Two Sample Bayesian Acceptance Sampling Plan

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Abstract

This article presents the optimal Bayesian acceptance sampling plan (BASP) for two sample cases. In constructing the BASP, the decision-theoretic approach is used with a specified loss function, and the Bayes decision rule is developed by minimizing the Bayes risk. Both batches of products from the production line are accepted or rejected simultaneously based on the observed sample. Such implementation has a significant advantage in reducing the cost and time by deciding on the acceptance or rejection of both the batches in a single-life testing experiment. The optimal BASP is derived under the assumption of Weibull and exponential lifetimes, although it can be extended for other lifetime distributions also. Some numerical results have been presented to show the performances of the proposed BASP.

Keywords and Phrases: Acceptance sampling, Bayesian acceptance sampling plan,Bayes decision function, Bayes risk, Joint censoring scheme.

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1 Introduction

In statistical quality control, the acceptance sampling plan has been quite extensively used to decide on product quality. A standard method in the acceptance sampling plan is to control the probability of making unfavorable decisions by setting constraints on the operating characteristic curve. From the economic perspective, the decision-theoretic approach is the most reasonable method because the sampling plan is determined by making an optimal decision based on some economic factors such as maximizing the profit or minimizing the loss. Reader may see the work of Lam [16], Lin et al. [19, 20, 21], Huang et al. [11] in this aspect. The censoring schemes are used in the life testing experiment whenever the test is destructive, lacks the time, or the inspection cost of items is high. In the literature, applications of decision-theoretic sampling plans are found under the different censoring schemes. For Type-I and type-II censoring schemes Lam [15, 16] had developed the decision-theoretic sampling plans. Lam and Choy [17] considered the exponential distribution with the uniformly distributed random censoring to obtained the decision-theoretic sampling plan. For the mixed censoring schemes Chen et al. [10] design the acceptance sampling plan under exponential and Weibull distributions. For more details the readers may refer to Lin et al. [20], Liang and Yang [18], Lin et al. [21], Prajapati et al. [28, 29] and the references cited therein.

Recently, Prajapati et al. [31] have developed the decision-theoretic sampling plan for the balanced joint censoring scheme (BJPC) introduced by Mondal and Kundu [24, 26], with product lifetime exponentially distributed. This type of work has vast applications where one needs to investigate more than one kind of product simultaneously. For example, to meet rapidly changing markets and various customer demands, it is expected that a single machine makes multiple products. Computers with different configurations, sizes, and cases are being constructed using a single machine. Compared with the single product machine, such a multi-product machine is more flexible, quick, can reduce production cost and time, and improve demand satisfaction. Therefore, investigating the quality of a multi-product machine has significant importance. Let a multi-product machine produce the two products, and the machine quality is measured by the quality of the batches of two kinds of products. The decision-making about the quality of a multi-product machine based on inspecting the produced items, usually an application of an acceptance sampling plan under multi sample life testing. In this situation, we can effectively apply joint censoring schemes.

The decision function proposed by Prajapati et al. [31], is not the Bayes decision function. Therefore the decision-theoretic sampling plan is not the optimal Bayesian acceptance sampling plan as it will not minimize Bayes risk. Lin et al. [22] have shown that the Bayesian sampling plan with the Bayes decision function performs better than the decision-theoretic sampling plan under Type-I censoring scheme. Chen et al. [9] obtained the Bayesian sampling plan for exponential distribution with random censoring. Huang and Lin [12] design Bayesian sampling plans for exponential distribution based on uniform random censored data. For hybrid censored samples Liang and Yang [18] obtained the optimal Bayesian sampling plans under exponential distributions. Bayesian sampling plans for exponential distributions with interval censored samples is design by Chen et al. [7]. Yang et al. [36] obtained the optimal Bayesian sampling plans for exponential distributions with modified type-II hybrid censored data. Chen et al. [4, 5] developed the efficient Bayesian sampling plans for step-stress of accelerated life test on censored data. Prajapati and Kundu [33] constructed the Bayesian acceptance sampling plan for random stress-change time model. For competing risks data optimal bayesian sampling plan is obtained by Prajapati et al. [32]. Such a procedure is also used by Tsai et al. [35], Chen et al. [6], and Prajapati et al. [30] to obtained the Bayesian sampling plan with different censoring schemes. Most works are done using the exponential distribution to obtain the

Bayesian sampling plan. The exponential distribution has limitations in modelling the lifetime data due to its constant failure rate. The significance of using the Weibull distribution over the exponential distribution comes from its popularity and flexibility in modelling lifetime data with increasing or decreasing failure rates. Due to the presence of the shape parameter, it has lots of flexibility compared to an exponential distribution.

No attempt has been made so far, to construct the Bayes decision function for obtaining the acceptance sampling plan on the two-sample problem under Weibull distribution. This work designs the Bayesian acceptance sampling plan, which we denote by BASP, using the Bayes decision function for two-sample problems under the BJPC scheme. Here we develop our methods when the product's lifetime follows two-parameter Weibull distribution. The Bayes decision rule is considered based on the 0-1 decision function. Therefore if we deal with a multi-component machine, the machine is accepted for production when both the batches are accepted, and if any one of the batches is rejected, we will reject the machine for production. The work is novel in the sense that, according to our knowledge, it is the first attempt to develop the Bayesian sampling plan for a two-sample problem under Weibull distribution. The significant advantage of such BASP is that the plan is optimal whenever both batches are accepted or rejected simultaneously.

In this work, under the assumption of the Weibull lifetime, the well-defined loss function and suitable prior distribution are used to obtain the BASP. The closed-form of the Bayes risk cannot be obtained analytically. Therefore, we rely on the Monte Carlo simulation. In the special case, the closed-form of the Bayes risk can be obtained analytically under the assumption of the exponential distribution. Therefore we derive the BASP for the exponential distribution by constructing the Bayes decision function. The optimal BASP is obtained by minimizing the Bayes risk.

The organization of the rest of the paper is as follows. In Section 2, Weibull, exponential, and prior distribution are defined, and the BJPC scheme is described. In Section 3, under the Weibull lifetime assumptions, BASP is constructed. For the exponential lifetime, the BASP is obtained in Section 4. A numerical study is performed in Section 5. The real-life application is discussed in Section 6, followed by the conclusion of this work in Section 7. All the derivations are provided in the Appendix.

2 Assumptions

Suppose there are two different types of products, say P_1 and P_2, coming from a production line, and we are interested in studying the relative merits of these two kinds of products. The quality of the product is measured by its lifetime. It is assumed that the lifetime of the product follows Weibull and exponential distribution. Let us recall that if a random variable X follows a Weibull distribution $WE(\alpha, \beta)$ with shape parameter α and scale parameter β, the probability density function (PDF) will be

$$f_X(x|\alpha,\beta) = \begin{cases} \alpha\beta x^{\alpha-1} e^{-\beta x^{\alpha}}, \text{ if } x > 0, \\ 0, \quad\quad\quad\quad \text{otherwise.} \end{cases} \tag{1}$$

Next, if random variable X follows an exponential distribution $\text{Exp}(\beta)$ with scale parameter β, then the corresponding PDF is given by

$$f_X(x|\beta) = \begin{cases} \beta e^{-\beta x}, \text{ if } x > 0, \\ 0, \quad\quad\quad\quad \text{otherwise.} \end{cases} \tag{2}$$

Note that exponential distribution is the special case of Weibull distribution when the shape parameter $\alpha = 1$.

A sample of size n is drawn from product type P_1 and similarly, another sample of size n is drawn from product type P_2. These two samples are put on the life testing experiment to take decision on product quality and a BJPC scheme, $\mathscr{R} = \{R_1, \ldots, R_{k-1}\}$ is applied. Here k be the total number of failures to be observed from the life testing experiment and $R_1, \ldots, R_{k-1}$ be the number of units censored at the intermediate stage of the experiment. Therefore, the possible values of $R_1, \ldots, R_{k-1}$ are non-negative integers such that $\sum_{i=1}^{k-1}(R_i + 1) < n$. The BJPC scheme can be described as follows. We place those two independent samples simultaneously on the life testing experiment. Suppose the first failure takes place at the time point W_1 and it comes from sample of P_1, then R_1 units are randomly chosen from the remaining $(n-1)$ surviving units of sample of P_1 and they are removed. At the same time, $(R_1 + 1)$ units are randomly chosen from the n surviving units of sample of P_2 and they are removed. Suppose the next failure takes place at the time point W_2 and it comes from sample of P_2, then $(R_2 + 1)$ units are chosen at random from the remaining $(n - 1 - R_1)$ surviving units of sample of P_1, and they are removed. At the same time, R_2 units are chosen at random from the remaining $(n - 2 - R_1)$ surviving units of sample of P_2, and they are removed, and so on we continue the test. Finally, at the time of the k-th failure, it may be either from sample of P_1 or from sample of P_2, all the remaining surviving items from both the samples are removed and the experiment stops. Let us introduce the indicator variable Z_i for $i = 1, \ldots, k$ where

$$Z_i = \begin{cases} 1, & \text{if ith failure comes from the sample of } P_1 \\ 0, & \text{if ith failure comes from the sample of } P_2. \end{cases}$$

Hence, the censored sample will be of the form $(\mathbf{W}, \mathbf{Z})$ where $\mathbf{W} = (W_1, \ldots, W_k)$ and $\mathbf{Z} = (Z_1, \ldots, Z_k)$. Let $K_1 = \sum_{i=1}^{k} Z_i$ be the number of failures from the sample of P_1 and $K_2 = \sum_{i=1}^{k}(1 - Z_i)$ be the number of failures from the sample of P_2. The schematic diagram of the BJPC scheme is given in Figure (1). For detailed discussion on the BJPC scheme, readers may refer to Mondal and Kundu [24].

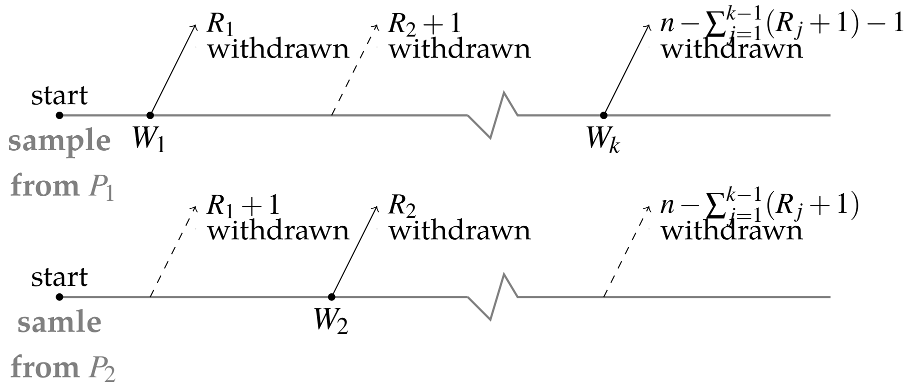

Figure 1: Illustration on BJPC scheme on two samples

2.1 Prior Assumption

We make the following notations through out the paper. A gamma distribution with the shape parameter $a > 0$ and scale parameter $b > 0$ will be denoted by $G(a,b)$, and it has the following PDF

$$f_G(x;a,b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}; \quad \text{if} \quad x > 0,$$

and zero, otherwise. Similarly, a beta distribution with parameters a and b will be denoted by $Beta(a,b)$ and it has the PDF

$$f_{Beta}(y;a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1}(1-y)^{b-1}; \quad \text{if} \quad 0 < y < 1,$$

and zero, otherwise. Following the idea of Pena and Gupta [27], Kundu [13], Kundu and Pradhan [14], we assume that $\lambda = \lambda_1 + \lambda_2 \sim G(a_0, b_0)$, with $a_0 > 0$, $b_0 > 0$ and $\frac{\lambda_1}{\lambda} \sim Beta(a_1, a_2)$, with $a_1 > 0$, $a_2 > 0$, and they are independently distributed. The joint PDF of $\lambda_1,\ \lambda_2$ can be derived as,

$$\pi(\lambda_1, \lambda_2) = \frac{\Gamma(a_1+a_2) b_0^{a_0}}{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)} (\lambda)^{a_0-a_1-a_2} \lambda_1^{a_1-1} \lambda_2^{a_2-1} e^{-\lambda b_0}; \quad \lambda_1,\ \lambda_2 > 0. \tag{3}$$

The joint prior is known as the Beta-Gamma prior distribution. It will be denoted by $BG(a_0, b_0, a_1, a_2)$. Such prior ensures that the dependence between λ_1 and λ_2 has a positive or negative correlation depending on $a_0 > a_1 + a_2$ or $a_0 < a_1 + a_2$. For $a_0 = a_1 + a_2$, λ_1 and λ_2 become independent. Pena and Gupta [27] first proposed this prior, see also Kundu and Pradhan [14], Mondal and Kundu [25] in this regard.

In the BJPC set up, sample size n, the number of failures k, $R_1, \ldots, R_{k-1}$, in combination with the decision function δ, i.e. $(n, k, R_1, \ldots, R_{k-1}, \delta)$, becomes a acceptance sampling plan. We aim to obtain the optimal BASP under the BJPC scheme for two sample problems. For this, we discover the Bayes decision function under the assigned loss function L, which minimizes the Bayes risk among all reasonable acceptance sampling plans.

3 Weibull Distribution

This section provides the optimal BASP for two Weibull populations under the BJPC scheme. Suppose a sample of size n is taken from product P_1 which follows $WE(\alpha, \lambda_1)$ and a sample of size n is from product P_2 which follows $WE(\alpha, \lambda_2)$. Applying a BJPC scheme on these two samples simultaneously as before, we observe the censored data $(\mathbf{w}, \mathbf{z}) = ((w_i, z_i) : i = 1, \ldots, k)$ and $k_1 = \sum_{i=1}^{k} z_i$, $k_2 = \sum_{i=1}^{k}(1 - z_i)$. Based on the data, the joint PDF can be obtained as

$$f(\mathbf{w}, \mathbf{z}|\alpha, \lambda_1, \lambda_2) \propto \alpha^k \lambda_1^{k_1} \lambda_2^{k_2} \prod_{i=1}^{k} w_i^{\alpha-1} e^{-(\lambda_1+\lambda_2)U(\alpha)}, \text{ if } 0 < w_1 < w_2 < \ldots < w_k < \infty,$$

and zero otherwise, where $U(\alpha) = \sum_{i=1}^{k-1}(R_i + 1)w_i^{\alpha} + (n - \sum_{i=1}^{k-1}(R_i + 1)w_k^{\alpha}$. We consider the significant case when the common shape parameter is unknown, which is most expected to occur in practice. But when the shape parameter is not known, the conjugate priors do not exist. In this situation the following priors have been considered: $(\lambda_1, \lambda_2) \sim BG(a_0, b_0, a_1, a_2)$, $\alpha \sim G(c, d)$ and they are independently distributed, see for example Kundu and Pradhan [14] in this regard. Let $\{1\}$ denote the action of accepting both the batches and $\{0\}$ denote the action of rejecting them. For the provided sample size, censoring time and parameter λ_1 and λ_2, the loss of taking decision δ is defined as

$$L(\delta, \alpha, \lambda_1, \lambda_2) = \begin{cases} nC_s - (n - K_1)r_{s_1} - (n - K_2)r_{s_2} + W_k C_\tau + g(\alpha, \lambda_1, \lambda_2), & \text{if } \delta(\mathbf{W}, \mathbf{Z}) = 1 \\ nC_s - (n - K_1)r_{s_1} - (n - K_2)r_{s_2} + W_k C_\tau + C_r, & \text{if } \delta(\mathbf{W}, \mathbf{Z}) = 0, \end{cases} \tag{4}$$

where C_s is the sum of inspection cost per product of P_1 and P_2, C_τ is the cost per unit time of the experiment, $r_{s_1}(r_{s_2})$ is the salvage value per product of $P_1(P_2)$ which does not fail and $C_s > r_{s_1} + r_{s_2}$, C_r is the rejection cost of the batches, which is fixed because the products are discarded or return if they are rejected. The acceptance cost $g(\alpha, \lambda_1, \lambda_2)$ is positive and increasing function of λ_1 and λ_2, for given α. Therefore, acceptance cost depends on the mean lifetime of the product P_1 and P_2 and it is assumed for simplicity that $g(\alpha, \lambda_1, \lambda_2) = g(\alpha, \lambda_1) + g(\alpha, \lambda_2)$. Here the cost $g(\alpha, \lambda_1)$ occurs due to the product from P_1 and similarly $g(\alpha, \lambda_2)$ occurs due to the product from P_2. Therefore, we have taken the acceptance cost $g(\alpha, \lambda_1, \lambda_2) = C_1 P(X_1 < L_1) + C_2 P(X_2 < L_2) = C_1\left[1 - e^{\lambda_1 L_1^\alpha}\right] + C_2\left[1 - e^{\lambda_2 L_2^\alpha}\right]$ and $C_1 > 0$ $(C_2 > 0)$ is the cost on accepting the product $P_1(P_2)$ whose lifetime is less then $L_1(L_2)$. Then the Bayes risk can be written as

$$\begin{aligned} R_B^\delta(n, k, R_1, \ldots, R_{k-1}) &= E(L(\delta(\mathbf{W}, \mathbf{Z}), \alpha, \lambda_1, \lambda_2)) \\ &= n(C_s - r_{s_1} - r_{s_2}) + E(K_1)r_{s_1} + E(K_2)r_{s_2} + E(W_k)C_\tau \\ &\quad + E(g(\alpha, \lambda_1, \lambda_2)) + R_1(\delta | n, k, R_1, \ldots, R_{k-1}), \end{aligned} \tag{5}$$

where

$$R_1(\delta|n,k,R_1,\ldots,R_{k-1}) \quad = \quad E[\delta(\mathbf{W},\mathbf{Z})(C_r - g(\alpha,\lambda_1,\lambda_2))] = E_{\mathbf{W},\mathbf{Z}}[\delta(\mathbf{W},\mathbf{Z})(C_r - \phi(\mathbf{W},\mathbf{Z}))],$$

and $\phi(\mathbf{w},\mathbf{z}) = E_{(\alpha,\lambda_1,\lambda_2)|(\mathbf{w},\mathbf{z})}[g(\alpha,\lambda_1,\lambda_2)]$. Hence for a fixed n, k, and $(R_1,\ldots,R_{k-1})$, the Bayes decision function which minimizes the Bayes risk $R_B^{\delta}(n,k,R_1,\ldots,R_{k-1})$ among all decision functions, is given by

$$\delta_B(\mathbf{w},\mathbf{z}|n,k,R_1,\ldots,R_{k-1}) = \begin{cases} 1, & \text{if } \ \phi(\mathbf{w},\mathbf{z}) < C_r \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

where

$$\begin{aligned} \phi(\mathbf{w},\mathbf{z}) &= E_{(\alpha,\lambda_1,\lambda_2)|(\mathbf{w},\mathbf{z})}[g(\alpha,\lambda_1,\lambda_1)] \\ &= C_1 + C_2 - C_1 E_{\alpha,\lambda_1,\lambda_2}[e^{-\lambda_1 L_1^{\alpha}}|(\mathbf{w},\mathbf{z})] - C_2 E_{\alpha,\lambda_1,\lambda_2}[e^{-\lambda_2 L_2^{\alpha}}|(\mathbf{w},\mathbf{z})]. \end{aligned} \tag{7}$$

Here for $j = 1,2$,

$$\begin{aligned} E_{\alpha,\lambda_1,\lambda_2}[e^{-\lambda_j L_j^{\alpha}}|(\mathbf{w},\mathbf{z})] &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-\lambda_j L_j^{\alpha}} \pi(\alpha,\lambda_1,\lambda_2|(\mathbf{w},\mathbf{z}))d\lambda_1 d\lambda_2 d\alpha \qquad (8) \\ &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-\lambda_j L_j^{\alpha}} \frac{f(\mathbf{w},\mathbf{z}|\alpha,\lambda_1,\lambda_2)\pi(\alpha,\lambda_1,\lambda_2)}{f(\mathbf{w},\mathbf{z})} d\lambda_1 d\lambda_2 d\alpha \\ &= \frac{\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-\lambda_j L_j^{\alpha}} \alpha^k \lambda_1^{k_1}\lambda_2^{k_2}\prod_{i=1}^k w_i^{\alpha-1} e^{-(\lambda_1+\lambda_2)u(\alpha)}\pi(\alpha,\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 d\alpha}{\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \alpha^k \lambda_1^{k_1}\lambda_2^{k_2}\prod_{i=1}^k w_i^{\alpha-1} e^{-(\lambda_1+\lambda_2)u(\alpha)}\pi(\alpha,\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 d\alpha} \\ &= \frac{I_j(\mathbf{w},\mathbf{z})}{I(\mathbf{w},\mathbf{z})} \quad (\text{say}). \qquad (9) \end{aligned}$$

Therefore,

$$\phi(\mathbf{w},\mathbf{z}) \quad = \quad C_1 + C_2 - C_1\frac{I_1(\mathbf{w},\mathbf{z})}{I(\mathbf{w},\mathbf{z})} - C_2\frac{I_2(\mathbf{w},\mathbf{z})}{I(\mathbf{w},\mathbf{z})}, \tag{10}$$

where computation of $I(\mathbf{w,z})$, $I_1(\mathbf{w,z})$, and $I_2(\mathbf{w,z})$ are given in Appendix. The acceptance sampling plan $(n,k,R_1,\ldots,R_{k-1},\delta_B)$ is denoted by BASP. Next, for the loss function (4) and for Bayes decision function (6), the Bayes risk (5) can be written as

$$\begin{aligned}
R_B^{\delta_B}(n,k,R_1,\ldots,R_{k-1}) &= n(C_s-r_{s_1}-r_{s_2})+E(K_1)r_{s_1}+E(K_2)r_{s_2} \\
&\quad +E(W_k)C_\tau+C_r+E_{\alpha,\lambda_1,\lambda_1}((g(\alpha,\lambda_1,\lambda_2)-C_r)P(\phi(\mathbf{W,Z})<C_r)) \\
&= n(C_s-r_{s_1}-r_{s_2})+E(K_1)r_{s_1}+E(K_2)r_{s_2}+E(W_k)C_\tau \\
&\quad C_r+R_1(\delta_B|n,k,R_1,\ldots,R_{k-1}), \qquad (11)
\end{aligned}$$

where $g(\alpha,\lambda_1,\lambda_2)=(C_1+C_2-C_1e^{-\lambda_1 L_1^\alpha}-C_2e^{-\lambda_2 L_2^\alpha})$. The exact expression of Bayes risk cannot be obtained as $E(W_k)$ and $R_1(\delta_B|n,k,R_1,\ldots,R_{k-1})$ are not in closed forms. We will be using the following simulation approach to obtain $E(W_k)$ and $R_1(\delta_B|n,k,R_1,\ldots,R_{k-1})$.

Step 1 Generate α from $\pi(\alpha)$ and $(\lambda_1,\ \lambda_2)$ from $\pi(\lambda_1,\lambda_2)$.

Step 2 Given n, k and $(R_1,\ldots,R_{k-1})$ and given a generated $(\alpha,\lambda_1,\lambda_2)$, generate the data **(W, Z)** under BJPC from two Weibull populations, $WE(\alpha,\lambda_1)$ and $WE(\alpha,\lambda_2)$, M times, say $\{(\mathbf{W}_1,\mathbf{Z}_1),\ldots,(\mathbf{W}_M,\mathbf{Z}_M)\}$.

Step 3 To obtain probability $P(\phi(\mathbf{W,Z})<C_r)$, compute $(\phi_1,\ldots,\phi_M)$, where $\phi_i=\phi(\mathbf{W}_i,\mathbf{Z}_i)$ given in (10), and probability can be approximated as $\frac{\sum_{i=1}^M \mathrm{I}(\phi_i<C_r)}{M}$, where $\mathrm{I}(A)$ is indicator function on set A, if A occurs then $\mathrm{I}(A)=1$, otherwise $\mathrm{I}(A)=0$.

Step 4 Use $(g(\alpha,\lambda_1,\lambda_2)-C_r)\frac{\sum_{i=1}^M \mathrm{I}(\phi_i<C_r)}{M}$ to yield the simulated value of $R_1(\delta_B|n,k,R_1,\ldots,R_{k-1})$ and also simulate $E(W_k)$ by $\frac{\sum_{i=1}^M W_{ki}}{M}$.

Step 5 Repeat Steps 1-4, N times. Denote the simulated values $R_1(\delta_B|n,k,R_1,\ldots,R_{k-1})$ by R_l and $E(W_k)$ by T_l, $l=1,2,\ldots,N$.

Step 6 Compute the $\frac{\sum_{l=1}^{N} R_l}{N}$ and $\frac{\sum_{l=1}^{N} T_l}{N}$, to obtain the estimated $R_1(\delta_B|n,k,R_1,\ldots,R_{k-1})$ and $E(W_k)$, respectively.

Therefore, the Bayes risk $R_B^{\delta_B}(n,k,R_1,\ldots,R_{k-1})$ given in (11) can be obtained by putting the estimated values of $R_1(\delta_B|n,k,R_1,\ldots,R_{k-1})$ and $E(W_k)$. To obtain the optimal BASP $(n_B,k_B,R_{1_B},\ldots,R_{k_B-1},\delta_B)$ we need to minimize the Bayes risk w.r.t. n,k, and all possible choices of $(R_1,\ldots,R_{k-1})$. But even for the moderate n and k, the set of admissible $R_1,\ldots,R_{k-1}$ is quite large. Consequently, finding the optimal solution demands a very high computational time. A suitable meta-heuristic search algorithm can be applied, leading to locally optimal solutions employed in a comparatively limited searching space. As a result, search complexity and duration can be reduced. For instance, a variable neighborhood search algorithm can be employed, which performs the searching process in the connected neighborhoods. These connected neighborhoods can produce the local optimal, and the global optimal is one of the local optimal. Under the BJPC scheme, Mondal et al. [23] applied a variable neighborhood search algorithm to find the optimal life testing plan. Then the optimal BASP $(n_B,k_B,R_{1_B},\ldots,R_{k_B-1},\delta_B)$ can be developed by using a variable neighbourhood search algorithm, which will minimize the Bayes risk $R_B^{\delta_B}(n,k,R_1,\ldots,R_{k-1})$, w.r.t. n,k and all possible choices of $(R_1,\ldots,R_{k-1})$. This algorithm is finite; that is, we can find an optimal sampling plan in a finite number of steps in the search. It is true by the following theorem.

Theorem 3.1. *Let n_B, k_B, and $(R_{1_B},\ldots,R_{k_B-1})$ be the optimal values of n, k, and $(R_1,\ldots,R_{k-1})$, respectively. Then,*

$$n_B \leq min\left\{\frac{E_{\alpha,\lambda_1,\lambda_2}[g(\alpha,\lambda_1,\lambda_2)]}{C_s - r_{s_1} - r_{s_2}}, \frac{C_r}{C_s - r_{s_1} - r_{s_2}}\right\}$$

which implies that $0 \leq k_B \leq n_B$ and all optimal values $R_{1_B},\ldots,R_{k_B-1}$ are strictly less then n_B.

Proof. See Appendix. ∎

4 Exponential Distribution

In this section, we provide the BASP for the two exponential populations under the BJPC scheme. Let a sample of size n is taken from product P_1 which follows Exp(λ_1) and a sample of size n is from product P_2 which follows Exp(λ_2). Based on the BJPC scheme data, the joint PDF is given by

$$f(\mathbf{w}, \mathbf{z}|\lambda_1, \lambda_2) \propto \lambda_1^{k_1} \lambda_2^{k_2} e^{-(\lambda_1+\lambda_2)U}, \text{ if } 0 < w_1 < w_2 < \ldots < w_k < \infty,$$

and zero otherwise, where $U = \sum_{i=1}^{k-1}(R_i+1)w_i + (n - \sum_{i=1}^{k-1}(R_i+1)w_k$. Let us considered that the prior distribution of $(\lambda_1, \lambda_2) \sim BG(a_0, b_0, a_1, a_2)$ and the loss function as follows

$$L(\delta, \lambda_1, \lambda_2) = \begin{cases} nC_s - (n-K_1)r_{s_1} - (n-K_2)r_{s_2} + W_k C_\tau + g(\lambda_1, \lambda_2), & \text{if } \delta(\mathbf{W}, \mathbf{Z}) = 1 \\ nC_s - (n-K_1)r_{s_1} - (n-K_2)r_{s_2} + W_k C_\tau + C_r, & \text{if } \delta(\mathbf{W}, \mathbf{Z}) = 0, \end{cases} \tag{12}$$

where the costs C_s, r_{s_1}, r_{s_2}, C_τ, C_r are defined in Section 3 and the acceptance cost $g(\lambda_1, \lambda_2) = g(\lambda_1) + g(\lambda_2)$ which should be positive and increasing function of λ_1 and λ_2. Lam [15] has used $g(\lambda_i)$ as a quadratic polynomial with positive coefficients, for $i = 1$ and 2. Therefore, let us take $g(\lambda_1) = c_0' + c_1\lambda_1 + c_3\lambda_1^2$ and $g(\lambda_2) = c_0'' + c_2\lambda_2 + c_4\lambda_2^2$. Then $g(\lambda_1, \lambda_2) = c_0 + c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_1^2 + c_4\lambda_2^2$, where coefficients $c_0 = c_0' + c_0'', c_i, i = 1, 2, 3, 4$, are such that $g(\lambda_1, \lambda_2)$ is positive and increasing in λ_1 and λ_2. In order to determine the Bayes decision function first we obtain the Bayes risk of the acceptance sampling plan $(n, k, R_1, \ldots, R_{k-1}, \delta)$

$$\begin{aligned} R_B^\delta(n, k, R_1, \ldots, R_{k-1}) &= nC_s - (n - E[K_1])r_{s_1} - (n - E[K_2])r_{s_2} + E[W_k]C_\tau + E\left[g(\lambda_1, \lambda_2)\right] \\ &\quad + E\left[(C_r - g(\lambda_1, \lambda_2))(1 - \delta(\mathbf{W}, \mathbf{Z}))\right] \\ &= nC_s - (n - E[K_1])r_{s_1} - (n - E[K_2])r_{s_2} + E[W_k]C_\tau + E\left[g(\lambda_1, \lambda_2)\right] + R^\delta, \end{aligned}$$

where

$$R^{\delta} = E_{(\mathbf{W},\,\mathbf{Z})}\big[(C_r - \phi(k_1, k_2, U))(1 - \delta(\mathbf{W}, \mathbf{Z}))\big], \tag{13}$$

and $\phi(k_1, k_2, u) = E_{(\lambda_1, \lambda_2)|\{(\mathbf{W},\,\mathbf{Z})=(\mathbf{w},\,\mathbf{z})\}}[g(\lambda_1, \lambda_2)]$ is the posterior expectation of $g(\lambda_1, \lambda_2)$ given $(\mathbf{W}, \mathbf{Z}) = (\mathbf{w}, \mathbf{z})$. Since (K_1, K_2, U) is a joint sufficient statistic for λ_1 and λ_2, therefore ϕ can be written in terms of k_1, k_2, and u. Therefore, for a given $(n, k, R_1, \ldots, R_{k-1})$, the Bayes decision function is the one which minimizes R^{δ} as in (13) among all decision functions. If we take $\delta_B = 1$ if $C_r - \phi(k_1, k_2, U) > 0$; and $\delta_B = 0$, otherwise, then the R^{δ} is minimize. Thus, the Bayes decision function for a given $(n, k, R_1, \ldots, R_{k-1})$, is defined as

$$\delta_B(k_1, k_2, u | n, k, R_1, \ldots, R_{k-1}) = \begin{cases} 1, & \text{if } \phi(k_1, k_2, u) < C_r \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

4.1 Alternative Bayes Decision Function and Bayes Risk

In this section we will discuss some nice properties of Bayes decision function by which we can present it in an alternative form.

Theorem 4.1. *Let $g(\lambda_1, \lambda_2)$ be a positive function in λ_1 and λ_2. Then,*

(a) For each fixed k_1 and k_2, $\phi(k_1, k_2, u)$ is strictly decreasing in u.

(b) For each fixed k_1 and k_2, δ_B is strictly increasing in u.

Proof. See Appendix. ∎

By Theorem 4.1, for a fixed k_1 and k_2 the $\phi(k_1, k_2, u)$ is strictly decreasing in u. Then the

Bayes decision function given in Equation (14) can be represented as

$$\delta_B(k_1,k_2,u|n,k,R_1,\ldots,R_{k-1}) = \begin{cases} 1, & \text{if } u \geq \xi(k_1,k_2) \\ 0, & \text{otherwise,} \end{cases} \tag{15}$$

where $\xi(k_1,k_2)$ is the solution of the equation $C_r - \phi(k_1,k_2,u) = 0$. In particular, for $g(\lambda_1,\lambda_2) = c_0 + c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_1^2 + c_4\lambda_2^2$ and for fixed k_1 and k_2,

$$\begin{aligned}
\phi(k_1,k_2,u) &= E_{(\lambda_1,\lambda_2)|(\mathbf{W},\mathbf{Z})=(\mathbf{w},\mathbf{z})}[g(\lambda_1,\lambda_2)] \\
&= c_0 + c_1\frac{(a_0+k_1+k_2)(a_1+k_1)}{(a_1+a_2+k_1+k_2)(b_0+u)} + c_2\frac{(a_0+k_1+k_2)(a_2+k_2)}{(a_1+a_2+k_1+k_2)(b_0+u)} \\
&\quad + c_3\frac{(a_0+k_1+k_2+1)(a_0+k_1+k_2)(a_1+k_1+1)(a_1+k_1)}{(a_1+a_2+k_1+k_2+1)(a_1+a_2+k_1+k_2)(b_0+u)^2} \\
&\quad + c_4\frac{(a_0+k_1+k_2+1)(a_0+k_1+k_2)(a_2+k_2+1)(a_2+k_2)}{(a_1+a_2+k_1+k_2+1)(a_1+a_2+k_1+k_2)(b_0+u)^2}.
\end{aligned}$$

To obtain $\xi(k_1,k_2)$ we need to solve $C_r - \phi(k_1,k_2,u) = 0$, which can be written as

$$A(b_0+u)^2 - B(b_0+u) - C = 0, \tag{16}$$

where

$$A = (C_r - c_0)(a_1+a_2+k_1+k_2+1)(a_1+a_2+k_1+k_2)$$

$$B = (a_1+a_2+k_1+k_2+1)(a_0+k_1+k_2)[c_1(a_1+k_1)+c_2(a_2+k_2)]$$

$$C = (a_0+k_1+k_2+1)(a_0+k_1+k_2)[c_3(a_1+k_1+1)(a_1+k_1)+c_4(a_2+k_2+1)(a_2+k_2)].$$

Now if $C_r > c_0$, then the root is $\frac{B+\sqrt{B^2+4AC}}{2A} - b_0$, and

$$\xi(k_1,k_2) = \max\left\{\frac{B+\sqrt{B^2+4AC}}{2A} - b_0, 0\right\}.$$

Therefore, the Bayes risk of the BASP $(n,k,R_1,\ldots,R_{k-1},\delta_B)$ is given by

$$\begin{aligned} R_B^{\delta_B}(n,k,R_1,\ldots,R_{k-1}) &= n(C_s-r_{s_1}-r_{s_2})+E[K_1]r_{s_1}+E[K_2]r_{s_2}+E[W_k]C_\tau \\ &\quad +E[g(\lambda_1,\lambda_2)]+E_{\lambda_1,\lambda_2}[(C_r-g(\lambda_1,\lambda_2))P(U\le\xi(K_1,K_2))], \end{aligned}$$

Now to compute the Bayes risk R^{δ_B} of Bayes decision function we need the distribution of U for given k_1 and k_2. Mondal and Kundu [24], had easily shown that for given k_1 and k_2, $2\lambda U$ follows a Chi-square distribution with $2k$ degrees of freedom. Therefore, the exact expression of the Bayes risk is given by

$$\begin{aligned} & R_B^{\delta_B}(n,k,R_1,\ldots,R_{k-1}) \\ & = n(C_s-r_{s_1}-r_{s_2})+E[K_1]r_{s_1}+E[K_2]r_{s_2}+E[W_k]C_\tau \\ & \quad +\frac{\Gamma(a_1+a_2)}{\Gamma(a_0)\Gamma(a_2)\Gamma(a_2)}\sum_{i=0}^{4}c_i\frac{\Gamma(a_1+p_i)\Gamma(a_2+q_i)\Gamma(a_0+p_i+q_i)}{\Gamma(a_1+a_2+p_i+q_i)b_0^{p_i+q_i}} \\ & \quad +\sum_{i=0}^{4}\sum_{r=0}^{k}\binom{k}{r}\frac{C_i\Gamma(p_i+q_i+a_0)B(r+p_i+a_1,k-r+q_i+a_2)}{\Gamma(a_0)B(a_1,a_2)b_0^{p_i+q_i}}I_{S_r}(k,p_i+q_i+a_0), \quad (17) \end{aligned}$$

where derivation of $E[K_1]$, $E[K_2]$, $E[W_k]$, and Bayes risk is given in Appendix. In the BJPC scheme, when the lifetime distributions follow exponential distribution and two sample sizes are equal (i.e. $m=n$), the distributions of the U and $K_1(K_2)$ are independent of the removals $R_1,\ldots,R_{k-1}$, for any n and k, see Mondal and Kundu [24] for details. Therefore, the expected total salvage values per sample and the expected cost of acceptance will not be affected by $R_1,R_2,\ldots,R_{k-1}$, only $E[W_k]$ involves $R_1,R_2,\ldots,R_{k-1}$. Moreover, it is known that for any given n and k, $E[W_k]$ is minimized when $R_1=\cdots,R_{k-1}=0$. It may be mentioned that this becomes the famous Self Reallocated Design (SRD) proposed by Srivastava [34]. Therefore, the optimal BASP is a choice of only n,k say,(n_B,k_B) which minimizes the Bayes risk $R_B^{\delta_B}(n,k,\underbrace{0,\ldots,0}_{(\text{ k-1}) \ 0'\text{s}})$. Since n and k can take only integers values, similar algorithm which is proposed by Lam [16] can be adopted to obtain the optimal BASP by minimizing the

Bayes risk. Therefore, the optimal BASP $(n_B, k_B, \underbrace{0, \ldots, 0}_{\text{(k-1) 0's}})$ will satisfy

$$R_B^{\delta_B}(n_B, k_B, \underbrace{0, \ldots, 0}_{\text{(k-1) 0's}}) = \min_{n,k}\{R_B^{\delta_B}(n, k, \underbrace{0, \ldots, 0}_{\text{(k-1) 0's}})\}. \tag{18}$$

5 Numerical Study

In this section we present some optimal BASPs for different costs, coefficients, and for different hyperparameters. In Tables, the notation $(r * 0) = (\underbrace{0, \ldots, 0}_{\text{(r) 0's}})$.

5.1 Weibull Case

In this section, we present some optimal BASPs for Weibull distribution. Due to the complexity of the meta-heuristic search approach and since even for the moderate n and k, the set of admissible $R_1, \ldots, R_{k-1}$ is quite large, we have first assumed that $R_1 = \ldots = R_{k-1} = 0$ and obtained the optimal values of n and k. The results are presented in Table 1. Further, we consider the corner points of $(R_1, \ldots, R_{k-1})$, in which the censoring takes place only at one failure time (that is, exactly one R_i is positive). Such points are of the form $(0, \ldots, 0, \underbrace{n-k}_{j\text{th position}}, 0, \ldots, 0)$, i.e. the censoring is carried out at the jth failure time of the life testing experiment. For fixed n and k we calculate the Bayes risk at each corner points and present the results in Table 2. We have considered costs as $C_s = 0.3$, $C_\tau = 0.05$, $r_{s_1} = 0.1$, $r_{s_2} = 0.1$, $C_r = 40$, $C_1 = 25$, $C_2 = 75$, $L_1 = 1.0$, and $L_2 = 0.9$, and the hyperparameters as $a_0 = 2.0$, $b_0 = 1.5$, $a_1 = 2.0$, $a_2 = 2.5$, $c = 0.8$, and $d = 4.0$. To obtain the BASP the simulation approach has been used with $N = 1000$ and $M = 100$. Some of the points are quite clear from these two tables. In Table 1, we observe that as a_0 or b_0 increases the Bayes risk decreases. When a_1 increases and other are fixed the Bayes risk decreases and if a_2 increases then the Bayes risk increases. In case of c or d no specific pattern has been noticed. In Table

Table 1: BASP and there corresponding Bayes risk with different hyper parameters

parameters						costs					BSP			
a_0	b_0	a_1	a_2	c	d	C_s	r_{s_1}	r_{s_2}	C_τ	C_r	n_B	k_B	$(R_{1_B},\ldots,R_{k_B-1})$	Bayes risk
2.5	1.5	2.0	2.5	8.0	4.0	0.3	0.1	0.1	0.05	40	6	5	$(4*0)$	36.6817
2.2											7	7	$(6*0)$	35.1189
2.0											7	6	$(5*0)$	33.8757
1.5											5	5	$(4*0)$	28.1724
2.0	1.0	2.0	2.5	8.0	4.0	0.3	0.1	0.1	0.05	40	5	5	$(4*0)$	38.1003
	1.5										7	6	$(5*0)$	33.8757
	2.0										8	8	$(7*0)$	29.8889
	2.5										4	2	$(1*0)$	26.6666
2.0	1.5	1.5	2.5	8.0	4.0	0.3	0.1	0.1	0.05	40	6	6	$(5*0)$	33.9856
		2.0									7	6	$(5*0)$	33.8757
		2.5									8	4	$(3*0)$	32.7278
		3.0									5	3	$(2*0)$	32.2978
2.0	1.5	2.0	1.5	8.0	4.0	0.3	0.1	0.1	0.05	40	6	6	$(5*0)$	31.5359
			2.0								3	3	$(2*0)$	32.4660
			2.5								7	6	$(5*0)$	33.8757
			3.0								5	4	$(3*0)$	33.9719
2.0	1.5	2.0	2.5	7.8	4.0	0.3	0.1	0.1	0.05	40	5	3	$(2*0)$	33.6959
				8.0							7	6	$(5*0)$	33.8757
				8.5							7	6	$(5*0)$	33.0057
				9.0							6	6	$(5*0)$	33.1719
2.0	1.5	2.0	2.5	8.0	3.8	0.3	0.1	0.1	0.05	40	7	7	$(6*0)$	33.4035
					4.0						7	6	$(5*0)$	33.8757
					4.2						7	7	$(6*0)$	33.5392
					4.5						7	7	$(6*0)$	33.5954

2 it is observed that the Bayes risks are quite close to each other among the corner points. Therefore, among the corner points anyone of them can be taken for practical purposes. Due to this reason we suggest to take the sampling scheme so that $R_1 = \ldots = R_{k-1} = 0$, and we can refer to this as the solution to be the near optimum.

5.2 Exponential Case

It has already been mentioned that in case of exponential distribution, for fixed n and k, the optimal R_i's are $R_1 = \ldots = R_{k-1} = 0$. In Table 3 we have presented the results when the costs are kept constant, the hyperparameters are changing, and in Table 4 it is the other way. Some of the points are quite clear from these two tables. It has been observed in

Table 2: BASP and there corresponding Bayes risk for corner points

parameters						costs					BSP			
a_0	b_0	a_1	a_2	c	d	C_s	r_{s_1}	r_{s_2}	C_τ	C_r	n_B	k_B	$(R_{1_B},\ldots,R_{k_B-1})$	Bayes risk
2.5	1.5	2.0	2.5	8.0	4.0	0.3	0.1	0.1	0.05	40	6	5	$(1,0,0,0)$	37.1101
													$(0,1,0,0)$	37.1269
													$(0,0,1,0)$	37.4221
													$(0,0,0,1)$	36.8402
2.0	1.5	2.0	2.5	8.0	4.0	0.3	0.1	0.1	0.05	40	7	6	$(1,0,0,0,0)$	33.7763
													$(0,1,0,0,0)$	33.7035
													$(0,0,1,0,0)$	33.9698
													$(0,0,0,1,0)$	34.1449
													$(0,0,0,0,1)$	33.8464
2.0	2.5	2.0	2.5	8.0	4.0	0.3	0.1	0.1	0.05	40	4	2	(0)	26.6666
													(2)	27.4035
2.0	1.5	2.5	2.5	8.0	4.0	0.3	0.1	0.1	0.05	40	8	4	$(4,0,0)$	33.6301
													$(0,4,0)$	33.7238
													$(0,0,4)$	33.7239
2.0	1.5	3.0	2.5	8.0	4.0	0.3	0.1	0.1	0.05	40	5	3	$(2,0)$	33.3072
													$(0,2)$	33.0899
2.0	1.5	2.0	3.0	8.0	4.0	0.3	0.1	0.1	0.05	40	5	4	$(1,0,0)$	33.7212
													$(0,1,0)$	34.6074
													$(0,0,1)$	34.5108
2.0	1.5	2.0	2.5	7.8	4.0	0.3	0.1	0.1	0.05	40	5	3	$(2,0)$	33.9201
													$(0,2)$	34.1449
2.0	1.5	2.0	2.5	8.0	3.8	0.3	0.1	0.1	0.05	40	6	5	$(1,0,0,0)$	34.3461
													$(0,1,0,0)$	34.4563
													$(0,0,1,0)$	34.2605
													$(0,0,0,1)$	34.0201

Table 3 that when the costs remain constant, the optimal BASP is robust with respect to a_1 and b_1, and it is quite sensitive with respect to a_0 and b_0. Therefore, one has to be quite careful in choosing the hyperparameters a_0 and b_0 to find the optimal BASP. Similarly, in Table 4 it is observed that when the inspection cost C_s increases the optimal values of n and k decrease and the optimal Bayes risk increases. It implies that if the inspection cost is high we cannot afford to put more sample units for testing purposes, as expected. When the salvage value r_{s_1} or r_{s_2} increases the optimal values of n and k increase. If the cost on time C_τ increases the optimal value of n increases and optimal value of k remains fixed.

Table 3: BASP and there corresponding Bayes risk with different hyper parameters

parameters				costs					BSP			
a_0	a_1	a_2	b_0	C_s	r_{s_1}	r_{s_2}	C_τ	C_r	n_B	k_B	$(R_{1_B}, \ldots, R_{k_B-1})$	Bayes risk
3.0	5.0	5.0	2.0	0.16	0.07	0.07	0.1	25	10	7	$(6*0)$	20.3445
4.0									9	7	$(6*0)$	23.0994
4.5									8	6	$(5*0)$	24.0102
5.0									6	5	$(4*0)$	24.6265
3.0	2.0	5.0	2.0	0.16	0.07	0.07	0.1	25	10	7	$(6*0)$	20.7272
	4.0								10	7	$(6*0)$	20.3871
	5.0								10	7	$(6*0)$	20.3445
	6.0								10	7	$(6*0)$	20.3442
3.0	5.0	2.0	2.0	0.16	0.07	0.07	0.1	25	10	7	$(6*0)$	20.7272
		4.0							10	7	$(6*0)$	20.3871
		5.0							10	7	$(6*0)$	20.3445
		6.0							10	7	$(6*0)$	20.3442
3.0	5.0	5.0	1.0	0.16	0.07	0.07	0.1	25	6	5	$(4*0)$	24.3269
			1.5						9	7	$(6*0)$	22.4990
			2.0						10	7	$(6*0)$	20.3445
			2.2						9	6	$(5*0)$	19.4940

When the rejection cost C_r increases, the optimal values of n and k increase, and the Bayes risk also increases. It indicate that it is difficult to reject the batches when the rejection cost is high.

6 Real Life Examples

In this section, we will consider two data sets: the real data set and the simulated data set for explaining the real-life application of the BASP. Let us assume that two different batches of products are coming from production lines P_1 and P_2. We want to accept or reject both batches simultaneously.

Table 4: BASP and there corresponding Bayes risk with different costs

parameters				costs					BSP			
a_0	a_1	a_2	b_0	C_s	r_{s_1}	r_{s_2}	C_τ	C_r	n_B	k_B	$(R_{1_B},\ldots,R_{k_B-1})$	Bayes risk
3.0	5.0	5.0	2.0	0.15	0.07	0.07	0.1	25	12	7	$(6*0)$	20.2355
				0.16					10	7	$(6*0)$	20.3445
				0.18					7	6	$(5*0)$	20.5138
				0.20					6	5	$(4*0)$	20.6485
3.0	5.0	5.0	2.0	0.16	0.03	0.07	0.1	25	7	6	$(5*0)$	20.5338
					0.05				7	6	$(5*0)$	20.4538
					0.07				10	7	$(6*0)$	20.3445
					0.08				12	7	$(6*0)$	20.2705
3.0	5.0	5.0	2.0	0.16	0.07	0.03	0.1	25	7	6	$(5*0)$	20.5338
						0.05			7	6	$(5*0)$	20.4538
						0.07			10	7	$(6*0)$	20.3445
						0.08			12	7	$(6*0)$	20.2705
3.0	5.0	5.0	2.0	0.16	0.07	0.07	0.05	25	8	7	$(6*0)$	20.2773
							0.10		10	7	$(6*0)$	20.3445
							0.15		11	7	$(6*0)$	20.3988
							0.20		12	7	$(6*0)$	20.4452
3.0	5.0	5.0	2.0	0.16	0.07	0.07	0.1	15	4	2	(0)	14.9564
								20	8	5	$(4*0)$	18.3216
								25	10	7	$(6*0)$	20.3445
								30	10	7	$(6*0)$	21.5282

6.1 Application 1: (Simulated data)

In this section, we considered the simulated data set coming from the Weibull distribution with $\alpha = 2.5473$, $\lambda_1 = 0.2554$ and $\lambda_2 = 0.3969$ under the production line P_1 and P_2. The manufacturer wants to know whether the batch of products from production lines P_1 and P_2 are acceptable. Lets assume that manufacturer comes with the costs as $C_s = 0.3$, $C_\tau = 0.05$, $r_{s_1} = 0.1$, $r_{s_2} = 0.1$, $C_r = 40$, $C_1 = 25$, $C_2 = 75$, $L_1 = 1.0$, and $L_2 = 0.9$, and the hyperparameters as $a_0 = 2.0$, $b_0 = 1.5$, $a_1 = 2.0$, $a_2 = 2.5$, $c = 8.0$, and $d = 4.0$. Then from Table 2 we can suggest the BASP $n = 7, k = 6$ with the scheme $R = (1,0,0,0,0)$ and Bayes risk 33.7763.

- First select a random sample of size $n = 7$ from both the batches and put the selected samples on the BJPC experiment till pre-fixed number of failures $k = 6$ are observed with the scheme $R = (1,0,0,0,0)$. Let we observed the censored sample $(\mathbf{w}, \mathbf{z}) = \{(0.5309,0),(0.6723,1),(1.1599,0),(1.2912,0),(1.4813,1),(1.5641,1))\}$

- Calculate $\phi(\mathbf{w}, \mathbf{z})$ given in equation (6). If $\phi(\mathbf{w}, \mathbf{z}) < C_r$ we will accept both the batches otherwise it will be rejected. Here $\phi(\mathbf{w}, \mathbf{z}) = 23.8077$, which is less then $C_r = 40$, so both the batches are accepted with the Bayes risk of 33.7763.

6.2 Application 2: (Real-life data)

The first data set we considered is the well-known air-conditioner failure data of Proschan (1963). The data set consists of the number of successive failures for the air conditioning system of Jet 7912 (which we considered as production line P_1) and Jet 7913 (production line P_2) given in Tables 5 and 6. First, we check whether the considered real data sets actually come from the Weibull distribution or not by the goodness-of-fit test. The goodness-of-fit test is based on the K–S statistic along with the corresponding p-values. Using the failure times of Jet 7912 and failure times of Jet 7913, the MLEs of the model parameters α, λ_1 and λ_2 become $\hat{\alpha} = 0.9503$, $\hat{\lambda}_1 = 0.0211$, and $\hat{\lambda}_2 = 0.0164$, respectively.

Table 5: Time of successive failures of air-conditioning system of Jet 7912

23	261	87	7	120	14	62	47	225	71	246	21	42	20	5
12	120	11	3	14	71	11	14	11	16	90	1	16	52	95

Table 6: Time of successive failures of air-conditioning system of Jet 7913

97	51	11	4	141	18	142	68	77	80	1	16	106	206
82	54	31	216	46	111	39	63	18	191	18	163	24	

Hence, the K-S distance (with associate p-value) for the data coming from Jet 7912 and Jet 7913 is 0.1926 (p-value = 0.2156) and 0.0942 (p-value = 0.9703), respectively. These results indicate that when shape parameter are same then the Weibull distribution fits Proschan's data-sets quite well. Next, the manufacturing company wants to know whether the air-conditioning systems from Jet 7912 and Jet 7913 are acceptable (working properly). They come with the costs as $C_s = 0.3$, $C_\tau = 0.05$, $r_{s_1} = 0.1$, $r_{s_2} = 0.1$, $C_r = 40$, $C_1 = 25$, $C_2 = 75$, $L_1 = 1.0$, and $L_2 = 0.9$, and the hyperparameters as $a_0 = 2.0$, $b_0 = 1.5$, $a_1 = 2.0$, $a_2 = 2.5$, $c = 8.0$, and $d = 4.0$. Then from Table 2 we can suggest the BASP $n = 7, k = 6$ with the scheme $R = (1, 0, 0, 0, 0)$ and Bayes risk 33.7763. Now, they will experiment to see whether both air-conditioning systems are accepted (working properly) or rejected (not working properly). The experiment goes as follows:

- First select a random sample of size $n = 7$ from both the batches:
  $P_1 : 23, 87, 62, 120, 90, 42, 225,$
  $P_2 : 51, 141, 68, 80, 106, 31, 39$
  and put the selected samples on the BJPC experiment till pre-fixed number of failures $k = 6$ are observed with the scheme $R = (1, 0, 0, 0, 0)$. Let we observed the censored sample $(\mathbf{w}, \mathbf{z}) = \{(23, 1), (39, 0), (42, 1), (51, 0), (62, 1), (80, 0)\}$

- Calculate $\phi(\mathbf{w}, \mathbf{z})$. If $\phi(\mathbf{w}, \mathbf{z}) < C_r$ we will accept both the batches otherwise it will be rejected. Here $\phi(\mathbf{w}, \mathbf{z}) = 1.11525$ which is less then $C_r = 40$, so both the batches are accepted (air-conditioning system working properly) with the Bayes risk of 33.7763.

7 Conclusion

This article addresses the optimal decision in a two-sample problem using the Bayesian approach. First, we considered the optimal BASP for Weibull distribution. In this case, we have shown that the optimal decision function is a Bayes rule. The closed-form of the Bayes decision function cannot be obtained. So we considered the simulation approach to get the Bayes risk and the associated BASP. In particular, for the exponential case, the closed-form of the Bayes decision function can be obtained. So next, we study the optimal BASP for the exponential distribution. The significant advantage of such BASP is that the plan is optimal whenever both the batches are accepted or rejected simultaneously and the disadvantage is that if any one of the batches is rejected, we will reject both the batches. Some optimal BASPs have been provided for both cases, which may be helpful for practitioners. Neutrosophic statistics is the extension of classical statistics and is applied when the data is coming from a complex process or from an uncertain environment. Aslam [1] and Aslam et al. [2, 3] have discussed the acceptance sampling plan using neutrosophic statistics. The proposed study can be extended for neutrosophic statistics under the censored data. It may be mentioned that although we have provided the results for two samples only, the results may be extended for more than two samples. More work is needed along that direction.

Appendix

Calculation of $\phi(\mathbf{W},\mathbf{Z})$ in Weibull distribution

$$\phi(\mathbf{w},\mathbf{z}) \;=\; C_1 + C_2 - C_1\frac{I_1(\mathbf{w},\mathbf{z})}{I(\mathbf{w},\mathbf{z})} - C_2\frac{I_2(\mathbf{w},\mathbf{z})}{I(\mathbf{w},\mathbf{z})}$$

where

$$I(\mathbf{w},\mathbf{z}) \;=\; \int_0^\infty\int_0^\infty\int_0^\infty \alpha^k \lambda_1^{k_1}\lambda_2^{k_2}\prod_{i=1}^{k} w_i^{\alpha-1} e^{-(\lambda_1+\lambda_2)u(\alpha)}\pi(\lambda_1,\lambda_2)\pi(\alpha)\,d\lambda_1 d\lambda_2 d\alpha$$

$$= \frac{\Gamma(a_1+a_2)\Gamma(a_0+k)\Gamma(a_1+k_1)\Gamma(a_2+k_2)b_0^{a_0}d^c}{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)\Gamma(a_1+a_2+k)\Gamma(c)}\int_0^\infty \frac{\alpha^{k+c-1}\prod_{i=1}^k w_i^{\alpha-1}e^{-d\alpha}}{(b_0+u(\alpha))^{a_0+k}}d\alpha,$$

$$\begin{aligned}
&I_1(\mathbf{w,z})\\
&= \int_0^\infty\int_0^\infty\int_0^\infty e^{-\lambda_1 L_1^\alpha}\alpha^k\lambda_1^{k_1}\lambda_2^{k_2}\prod_{i=1}^k w_i^{\alpha-1}e^{-(\lambda_1+\lambda_2)u(\alpha)}\pi(\alpha,\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 d\alpha\\
&= \sum_{l=0}^\infty \frac{(-1)^l}{l!}\int_0^\infty\int_0^\infty\int_0^\infty L_1^{l\alpha}\alpha^k\lambda_1^{k_1+l}\lambda_2^{k_2}\prod_{i=1}^k w_i^{\alpha-1}e^{-(\lambda_1+\lambda_2)u(\alpha)}\pi(\lambda_1,\lambda_2)\pi(\alpha)d\lambda_1 d\lambda_2 d\alpha\\
&= \sum_{l=0}^\infty \frac{(-1)^l}{l!}\frac{\Gamma(a_1+a_2)\Gamma(a_0+k+l)\Gamma(a_1+k_1+l)\Gamma(a_2+k_2)b_0^{a_0}d^c}{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)\Gamma(a_1+a_2+k+l)\Gamma(c)}\int_0^\infty \frac{\alpha^{k+c-1}\prod_{i=1}^k w_i^{\alpha-1}L_1^{l\alpha}e^{-d\alpha}}{(b_0+u(\alpha))^{a_0+k+l}}d\alpha.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&I_2(\mathbf{w,z})\\
&= \sum_{l=0}^\infty \frac{(-1)^l}{l!}\frac{\Gamma(a_1+a_2)\Gamma(a_0+k+l)\Gamma(a_1+k_1)\Gamma(a_2+k_2+l)b_0^{a_0}d^c}{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)\Gamma(a_1+a_2+k+l\Gamma(c))}\int_0^\infty \frac{\alpha^{k+c-1}\prod_{i=1}^k w_i^{\alpha-1}L_2^{l\alpha}e^{-d\alpha}}{(b_0+u(\alpha))^{a_0+k+l}}d\alpha
\end{aligned}$$

Proof of Theorem 4.1

(a) For the fixed k_1 and k_2, let us denote $\phi(k_1,k_2,u) = E[g(\lambda_1,\lambda_2)|k_1,k_2,u]$. The joint posterior PDF of λ_1 and λ_2 can be obtained as

$$\pi(\lambda_1,\lambda_2|k_1,k_2,u) = \frac{\lambda_1^{k_1}\lambda_2^{k_2}e^{-(\lambda_1+\lambda_2)u}\pi(\lambda_1,\lambda_2)}{\int_0^\infty\int_0^\infty \lambda_1^{k_1}\lambda_2^{k_2}e^{-(\lambda_1+\lambda_2)u}\pi(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2}. \tag{19}$$

Simplifying we obtain $(\lambda_1,\lambda_2)|(k_1,k_2,u) \sim BG(a_0+k, b_0+u, a_1+k_1, a_2+k_2)$ which implies that the Beta-Gamma is a conjugate prior. Therefore it is straight forward that

$$\left.\frac{\lambda_1}{\lambda_1+\lambda_2}\right|(k_1,k_2,u) \sim Beta(a_1+k_1,a_2+k_2) \text{ and } \lambda_1+\lambda_2|(k_1,k_2,u) \sim G(a_0+k,b_0+u).$$

Let us define $\eta = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$ and $\lambda = \lambda_1 + \lambda_2$ then by the following one-to-one transformation $\lambda_1 = \lambda\eta$ and $\lambda_2 = \lambda(1-\eta)$, the joint posterior density function of λ_1 and λ_2 is

$$\pi(\lambda_1, \lambda_2 | k_1, k_2, u) = \frac{1}{\lambda} \times \pi(\eta | k_1, k_2, u) \times \pi(\lambda | k_1, k_2, u) \;\; \forall \lambda_1, \lambda_2 > 0. \tag{20}$$

Note that for a fixed k_1 and k_2, the posterior density function of $\eta|(k_1, k_2, u)$ is constant w.r.t. u, and the posterior density function of $\lambda|(k_1, k_2, u)$ is a function of u. Since the distribution function of $\lambda|(k_1, k_2, u)$ is gamma which belongs to the exponential family, it has Monotone Likelihood Ratio (MLR) property. Then it is straightforward to show that for $u_1 < u_2$, the likelihood ratio $\frac{\pi(\lambda|k_1,k_2,u_1)}{\pi(\lambda|k_1,k_2,u_2)}$ is strictly increasing in λ. Therefore, the posterior distribution of $\lambda|(k_1, k_2, u)$ is stochastically decreasing function of u, and hence due to (20) the joint posterior distribution of $(\lambda_1, \lambda_2)|(k_1, k_2, u)$ is stochastically decreasing function of u. So, if $g(\lambda_1, \lambda_2)$ is a positive function in λ_1, λ_2 and $u_1 < u_2$, then

$$E[g(\lambda_1, \lambda_2)|k_1, k_2, u_1] > E[g(\lambda_1, \lambda_2)|k_1, k_2, u_2],$$

which implies that $\phi(k_1, k_2, u_1) > \phi(k_1, k_2, u_2)$. Hence $\phi(k_1, k_2, u)$ is strictly decreasing in u for fixed k_1 and k_2.

(b) From (a), it is straightforward that δ_B is strictly increasing in u for fixed k_1 and k_2.

Bayes risk derivation in Exponential distribution

The Bayes risk is given by

$$\begin{aligned} R_B^{\delta_B}(n, k, R_1, \ldots, R_{k-1}) &= n(C_s - r_{s_1} - r_{s_2}) + E[K_1] r_{s_1} + E[K_2] r_{s_2} + E[W_k] C_\tau \\ &+ \frac{\Gamma(a_1 + a_2)}{\Gamma(a_0)\Gamma(a_2)\Gamma(a_2)} \sum_{i=0}^{4} c_i \frac{\Gamma(a_1 + p_i)\Gamma(a_2 + q_i)\Gamma(a_0 + p_i + q_i)}{\Gamma(a_1 + a_2 + p_i + q_i) b_0^{p_i + q_i}} + R^{\delta_B}, \end{aligned}$$

where

$$
\begin{aligned}
R^{\delta_B} &= E_{\lambda_1,\lambda_2}[(C_r - c_0 - c_1\lambda_1 - c_2\lambda_2 - c_3\lambda_1^2 - c_4\lambda_2^2)P(U \le \xi(K_1,K_2))] \\
&= \sum_{i=0}^{4} C_i E_{\lambda_1,\lambda_2}[\lambda_1^{p_i}\lambda_2^{q_i} P(U \le \xi(K_1,K_2))] \\
&= \sum_{i=0}^{4}\sum_{r=0}^{k} \binom{k}{r} \frac{C_i}{\Gamma(k)} E_{\lambda_1,\lambda_2}\left[\lambda_1^{p_i}\lambda_2^{q_i} \int_0^{\xi(r,k-r)} \lambda_1^k \lambda_2^{k-r} u^{k-1} e^{-\lambda u} du\right] \\
&= \sum_{i=0}^{4}\sum_{r=0}^{k} \binom{k}{r} \frac{C_i}{\Gamma(k)} \int_0^{\xi(r,k-r)} E_{\lambda_1,\lambda_2}\left[\left(\frac{\lambda_1}{\lambda}\right)^{r+p_i}\left(1-\frac{\lambda_1}{\lambda}\right)^{k-r+q_i} \lambda^{k+p_i+q_i} e^{-\lambda u}\right] u^{k-1} du
\end{aligned}
$$

Since $\frac{\lambda_1}{\lambda}$ and λ is independent we can write

$$
\begin{aligned}
R^{\delta_B} &= \sum_{i=0}^{4}\sum_{r=0}^{k} \binom{k}{r} \frac{C_i}{\Gamma(k)} \int_0^{\xi(r,k-r)} E_{\lambda_1,\lambda_2}\left[\left(\frac{\lambda_1}{\lambda}\right)^{r+p_i}\left(1-\frac{\lambda_1}{\lambda}\right)^{k-r+q_i}\right] E_{\lambda_1,\lambda_2}\left[\lambda^{k+p_i+q_i} e^{-\lambda u}\right] u^{k-1} du \\
&= \sum_{i=0}^{4}\sum_{r=0}^{k} \binom{k}{r} \frac{C_i\Gamma(k+p_i+q_i+a_0)B(r+p_i+a_1,k-r+q_i+a_2)}{\Gamma(k)\Gamma(a_0)B(a_1,a_2)b_0^{-a_0}} \int_0^{\xi(r,k-r)} \frac{u^{k-1}}{(u+b_0)^{k+p_i+q_i+a_0}} du \\
&= \sum_{i=0}^{4}\sum_{r=0}^{k} \binom{k}{r} \frac{C_i\Gamma(k+p_i+q_i+a_0)B(r+p_i+a_1,k-r+q_i+a_2)}{\Gamma(k)\Gamma(a_0)B(a_1,a_2)b_0^{p_i+q_i}} \int_0^{\frac{\xi(r,k-r)}{b_0}} \frac{u^{k-1}}{(1+u)^{k+p_i+q_i+a_0}} du \\
&= \sum_{i=0}^{4}\sum_{r=0}^{k} \binom{k}{r} \frac{C_i\Gamma(p_i+q_i+a_0)B(r+p_i+a_1,k-r+q_i+a_2)}{\Gamma(a_0)B(a_1,a_2)b_0^{p_i+q_i}} I_{S_r}(k,p_i+q_i+a_0),
\end{aligned}
$$

where

$$
\int_0^{\frac{\xi(r,k-r)}{b_0}} \frac{u^{k-1}}{(1+u)^{k+p_i+q_i+a_0}} du = \int_0^{S_r} v^{k-1}(1-v)^{p_i+q_i+a_0-1} dv = B_{S_r}(k,p_i+q_i+a_0),
$$

$S_r = \frac{\xi(r,k-r)}{b_0}$and $I_{S_r}(k,p_i+q_i+a_0) = \frac{B_{S_r}(k,p_i+q_i+a_0)}{B(k,p_i+q_i+a_0)}$, and $B(a,b) = \int_0^1 z^{a-1}(1-z)^{b-1}dz = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function, and $B_x(a,b) = \int_0^x u^{a-1}(1-u)^{b-1}du,$ where $0 \le x \le 1$, is the incomplete beta function. We denote the cumulative distribution function of beta as, $I_x(a,b) = \frac{B_x(a,b)}{B(a,b)}$. Then Bayes risk is given as

$$
R_B^{\delta_B}(n,k,R_1,\ldots,R_{k-1})
$$

$$
\begin{aligned}
&= n(C_s - r_{s_1} - r_{s_2}) + E[K_1]r_{s_1} + E[K_2]r_{s_2} + E[W_k]C_\tau \\
&\quad + \frac{\Gamma(a_1+a_2)}{\Gamma(a_0)\Gamma(a_2)\Gamma(a_2)} \sum_{i=0}^{4} c_i \frac{\Gamma(a_1+p_i)\Gamma(a_2+q_i)\Gamma(a_0+p_i+q_i)}{\Gamma(a_1+a_2+p_i+q_i)b_0^{p_i+q_i}} \\
&\quad + \sum_{i=0}^{4}\sum_{r=0}^{k}\binom{k}{r}\frac{C_i\Gamma(p_i+q_i+a_0)B(r+p_i+a_1,k-r+q_i+a_2)}{\Gamma(a_0)B(a_0,a_1)b_0^{p_i+q_i}} I_{S_r}(k,p_i+q_i+a_0),
\end{aligned}
$$

where $K_1(K_2)$ follows a Binomial distribution with parameters k and $\frac{\lambda_1}{\lambda}(\frac{\lambda_2}{\lambda})$, therefore

$$
E[K_1] \;=\; E_{(\lambda_1,\lambda_2)}E_{(\mathbf{W},\mathbf{Z})|(\lambda_1,\lambda_2)}[K_1] = E_{(\lambda_1,\lambda_2)}\left[k\frac{\lambda_1}{\lambda}\right] = k\frac{B(a_1+1,a_2)}{B(a_1,a_2)}
$$

similarly, $E(K_2) = k\dfrac{B(a_1,a_2+1)}{B(a_1,a_2)}$, and from Mondal and Kundu [24] for all $i = 1,\ldots,k$, $W_i = \sum_{s=1}^{i} V_s$, where $V_s \sim Exp(\frac{1}{E_s})$ independently and $E_s = \lambda(n - \sum_{j=1}^{s-1}(R_j+1))$, using this the $E[W_k]$ is given as

$$
E[W_k] \;=\; b_0\frac{\Gamma(a_0-1)}{\Gamma(a_0)} \times \sum_{s=1}^{k} \frac{1}{(n - \sum_{j=1}^{s-1}(R_j+1))}
$$

Therefore, $E(W_k)$ exists only if $a_0 > 1$.

Proof of Theorem 3.1

Let $(n_B, k_B, R_{1_B}, \ldots, R_{k_B-1}, \delta_B)$ be the optimal BASP, then minimum Bayes risk is given by $R_B^{\delta_B}(n_B, k_B, R_{1_B}, \ldots, R_{k_B-1})$. Let $(0,0,0,\ldots,0,\delta_B = 0)$ denote the sampling plan, when we reject the batch without sampling with Bayes risk $R_B^{\delta_B=0}(0,0,0,\ldots,0) = C_r$ and $(0,0,0,\ldots,0,\delta_B = 1)$ denote the sampling plan when we accept the batch without sampling with Bayes risk $R_B^{\delta_B=1}(0,0,0,\ldots,0) = E_{\alpha,\lambda_1,\lambda_2}[g(\alpha,\lambda_1,\lambda_2)]$. Then the Bayes risk of optimal BASP is

$$
R_B^{\delta_B}(n_B, k_B, R_{1_B}, \ldots, R_{k_B-1}) \leq \min\left\{E_{\alpha,\lambda_1,\lambda_2}[g(\alpha,\lambda_1,\lambda_2)], C_r\right\}. \tag{21}
$$

From (5) it is clear that

$$R_B^{\delta_B}(n_B, k_B, R_{1_B}, \ldots, R_{k_B-1}) \geq n_B(C_s - r_{s_1} - r_{s_2}). \tag{22}$$

Now from (22) and (21), it follows that

$$n_B(C_s - r_{s_1} - r_{s_2}) \leq \min\left\{E_{\alpha,\lambda_1,\lambda_2}[g(\alpha,\lambda_1,\lambda_2)], C_r\right\},$$

which implies that

$$n_B \leq \min\left\{\frac{E_{\alpha,\lambda_1,\lambda_2}[g(\alpha,\lambda_1,\lambda_2)]}{C_s - r_{s_1} - r_{s_2}}, \frac{C_r}{C_s - r_{s_1} - r_{s_2}}\right\},$$

and $0 \leq k_B \leq n_B$.

Compliance with Ethical Standards

Conflict of Interest: Author 1 declares that he/she has no conflict of interest. Author 2 declares that he/she has no conflict of interest. Author 3 declares that he/she has no conflict of interest.

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