks

In many situations, an investigator is often interested in the assessment of a specific risk, in presence of other risk factors. In the statistical literature, it is well known as the competing risks problem. In a competing risks problem, the data consist of failure times and an indicator denoting the cause of failure. There are several models to analyze the competing risks problem. Among them, the two most popular models are: (i) the latent failure times model proposed by Cox [18], and (ii) the cause-specific hazard functions model suggested by Prentice [63]. In latent failure model of Cox [18], it is assumed that the competing risks are independent of one another, i.e., if X_{ij} denotes the lifetime of the i^{th} unit under the j^{th} risk then X_{ij}'s are independent for each i. Let us assume that there are only two causes of failures. Suppose n units are put in the experiment and $\{X_{i1}, X_{i2}\}$ are the random variables denoting the lifetimes of i^{th} units due to cause 1 and 2, respectively. Suppose $I_i = 1,$ denotes that the i^{th} unit fails due to cause 1 and $I_i = 0$ denotes that the i^{th} unit fails due to cause 2, i.e.,

$$I_i = \begin{cases} 1, & \text{if} \quad X_{i1} \leq X_{i2} \\ 0, & \text{if} \quad X_{i1} > X_{i2}, \end{cases}$$

for all $i = 1, 2, \ldots, n$. It is assumed that if the i-th unit fails at Z_i, then $Z_i = min\{X_{i1}, X_{i2}\}$. Further, $\{Z_1, Z_2, \ldots, Z_n\}$ are statistically independent lifetimes of n units and $\{I_1, I_2, \ldots, I_n\}$ are associated indicator variables denoting the cause of failure of these n units, respectively. A typical competing risks data set is of the following form:

$$\mathbf{z} = \{(z_1, I_1), (z_2, I_2), \ldots, (z_n, I_n)\}. \tag{7}$$

A substantial amount of work has been done on different aspects related to competing risks. Interested readers are referred to the classical book of Crowder [19] for further reading in this area.

2.7 Lifetime Distributions

In this section we provide some brief reviews of some of the lifetime distributions which have been used in this article. Readers are referred to Johnson *et al.* [28, 29] for more details on these lifetime distributions.

Exponential Distribution

The exponential distribution serves as one of the basic lifetime models and plays a significant role in the field of life testing experiments. It is the simplest, most natural and widely used model in many areas of research, and is defined as follows:

Definition 2.1. *A continuous random variable X follows a one-parameter exponential distribution, denoted by $X \sim exp(\lambda)$, if its PDF is given by:*

$$f(x|\lambda) = \begin{cases} \lambda e^{-\lambda x}, & \textit{if} \quad x > 0,\ \lambda > 0 \\ 0, & otherwise, \end{cases} \tag{8}$$

where λ is the scale parameter. In this case it also represents the failure rate of items per unit of time.

If θ is the mean lifetime of the item then $\theta = \frac{1}{\lambda}$. Throughout this article in $exp(\lambda)$, we mean λ is the scale parameter as in (8) and θ as the mean lifetime of $exp(\lambda)$, unless otherwise mentioned. The more general model is a two parameter exponential model and will be denoted $exp(\mu, \lambda)$, it has the following PDF

$$f(x|\lambda) = \begin{cases} \lambda e^{-\lambda(x-\mu)} & \text{if } \ x \geq \mu,\ \lambda > 0, -\infty < \mu < \infty \\ 0, & otherwise. \end{cases} \tag{9}$$

Here, λ and μ are the scale and location parameters, respectively.

Gamma Distribution

A continuous random variable is said to have a gamma distribution with the shape parameter a and the scale parameter b, will be denoted by $G(a, b)$, if its PDF is given by:

$$f(x|a,b) = \begin{cases} \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}, & \text{if} \quad x > 0, a > 0, b > 0 \\ 0, & otherwise. \end{cases} \tag{10}$$

If $a = 1$, we obtain

$$f(x|b) = \begin{cases} be^{-bx}, & \text{if} \quad x > 0, b > 0 \\ 0, & otherwise, \end{cases}$$

i.e. $G(1,b) = exp(b)$. A more general form of a gamma distribution is known as the shifted gamma distribution with the shape parameter a, the scale parameter b and the location parameter c, and it has the following the PDF;

$$f_{SG}(y|a,b,c) = \begin{cases} \frac{b^a}{\Gamma(a)} (y-c)^{a-1} e^{-b(y-c)}, & \text{if} \quad y > c, a > 0, b > 0, -\infty < c < \infty \\ 0, & otherwise. \end{cases} \tag{11}$$

Beta Distribution

A continuous random variable is said to have a Beta distribution (more specially, Beta distribution of the first kind) if it has the following PDF:

$$f(y|\alpha,\beta) = \begin{cases} \frac{1}{B(\alpha,\beta)} y^{\alpha-1} (1-y)^{\beta-1}, & \text{if} \quad 0 < y < 1, \alpha > 0, \beta > 0 \\ 0, & otherwise, \end{cases} \tag{12}$$

where $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

Normal Distribution

A continuous random variable is said to follow a Normal distribution with mean μ and variance σ^2, will be denoted by $N(\mu,\sigma)$, if it has the following PDF:

$$f(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0. \tag{13}$$

3 Complete sample

In this section we provide a brief review on the Bayesian sampling plan for a complete sample. Fertig and Mann [22] has discussed the decision-theoretic approach to find the variable sampling plan for finite lots based on a Bayesian framework. It is assumed that the quality of a lot is determined by the proportion of defective items in the lot, and the lifetime distributions of the experimental units are either exponential or Gaussian. Fertig and Mann [22] used the loss function L, developed by Hald [24], which depends on k, the number of defects in the lot, the decision $\delta(\mathbf{x})$ and on the realization $\mathbf{x}$ of the random vector $\mathbf{X}$. It is given by

$$L(k,\delta(\mathbf{x})) = \begin{cases} na_s + (k-l), & \text{if } \delta(\mathbf{x}) = d_0, \\ na_s + (N-n)a_r, & \text{if } \delta(\mathbf{x}) = d_1, \end{cases}$$

where l represents the number of defects in a sample of size n, N is the lot size, a_s is the cost of sampling per item, and a_r is the cost of sorting the rejected portion of the lot per item. The unit of cost is taken as the cost of a defective item in an accepted lot. The term $d_0(d_1)$ represents the action "accept the lot" ("reject the lot"). We will be using this notation throughout this article. In some places we will use $d_0 = 1$ (accept the lot) and $d_1 = 0$ (reject the lot). It should be clear from the context. Let $X \sim F_X(x|\theta)$, then the distribution of k given θ, is binomial. Set $\delta(\mathbf{x}) = d_0$ if

$$E\{L(k,d_0)|\mathbf{x}\} \leq E\{L(k,d_1)|\mathbf{x}\} \tag{14}$$

equivalently,

$$E\{k-l|\mathbf{x}\} \leq (N-n)a_r \tag{15}$$

The expected value of $k-l$ given $\mathbf{x}$ is computed from

$$E\{k-l|\mathbf{x}\} = E_{\theta|\mathbf{x}}(E\{k-l|\mathbf{x},\theta\}), \tag{16}$$

where

$$E(k-l|\mathbf{x},\theta) = (N-n)[1-F_X(u|\theta)] = (N-n)p, \tag{17}$$

$p = [1-F_X(u|\theta)]$ is the process quality and u is the upper specification limit. The Equation (15) tells us to accept the lot if and only if the expected cost of accepting is less than the cost of rejecting. The acceptance region when observations are coming from an exponential or from a normal distribution can be obtained as follows:

Acceptance Region for an Exponential Process: Let $X \sim exp(\lambda)$, as defined in (8). Let us assume an improper prior $\pi(\lambda) = \frac{1}{\lambda}$, $\lambda > 0$ on the unknown parameter λ. Hence, the posterior becomes $G(n, n\bar{x})$. Therefore,

$$E\{k-l|\mathbf{x}\} = (N-n)\left(\frac{n\bar{x}}{n\bar{x}+u}\right)^n.$$

Using (15) the acceptance region of the lot becomes

$$\frac{\bar{x}}{u} \leq \frac{a_r^{1/n}(1-a_r^{1/n})}{n}.$$

Acceptance Region for a Normal Process: It is assumed that $X \sim N(\mu, \sigma)$. The posterior density of μ, given the sample, can be determined by using the fact that $(\bar{x}-\mu)/\sigma$ is $N(0, \frac{1}{\sqrt{n}})$, i.e., normal with mean zero and standard deviation $1/\sqrt{n}$. Thus, $\mu \sim N(\bar{x}, \frac{\sigma}{\sqrt{n}})$. This corresponds to a prior density for μ of the form $\pi(\mu) = 1, \ \ -\infty < \mu < \infty$. The prior density $\pi(\mu)$ is also a limiting member of the natural conjugate family of densities given by Raiffa and Schlaifer [64]. Therefore,

$$E\{k-l|\mathbf{x}, \sigma\} = (N-n)\left\{1 - \Phi\Big(\frac{u-\bar{x}}{\sigma\sqrt{\frac{n+1}{n}}}\Big)\right\}$$

where $\Phi(.)$ is the distribution of standard normal random variable. For known variance case by (15) acceptance region of the lot is

$$\frac{\bar{x}-u}{\sigma} \leq -\sqrt{\frac{n+1}{nz_{1-a_r}}}$$

where z_α is such that $\Phi(z_\alpha) = \alpha$. For unknown variance case,

$$E\{k-l|\mathbf{x}\} = (N-n)\left\{1 - T_{n-1}\Big(\frac{u-\bar{x}}{s\sqrt{\frac{n+1}{n}}}\Big)\right\}$$

where s^2 is the sample variance, T_r is the cumulative Student's-t distribution with r degrees of freedom and lot is accepted if

$$\frac{\bar{x}-u}{s} \leq -\sqrt{\frac{n+1}{nt_{1-a_r,n-1}}},$$

where $t_{\alpha,r}$ is the 100αth percentile of a Student's-t variate with r degrees of freedom. In determining the sample size, the classical procedure can be adopted, i.e. choose the smallest n such that the "risks" at the acceptable quality level (AQL) and the rejectable quality level (RQL) are less than some preassigned values.

Fertig and Mann [22] have considered sampling inspection plans based on a decision theory approach where they obtained the acceptance region by minimizing the posterior risk for fixed n. But for determining the optimal value of n they don't use Bayes risk minimization criterion and also they deal with only the case of linear loss which lead them to the optimal sample size which was usually not an integer. Lam [32] had discussed the single Bayesian variable sampling plans for polynomial loss functions in a different way by minimizing the Bayes risk. A single measurement is taken on each component from a batch of products, and is assumed to have a normal distribution $N(m,\tau^2)$, where τ^2 is known. Furthermore, m has a prior distribution $N(\mu,\sigma^2)$ with known μ and σ^2. A random sample of size n is taken from the batch. A decision function $\delta(\mathbf{x})$, either d_0 or d_1, is adopted. The one-sided and two sided decision functions are as follows:

$$\delta(\mathbf{x}) = \begin{cases} d_0, & 0 \leq \bar{x} - \mu < \delta \\ d_1, & \text{otherwise,} \end{cases} \tag{18}$$

and

$$\delta(\mathbf{x}) = \begin{cases} d_0, & \delta_1 \leq \bar{x} - \mu < \delta_2 \\ d_1, & \text{otherwise,} \end{cases} \tag{19}$$

where $\bar{x}$ is the sample mean, and δ, δ_1, and δ_2 are all non-negative real numbers. The loss function is given by:

$$L(m,\delta(\mathbf{x})) = \begin{cases} nC_s + C_0 + C_1(m-\mu_0) + \ldots + C_k(m-\mu_0)^k & \delta(\mathbf{x}) = d_0 \\ nC_s + C_r & \delta(\mathbf{x}) = d_1, \end{cases} \tag{20}$$

where μ_0 is the standard value of X specified by standard or contract, and C_1, C_1, $\ldots$, C_k, C_r and C_s, are constants all independent of m.

In Bayesian decision theory, the choice of a loss function plays a crucial role in determining optimal sampling plans. A commonly used class of loss functions in this context is the polynomial loss function. The preference for polynomial forms arises from several theoretical and practical considerations. First, polynomial loss functions provide substantial analytical tractability. When the prior and likelihood belong to conjugate families, the posterior distribution often admits closed-form expressions. In such cases, the expected posterior loss under a polynomial loss function can frequently be expressed in terms of moments of the posterior distribution. Since posterior moments are often available in closed form, this simplifies the computation of Bayes risk and facilitates the derivation of optimal decision rules. Second, polynomial loss functions are flexible and general. They can approximate a wide range of loss behaviors through suitable choice of degree and coefficients. For example, the squared error loss is a special case of a quadratic polynomial loss function. Higher-order polynomials allow asymmetric and more complex penalty structures, which may be useful in practical applications such as reliability testing and quality control. Third, these loss functions are compatible with moment-based summaries. In many sampling plan problems, especially in life-testing and acceptance sampling, decisions depend on summary statistics such as sample means or order statistics. Polynomial loss functions naturally align with such summaries, as their expectations can be expressed using moments of the underlying distribution. Fourth, polynomial loss functions often ensure computational efficiency. In Bayesian optimal sampling plans, one needs to evaluate expected losses repeatedly for different sample sizes and decision rules. Polynomial forms reduce computational burden compared to more complex nonlinear loss functions, particularly when numerical integration is required. Finally, from a modeling perspective, polynomial loss functions can be interpreted as providing smooth and continuous penalties for estimation or decision errors. This avoids abrupt changes in the loss structure and leads to stable and well-behaved optimal solutions.

Now the optimal decision function $\delta(\mathbf{x})$ is determined such that the Bayes risk is minimized under the given loss function. In other words, in one-sided case, a single sampling plan (n, δ) of size n with one specification limit δ is obtained, so that the decision function $\delta(\mathbf{x})$ minimizes the Bayes risk $r(n,\delta) = E[L(m,\delta(\mathbf{X}))]$. Similarly, in a two-sided case, a single sampling plan (n, δ_1, δ_2) of size n with two specification limits δ_1 and δ_2 is obtained, so that the decision function $\delta(\mathbf{x})$ minimizes the Bayes risk $r(n, \delta_1, \delta_2) = E[L(m, \delta(\mathbf{X}))]$. Explicit expressions of the Bayes risks in case of a quadratic loss function $(k = 2)$ can be obtained for the one-sided and two-sided cases. An optimal single sampling plan is obtained by choosing a pair (n_0, δ_0) which corresponds to the smallest value of the Bayes risk $r(n, \delta)$. Similarly for two-sided case also optimal sampling plan is obtained by minimizing the Bayes risk $r(n, \delta_1, \delta_2)$.

The Bayesian sampling plan for symmetric two-sided decision function, that is when $\delta_1 = \delta_2 = \delta$ becomes

$$\delta(\mathbf{x}) = \begin{cases} d_0 & |\bar{x} - \mu_0| < \delta \\ d_1 & |\bar{x} - \mu_0| \geq \delta \end{cases} \tag{21}$$

and for $\mu_0 = 0$, the associated loss function takes the following form;

$$L(m, \delta(\mathbf{x})) = \begin{cases} nC_s + C_2 m^2 & \delta(\mathbf{x}) = d_0 \\ nC_s + C_r & \delta(\mathbf{x}) = d_1 \end{cases} \tag{22}$$

The above Bayesian sampling plan is considered by Lam [33], when the τ^2 is known. Lam and Lau [37] generalizes this work for unknown τ^2.

Lam and Lam [36] has discussed the Bayesian double-sampling plans for complete sample with normal distribution. Suppose that the quality of an item in a batch is measured by a random variable X. Assume that $X \sim N(\lambda, \tau^2)$ with known variance τ^2 and λ has a prior distribution $N(\mu, \sigma^2)$ where μ and σ^2 are known. The double sampling plan can be briefly described as following. A random sample $\mathbf{X}_1 = (X_1, \ldots, X_{n_1})$ of size n_1 is taken from the batch. Let the observations be $\mathbf{x}_1 = (x_1, \ldots, x_{n_1})$ and let

$$\bar{x}_1 = \sum_{i=1}^{n_1} x_i / n_1$$

Then the batch is accepted if $-\delta_{a1} \leq \bar{x}_1 - \lambda_0 \leq \delta_{a2}$, and it is rejected if $\bar{x}_1 - \lambda_0 < -\delta_{r1}$ or $\bar{x}_1 - \lambda_0 > \delta_{r2}$, where $\delta_{rl} \geq \delta_{al} \geq 0, \delta_{r2} \geq \delta_{a2} \geq 0$ and λ_0 is the standard value. Otherwise, a second random sample $\mathbf{X}_2 = (X_{n_1+1}, X_{n_1+2}, \ldots, X_n)$ of size n_2 is taken independently of the first random sample with $n = n_1 + n_2$. Thus $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent. Let the observations for the second sample be $\mathbf{x}_2 = (x_{n_1+1}, x_{n_1+2}, \ldots, x_n)$. Also let $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$, $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$ and

$$\bar{x} = \sum_{i=1}^{n} x_i / n$$

Then, the batch is accepted if $-\delta_1 \leq \bar{x} - \lambda_0 \leq -\delta_2$ and it is rejected otherwise; here $\delta_1, \delta_2 \geq 0$. The double-sampling plan is denoted by $(n_l, n_2, \delta_{al}, \delta_{a2}, \delta_{r1}, \delta_{r2}, \delta_1, \delta_2)$. Now, write

$$\begin{aligned}
V_{1a} &= \{\mathbf{x} : \mathbf{x} = \mathbf{x}_1, -\delta_{a1} \leq \bar{x}_1 - \lambda_0 \leq \delta_{a2}\} \\
V_{2a} &= \{\mathbf{x} : -\delta_{r1} \leq \bar{x}_1 - \lambda_0 < \delta_{a1} \text{ and } -\delta_1 \leq \bar{x} - \lambda_0 \leq \delta_2\} \\
&\quad \cup \{\mathbf{x} : -\delta_{a1} < \bar{x}_1 - \lambda_0 \leq \delta_{r2} \text{ and } -\delta_1 \leq \bar{x} - \lambda_0 \leq \delta_2\} \\
V_{1r} &= \{\mathbf{x} : \mathbf{x} - \mathbf{x}_1, \bar{x}_1 - \lambda_0 < -\delta_{r1} \text{ or } \bar{x}_1 - \lambda_0 > \delta_{r2}\}, \\
V_{2r} &= \{\mathbf{x} : -\delta_{r1} \leq \bar{x}_1 - \lambda_0 < \delta_{a1} \text{ and } (\bar{x} - \lambda_0 < -\delta_1 \text{ or } \bar{x} - \lambda_0 > \delta_2)\} \\
&\quad \cup \{\mathbf{x} : -\delta_{a1} < \bar{x}_1 - \lambda_0 \leq \delta_{r2} \text{ and } (\bar{x} - \lambda_0 < -\delta_1 \text{ or } \bar{x} - \lambda_0 > \delta_2)\}
\end{aligned}$$

The meaning of this notation is as follows: V_{ia} means the event that the batch is accepted after taking the i-th sample, whereas V_{ir} represents the event of rejecting the batch after taking the i-th sample $(i = 1, 2)$. Thus the decision function is of the form:

$$\delta(\mathbf{x}) = \begin{cases} d_0 & \mathbf{x} \in V_{1a} \cup V_{2a}, \\ d_1 & \mathbf{x} \in V_{1r} \cup V_{2r}. \end{cases}$$

The following loss function has been used:

$$L(m, \delta(\mathbf{x})) = \begin{cases} n_1 C_s + a_0 + a_1 \lambda + \ldots + a_k \lambda^k & \text{if } \mathbf{x} \in V_{1a}, \\ n C_s + a_0 + a_1 \lambda + \ldots + a_k \lambda^k & \text{if } \mathbf{x} \in V_{2a}, \\ n_1 C_s + C_r & \text{if } \mathbf{x} \in V_{1r}, \\ n C_s + C_r & \text{if } \mathbf{x} \in V_{2r} \end{cases} \tag{23}$$

where the coefficients $a_0, a_1, \ldots, a_k$ and the parameters C_r and C_s are constants, C_s refers to the inspection cost per item and C_r is the cost of rejecting a batch. Assume further that $C_s > 0$ and

$$a_0 + a_1\lambda + \ldots + a_k\lambda^k \geq 0 \quad \text{for all real } \lambda > 0$$

A double-sampling plan will reduce to a single-sampling plan if $\delta_{al} = \delta_{r1}$, and $\delta_{a2} = \delta_{r2}$. In this case, we can take $n_2 = 0$. Hence the plan $(n_1, 0, \delta_{al}, \delta_{a2}, \delta_{al}, \delta_{a2}, \delta_1\delta_2)$ will represent a single-sampling plan. The Bayes risk of double-sampling plan $(n_l, n_2, \delta_{al}, \delta_{a2}, \delta_{r1}, \delta_{r2}, \delta_1, \delta_2)$ is obtained by computing $E(L(m, \delta(\mathbf{x})))$. The optimal double sampling plan, let us denote it by $(n_{l0}, n_{20}, \delta_{al0}, \ \delta_{a20}, \delta_{r10}, \delta_{r20}, \delta_{10}, \delta_{20})$, is obtained by minimizing the associated Bayes risk. For the quadratic loss function, i.e., when $k = 2$, explicit expressions of the Bayes risk and the corresponding optimal double sampling plan can be obtained.

4 Type-I and Type-II censoring

In this section, we will discus the Bayesian variable sampling plan for Type-I and Type-II censoring schemes. Lam [33] has obtained the optimal single variable sampling plan for Type-II censoring scheme. It is assumed that the lifetime $X \sim exp(\lambda)$. The property of an item is determined by the lifetime X. Moreover, λ has a gamma conjugate prior distribution $G(a, b)$. A random sample of size n is taken from the batch and the decision function is as follows:

$$\delta(\mathbf{X}) = \begin{cases} d_0 & \text{if} \quad \hat{\theta} \geq T \\ d_1 & \qquad \text{otherwise.} \end{cases} \tag{24}$$

Here, T is the minimum acceptable survival time, based on which the batch will be accepted or rejected and $\hat{\theta}$, the maximum likelihood estimator (MLE) of the mean lifetime θ, is given by

$$\hat{\theta} = \frac{\sum_{i=1}^{r} X_{(r)} + (n - r)X_{(r)}}{r}. \tag{25}$$

The following polynomial loss function has been considered,

$$L(\lambda, \delta(\mathbf{X})) = \begin{cases} nC_s + a_0 + a_1\lambda + \ldots + a_k\lambda^k & \text{if} \quad \delta(\mathbf{X}) = d_0 \\ nC_s + C_r & \text{if} \quad \delta(\mathbf{X}) = d_1, \end{cases} \tag{26}$$

where $a_0, \ldots, a_k, C_r$ and C_s are constants with a natural constraint

$$a_0 + a_1\lambda_1 + \ldots + a_k\lambda^k \geq 0, \qquad \lambda > 0.$$

Under this set up the Bayes risk can be obtained in explicit forms for the quadratic loss function. The optimal sampling plan (n_0, r_0, T_0) can be obtained by minimizing the Bayes risk $r(n, r, T)$ w.r.t. n, r, and T numerically.

The result has been extended when the lifetime of an item in a batch follows a two-parameter exponential distribution with the scale parameter λ and the location parameter μ. A four-parameter conjugate prior distribution for (λ, μ) has been assumed, and it has the following joint density function:

$$p(\lambda, \mu) = \frac{1}{A}\lambda^{\alpha-1} exp[-\lambda(\beta - \lambda\mu)], \qquad \lambda > 0, 0 < \mu \leq \eta,$$

where $\alpha > 0$, $\beta > 0$, $\alpha \leq \gamma < \beta/\eta$.

$$A = \begin{cases} \frac{\alpha-1}{\gamma}\left[\frac{1}{(\beta-\gamma\eta)^{\alpha-1}} - \frac{1}{\beta^{\alpha-1}}\right] & \text{if} \quad \alpha \neq 1, \\ \frac{1}{\gamma}[ln\beta - ln(\beta - \gamma\eta)] & \text{if} \quad \alpha = 1 \end{cases}$$

with known α, β, γ, and η. This conjugate prior distribution was introduced and studied by Varde [69]. Varde restricted α and γ to be positive integers, but this is not necessary. The loss function in this case is also a polynomial as follows;

$$L(\lambda, \mu, \delta(\mathbf{X})) = \begin{cases} nC_s + \sum_{0 \leq i+j \leq k} C_{ij}\lambda^i\mu^j & \text{if} \quad \delta(\mathbf{X}) = d_0 \\ nC_s + C_r & \text{if} \quad \delta(\mathbf{X}) = d_1. \end{cases} \tag{27}$$

where C_{ij} are constants. The decision function is as follow:

$$\delta(\mathbf{X}) = \begin{cases} d_0, & \text{if} \quad \widehat{W} \geq T \\ d_1, & \quad \text{otherwise} \end{cases} \tag{28}$$

where $T \geq \mu$ and

$$\widehat{W} = \begin{cases} \hat{\mu} & \text{if} \quad r = 1, \\ \hat{\mu} + \hat{\theta} & \text{if} \quad r \geq 2. \end{cases} \tag{29}$$

is the MLE of the average lifetime W, where $\hat{\mu}$ and $\hat{\theta}$ are the MLEs of μ and θ, respectively. For degree $k = 2$ the Bayes risk $r(n, r, T)$ can be obtained in explicit forms, and the optimal sampling plan (n_0, r_0, T_0) can be obtained by minimizing the Bayes risk w.r.t. n, r, and T.

Lam [34] has provided the Bayesian sampling plan for Type-I censoring scheme. In this case also it is assumed that the lifetime random variable $X \sim exp(\lambda)$. The scale parameter λ has a conjugate Gamma prior $G(a, b)$ prior. The loss function (26) is considered with the decision function (24) under Type-I censoring, where the estimator $\hat{\theta}$ is defined as:

$$\hat{\theta} = \begin{cases} \hat{\theta}_M & \text{if} \quad M \geq 1 \\ n\tau & \text{if} \quad M = 0. \end{cases} \tag{30}$$

In this case $\hat{\theta}_M$ is the MLE of θ under Type-I censoring and it has the following form;

$$\hat{\theta}_M = \begin{cases} \dfrac{\sum_{i=1}^{m} X_{(i)} + (n-m)\tau}{M} & \text{if} \quad M \geq 1 \\ \infty & \text{if} \quad M = 0. \end{cases} \tag{31}$$

Under this setup also for a quadratic loss function, the Bayes risk can be obtained in explicit forms. Further, the Bayesian optimal sampling plan, (n_0, τ_0, T_0), which minimizes the Bayes risk $r(n, \tau, T)$ w.r.t. n, τ, and T can be obtained in a finite discrete search algorithm.

Chen and Lam [7] have developed the Bayesian sampling plan for the Weibull distribution when the data are Type-I censored. The quality of an item in a batch is measured by a random variable which follows a Weibull distribution with the scale parameter λ and the shape parameter m. It is assumed that the scale parameter has a gamma prior and the shape parameter has an independent uniform prior. The decision function in this case depends on the Kaplan-Meier estimator of the survival function:

$$\delta(\mathbf{X}) = \begin{cases} d_0, & \hat{S}_n(\tau) \geq p, \\ d_1, & \text{otherwise}, \end{cases} \tag{32}$$

where under Type I censoring, the Kaplan-Meier estimator of the survival function is given by:

$$\hat{S}_n(\tau) = \begin{cases} 1, & \tau < X_{(1)} \\ \prod_{X_{(i)} \leq \tau} \Big(\frac{n-i}{n-i+1}\Big), & \text{otherwise}. \end{cases} \tag{33}$$

The single sampling plan in this case is defined by (n, τ, p), where n is the sample size, τ is the censoring time and p is the minimum acceptance reliability. The following exponential loss function has been used;

$$L(\lambda, m, \delta(\mathbf{X})) = \begin{cases} nC_s - (n-M)r_s + \tau a_s + C_0[p_s - e^{-\lambda t_s^m}]I(e^{-\lambda t_s^m} < p_s), & \delta(\mathbf{X}) = d_0 \\ nC_s - (n-M)r_s + \tau a_s + C_r & \delta(\mathbf{X}) = d_1, \end{cases} \tag{34}$$

where C_s, r_s, a_s, C_r, C_0, p_s, and t_s are pre-assigned values. At the end of the experiment, there are $(n-M)$ surviving items each of which can be sold at a reduced price r_s and a_s is the rate of time-consuming cost. To determine an optimal sampling plan (n_0, τ_0, p_0) the Bayes risk $R(n, \tau, p)$ is minimized w.r.t. n, τ, and p. It can be shown that the optimal sampling plan can be obtained in a finite search algorithm.

Huang and Lin [26] has provided an improved Bayesian sampling plan under Type-I censoring when the lifetime distribution of an experiment unit follows exponential distribution. The following estimator of the mean lifetime has been used:

$$\hat{\theta}_M = \begin{cases} \frac{\sum_{i=1}^{M} X_{(i)}}{M} + \frac{[(n-M)\tau]}{M}(1 - \frac{1}{M} + \frac{1}{M^2} - \frac{1}{M^3}), & \text{if} \quad M \neq 0, \\ n\tau, & \text{otherwise} \end{cases} \tag{35}$$

The estimator $\hat{\theta}_M$ dominates the MLE of θ asymptotically both in bias and in mean square error (see Huang and Chen [25]). The decision function is defined as:

$$\delta(\mathbf{X}) = \begin{cases} d_0, & \text{if } \hat{\theta}_M \geq T + \varepsilon, \\ d_0, & \text{with probability } p \text{ if } T - \varepsilon \leq \hat{\theta}_M < T + \varepsilon, \\ d_1, & \text{with probability } 1-p \text{ if } T - \varepsilon \leq \hat{\theta}_M < T + \varepsilon, \\ d_1, & \text{if } \hat{\theta}_M < T - \varepsilon. \end{cases} \tag{36}$$

The sampling plan is given by $(n, \tau, T, \varepsilon, p)$ and the associated loss function becomes:

$$L(\lambda, \delta(\mathbf{X})) = nC_s + \tau C_\tau + \delta(\mathbf{X})h(\lambda) + (1 - \delta(\mathbf{X}))C_r \tag{37}$$

where C_τ is cost on per unit time and $h(\lambda)$ is acceptance cost which will be positive and increasing in λ for $\lambda > 0$. The explicit expressions of Bayes risk can be derived under the loss function with $h(\lambda) = a_0 + a_1\lambda + a_2\lambda^2$, and the corresponding optimal sampling plan $(n_0, \tau_0, T_0, \varepsilon_0, p_0)$ can be obtained by minimizing the Bayes risk $r(n, \tau, T, \varepsilon, p)$ w.r.t. n, τ, T, ε, and p.

Lin et al. [49] has observed that the sampling plan proposed by Lam [34] has some defects. Although, the MLE of the mean lifetime has certain optimal property and the decision function based on the MLE is a natural decision function, the decision function is not the Bayes decision function. The main reason is that the cost on the time τ is not included in the loss function of Lam [34]. Therefore, when the sample size n is fixed then the best choice of the censoring time becomes at $\tau = \infty$, from which one observes the complete sample and this will provide a better decision rule. Lin et al. [49] has obtained the Bayesian sampling plan for Type-I censoring by obtaining the Bayes decision function and including cost on time of the experiment in the loss function. The following loss function including the cost of time has been considered:

$$L(\lambda, \delta(\mathbf{X})) = \begin{cases} nC_s + \tau C_\tau + h(\lambda), & \delta(\mathbf{X}) = 1 \\ nC_s + \tau C_\tau + C_r, & \delta(\mathbf{X}) = 0, \end{cases} \tag{38}$$

where $h(\lambda) = a_0 + a_1\lambda + a_2\lambda^2$. Under Type-I censoring scheme, the Bayes decision function takes the following form;

$$\delta_B(\mathbf{X}) = \begin{cases} 1, & \text{if } \phi(m, y(n,\tau,m)) \leq C_r \\ 0, & \text{otherwise,} \end{cases} \tag{39}$$

where $\phi(m,y(n,\tau,m)$ is the posterior expectation of $h(\lambda)$ and $y(n,\tau,m) = \sum_{i=1}^{m} x_i + (n-m)\tau$. It can be shown that for a fixed m, $\phi(m,y(n,\tau,m)$ is strictly decreasing in $y(n,\tau,m)$ for $y(n,\tau,m) > 0$. Therefore, an alternative form of the Bayes decision function becomes:

$$\delta_B(\mathbf{X}) = \begin{cases} 1, & \text{if } y(n,m,\tau) \geq D_n(m) - \beta \\ 0, & \text{otherwise,} \end{cases} \tag{40}$$

where

$$D_n(m) = \frac{a_1(m+\alpha) + \sqrt{a_1^2(m+\alpha^2)^2 + 4(C_r - a_0)a_2(m+\alpha)(m+\alpha+1)}}{2(C_r - a_0)}.$$

Note that if the acceptance cost $h(\lambda)$ is a higher degree polynomial function or a non-polynomial function then $D_n(m)$ cannot be obtained analytically. The optimal Bayesian sampling plan (n_B, τ_B) is obtained by minimizing the Bayes risk $r(n,\tau)$, w.r.t. n and τ.

For $M = 0$, the MLE of the mean life time does not exist. Due to this reason, Lin et al. [45] obtained the exact Bayesian sampling plan for Type-I censoring based on the conditional MLE. But due to conditioning on at least one failure in order for the MLE to exist, the proposed Bayesian sampling plan has a higher Bayes risks than the minimum Bayes risks. Prajapati et al. [54] has used a shrinkage estimator of the mean lifetime which always exists and had provided a decision-theoretic sampling plan for Type-I censoring scheme which is as good as the Bayesian sampling plan in terms of the Bayes risk. The shrinkage estimator of the mean lifetime θ has the following form:

$$\hat{\theta}_{M+c} = \frac{\sum_{i=1}^{M} X_i + (n-M)\tau}{M+c}.$$

The decision function is;

$$\delta(\mathbf{X}) = \begin{cases} 1, & \text{if} \quad \hat{\theta}_{M+c} \geq \xi \\ 0, & \text{if} \quad \hat{\theta}_{M+c} < \xi, \end{cases} \tag{41}$$

and the associated sampling becomes (n, τ, ξ, c). Based on the loss function

$$L(\lambda, \delta(\mathbf{X})) = \begin{cases} nC_s - (n-M)r_s + \tau C_\tau + h(\lambda) & \delta(\mathbf{X}) = 1, \\ nC_s - (n-M)r_s + \tau C_\tau + C_r & \delta(\mathbf{X}) = 0, \end{cases}$$

the Bayes risk of decision theoretic sampling plan (n, τ, ξ, c) can be obtained. The optimal decision theoretic sampling plan is obtained by minimizing the Bayes risk with respect to n,τ, ξ and c. The main advantage of this method is that even when the loss function is a higher degree polynomial or a non-polynomial function, the optimal Bayesian sampling plan can be obtained. Prajapati et al. [55] had also provided another decision theoretic sampling plan based on the estimator of λ, for Type-I censoring scheme. The estimator of λ is

$$\hat{\lambda} = \begin{cases} 0, & \text{if} \quad M = 0 \\ \hat{\lambda}_{MLE}, & \text{if} \quad M \geq 1 \end{cases} \tag{42}$$

where $\hat{\lambda}_{MLE}$ is the MLE of λ. It is clear that $\hat{\lambda}$ always exists. Based on $\hat{\lambda}$, the decision function becomes:

$$\delta(\mathbf{X}) = \begin{cases} 1, & \text{if} \quad \hat{\lambda} < \zeta \\ 0, & \text{otherwise} \end{cases} \tag{43}$$

where ζ is the threshold point which is used to determine the quality of the lot, and it is a part of the sampling plan. In this case a decision theoretic sampling plan is (n, τ, ζ). The optimal decision-theoretic sampling plan is obtained by minimizing the Bayes risk. Prajapati et al. [56] further proposed a decision function based on the Bayes estimator of the mean lifetime of θ. It always exists for all values of M, under Type-I censoring scheme. The optimal sampling plan can be obtained by minimizing the Bayes risks.

The quality control of products is essential to the manufacturer since it immediately influences the market sales and also the manufacturer's profits. If the batch of items which will be sold in the market is under a warranty policy, the study must include a warranty cost. A manufacturer's warranty gives certain levels of quality assurance of a product during a specific periods. However, the costs incurred to the manufacturer due to this policy, reduces the profits. The commonly used warranty policies are the failure-free policy and the prorated rebate policy as introduced by Thomas [67]. In this case, the failure-free policy is used during the period $(0, w_1)$ and the prorated rebate policy is used during (w_1, w_2) where $w_1 < w_2$ are pre-specified positive constants. Chen et al. [9] first considered this and designed the Bayesian sampling plans for Type-II censored samples when the lifetime distribution of the item follows an exponential distribution with the parameter λ. The following loss function has been considered:

$$L(\delta, n, r, \lambda) = nC_s - (n-M)r_s + \tau C_\tau + \delta \sum_{i=1}^{N} C(X_i) + (1-\delta)C_r,$$

where δ is an action taken on the batch, when $\delta = 1$, it means to accept the batch; and when $\delta = 0$, it means to reject the batch, $C_r = Nc_p$ is the cost of rejection the batch, c_p is the cost of rejecting an item in the batch, and according to the general rebate warranty policy, the cost of an accepted item with lifetime X is:

$$C(X) = \begin{cases} c_a & \text{if } X < w_1 \\ c_a(w-X)/(w_2-w_1), & \text{if } w_1 \leq X < w_2, \\ 0, & \text{if } X \geq w_2, \end{cases} \tag{44}$$

where w_1 is time limit of total rebate warranty, w_2 is time limit of prorated warranty ($w_2 > w_1$), and c_a is the cost associated with the failure of an item in the accepted batch during the warranty period. The Bayes decision function, and an efficient optimal Bayesian sampling plan have been obtained by using the curtailment procedure.

The curtailment procedure was first introduced by Chen et al. [13]. Sometimes, it is necessary to make a proper decision as soon as the cumulative information regarding the lifetime of the experiments units is sufficient. An online inspection design can enhance the efficiency of the conventional censoring schemes. Chen et al. [13] discussed the optimal curtailed censoring scheme for Type-II censoring scheme. Similar procedure has been followed by Tsai et al. [68] for Type-I censored data. Bayes decision functions, and optimal censoring schemes have been provided in both the cases. In both the cases the resulting decision functions can reduce the experimental times, which lead to smaller Bayes risks.

5 Type-I and Type-II hybrid censoring

In the last section we have discussed about the Bayesian sampling plans under the traditional Type-I and Type-II censoring schemes. In recent times Bayesian sampling plans under different other censoring schemes have also been developed. In this section we will briefly review Bayesian sampling plans for Type-I and Type-II hybrid censoring schemes. Chen et al. [5] first designed Bayesian acceptance sampling plan for Type-I hybrid censoring scheme. The decision function is based on the survival function given in (32), and it has the following form:

$$\delta(\mathbf{X}) = \begin{cases} d_0, & \text{if} \quad M \leq [n(1-p)], \\ d_1, & \quad \text{otherwise} \end{cases} \tag{45}$$

here M denotes the number of failures observed in the experiment and $[a]$ is the integer part of a. The loss function has the form;

$$L(\lambda, \delta(\mathbf{X})) = \begin{cases} nC_s - (n-M)r_s + \tau^* C_\tau + h(\lambda) & \text{if} \quad \delta(\mathbf{X}) = d_0, \\ nC_s - (n-M)r_s + \tau^* C_\tau + C_r & \text{if} \quad \delta(\mathbf{X}) = d_1 \end{cases} \tag{46}$$

where r_s is the salvage value. In case of exponential lifetime distribution, $h(\lambda) = a_0 + a_1\lambda + \ldots + a_k\lambda^k$ and for Weibull lifetime $h(\lambda) = (N-n)C[p_s - S(t_s)]I(S(t_s) < p_s)$, where t_s and p_s are known pre-assigned values and C is a proportional constant. An optimal sampling plan (n_0, r_0, τ_0, p_0) is obtained by minimizing the Bayes risk. Lin et al. [46] obtained the exact Bayesian variable sampling plans for an exponential distribution based on Type-I and Type-II hybrid censored samples. The decision function is based on the MLE of the mean lifetime of θ, and it is defined as:

$$\delta(\mathbf{X}) = \begin{cases} 1, & \text{if } \hat{\theta} > \xi \\ 0, & \text{otherwise} \end{cases} \tag{47}$$

For $M=0$ the MLE does not exist in case of Type-I hybrid censoring. The exact distribution of the conditional MLE of the mean lifetime has been used to compute the Bayes risk of the decision function under the quadratic loss function:

$$L(\delta(\mathbf{X}),\lambda,n) = nC_s + \delta(\mathbf{X}(a_0 + a_1\lambda + a_2\lambda^2) + (1-\delta(\mathbf{X})C_r. \tag{48}$$

Optimal sampling plans have been obtained by minimizing the Bayes risks.

Liang and Yang [41] criticized the method proposed by Lin et al. [46] and mentioned that not including the cost on time in the loss function the exact Bayesian sampling plan obtained by Lin et al. [46] has Bayes risk lager then the minimum Bayes risk. They have considered the loss function which has the cost on time component and obtained the Bayes risks in explicit forms. Finally optimal censoring schemes have been provided by minimizing the Bayes risks.

The Bayesian sampling plan proposed Liang and Yang [41] cannot be obtained quite easily when the loss function is a higher degree polynomial or a non-polynomial function. Prajapati et al. [54] had mention this limitation of the Bayesian sampling plan proposed by Liang and Yang [41]. Instead of using the conditional MLE of the mean lifetime θ, Prajapati et al. [54] proposed a shrinkage estimator of the mean lifetime θ as:

$$\hat{\theta}_{M+c} = \begin{cases} \frac{\sum_{i=1}^{M} X_i + (n-M)\tau}{M+c} & M = 0,\ldots,r-1 \\ \frac{\sum_{i=1}^{r} X_i + (n-r)X_{(r)}}{r+c} & M = r. \end{cases} \tag{49}$$

It always exists even when there is no failure has been observed in the experiment. The corresponding decision function takes the form

$$\delta(\mathbf{X}) = \begin{cases} 1, & \hat{\theta}_{M+c} \geq \xi \\ 0, & \hat{\theta}_{M+c} < \xi. \end{cases} \tag{50}$$

Based on the same loss function as in Liang and Yang [41], Prajapati et al. [54] obtained the Bayes decision function, and by minimizing the Bayes risks Bayes optimal sampling plan has been obtained. It has been shown by Prajapati et al. [54] that the two Bayesian sampling plans have the same Bayes risks if the acceptance cost $h(\lambda) = a_0 + a_1\lambda + a_2\lambda^2$, and they could extend their results for higher degree polynomial and non-polynomial loss functions also. The optimal decision-theoretic sampling plan is obtained by determining the optimal decision function which minimizes the Bayes risk among all decision-theoretic sampling plans (n, r, τ, ξ, c). Due to the presence of shrinkage estimator the dimension of the sampling plan increases. To tackle this issue Prajapati et al. [55] had suggested a decision theoretic sampling plan based on the estimator of the scale parameter λ instead of θ, which always exists. The optimal decision-theoretic sampling plan is obtained by minimizing the Bayes risk. It has been shown that the optimum decision-theoretic sampling plan $(n_0, r_0, \tau_0, \zeta_0)$ is as good as Bayesian sampling plan of Liang and Yang [41] in terms of Bayes risk and it can be easily used even for higher degree or non-polynomial loss functions also.

Prajapati et al. [56] discussed the Bayesian sampling plan for Type-I hybrid censoring using the Bayes estimator of mean lifetime which always exists for all values of M. The Bayes estimator of mean lifetime θ for squared error loss function under Type-I hybrid censoring scheme is given by:

$$\hat{\theta}_B = \begin{cases} \frac{b+\sum_{i=1}^{M} X_i + (n-M)\tau}{M+a} & \text{if } X_{(r)} \geq \tau \\ \frac{b+\sum_{i=1}^{r} X_i + (n-r)X_r}{r+a} & \text{if } X_{(r)} < \tau \end{cases} \tag{51}$$

The decision function is:

$$\delta(\mathbf{X}) = \begin{cases} 1, & \hat{\theta}_B > \xi \\ 0, & \text{otherwise} \end{cases} \tag{52}$$

The optimal sampling plan $(n_0, r_0, \tau_0, \xi_0)$ is obtained by minimizing the Bayes risk w.r.t. n, r, τ, and ξ.

Chen et al. [8] constructed a curtailed Bayesian sampling plan with Type-I hybrid censored samples. The curtailed decision function possesses the property that it is equivalent to the Bayes decision function in the sense that the decisions made by the two decision functions on whether accepting or rejecting the batch are the same. It reduces the duration time of the life test experiment, the resulting Bayes risk is always less than or equal to that of Bayes risk of Bayesian sampling plan.

6 Type-I and Type-II progressive hybrid censoring

Lin et al. [48] generalized the results of Lin et al. [46] for progressively hybrid censored data. It is assumed that the lifetime of the item follows an exponential distribution with scale parameter λ and the prior is the conjugate gamma prior. The goal is to find an optimal sampling plan $(n,m,R_1,\ldots,R_m,T,\xi)$ possessing the property that the Bayes risk is minimum among the class of all sampling plans $(n,m,R_1,\ldots,R_m,T,\xi)$ based on Type-I and Type-II PHCS, respectively. The decision function for acceptance sampling is

$$\delta(\mathbf{X}) = \begin{cases} 1, & \hat{\theta} > \xi \\ 0, & \text{otherwise}, \end{cases} \tag{53}$$

where $\hat{\theta}$ is MLE of mean lifetime θ under Type-I and Type-II PHCS, respectively and the following loss function has been considered:

$$L(\delta\mathbf{X},\lambda,n) = nC_s + \delta(\mathbf{X})(a_0 + a_1\lambda + a_2\lambda^2) + (1-\delta(\mathbf{X}))C_r. \tag{54}$$

Based on simulated annealing algorithm of Corana et al. [17], an optimal sampling plan is obtained by minimizing the associated Bayes risk. The optimal sampling plans based on progressive hybrid censoring scheme and ordinary hybrid censoring scheme have been compared. It is observed that the former performs better in terms of lower Bayes risks.

Lin and Huang [44] have applied the same procedure to investigate the optimal sampling plans from an exponential distribution under adaptive progressive censoring schemes (APCS). The APCS as introduced by Ng et al. [52] can be briefly described as follows. In a Type-I APCS, during the experiment, for each $j = 1,\ldots,m-1$, when the

jth failure is observed, immediately after the failure, R_j functioning items are randomly removed from the life test. Let $X_{j:m,n}$ and $X_{m:m,n}$ denote the lifetime of the jth and mth failed item, respectively. When $X_{m:m,n} \geq \tau$, the experiment terminates at time τ. When $X_{m:m,n} < \tau$, at the mth failure, do not follow the pre-specified censoring scheme to remove the remaining R_m functioning items; instead, continue to observe failures without any further withdrawals up to time τ. For the Type-II APCS, during the experiment, for each $j = 1, \ldots, m$ if $X_{j:m:n} \leq T$, immediately after the failure, R_j functioning items are randomly removed from the life test; if $X_{j:m:n} > T$ and $j < m$, do not follow the pre-specified censoring scheme to remove R_j functioning items from the life test; instead, continue to observe failures without any further withdrawals; and if $X_{m:m:n} > T$, immediately remove all the remaining functioning items and terminate the experiment. Let D denote the number of failures before time τ. Thus, $0 \leq D \leq m$. The duration of the life test experiment is $\tau^* = X_{m:m:n}$. Lin and Huang [44] have obtained the optimal sampling plan for Type-I APCS and Type-II APCS under a fairly general class of loss functions, and compared them with Type-I PHCS and Type-II PHCS.

Liang [39] observed that the sampling plans proposed by Lin and Huang [44] are not Bayesian sampling plans. Liang [39] provided the Bayesian sampling plans for quadratic loss function, and numerically it has been shown that it performs better than the sampling plans proposed by Lin and Huang [44] in terms of lower Bayes risks.

7 Generalized Hybrid Censoring Schemes

Usually, any life testing experiment incorporates different types of costs, for example, cost on time, cost on items, etc. Standard life testing experiments are usually censored due to the unavailability of time or money. The choice of the censoring scheme also becomes essential for getting the maximum information from the observed sample. In a Type-I hybrid censoring scheme, experiment time is bounded, but it may so happen that very few failures occur in that time, which affects the efficiency of the estimators.

On the other hand, the Type-II hybrid censoring scheme ensures $'r'$ number of failures, but it may take a long time to observe those 'r' failures. To overwhelm the deficiency of the hybrid censoring scheme, Chandrasekar et al. [4] proposed the generalized hybrid censoring schemes. Most of the work on the Bayesian sampling plan are based on an estimator of the mean lifetime. However, if the estimator is not efficient, it affects the optimal sampling plan, leading to a wrong decision. Therefore, it is reasonable to include the efficiency of the estimator of the mean lifetime of an experimental unit, as a component of the loss function. Prajapati et al. [57] has considered such a loss function in deriving the Bayesian sampling plan for the generalized Type-II hybrid censoring scheme (Type-II GHCS).

Let us introduce briefly a Type-II GHCS as introduced by Chandrasekar et al. [4]. In a Type-II GHCS, for a fixed integer $r \in \{1, 2, \ldots, n\}$ and time points $\tau_1, \tau_2 \in (0, \infty)$, with $\tau_1 < \tau_2$, if the r^{th} failure occurs before the time point τ_1, the experiment is terminated at τ_1. If the r^{th} failure occurs between τ_1 and τ_2, the experiment is terminated at $X_{(r)}$. Finally, if the r^{th} failure occurs after τ_2, the experiment is terminated at τ_2. This censoring guarantees that the experiment will be completed by time τ_2. It is assumed that $X_1, X_2, \ldots, X_n$ are mutually independent and identically follow an exponential distribution with parameter λ and the parameter λ has a prior distribution gamma $G(a, b)$. In Type-II GHCS, let the experiment gets terminated at random time τ^*. Let M_1 and M_2 be the number of failures among n items up to time τ_1 and τ_2, respectively. Then the MLE of λ is given by:

$$\begin{aligned}
\widehat{\lambda}_{MLE} &= \begin{cases} \frac{M_1}{\sum_{i=1}^{M_1} X_{(i)} + (n-M_1)\tau_1}, & \text{if } \ M_1 = r, r+1, \ldots, n \\ \frac{r}{\sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}}, & \text{if } \ M_1 = 0, 1, \ldots, r-1 \ \ \& \ \ M_2 = r \\ \frac{M_2}{\sum_{i=1}^{M_2} X_{(i)} + (n-M_2)\tau_2}, & \text{if } \ M_2 = 1, \ldots, r-1, \end{cases} \\
&= \frac{M}{\sum_{i=1}^{M} X_{(i)} + (n-M)\tau^*} = \frac{M}{Y(n, r, \tau_1, \tau_2, M)}, \ \ \forall\, M \geq 1,
\end{aligned}$$

where $Y(n, r, \tau_1, \tau_2, M) = \sum_{i=1}^{M} X_{(i)} + (n-M)\tau^*$. Clearly for $M = M_2 = 0$, the MLE does not exist. In place of the MLE of λ, the following estimator has been used, which always exists.

$$\widehat{\lambda} = \begin{cases} 0, & \text{if } M = 0 \\ \widehat{\lambda}_{MLE}, & \text{if } M > 0. \end{cases} \tag{55}$$

The associated Bayes decision function becomes:

$$\delta_B = \delta_B(m, y(n, r, \tau_1, \tau_2, m)|n, r, \tau_1, \tau_2) = \begin{cases} 1, & \text{if } \phi_\pi(m, y(n, r, \tau_1, \tau_2, m)) \leq C_r \\ 0, & \text{otherwise,} \end{cases} \tag{56}$$

where $\phi_\pi(m, y(n, r, \tau_1, \tau_2, m))$ is the posterior expectation of the acceptance cost $h(\lambda)$. It has been shown that the Bayes decision function can be written as:

$$\delta_B(m, y(n, r, \tau_1, \tau_2, m)|n, r, \tau_1, \tau_2) = \begin{cases} 1, & \text{if } \hat{\lambda} < c(n, r, \tau_1, \tau_2, m) \\ 0, & \text{otherwise,} \end{cases} \tag{57}$$

where for $m \geq 1$, $c(n, r, \tau_1, \tau_2, m) = m/T(n, r, \tau_1, \tau_2, m)$ and for $m = 0$, $c(n, r, \tau_1, \tau_2, 0) = n\tau_2 - T(n, r, \tau_1, \tau_2, 0)$, and $T(n, r, \tau_1, \tau_2, m = 0) = T_0$; $T(n, r, \tau_1, \tau_2, m) = min\{T_m, n\tau_2\}$, for $m = 1, 2, \ldots, n$, where if $A(m) = \{y > 0 | \phi_\pi(m, y) \leq C_r\}$ then

$$T_m = \begin{cases} \inf A(m), & \text{if } A(m) \text{ is nonempty} \\ \infty, & \text{otherwise.} \end{cases} \tag{58}$$

The Bayesian decision function (57) can be written in terms of the estimator $\hat{\lambda}$ indicates that the estimator plays an important role in making decision of the batch. The following loss function has been used which has included the efficiency of the estimator of λ as a component in it:

$$L(\delta(\mathbf{X}), \lambda) = \begin{cases} nC_s - (n-M)r_s + \tau^* C_\tau + MSE(\hat{\lambda})C_v + h(\lambda), & \text{if } \delta(\mathbf{X}) = 1 \\ nC_s - (n-M)r_s + \tau^* C_\tau + MSE(\hat{\lambda})C_v + C_r, & \text{if } \delta(\mathbf{X}) = 0, \end{cases} \tag{59}$$

where n is the sample size, M denotes the number of failures in the experiment, τ^* is the duration of the experiment, $MSE(\hat{\lambda})$ is the MSE of the estimator $\hat{\lambda}$, C_v is the cost associated with the imprecision in the estimator $\hat{\lambda}$, and $h(\lambda)$ is the cost on accepting the batch, which is positive and an increasing function in λ for $\lambda > 0$. Finally, the Bayes decision function, and the optimal Bayes sampling plans have been obtained. The results have been extended for the generalized Type-I censoring scheme (Type-I GHCS) also, see for example Prajapati et al. [58].

8 Interval Censoring Schemes

Chen et al. [12] discussed the Bayesian sampling plans for exponential distributions with interval censored samples as discussed in Section 2.5. In an interval censoring, n items are placed in a life testing experiment at the initial time 0. The items are inspected at predetermined times $0 < \tau_1 < \ldots < \tau_k < \infty$, where $\tau_j = j\Delta$, $\Delta > 0$, $j = 1, \ldots, k$ and τ_k is the censoring time. Δ is the time duration between two successive inspection times and it is assumed to be known. The life testing experiment terminates either when all the n items fail before time τ_k or at the time τ_k. When the latter case occurs, the remaining functioning items are censored at τ_k. It is assumed that the lifetime of the n items are independent identically distributed exponential distribution, having the expected lifetime θ. The scale parameter λ has the usual conjugate gamma prior. The loss function is as follows:

$$L(\delta, \lambda, n, k) = \delta h(\lambda) + (1-\delta)C_r + nC_s + \tau(n,k,\Delta)C_\tau \tag{60}$$

where δ denotes an action on the problem of acceptance sampling. When $\delta = 1$, it means that the batch is accepted; and when $\delta = 0$, it means to reject the batch and C_r is the cost of rejecting the batch, $h(\lambda)$ is the cost of accepting the batch, $h(\lambda) > 0$ and increases in λ for $\lambda > 0$, C_s the cost per sampling unit, C_τ the cost per unit of the life test experiment, and $\tau(n,k,\Delta)$ denotes the duration of the experiment. Based on the loss function (60) the Bayes decision function takes the following form:

$$\delta(\mathbf{x}|n,k) = \begin{cases} 1, & \text{if } \phi(\mathbf{x}) - C_r < 0, \\ 0, & \text{otherwise,} \end{cases} \tag{61}$$

where $\phi(\mathbf{x})$ is the posterior expectation of $h(\lambda)$ given $\mathbf{X} = \mathbf{x}$. The Bayes decision function cannot be obtained in closed forms, hence, the optimal Bayesian sampling plan has been obtained numerically.

Chen et al. [12] under this same set up obtained the curtailed Bayes decision function. It has been shown that the curtailed Bayes decision function is equivalent to the Bayes decision function. The optimal curtailed Bayesian sampling plan is obtained by minimizing the corresponding Bayes risk. It is observed that this curtailed Bayes decision function can reduce the duration time of the experiment, leading to a reduction in the Bayes risk. Therefore, the Bayes risk of a curtailed Bayesian sampling plan is smaller than the Bayes risk of the Bayesian sampling plan.

9 Joint Censoring Schemes

An extensive amount of work has been done in developing decision theoretic sampling plans for one sample problem, although not much work has been done for more than one sample. Prajapati et al. [60] first considered the optimal Bayesian sampling plans for more than one sample based on a joint censoring scheme proposed by Mondal and Kundu [51]. Mondal and Kundu [51] have introduced a joint progressive censoring scheme applied on two or multiple samples called the balanced joint progressive censoring (BJPC) scheme.

The BJPC scheme as proposed by Mondal and Kundu [51] can be briefly described as follows. Suppose there are two different kinds of products from product lines A and B, and it is required to study the relative merits of these two kinds of products. A sample of size m is drawn from the product line A and it is called sample A. Similarly, another sample of size n is drawn from the product line B and it is called sample B. Let k be the total number of failures to be observed from the life testing experiment and $R_1, \ldots, R_{k-1}$ be pre-specified non-negative integers such that $\sum_{i=1}^{k-1}(R_i+1) < \min(m,n)$. The units of these two samples are simultaneously put on the test. Suppose the first failure comes from sample A and the failure time is denoted by W_1. Then at W_1 time point, R_1 units are removed randomly from the remaining $m-1$ surviving units of sample A, and R_1+1 units are chosen randomly from the remaining n surviving units of sample B, and they are withdrawn from the experiment. Next, if the second failure comes from sample B at time point W_2, $R_2 + 1$ units are withdrawn from the remaining $m - R_1 - 1$ units of sample A, and R_2 units are withdrawn from the remaining $n - R_1 - 2$ units of sample B randomly at W_2 time point. The experiment is continued until the k-th failure occurs. At the k-th failure time point W_k, the experiment is terminated with the removal of all the remaining surviving units from both the samples. Along with the ordered failure time W_i, we also record the indicator variable Z_i, which is 1, if i-th failure comes from sample A and 0 if it comes from sample B. Let us define, $K_1 = \sum_{i=1}^{k} Z_i$ denoting the number of failures from sample A and $K_2 = \sum_{i=1}^{k}(1 - Z_i)$ be the number of failures from sample B, in this experiment. Suppose n items from product line A follow exponential distribution with mean θ_1 and n items from product line B follow exponential distribution with mean θ_2. Applying the BJPC scheme and from the likelihood function, it is evident that the MLEs of θ_1 and θ_2 exist only when $0 < k_1 < k$.

Prajapati et al. [60] used the following shrinkage estimators of θ_1 and θ_2 to obtain the optimal sampling plan.

$$\widehat{\theta}_{1Sh} = \frac{U}{k_1 + c}, \quad \widehat{\theta}_{2Sh} = \frac{U}{k_2 + c},$$

for some $c > 0$. The shrinkage parameter c is included in the part of the sampling plan. It is possible to choose different c values for two different estimators, but it makes the optimization problem more complex, so it is not recommended. The shrinkage estimators exist even when $k_1 = 0$ or k, for any $c > 0$. The decision function based on shrinkage estimator becomes:

$$\delta(\mathbf{W},\mathbf{Z}) = \begin{cases} (d_{10}, d_{20}), \text{ if } \widehat{\theta}_{1Sh} > \xi_1, \ \widehat{\theta}_{2Sh} > \xi_2 \\ (d_{10}, d_{21}), \text{ if } \widehat{\theta}_{1Sh} > \xi_1, \ \widehat{\theta}_{2Sh} < \xi_2 \\ (d_{11}, d_{20}), \text{ if } \widehat{\theta}_{1Sh} < \xi_1, \ \widehat{\theta}_{2Sh} > \xi_2 \\ (d_{11}, d_{21}), \text{ if } \widehat{\theta}_{1Sh} < \xi_1, \ \widehat{\theta}_{2Sh} < \xi_2, \end{cases} \tag{62}$$

where $d_{10}(d_{11})$ denotes the action of accepting (rejecting) a batch from product line A and $d_{20}(d_{21})$ denotes the action of accepting (rejecting) a batch from product line B. ξ_1 and ξ_2 denote the minimum acceptable mean survival time from product line A and B, respectively. The associated loss function is given by:

$$L(\delta,\lambda_1,\lambda_2)=\begin{cases} nC_s-(n-K_1)r_{s_1}-(n-K_2)r_{s_2}+W_kC_\tau & \\ \quad +\text{MSE}(\widehat{\theta}_{1Sh})C_{e_1}+\text{MSE}(\widehat{\theta}_{2Sh})C_{e_2}+g_1(\lambda_1)+g_2(\lambda_2), & \text{if } \delta(\mathbf{W},\mathbf{Z})=(d_{10},d_{20}) \\ nC_s-(n-K_1)r_{s_1}-(n-K_2)r_{s_2}+W_kC_\tau & \\ \quad +\text{MSE}(\widehat{\theta}_{1Sh})C_{e_1}+\text{MSE}(\widehat{\theta}_{2Sh})C_{e_2}+g_1(\lambda_1)+C_{r_2}, & \text{if } \delta(\mathbf{W},\mathbf{Z})=(d_{10},d_{21}) \\ nC_s-(n-K_1)r_{s_1}-(n-K_2)r_{s_2}+W_kC_\tau & \\ \quad +\text{MSE}(\widehat{\theta}_{1Sh})C_{e_1}+\text{MSE}(\widehat{\theta}_{2Sh})C_{e_2}+C_{r_1}+g_2(\lambda_2), & \text{if } \delta(\mathbf{W},\mathbf{Z})=(d_{11},d_{20}) \\ nC_s-(n-K_1)r_{s_1}-(n-K_2)r_{s_2}+W_kC_\tau & \\ \quad +\text{MSE}(\widehat{\theta}_{1Sh})C_{e_1}+\text{MSE}(\widehat{\theta}_{2Sh})C_{e_2}+C_r, & \text{if } \delta(\mathbf{W},\mathbf{Z})=(d_{11},d_{21}). \end{cases} \tag{63}$$

Here, $C_s=C_{s_1}+C_{s_2}$ where $C_{s_1}(C_{s_2})$ is the inspection cost per item from product line A (line B), C_τ is the cost per unit time for conducting the experiment, $C_{r_1}(C_{r_2})$ is the cost of rejecting the batch from product line A (line B), and C_r is the cost of rejecting both the batches. Further, $g_1(\lambda_1)=l'_0+l'_1\lambda_1+l'_2\lambda_1^2$ is the cost on accepting the batch from product line A, which has to be positive and an increasing function in λ_1 for $\lambda_1>0$, and $g_2(\lambda_2)=l''_0+l''_1\lambda_2+l''_2\lambda_2^2$ is the cost on accepting the batch from product line B, which is also positive and an increasing function in λ_2 for $\lambda_2>0$. If an item from product line A (line B) does not fail, it can be reused with a salvage value $r_{s_1}(r_{s_2})$ such that $C_{s_1}>r_{s_1}\geq 0$ ($C_{s_2}>r_{s_2}\geq 0$). If the mean square errors (MSE) of the estimators are high, then from sample to sample the decision may vary significantly even when the production process is under control. Therefore, we consider cost C_{e_1} and C_{e_2} on the mean square errors of the estimators $\widehat{\theta}_{1Sh}$ and $\widehat{\theta}_{2Sh}$, respectively. Note that when we reject a batch, the whole batch is discarded or returned to the supplier, and the loss is fixed, so we presume that C_{r_1}, C_{r_2} and C_r are fixed. Under this set-up the decision-theoretic sampling plan can be defined as $(n,k,R_1,\ldots,R_{k-1},\xi_1,\xi_2,c)$.

Based on the following joint beta-gamma prior on λ_1 and λ_2

$$\pi(\lambda_1,\lambda_2) = \frac{\Gamma(a_1+a_2)}{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)} {b_0}^{a_0}(\lambda_1+\lambda_2)^{a_0-a_1-a_2}\lambda_1^{a_1-1}\lambda_2^{a_2-1}e^{-b_0(\lambda_1+\lambda_2)}; \quad \lambda_1,\lambda_2 > 0 \quad (64)$$

and the loss function (63), the explicit expression of the Bayes risk of the decision function (62), which is a function of the sampling plan $(n,k,R_1,\dots,R_{k-1},\xi_1,\xi_2,c)$ has been derived. The optimal sampling plan $(n_0,k_0,R_{10},\dots,R_{k-10},\xi_{10},\xi_{20},c_0)$ can be obtained by minimizing the Bayes risk w.r.t. $n,k,R_1,\dots,R_{k-1},\xi_1,\xi_2$, and c. Prajapati et al. [61] extended this work by constructing the Bayes decision fucntion under BJPC scheme and obtained the two sample Bayesian acceptance sampling plan.

10 Competing Risks Problem

Prajapati et al. [59] considered the Bayesian sampling plan under competing risks set up. They have mainly considered Type-II and Type-I hybrid censoring schemes based on the latent failure model assumption of Cox [18]. In which it is assumed that the competing risks are independent of one another, i.e., if X_{ij} denotes the lifetime of the i^{th} unit under the j^{th} risk then X_{ij}'s are independent for each i. Let n identical test units be simultaneously put on a test. Let there be only two competing causes of failures, and lifetimes X_{ij} of the i^{th} test unit due to j^{th} competing cause follow exponential distribution with the scale parameter λ_j. Since X_{i1} and X_{i2} are independent and if Z_i denotes the lifetime of the i^{th} unit, then $Z_i = min\{X_{i1},X_{i2}\}$ with $\{Z_1,Z_2,\dots,Z_n\}$ being statistically independent lifetimes of n units. If X_{ij} has an exponential distribution then clearly Z_i's are also exponentially distributed.

Suppose $I_i = 1$ denotes that the i^{th} unit fails due to cause 1 and $I_i = 0$ denotes that the i^{th} unit fails due to cause 2, i.e.,

$$I_i = \begin{cases} 1, & \text{if} \quad X_{i1} \le X_{i2} \\ 0, & \text{if} \quad X_{i1} > X_{i2}, \end{cases}$$

for all $i = 1,2,\dots,n$. Then $\{I_1,I_2,\dots,I_n\}$ are indicator variables denoting the cause of failure of $\{Z_1,Z_2,\dots,Z_n\}$, respectively. Let $\{Z_{(1)},Z_{(2)},\dots,Z_{(n)}\}$ be the ordered lifetimes of n units, and $I_{[i]}$ is the associated indicator variable denoting the cause of failure of $Z_{(i)}$. Let δ denote the decision function based on the observations. For $\delta = 1$, the batch is accepted and $\delta = 0$, the batch is rejected. The following loss function has been considered:

$$L(\delta(\mathbf{Z}), \lambda_1, \lambda_2) = \begin{cases} nC_s + \tau^* C_\tau - (n-M)r_s + g(\lambda_1, \lambda_2), & \text{if} \quad \delta(\mathbf{Z}) = 1 \\ nC_s + \tau^* C_\tau - (n-M)r_s + C_r, & \text{if} \quad \delta(\mathbf{Z}) = 0. \end{cases} \tag{65}$$

Here C_s be the inspection cost per item, C_τ cost per unit time, C_r rejection cost, $g(\lambda_1, \lambda_2)$ acceptance cost. Items which do not fail in the experiment are reused with a salvage value r_s. It is assumed that $C_s,\ C_\tau,\ C_r,\ r_s$ are non-negative, $C_s > r_s \geq 0$, and acceptance cost $g(\lambda_1, \lambda_2)$ is positive and non-decreasing function of λ_1 and λ_2. Based on independent gamma priors of λ_1 and λ_2 the following Bayes decision function can be obtained:

$$\delta_B = \delta_B(d_1, d_2, w|\mathbf{S}) = \begin{cases} 1, & \text{if} \ \ \phi_\pi(d_1, d_2, w) \leq C_r \\ 0, & \text{otherwise}, \end{cases} \tag{66}$$

where $\phi_\pi(d_1, d_2, w)$ is posterior expectation of acceptance cost $g(\lambda_1, \lambda_2)$ and

$$\mathbf{S} = \begin{cases} (n, r, \tau), & \text{under Type-I hybrid censoring} \\ (n, r), & \text{under Type-II censoring.} \end{cases} \tag{67}$$

It is also observed that if the acceptance cost $g(\lambda_1, \lambda_2) = a_0 + a_1\lambda_1 + a_2\lambda_2$, or $g(\lambda_1, \lambda_2) = a_{00} + a_{10}\lambda_1 + a_{01}\lambda_2 + a_{20}\lambda_1^2 + a_{02}\lambda_2^2 + a_{11}\lambda_1\lambda_2$, with positive coefficients, then explicit forms of the Bayes decision functions can be obtained. Under this set-up the optimal Bayesian sampling plans have been reported by Prajapati et al. [59].

11 Numerical Results

In this section, we present some numerical results mainly to compare different Bayesian sampling plans available under different censoring schemes.

11.1 Type-I Censoring

In this section first we compare the Bayesian sampling plans proposed by Lam [34], Lin et al. [49] and Huang and Lin [26] under Type-I censoring schemes. It may be recalled that the loss function considered by Lam [34] did not have the cost on time component. Since there is no cost on time component in the loss function, it is natural to run the experiment as long as possible to make a better decision, and it results smaller Bayes risks. In Table 1 we have presented the optimal sampling plan as presented by Lam [34] along with optimal sampling plan when $\tau = \infty$. Here τ denotes the stopping time for a Type-I censoring scheme. We have used the following coefficients in the cost function $C_s = 0.5$, $C_\tau = 0, C_r = 30$ $a_0 = a_1 = a_2 = 2.0$.. Further, a and b denote the hyperparameters of the conjugate gamma prior. From Table 1, it is clear that the optimal sampling plan as presented by Lam [34], is actually not optimal because it does not minimize the Bayes risk.

Table 1: Lam's sampling plan

a	b	$r(n_0, \tau_0, T_0)$	$r(n_0, \infty, T^*)$
0.2	0.2	$r(4, 0.0270, 0.1080) = 12.1499$	$r(4, \infty, 0.3020) = 9.2253$
1.5	0.8	$r(3, 0.5262, 0.2631) = 16.6233$	$r(3, \infty, 0.2333) = 16.5825$
2.5	0.8	$r(3, 0.7077, 0.3539) = 24.9366$	$r(3, \infty, 0.3336) = 24.8847$
3.0	0.8	$r(3, 0.8170, 0.4085) = 27.6136$	$r(3, \infty, 0.3865) = 27.5581$

Lin et al. [49] and Huang and Lin [26] also observed this, and they have included the cost on time component in the loss function. In Table 2 we have provided the optimal sampling plans and the associated Bayes risks based on the decision functions proposed by Lin et al. [49] and Huang and Lin [26]. In both the cases the cost on time $c_\tau = 0.5$. It is clear from Table 2 that the decision function proposed by Lin et al. [49] provides a better result than Huang and Lin [26] in terms of smaller Bayes risks.

Let us recall that Prajapati et al [54, 55, 56] proposed three different decision functions to compute the optimal sampling plans for Type-I censored samples. Now in

Table 2: Comparison between the Lin et al. [49] and Huang and Lin [26] Bayesian sampling plans.

a	b	$r(n_B, \tau_B, T_B, \varepsilon_B, p_B)$ (Huang and Lin [26])	$r(n_B, \tau_B, \delta_B)$ (Lin et al. [49])
0.1	0.2	$r(2, 0.0576, 0.1729, 0.0005, 0.5) = 8.1382$	$r(2, 0.4038) = 6.1832$
2.5	0.8	$r(3, 1.4611, 0.7305, 0.0040, 0.1) = 25.9132$	$r(3, 0.6750) = 25.2777$
2.5	1.0	$r(4, 1.2546, 0.6273, 0.0035, 0.3) = 22.6044$	$r(4, 0.6738) = 22.0361$
3.0	0.8	$r(3, 1.2800, 1.2800, 0.0090, 0.7) = 28.4507$	$r(3, 0.8250) = 28.0087$

Table 3 we compare the Bayes risks of the optimal sampling plans of Lin et al. [49] and Prajapati et al [54, 55, 56]. We have considered the same set of coefficients as before.

Table 3: Bayes risks comparison of the Bayesian sampling plans between the Lin et al. [49] and Prajapati et al. [54, 55, 56] for Type-I censoring scheme.

a	b	Lin et al. [49]	Prajapati et al. [54]	Prajapati et al. [55]	Prajapati et al. [56]
0.1	0.2	6.1832	6.1832	6.1832	6.1832
2.5	0.8	25.2777	25.2777	25.2777	25.2777
2.5	1.0	22.0361	22.0361	22.0361	22.0361
3.0	0.8	28.0087	28.0087	28.0087	28.0087

In Figure 1 we have provided the Bayes risks of the different optimal sampling plans for Type-I censored cases. Some of the points are clear from Table 3 and Figure 1. It is immediate that the optimal sampling plan proposed by Lam [34] is not optimal. The plan proposed by Lin et al. [49] is optimal. The optimal sampling plans proposed by Prajapati et al. [54, 55, 56] perform as good as the sampling plans of Lin at el. [49] in terms of the Bayes risks. The main advantage of the optimal sampling plans proposed by Prajapati et al. [54, 55, 56] is that it can be used quite easily in practice and also for higher degree polynomial and for non-polynomial loss functions also.

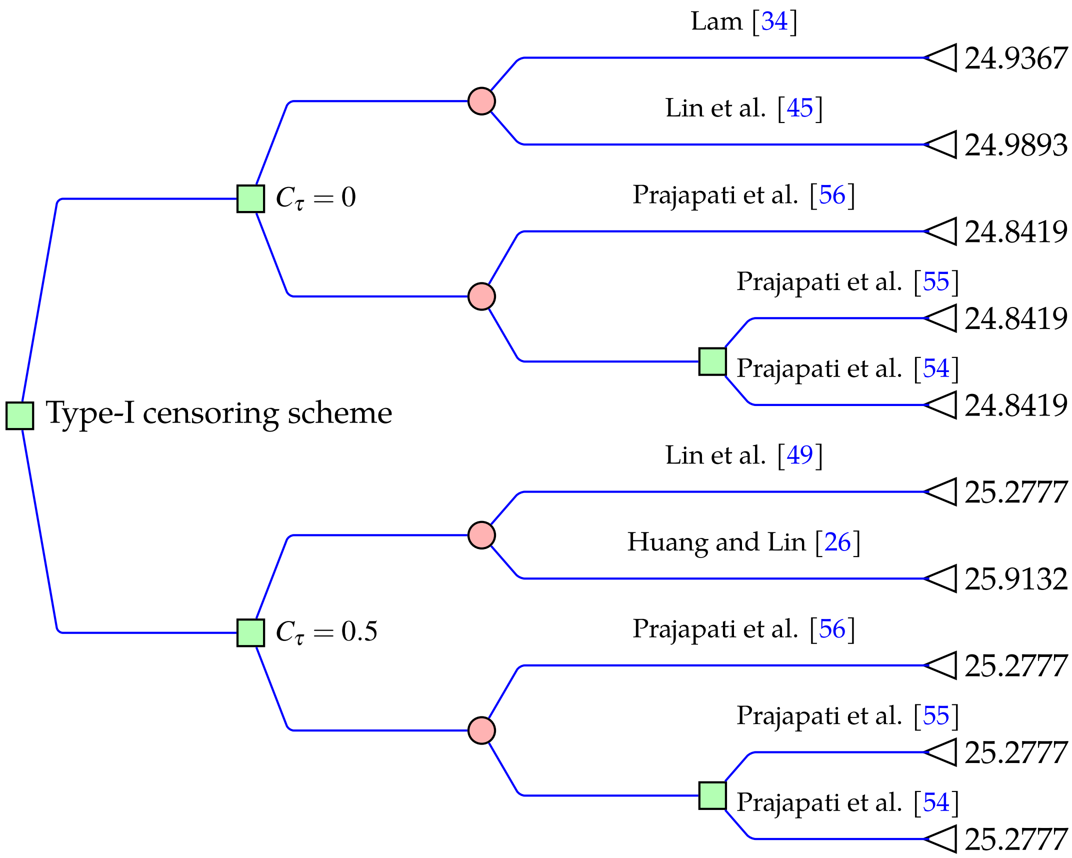

Figure 1: Comparison of optimal sampling plan under Type-I censoring scheme

11.2 Type-I Hybrid Censoring

In this section we provide a comparison between the different optimal sampling plans proposed by Chen et al. [5], Lin et al. [46, 47], Liang and Yang [41], and Prajapati et al. [54, 55] for Type-I hybrid censoring scheme. Lin et al. [46, 47] presented a sampling plan based on the conditional MLE (conditioning on $M \geq 1$). Liang and Yang [41] pointed out that the decision function proposed by Lin et al. [46, 47] is not Bayes. Further, Lin et al. [46, 47] did not consider the cost on time component in the loss function, hence the Bayes sampling plans obtained by Lin et al. [46, 47] are not optimal. In Table 4 we have presented the Bayes risks of the optimal sampling plans proposed by Lin et al. [46, 47] and Liang and Yang [41]. Here a and b are the hyperparameters, and the other coefficients are $a_0 = a_1 = a_2 = 2.0$, $C_s = 0.5$, $C_r = 30, r_s = 0$. It is clear that the optimal sampling plans

Table 4: Bayes risks comparison of the Bayesian optimal sampling plan between the Lin et al. [46, 47] and Liang and Yang [41] for Type-I hybrid censoring scheme.

a	b	Lin et al. [46, 47]	Liang and Yang [41]
2.5	0.4	29.7850	29.7505
2.5	0.6	27.7279	27.7266
2.5	0.8	24.8419	24.8418
3.0	0.8	27.5593	27.5581
3.5	0.8	29.3031	29.2788

proposed by Liang and Yang [41] perform better than Lin et al. [46, 47] in terms of smaller Bayes risks.

Let us recall that Liang and Yang [41], Chen et al. [5] and Prajapati et al. [54, 55] provided optimal sampling plans based on a loss function which has cost on time component. In Table 5 we have compared different optimal sampling plans with different cost on time coefficient C_τ and for $a = 2.5$ and $b = 0.8$. Rest of the coefficients remain same. In Figure 2, we have provided the Bayes risks of the different optimal sampling plans for

Table 5: Bayes risks comparison between the Chen et al. [5], Liang and Yang [41], and Prajapati et al. [54, 55] Bayesian sampling plans.

C_τ	Chen et al. [5]	Liang and Yang [41]	Prajapati et al. [54]	Prajapati et al. [55]
0.0	24.6827	24.6354	24. 6736	24.6754
8.0	26.4962	26.4662	24.4672	26.4672
16.0	27.2695	27.2513	27.2513	27.2513

Type-I hybrid censored cases. Some of the points are clear from this comparison. It is clear that the optimal sampling plan proposed by Chen et al. [5] is not optimal. The optimal sampling plan proposed by Liang and Yang [41] is optimal. The sampling plans proposed by Prajapati et al. [54, 55] behave as good as Liang and Yang [41], and they behave better than Chen et al. [5]. Similar to the Type-I censoring case, the sampling plans proposed by Prajapati et al. [54, 55] can be implemented very easily in practice, and it can be used for higher degree polynomial and non-polynomial loss function also.

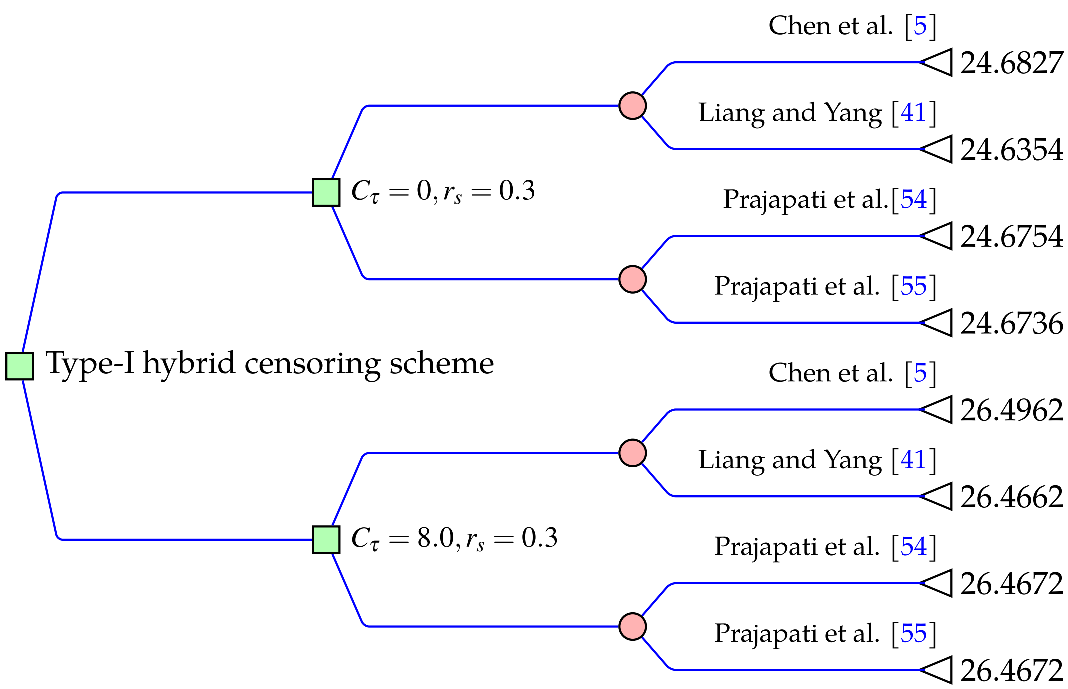

Figure 2: Comparison of optimal sampling plans under Type-I hybrid censoring scheme

11.3 Type-I and Type-II APCS and PHCS

In this section we compare the the Bayesian sampling plans proposed by Lin and Huang [44] and Liang [39] under Type-I and Type-II APCS and PHCS. In Table 6 we present the results for Type-I APCS and PHCS cases, and similarly in Table 7 we present the results for Type-II APCS and PHCS. In both the cases, we have considered different hyperparameters, and the other coefficients are as the following: $C_s = 1.0$, $C_\tau = 2.0$, $C_r = 5$, $r_s = 0.5$, $a_0 = 1$, $a_1 = 1$, $a_2 = 1$, and $N = 1000$. It is clear from Tables 6 and 7 that the Bayesian sampling plans of Liang [39] perform better than Lin and Huang [44].

Table 6: Bayes risks comparison between the Lin and Huang [44] and Liang [39] Bayesian sampling plans Type-I APCS and PHCS.

a	b		Scheme	Bayes risk
1.5	2.0	Lin and Huang [44]	APCS	3.7787
			PHCS	4.8883
		Liang [39]	APCS	2.6875
1.5	0.8	Lin and Huang [44]	APCS	5.5397
			PHCS	6.3410
		Liang [39]	APCS	5.0000
2.0	2.0	Lin and Huang [44]	APCS	4.4285
			PHCS	5.5777
		Liang [39]	APCS	3.5000
1.5	3.0	Lin and Huang [44]	APCS	3.0255
			PHCS	4.2257
		Liang [39]	APCS	1.9167

Table 7: Bayes risks comparison between the Lin and Huang [44] and Liang [39] Bayesian sampling plans under Type-II APCS and PHCS.

a	b		Scheme	Bayes risk
1.5	2.0	Lin and Huang [44]	APCS	7.1844
			PHCS	7.1011
		Liang [39]	APCS	2.6875
1.5	0.8	Lin and Huang [44]	APCS	7.4685
			PHCS	3.7891
		Liang [39]	APCS	5.0000
2.0	2.0	Lin and Huang [44]	APCS	6.7917
			PHCS	5.6288
		Liang [39]	APCS	3.5000
1.5	3.0	Lin and Huang [44]	APCS	7.9389
			PHCS	7.8989
		Liang [39]	APCS	1.9167

11.4 Generalized Hybrid Censoring Schemes

Prajapati et al. [58] obtained the Bayesian sampling plan for Type-I GHCS under a quadratic loss function and also under a fifth degree polynomial loss function. In case of the quadratic loss function the coefficients are $a_0 = 2$, $a_1 = 2$, $a_2 = 2$, C_s =0.3, r_s =0.1, C_τ =0.8, $C_r = 30$. Different optimal sampling plans have been reported in Table 8 for different hyperparameters and for different C_v values. Prajapati et al. [60] provided similar results for Type-II GHCS under a quadratic loss function. In Table 9 we have presented some optimal sampling plans for Type-II GHCS cases. In this case the coefficients are $a_0 = 2$, $a_1 = 3$, $a_2 = 4$, $C_s = 1.5$, $r_s = 1.2$, $C_v = 0.5$, $C_\tau = 0.1$, and $C_r = 75$. We have reported the results for different hyperparameters and for different C_v values. It is clear from both the tables that if the cost on MSE is high, more sample size is needed to make a decision about the batch.

Table 8: Optimum Bayesian sampling plans and their Bayes risks when a, b and C_s vary with $a_0 = 2,\ a_1 = 2,\ a_2 = 2,\ r_s = 0.1,\ C_s = 0.3,\ C_\tau = 0.8,\ C_r = 30$

a	b	C_v	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B	a	b	C_v	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B
2.2	0.8		24.1538	8	7	6	0.6281			0.5	24.6611	6	5	2	0.5070
2.5	0.8		25.1849	7	6	5	0.6234			1.5	24.8292	6	5	3	0.5234
3.0	1.0	3.0	25.0148	7	6	4	0.5633	3.0	1.0	3.0	25.0148	7	6	4	0.5633
3.0	1.2		22.3891	7	6	4	0.4898			5.0	25.2457	7	6	5	0.6039
3.5	0.8		29.2754	4	4	3	1.1500			8.0	25.5379	8	7	6	0.6414

Table 9: Minimum Bayes risks and their optimum Bayesian sampling plans when a, b and C_s vary with $a_0 = 2,\ a_1 = 3,\ a_2 = 4,\ r_s = 1.2,\ C_s = 1.5,\ C_\tau = 0.1,\ C_r = 75$

a	b	C_v	$R_B^{\delta_B}$	n_B	r_B	τ_{1B}	τ_{2B}	a	b	C_v	$R_B^{\delta_B}$	n_B	r_B	τ_{1B}	τ_{2B}
1.55	0.45		57.2735	9	6	0.1625	0.2538			0.2	51.8132	7	5	0.1580	0.2761
1.55	0.50		53.3555	8	6	0.2041	0.2843			1.0	55.2654	10	7	0.2077	0.2226
2.40	0.80	0.5	55.1567	8	8	0.1037	0.2173	1.55	0.50	1.5	56.8266	11	11	0.0998	0.2278
2.50	0.80		57.3335	8	8	0.1166	0.2235			2.0	58.1953	12	12	0.1165	0.2164
4.50	1.25		68.5803	8	5	0.1803	0.2241			2.5	59.4396	13	13	0.1116	0.2184

11.5 Joint Censoring Scheme

Prajapati et al. [60] developed optimal decision theoretic sampling plans for two samples based on BJPC scheme. In the optimal sampling plan based on the BJPC scheme, it is observed that $R_1 = \ldots = R_{k-1} = 0$. In this section we compare the Bayes risks of the optimal decision-theoretic sampling plan based on the BJPC scheme with the sum of optimal Bayes risks based on two Type-II censoring schemes applied on the two samples separately. For single sample case, we have considered the optimal sampling plans proposed by Lam [33]. Independent gamma priors have been used with hyperparameters (a_1, b_0) and (a_2, b_0). In order to make the comparison meaningful, the following coefficients have been used: $C_s = C_{s_1} + C_{s_2} = 0.08 + 0.08,\ r_{s_1} = 0.07,\ r_{s_1} = 0.07,\ C_\tau = 0.1,\ C_{r_1} = 9,\ C_{r_2} = 16$ $C_{e_1} = 0.5,\ C_{e_2} = 0.5,\ C_r = C_{r_1} + C_{r_3} = 25$ and $l_0' = 4,\ l_1' = 5,\ l_2' = 1,\ l_0'' = 4,\ l_1'' = 5,\ l_2'' = 4$. The optimal sampling plans and the associated Bayes risk under Type-II censoring for each samples based on Lam's [33] decision functions are obtained. The optimal Bayes risks for Sample A and Sample B are denoted by R_1^L and R_2^L, respectively. The optimal sampling plans for Sample A, Sample B, and the associated Bayes risks are provided in Tables 10 and 11. In Table 12, the Bayes risk of the optimal decision-theoretic sampling plan under the BJPC scheme, denoted by R_B^δ and compared it with the sum of the two optimal Bayes risks R_1^L and R_2^L, based on single sample Type-II censoring scheme. From Table 12, it is evident that the Bayes risk R_B^δ of the optimal decision-theoretic sampling plan is smaller than the sum of the Bayes risks R_1^L and R_2^L. Therefore, it can be concluded that joint censoring scheme is preferable than single censoring schemes if there are more than one samples are present.

Table 10: Lam's sampling plan for Sample A

a_1	b_0	R_1^L	n_0	k_{10}	ξ_{10}^L
5.0	5.0	8.9303	6	6	1.3375
5.0	5.5	8.7875	6	6	1.2543
4.0	5.0	8.4689	8	8	1.1500
3.5	4.0	8.6000	8	8	1.2015
4.7	5.0	8.8173	6	6	1.2789

Table 11: Lam's sampling plan for Sample B

a_2	b_0	R_2^L	n_0	k_{20}	ξ_{20}^L
5.0	5.0	13.1262	8	8	0.7441
5.0	5.5	12.1374	8	8	0.6816
6.0	5.0	14.5789	8	8	0.8471
4.5	4.0	13.8375	8	8	0.8179
4.3	5.0	11.8701	8	8	0.6725

Table 12: Comparison between optimal decision-theoretic sampling plan and Lam's sampling plan in terms of Bayes risks under two samples case

a_0	a_1	a_2	b_0	$R_1^L + R_2^L$	R_B^δ	n_0	k_0	ξ_{10}	ξ_{20}	c_0
10.0	5.0	5.0	5.0	8.9303+13.1262=22.0565	20.3742	5	4	6.9407	0.0036	1.2445
10.0	5.0	5.0	5.5	8.7876+12.1374=20.9250	19.8696	6	4	6.4416	0.0042	1.2451
10.0	4.0	6.0	5.0	8.4689+14.5789=23.0478	19.3091	6	5	10.0423	0.0062	1.1807
8.0	3.5	4.5	4.0	8.6000+13.8375=22.4375	20.0316	7	5	8.8095	0.0013	1.1712
9.0	4.7	4.3	5.0	8.8173+11.8701=20.6874	20.2678	7	5	6.5293	0.0030	1.2919

11.6 Competing Risks

Prajapati et al. [59] provided the optimum Bayesian sampling plan in presence of competing risk for Type-II censoring and also for Type-I hybrid censoring. It is observed that when the loss function is quadratic, the Bayes decision function can be obtained in explicit forms. It is assumed that there are two competing causes, and the lifetime of these two causes follow independent exponential distributions. In this section we present some optimal Bayesian sampling plans based on the assumptions that the priors are independent gamma priors with hyperparameters (a, b) and (c, d). The other coefficients are as follows: $a_{00} = 2,\ a_{10} = 4,\ a_{01} = 4,\ a_{20} = 4,\ a_{02} = 4,\ a_{11} = 4, C_s = 0.3,\ r_s = 0.25,\ C_\tau = 0.3$, and $C_r = 40$ In the Table 13, optimal censoring plans and the associated Bayes risks are presented. Similarly, the results for Type-I hybrid censoring scheme are presented in Table 14. In this case the following coefficients are chosen: $a_{00} = 5,\ a_{10} = 6,\ a_{01} = 6,\ a_{20} = 6,\ a_{02} = 6,\ a_{11} = 6,$ $C_s = 0.5,\ r_s = 0.2,\ C_\tau = 0.3$, and $C_r = 130$. In both the cases it has been observed that although the optimal sampling schemes are more or less robust with respect to the hyperparameters, they vary significantly as C_s changes.

Table 13: Minimum Bayes risks and their optimum Bayesian sampling plans for different values of $a, b, c,\ d$ and C_s when $a_{00} = 2,\ a_{10} = 4,\ a_{01} = 4,\ a_{20} = 4,\ a_{02} = 4,\ a_{11} = 4,\ r_s = 0.25,\ C_\tau = 0.3,\ C_r = 40$

a	b	c	d	C_s	$R_B^{\delta_B}(n_B, r_B)$	n_B	r_B	a	b	c	d	C_s	$R_B^{\delta_B}(n_B, r_B)$	n_B	r_B
1.5	0.8	1.5	0.8		37.3147	6	5					0.27	34.8397	10	6
1.5	0.6	1.5	0.8		38.5331	5	4					0.30	35.0911	8	6
2.0	0.8	1.5	0.8	0.3	39.2008	5	4	2.5	1.5	2.2	2.0	0.35	35.4204	6	5
1.5	0.8	2.0	0.8		39.2008	5	4					0.40	35.6856	5	5
2.5	1.5	2.2	1.2		38.5062	6	5					0.50	36.1067	4	4
2.5	1.5	2.2	2.0		35.0911	8	6					1.00	37.5068	2	2

12 Conclusions, Open Problems and Further Reading

In this article we have presented briefly different acceptance sampling plans based on Bayesian decision functions. The Bayesian decision function has some advantages, as it minimizes the Bayes risks, and it can be implemented quite easily if the loss function is quadratic. It should be mentioned that most of the work which has been done during the last two decades are based on the assumption that the lifetime of the experimental units follow one parameter exponential distribution. While this choice simplifies the analysis,

Table 14: Minimum Bayes risks and their optimum Bayesian sampling plans for different values of $a, b, c,\ d$ and C_s when $a_{00} = 5,\ a_{10} = 6,\ a_{01} = 6,\ a_{20} = 6,\ a_{02} = 6,\ a_{11} = 6,\ \ r_s = 0.2,\ C_\tau = 0.3$, and $C_r = 130$

a	b	c	d	C_s	$R_B^{\delta_B}(n_B, r_B, \tau_B)$	n_B	r_B	τ_B	a	b	c	d	C_s	$R_B^{\delta_B}(n_B, r_B, \tau_B)$	n_B	r_B	τ_B
1.5	0.5	2.5	1.0		116.2559	9	9	1.0031					0.3	114.0845	12	12	0.8891
2.5	1.0	1.5	0.5		116.2559	9	9	1.0031					0.4	115.2524	11	11	0.9313
2.5	0.7	1.5	0.5	0.5	123.9614	8	8	1.1688	1.5	0.5	2.5	1.0	0.5	116.2559	9	9	1.0031
2.0	0.7	1.5	0.5		117.8878	9	9	1.0375					0.8	118.6031	7	7	1.0344
2.5	1.0	2.5	0.8		121.8392	9	9	1.1000					1.0	119.8281	6	6	1.0250
3.0	1.5	2.5	0.8		117.8483	10	10	0.9875					1.5	122.1700	4	4	0.9281

it restricts attention to more flexible families such as Weibull, log-normal, generalized exponential etc. which are more realistic in practice. There are some issues related to these non-exponential lifetime distributions. Often in these cases it is not possible to obtain conjugate priors, hence, the Bayes decision function cannot be obtained explicitly. One can only obtain the Bayes decision function numerically. Due to this reason, finding the optimal Bayes decision theoretic sampling plan becomes computationally quite challenging. Some attempts have been made mainly for Weibull distribution by Prajapati et al. [59] and Prajapati et al. [61]. Efficient numerical methods need to be developed for practical implementation. More work is needed in this direction. Although as extensive amount of work has been done for different censoring schemes for one sample problem, not much work has been done for more than one sample. It will be interesting to develop optimal sampling plans for more than one sample also.

Now we will provide some open problems for future research.

Open Problem 1: In Section 3 Bayesian optimal sampling plans have been developed for exponential and normal distributions for complete sample. The problem is still remains open for other distributions like Weibull, log-normal, Pareto distributions etc.

Open Problem 2: In Section 4 it has been observed that the most of the work considered in

Type-I and Type-II censoring scheme is based on one parameter exponential distribution. Prajapat et al. [53] obtained the optimal Bayesian sampling plan in case of Type-I censored data for two-parameter exponential distribution based on conjugate priors. It will be interesting to develop optimal Bayesian sampling plans for two-parameter exponential distribution in case of Type-II censoring.

Open Problem 3: Under Type-I and Type-II hybrid censoring scheme problem of finding the Bayesian sampling plan for two or more parameter distribution function remains open.

Open Problem 4: Under Progressive censoring scheme finding the optimal Bayesian sampling plan problem is still growing. The problem remains open for other distributions except the one-parameter exponential distribution.

Open Problem 5: Finding optimal Bayesian sampling plan problem still remains open for other type of progressive censoring schemes under exponential and related distribution functions.

Open Problem 6: Generalized hybrid censoring schemes has certain advantages over the hybrid censoring schemes. Which open up an interesting problem; among all censoring schemes which censoring scheme will perform the best in a given scenario.

Open Problem 7: Finding the optimal Bayesian sampling plan for generalized hybrid censoring problem is still open for distributions other than the one-parameter exponential distribution.

Open Problem 8: Optimal Bayesian sampling plans for joint censoring schemes remains an open problem for more than two populations and also for distributions other than one-parameter exponential distribution.

Open Problem 9: Among different joint censoring schemes, which one will perform the best is an exciting problem need to be considered.

Open Problem 10: It will be interesting to generalize the results for interval censoring schemes for other lifetime distributions like two-parameter exponential, Weibull etc..

Open Problem 12: Optimal Bayesian sampling plan in presence of competing risk is an exciting problem and it remains open for other distributions except the exponential distribution case.

Open Problem 13: Optimal Bayesian sampling plans in presence of competing risk under the different censoring schemes like interval censoring, progressive censoring are some important problems to address in the future.

Open Problem 14: It will be interesting to develop optimal Bayesian sampling plans in presence of complementary risks for different lifetime distributions under various censoring schemes.

In a review article of this type it is not possible to cover every aspects of this problem. Interested readers are requested to see some of the related work which we have not covered here, for example Lam [35], Chen et al. [6], Huang and Lin [27], Liang et al. [40], Salem et al. [65], Chen et al. [10], Prajapat et al. [53], Prajapati and Kundu [62], Chen et al. [14], Chen et al. [11], Liang et al. [42], Liang et al. [43] and the references cited therein.

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