A Finite Mixture Model for Multiple Dependent Competing Risks with Applications of Automotive Warranty Claims Data

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Abstract

This paper introduces a parametric finite mixture model (FMM) approach to analyze the dependent competing risks data subjected to progressive first-failure censoring and multiple causes of failure. The cause-specific failure times are assumed to be flexibly modeled by the Lehmann family of distributions (also known as the exponentiated distributions) with variation in both distribution parameters. Application of the expectation maximization (EM) algorithm facilitates the maximum likelihood estimation of the model parameters and illuminates the contribution of the censored data. For interval estimation purposes, we resort to using the asymptotic confidence intervals based on the observed Fisher information matrix. Practitioners often prefer employing simpler lifetime distribution in order to facilitate the data modeling process while knowing the true distribution. In this context, the effects of model misspecification are studied based on the p-th quantile when the true distribution is misspecified. An extensive simulation study is performed to validate our proposed model. Finally, an automotive warranty claims data set is used as an illustration to study the effectiveness of our proposed model, assuming some important members of the Lehmann family, like generalized exponential and exponentiated Pareto distributions.

Keywords and Phrases: Dependent competing risks, Lehmann Family, Generalized exponential, Exponentiated Pareto, Progressive first-failure censoring scheme.

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1 Introduction

In real life instances, failures of devices or systems generally happen owing to the effect of multiple causes competing among each other. For instance, an electrical device may fail because of a defect made during production or a wear out caused by normal usage. In statistical literature, such failure time data along with the associated causes are referred to as the competing risks data. Competing risks data occur in diverse fields, such as engineering, biological, social science, or medical studies. Overlooking competing risks in statistical analysis can lead to biased risk estimates (see [1] in this respect). For a detailed review on competing risks problem, readers are referred to the monograph by [2], and the references cited therein. Several techniques are available now in the literature to model competing risks data. Among them, the most well-known approach is the latent failure times model proposed by [3]. The latent failure times model requires that the cause-specific failure time distributions are independent. However, the independence assumption among the failure causes is a rigid one and often turns out to be inappropriate in practical world. For instance, the breakdown of one aeroplane engine will lead to additional pressure on the remaining functional engines and thus enhances the risk of failure of both engines and aeroplane. Again, in medical treatment, if a patient suffering from severe eye disease turns blind in one eye, the other eye may experience the higher risk of blindness. Hence, it seems quite logical to assume the dependency among the causes of failure. Often, finite mixture models (FMMs) can be employed to analyze competing risks failure-time data and other scenarios in this context (see [4, 5, 6, 7, 8, 9, 10, 11]).

Lehmann family of distributions (also known as the exponentiated distributions) are well known in reliability literature for analyzing the failure-time data. A random variable T is said to belong to Lehmann family of distributions if the cumulative distribution function

(CDF) of T is given by:

$$F(t;\alpha,\lambda) = \begin{cases} [G_0(t;\lambda)]^{\alpha} & \text{if } t > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

where $G_0(t;\lambda)$ is some absolutely continuous CDF of the baseline random variable assumed to be specified completely except the unknown parameter $\lambda > 0$. The associated probability density function (PDF) is given by

$$f(t;\alpha,\lambda) = \begin{cases} \alpha g_0(t;\lambda)[G_0(t;\lambda)]^{\alpha-1} & \text{if } t > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

where $g_0(t;\alpha)$ is the PDF of the baseline random variable. The Lehmann family is quite flexible in reliability engineering in the sense that the hazard function of the members of this family can exhibit various shapes like increasing, decreasing, unimodal, bathtub shaped etc., depending on the choice of the base line distribution. Over the years, researchers have proposed various members of the Lehmann family - the exponentiated Weibull distribution [12] (some special cases of which are the generalized (exponentiated) exponential distribution [13] and the exponentiated Rayleigh distribution [14]) and exponentiated Pareto distribution [15], etc.

Recently, [16] introduced a new censoring scheme - the progressive first-failure censoring scheme (PFFCS), to analyze lifetime data. Let n independent groups with each group containing k units be subjected to a life test. Denoting the progressive censoring scheme as $\boldsymbol{R} = (R_1, R_2, \ldots, R_m)$ with $R_i \in \mathbb{N} \cup \{0\}$, a progressive first-failure censored sample of size m from a parent sample of size kn can be obtained as follows. As soon as the first failure (say $T^{\boldsymbol{R}}_{1:m:n:k}$) occurs, the group with this observed failure along with R_1 groups of the remaining $(n-1)$ groups are randomly removed from the experiment. Next, at the second failure (say $T^{\boldsymbol{R}}_{2:m:n:k}$), the group with this second observed failure along with R_2 groups of the remaining $(n - R_1 - 2)$ groups are randomly removed from the test. Lastly, at the m-th failure (say $T^{\boldsymbol{R}}_{m:m:n:k}$), the group with this observed failure along with the remaining R_m groups are removed from the experiment. The quantities m and $\boldsymbol{R} = (R_1, R_2, \ldots, R_m)$ are fixed known quantities and clearly $\sum_{i=1}^{m} R_i + m = n$. An illustration of the scheme is provided below in Figure 1. PFFCS turns out to be a generalization of several notable censoring

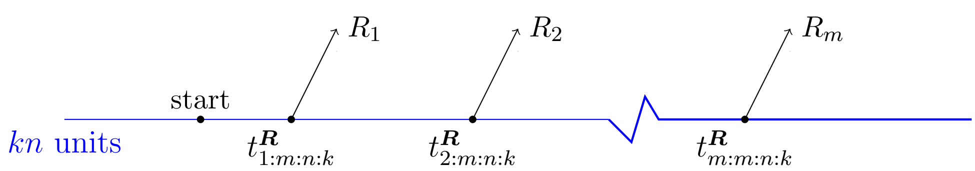

Figure 1: Illustration of PFFCS

schemes. If $\boldsymbol{R} = (0, \ldots, 0)$, PFFCS boils down to the first-failure censoring scheme. In case if $k = 1$, PFFCS reduces to the progressive Type-II censoring scheme. If $k = 1$, and $\boldsymbol{R} = (0, \ldots, 0)$, then clearly $n = m$ and it corresponds to the complete sample case. If $k = 1$ and $\boldsymbol{R} = (0, \ldots, 0, n - m)$, then one can obtain the usual Type-II censoring scheme. It has an advantage over the usual censoring schemes in terms of shorter test time as well as a saving of resources. Precisely, the scheme $R = (0, \ldots, 0, n - m)$ usually has the shortest expected test time, and the scheme $R = (n - m, 0, \ldots, 0)$ which has a longest expected test time is the one with less MSE which indicate the trade-off between time and accuracy in the estimation of PFFCS.

In this paper, we have developed the finite mixture model (FMM) for analyzing the dependent competing risks data with multiple causes of failure under PFFCS. A multinomial distribution is assumed for modeling the failure types (causes) and the cause-specific distributions are assumed to originate from the Lehmann family of distributions. The motivating example for the model and methodology discussed in this article is an interesting data set comprising of recurrent failure history of a fleet of automobiles (see [17] for the data set). The outcome of interest is repeated mileages at failure for 189 vehicles of a certain model and make obtained from a warranty claim database that also documents the labor code associated with the failure. In particular, these data focus on early failures from the time of production to a maximum mileage of 3000. Fourteen different labor codes (LC) for failure are recorded in the warranty claims and they are grouped into three broad causes of failure - FM1, FM2 and FM3. Clearly, these resulting groups of FMs are not independent. Analysis of this automobile warranty claims data set demands for a suitable dependent competing risks model. We have used this particular data set to see the performance of the proposed FMM.

Employing the expectation maximization (EM) based FMM formulation, maximum likelihood estimators (MLEs) of all the unknown model parameters can be obtained easily. However, in this set up, it is difficult to obtain the exact distribution of the MLEs and hence the exact confidence intervals (CIs). Hence, we resort to construct the CIs using the asymptotic normality of the MLEs for interval estimation purpose.

The rest of the article is organized as follows. In Section 2, the FMM formulation is described in detail along with the resulting likelihood function. The proposed EM algorithm is introduced in Section 3 to obtain the point estimates of all the model parameters. In addition, the associated interval estimation methodology for the model parameters is briefly outlined. In Section 4, the effect of misspecification analysis is discussed when the underlying true failure time distribution is misspecified. An extensive simulation study is carried out in Section 5 assuming generalized exponential and exponentiated Pareto distributions - the two notable members of the Lehmann family of distributions. The expected total times associated with the proposed method are also computed for both the distributions. Analysis of a real life automobile warranty claims example is carried out in Section 6 and finally we conclude the paper in Section 7.

2 Model Description and Likelihood Function

Suppose, the observed progressive first-failure censored data with the progressive censoring scheme $\boldsymbol{R} = (R_1, R_2, \ldots, R_m)$ is given by $t^{\boldsymbol{R}}_{1:m:n:k} < t^{\boldsymbol{R}}_{2:m:n:k} < \cdots < t^{\boldsymbol{R}}_{m:m:n:k}$. Now, assuming that failure times of all the kn units under study follow an absolutely continuous CDF $F(\cdot)$

with the corresponding PDF $f(\cdot)$, the joint PDF of $T^{\boldsymbol{R}}_{1:m:n:k}, T^{\boldsymbol{R}}_{2:m:n:k}, \ldots, T^{\boldsymbol{R}}_{m:m:n:k}$ is given by

$$f_{1,2,\ldots,m}(t^{\boldsymbol{R}}_{1:m:n:k}, t^{\boldsymbol{R}}_{2:m:n:k}, \cdots, t^{\boldsymbol{R}}_{m:m:n:k}) = Ck^m \prod_{i=1}^{m} f(t_{i:m:n:k}) \left[1 - F(t_{i:m:n:k})\right]^{k(R_i+1)-1}, \quad (2.1)$$

where

$$C = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \cdots (n - \sum_{i=1}^{m-1} R_i - m + 1). \quad (2.2)$$

It is to note here that $T^{\boldsymbol{R}}_{1:m:n:k}, T^{\boldsymbol{R}}_{2:m:n:k}, \ldots, T^{\boldsymbol{R}}_{m:m:n:k}$ can also be looked upon as a progressively Type-II censored sample from a population with CDF as $1-\{1-F(x)\}^k$. Hence, one can easily extend the results for the progressively Type-II censored order statistics to progressively first-failure censored order statistics. Although only m out of kn items have failed in PFFCS, it has significant advantages in terms of reducing test cost and test time (see [16] for more details). Let $T^{\boldsymbol{R}}_{i:m:n:k}$, $i = 1, 2, \ldots, m$, denote the progressively first-failure censored order statistics from Lehmann family of distributions with parameters α, λ and with the progressive censoring scheme $\boldsymbol{R}$. From now on, t_i will be used instead of $t^{\boldsymbol{R}}_{i:m:n:k}$ for notational simplicity. Using (2.1) and assuming $\boldsymbol{t} = (t_1, t_2, \cdots, t_m)$, the likelihood function of α and λ is given by

$$L(\alpha, \lambda|\boldsymbol{t}) = Ck^m\alpha^m \prod_{i=1}^{m} \Big[\{G_0(t_i;\lambda)\}^{\alpha-1} g_0(t_i;\lambda)\{1 - [G_0(t_i;\lambda)]^{\alpha}\}^{k(R_i+1)-1}\Big],$$

where C is defined as in (2.2).

In a competing risks failure time model, it is assumed that every unit of a certain target population fails owing to S competing causes. Let ϕ_j denotes the prior probability that a unit fails owing to $j-$th cause, $j = 1, 2, \ldots, S$. In a FMM formulation, the PDF of the survival time of a unit is then given by $\sum_{j=1}^{S} \phi_j f(t;\alpha_j,\lambda_j)$, where ϕ_1, ϕ_2, $, \ldots, \phi_S$ are the mixing weights of the FMM and $f(t;\alpha_j,\lambda_j)$ is the PDF of the unit's failure time owing to $j-$th cause. The associated survival function of the unit is similarly given by $\sum_{j=1}^{S} \phi_j S(t;\alpha_j,\lambda_j)$ where $S(t;\alpha_j,\lambda_j)$ is the $j-$th cause-specific survival function ($j = 1, 2, \ldots, S$). It is to note here that the dependency among the causes is accounted for by the restriction $\sum_{j=1}^{S} \phi_j = 1$.

Let Δ be a random variable defined by

$$\Delta = \begin{cases} 1, & \text{if the unit fails owing to 1st cause} \\ 2, & \text{if the unit fails owing to 2nd cause} \\ \vdots & \\ S, & \text{if the unit fails owing to } S\text{th cause.} \end{cases}$$

The resulting competing risks failure time data set then looks like

$$(\boldsymbol{t},\, \boldsymbol{\delta}) = (t_1, \delta_1), (t_2, \delta_2), \ldots, (t_m, \delta_m),$$

where (t_i, δ_i), $i = 1, 2, \ldots, m$ are the realizations of the random vector (T, Δ). The associated joint PDF is given by

$$g(t_i, \delta_i = j) \propto \phi_j f(t_i; \alpha_j, \lambda_j) \left[\sum_{q=1}^{S} \phi_q S(t_i; \alpha_q, \lambda_q) \right]^{k(R_i+1)-1}, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, S. \tag{2.3}$$

Now, for $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, S$, define an indicator random variable as

$$I_j(i) = \begin{cases} 1, & \text{if } \delta_i = j \\ 0, & \text{otherwise.} \end{cases}$$

From (2.3), the likelihood function based on $(\boldsymbol{t},\, \boldsymbol{\delta})$ is given by

$$\begin{aligned} L(\boldsymbol{\theta}|\boldsymbol{t},\, \boldsymbol{\delta}) \;\propto\; & \Big\{ \prod_{j=1}^{S-1} \phi_j^{m_j} \Big\} (1 - \sum_{i=1}^{S-1} \phi_i)^{m_S} \Big\{ \prod_{j=1}^{S} \alpha_j^{m_j} \Big\} \\ & \times \prod_{i=1}^{m} \prod_{j=1}^{S} \Big[\{G_0(t_i; \lambda_j)\}^{\alpha_j - 1} g_0(t_i; \lambda_j) \{ \sum_{j=1}^{S} \phi_j \big(1 - [G_0(t_i; \lambda_j)]^{\alpha_j}\big) \}^{\{k(R_i+1)-1\}} \Big]^{I_j(i)}, \end{aligned} \tag{2.4}$$

where $\boldsymbol{\theta} = (\phi_1, \phi_2, \ldots, \phi_{S-1}, \alpha_1, \alpha_2, \ldots, \alpha_S, \lambda_1, \lambda_2, \ldots, \lambda_S)$ is the set of model parameters of interest and $m_j = \sum_{i=1}^{m} I_j(i)$ is the number of failures owing to $j-$th cause, $j = 1, 2, \ldots, S$. The associated log-likelihood function including terms involving only $\boldsymbol{\theta}$ is given by

$$\begin{aligned} l(\boldsymbol{\theta}|\boldsymbol{t},\, \boldsymbol{\delta}) \;\propto\; & \sum_{j=1}^{S-1} m_j \ln \phi_j + m_S \ln(1 - \sum_{i=1}^{S-1} \phi_i) + \sum_{j=1}^{S} m_j \ln \alpha_j \\ & + \sum_{i=1}^{m} \sum_{j=1}^{S} I_j(i) \Big[\ln g_0(t_i; \lambda_j) + (\alpha_j - 1) \ln G_0(t_i; \lambda_j) \\ & + (k(R_i + 1) - 1) \ln \sum_{j=1}^{S} \phi_j \big(1 - [G_0(t_i; \lambda_j)]^{\alpha_j}\big) \Big]. \end{aligned} \tag{2.5}$$

$\widehat{\boldsymbol{\theta}}$, the MLE of $\boldsymbol{\theta}$, can be obtained by maximization of the log-likelihood function (2.5) using some standard maximization routine and it requires solving a $(3S - 1)$ dimensional optimization problem. In particular, the MLEs of any of the model parameters cannot be obtained in closed form and the complexity of the optimization problem grows as S increases. However, the maximization problem gets simplified if we treat it as a missing data problem. Hence, we propose to use the EM algorithm.

3 EM Algorithm

At first, let us denote the observed and the censored data by $\boldsymbol{T} = (T_1, \ldots, T_m)$ and $\boldsymbol{Z} = (\boldsymbol{Z_1}, \boldsymbol{Z_2}, \ldots, \boldsymbol{Z_m})$, respectively, where each $\boldsymbol{Z_j}$ is a $1 \times (kR_j + k - 1)$ vector with $\boldsymbol{Z_j} = (Z_{j1}, Z_{j2}, \ldots, Z_{j(kR_j+k-1)})$ for $j = 1, 2, \ldots, m$. It is to note here that the censored data vector $\boldsymbol{Z}$ is not observable and can be thought of as the missing data and

$$\gamma_j(i) = \begin{cases} 1 & \text{if the unit } Z_{il} \text{ fails owing to } j- \text{ th cause } \forall\ 1 \le i \le m,\ 1 \le l \le k(R_i + 1) - 1 \\ 0 & \text{otherwise.} \end{cases}$$

To apply the EM algorithm for dealing with dependent competing risks data under progressive first- failure censoring, we use the approach suggested by Kundu and Basu [18]. The combination of $(\boldsymbol{T},\boldsymbol{Z})=\boldsymbol{V}$ constitutes the complete dataset. If we denote the log-likelihood function of the complete data set by $l_c(\boldsymbol{v};\boldsymbol{\theta})$, then, ignoring the additive constant, we have

$$\begin{aligned}
l_c(\boldsymbol{\theta}) \;\propto\; & \sum_{j=1}^{S-1}\big[m_j+\sum_{i=1}^{m}(k(R_i+1)-1)\gamma_j(i)\big]\ln\phi_j \\
& +\big[m_S+\sum_{i=1}^{m}(k(R_i+1)-1)\gamma_S(i)\big]\ln(1-\sum_{i=1}^{S-1}\phi_i) \\
& +\sum_{j=1}^{S}\sum_{i=1}^{m_j}\big[\ln(\alpha_j)+\ln g_0(t_i,\lambda_j)+(\alpha_j-1)\ln G_0(t_i,\lambda_j)\big] \\
& +\sum_{j=1}^{S}\sum_{i=1}^{m}(k(R_i+1)-1)\gamma_j(i)\big[\ln(1-[G_0(t_i;\lambda_j)]^{\alpha_j})\big]
\end{aligned}$$

E Step: In the E-Step, the algorithm constructs 'pseudo data' where the failed observations are left as it is. The unit mass associated with each of the censored observations (say $Z_{ip}, p=1,2,\ldots,(kR_i+k-1)$) in $\boldsymbol{Z_i}(i=1,2,\ldots,m)$, is partitioned and assigned to S 'pseudo observations' of the form $(Z_{ip},\Delta=j)$. Specifically, the partitioned mass $w_{ij}(\phi_j,\alpha_j,\lambda_j)$, assigned to this 'pseudo observation' is the conditional probability that the unit fails from cause j given that the unit had survived after time t_i, which is $E(\gamma_j(i))$. Hence, for $i=1,2,\ldots,m$, we have

$$w_{ij}(\phi_j,\alpha_j,\lambda_j)=P(\Delta=j|Z_{ip}>t_i)=\frac{\phi_j\big(1-[G_0(t_i;\lambda_j)]^{\alpha_j}\big)}{\sum_{k=1}^{S}\phi_k\big(1-[G_0(t_i;\lambda_k)]^{\alpha_k}\big)},\quad j=1,2,\ldots,S \quad (3.1)$$

The log-likelihood of the 'pseudo data' can be written as

$$\begin{aligned}
l_c(\boldsymbol{\theta}) \;\propto\; & \sum_{j=1}^{S-1}\big[m_j+\sum_{i=1}^{m}(k(R_i+1)-1)w_{ij}(\phi_j,\alpha_j,\lambda_j)\big]\ln\phi_j \\
& +\big[m_S+\sum_{i=1}^{m}(k(R_i+1)-1)w_{iS}(1-\sum_{i=1}^{S-1}\phi_i,\alpha_S,\lambda_S)\big]\ln(1-\sum_{i=1}^{S-1}\phi_i) \\
& +\sum_{j=1}^{S}\sum_{i=1}^{m_j}\big[\ln(\alpha_j)+\ln g_0(t_i,\lambda_j)+(\alpha_j-1)\ln G_0(t_i,\lambda_j)\big] \\
& +\sum_{j=1}^{S}\sum_{i=1}^{m}(k(R_i+1)-1)\big[w_{ij}(\phi_j,\alpha_j,\lambda_j)\ln(1-[G_0(t_i;\lambda_j)]^{\alpha_j})\big].
\end{aligned}$$

M Step: The M-Step of the EM algorithm involves maximization of $l_c(\boldsymbol{\theta})$ iteratively. Assuming $w_{ij}(\phi_j, \alpha_j, \lambda_j)'$s to be known, one requires to maximize $l_c(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$. To be precise, if $\boldsymbol{\theta}^{(l)} = (\phi_1^{(l)}, \phi_2^{(l)}, \ldots, \phi_{S-1}^{(l)}, \alpha_1^{(l)}, \alpha_2^{(l)}, \ldots, \alpha_S^{(l)}, \lambda_1^{(l)}, \lambda_2^{(l)}, \ldots, \lambda_S^{(l)})$ is the $l-$th iterate, the $(l+1)-$th iterate $\boldsymbol{\theta}^{(l)} = (\phi_1^{(l+1)}, \phi_2^{(l+1)}, \ldots, \phi_{S-1}^{(l+1)}, \alpha_1^{(l+1)}, \alpha_2^{(l+1)}, \ldots, \alpha_S^{(l+1)}, \lambda_1^{(l+1)}, \lambda_2^{(l+1)}, \ldots, \lambda_S^{(l+1)})$ can be obtained by maximizing the following log-likelihood:

$$
\begin{aligned}
l_c(\boldsymbol{\theta}) \;\propto\; & \sum_{j=1}^{S-1} \Big[m_j + \sum_{i=1}^{m} (k(R_i+1)-1) w_{ij}(\phi_j^{(l)}, \alpha_j^{(l)}, \lambda_j^{(l)})\Big] \ln \phi_j \\
& + \Big[m_S + \sum_{i=1}^{m} (k(R_i+1)-1) w_{iS}(1 - \sum_{i=1}^{S-1} \phi_i^{(l)}, \alpha_S^{(l)}, \lambda_S^{(l)})\Big] \ln(1 - \sum_{i=1}^{S-1} \phi_i) \\
& + \sum_{j=1}^{S} \sum_{i=1}^{m_j} \Big[\ln(\alpha_j) + \ln g_0(t_i, \lambda_j) + (\alpha_j - 1) \ln G_0(t_i, \lambda_j)\Big] \\
& + \sum_{j=1}^{S} \sum_{i=1}^{m} (k(R_i+1)-1) \Big[w_{ij}(\phi_j^{(l)}, \alpha_j^{(l)}, \lambda_j^{(l)}) \ln(1 - [G_0(t_i;\lambda_j)]^{\alpha_j})\Big]
\end{aligned}
$$

At first, we obtain the improved estimates of the mixing weights at the $(l+1)-$th iterate given by

$$
\phi_d^{(l+1)} = \frac{m_d + \sum_{i=1}^{m} (k(R_i+1)-1) w_{id}(\phi_d^{(l)}, \alpha_d^{(l)}, \lambda_d^{(l)})}{kn}, \quad d = 1, 2, \ldots, S-1.
$$

For all $d = 1, 2, \ldots, S$, $(\alpha_d^{(l+1)}, \lambda_d^{(l+1)})$ is then obtained by solving the profile-likelihood:

$$
\begin{aligned}
l_d(\alpha_d, \lambda_d) \;\propto\; & \Big[m_d + \sum_{i=1}^{m} (k(R_i+1)-1) w_{id}(\phi_d^{(l)}, \alpha_d^{(l)}, \lambda_d^{(l)})\Big] \ln \phi_d^{(l+1)} \\
& + \sum_{i=1}^{m_d} \Big[\ln(\alpha_d) + \ln g_0(t_i, \lambda_d) + (\alpha_d - 1) \ln G_0(t_i, \lambda_d)\Big] \\
& + \sum_{i=1}^{m} (k(R_i+1)-1) \Big[w_{id}(\phi_d^{(l)}, \alpha_d^{(l)}, \lambda_d^{(l)}) \ln(1 - [G_0(t_i;\lambda_d)]^{\alpha_d})\Big]
\end{aligned}
$$

Therefore, we can use Algorithm 1 to proceed from the $l-$th iterate to $(l+1)-$th iterate.

Algorithm 1 : EM Algorithm

1: Set the initial guesses as $\phi_1^{(0)}, \dots, \phi_{S-1}^{(0)}$, $\alpha_1^{(0)}, \dots, \alpha_S^{(0)}$, $\lambda_1^{(0)}, \dots, \lambda_S^{(0)}$.

2: Obtain the improved estimates of the mixing weights at the $(l+1)-$th iterate given by

$$\phi_d^{(l+1)} = \frac{m_d + \sum_{i=1}^{m}(k(R_i+1)-1)w_{id}(\phi_d^{(l)}, \alpha_d^{(l)}, \lambda_d^{(l)})}{kn}, \quad d = 1, 2, \dots, S-1.$$

3: For all $d = 1, 2, \dots, S$, $(\alpha_d^{(l+1)}, \lambda_d^{(l+1)})$ is then obtained by solving the equation:

$$\begin{aligned} l_d(\alpha_d, \lambda_d) \propto \; & \Big[m_d + \sum_{i=1}^{m}(k(R_i+1)-1)w_{id}(\phi_d^{(l)}, \alpha_d^{(l)}, \lambda_d^{(l)})\Big] \ln \phi_d^{(l+1)} \\ & + \sum_{i=1}^{m_d}\Big[\ln(\alpha_d) + \ln g_0(t_i, \lambda_d) + (\alpha_d - 1)\ln G_0(t_i, \lambda_d)\Big] \\ & + \sum_{i=1}^{m}(k(R_i+1)-1)\Big[w_{id}(\phi_d^{(l)}, \alpha_d^{(l)}, \lambda_d^{(l)}) \ln(1 - [G_0(t_i;\lambda_d)]^{\alpha_d})\Big] \end{aligned}$$

4: Check the convergence of $(\phi_d^{(l+1)}, \alpha_d^{(l+1)}, \lambda_d^{(l+1)})$. If the convergence is met, terminate the iteration, otherwise go back to Step 2.

3.1 Special Cases

Generalized Exponential (GE) Distribution

If we take $G_0(t;\lambda) = 1 - e^{-\lambda t}$ in (1.1), we get the generalized exponential distribution with shape parameter α and scale parameter λ. Let $\boldsymbol{\theta}^{GE} = (\phi_1, \dots, \phi_{S-1}, \alpha_1, \dots, \alpha_S, \lambda_1, \dots, \lambda_S)$ be the parameters of interest. In E-Step, the conditional probability that the unit fails from Cause j given that the unit had survived after time t_i is given by,

$$w_{ij}(\phi_j, \alpha_j, \lambda_j) = \frac{\phi_j\big(1 - [1 - e^{-\lambda_j t_i}]^{\alpha_j}\big)}{\sum_{k=1}^{S}\phi_k\big(1 - [1 - e^{-\lambda_k t_i}]^{\alpha_k}\big)} \quad j = 1, 2, \dots, S \tag{3.2}$$

In M-step of the EM algorithm if the l-th stage estimate of (α_d, λ_d) are $(\alpha_d^{(l)}, \lambda_d^{(l)})$ then the corresponding $(l+1)$th stage estimate $(\alpha_d^{(l+1)}, \lambda_d^{(l+1)})$ is obtained by solving the following profile-likelihood

$$
\begin{aligned}
l_d(\alpha_d,\lambda_d) \;\propto\; & \Big[m_d + \sum_{i=1}^{m}(k(R_i+1)-1)w_{id}(\phi_d^{(l)},\alpha_d^{(l)},\lambda_d^{(l)})\Big]\ln\phi_d^{(l+1)} \\
& + \sum_{i=1}^{m_d}\big[\ln\alpha_d + \ln\lambda_d - \lambda_d t_i + (\alpha_d-1)\ln(1-e^{-\lambda_d t_i})\big] \\
& + \sum_{i=1}^{m}(k(R_i+1)-1)\big[w_{id}(\phi_d^{(l)},\alpha_d^{(l)},\lambda_d^{(l)})\ln(1-[1-e^{-\lambda_d t_i}]^{\alpha_d})\big]
\end{aligned}
$$

Exponentiated Pareto (EP) Distribution

If we take $G_0(t;\lambda) = 1-(1+t)^{-\lambda}$ in (1.1), we get the exponentiated Pareto distribution with shape parameters α and λ. Let $\boldsymbol{\theta}^{EP} = (\phi_1,\ldots,\phi_{S-1},\alpha_1,\ldots,\alpha_S,\lambda_1,\ldots,\lambda_S)$ be the parameters of interest. In E-Step, the conditional probability that the unit fails from Cause j given that the unit had survived after time t_i is given by

$$
w_{ij}(\phi_j,\alpha_j,\lambda_j) = \frac{\phi_j\big(1-[1-(1+t_i)^{-\lambda_j}]^{\alpha_j}\big)}{\sum_{k=1}^{S}\phi_k\big(1-[1-(1+t_i)^{-\lambda_k}]^{\alpha_k}\big)} \quad j=1,2,\ldots,S \tag{3.3}
$$

In M-step of the EM algorithm if the l-th stage estimate of (α_d,λ_d) are $(\alpha_d^{(l)},\lambda_d^{(l)})$ then the corresponding $(l+1)$th stage estimate $(\alpha_d^{(l+1)},\lambda_d^{(l+1)})$ is obtained by solving the following profile-likelihood

$$
\begin{aligned}
l_d(\alpha_d,\lambda_d) \;\propto\; & \Big[m_d + \sum_{i=1}^{m}(k(R_i+1)-1)w_{id}(\phi_d^{(l)},\alpha_d^{(l)},\lambda_d^{(l)})\Big]\ln\phi_d^{(l+1)} \\
& + \sum_{i=1}^{m_d}\big[\ln(\alpha_d) + \ln\lambda_d - (\lambda_d+1)\ln(1+t_i) + (\alpha_d-1)\ln(1-(1+t_i)^{-\lambda_d})\big] \\
& + \sum_{i=1}^{m}(k(R_i+1)-1)\big[w_{id}(\phi_d^{(l)},\alpha_d^{(l)},\lambda_d^{(l)})\ln(1-[1-(1+t_i)^{-\lambda_d}]^{\alpha_d})\big]
\end{aligned}
$$

For simulation aid, we provide here the algorithm (see Algorithm 2) to generate progressive first-failure censored competing risks failure time data from $F(;\alpha,\lambda)$. The algorithm is constructed based on the fact that the progressive first-failure censored sample $T_1,T_2,\ldots,T_m$ can be viewed as a progressively Type-II censored sample from a population with CDF

$1 - \{1 - F(x)\}^k$ and then using the simple and efficient algorithm provided by [19].

Algorithm 2 : Generating progressive first-failure censored competing risks data

1: Given parameter values : $\phi_1,\ \phi_2, \ldots, \phi_S,\ \alpha_1,\ \alpha_2, \ldots, \alpha_S,\ \lambda_1,\ \lambda_2, \ldots, \lambda_S$ and set $\phi_0 = 0$.

2: Generate m independent Uniform (0,l) observations $Z_1,\ Z_2,\ \ldots, Z_m$.

3: Compute $Y_i = Z_i^{\overline{i+R_m+R_{m-1}+\ldots+R_{m-i+1}}^{1}}$ for all $i = 1, 2, \ldots, m$. It can be shown that Y_i has a Beta$(i + \sum_{j=m-i+1}^{m} R_j, 1)$ distribution and that the variables $Y_1, Y_2, \ldots, Y_m$ are statistically independent (see [19] in this respect).

4: Let $V_i = 1 - Y_m Y_{m-1} \ldots Y_{m-i+1}$ for all $i = 1, 2, \ldots, m$. Then $V_1, V_2, \ldots, V_m$ constitute the required progressive Type-II censored sample from the Uniform (0,1) distribution (see [19] in this respect).

5: Let $G(t) = 1 - [1 - F(t; \alpha_i, \lambda_i)]^k$. Recall that progressive first-failure censored data $T_1, T_2, \ldots, T_m$ can also be viewed as a progressively Type-II censored sample from a population with CDF $1 - \{1 - F(t)\}^k$ (see [16] in this respect).

6: For each $i = 1, 2, \ldots, m$, find $(\sum_{j=0}^{h_i-1} \phi_j, \sum_{j=0}^{h_i} \phi_j]$ such that $\sum_{j=0}^{h_i-1} \phi_j < Z_i \leq \sum_{j=0}^{h_i} \phi_j$, $h_i = 1, 2, \ldots, S$. Note that $P(\sum_{j=0}^{h_i-1} \phi_j < Z_i \leq \sum_{j=0}^{h_i} \phi_j) = \phi_{h_i}$

7: **while do** $\sum_{j=0}^{h_i-1} \phi_j < Z_i \leq \sum_{j=0}^{h_i} \phi_j$

8: Set $T_i = G^{-1}(V_i)$ for all $i = 1, 2, \ldots, m$, where $G^{-1}(.)$ represents the inverse function of $G(.)$.

9: Hence,

$$(\boldsymbol{t},\ \boldsymbol{\delta}) = (t_1, \delta_{h_1}), (t_2, \delta_{h_2}), \ldots, (t_m, \delta_{h_m}),$$

is the progressive first-failure censored competing risks data, where δ_{h_i} is the failure cause associated with the mixing weight ϕ_{h_i}.

The confidence intervals of all the model parameters can be constructed based on the asymptotic normality property of the MLEs. We have provided the associated observed Fisher information matrix in the Appendix Section.

4 Misspecification Analysis

Lehmann family includes many important distributions like Generalized Exponential and Exponentiated Pareto distributions. There are situations when either the Generalized Exponential or the Exponentiated Pareto distribution provides an adequate description of failure times in dependent competing risks model. In this Section, we assume that the practitioner intends to fit a Generalized Exponential (Exponentiated Pareto) dependent competing risks model to data which could possibly be better described by a Exponentiated Pareto (Generalized Exponential) model. Therefore, the objective of this study is if the data originally comes from a Generalized Exponential (Exponentiated Pareto) distribution but wrongly fitted by Exponentiated Pareto (Generalized Exponential), then "what is the effect of model misspecification on estimating the properties of unit's failure time distribution?". For more details for the misspecification analysis reader may look [20].

Let $l_1(\boldsymbol{\theta}_1|\boldsymbol{t},\, \boldsymbol{\delta})$ and $l_2(\boldsymbol{\theta}_2|\boldsymbol{t},\, \boldsymbol{\delta})$ be the loglikelihoods under models M_1 (Generalized Exponential) and M_2 (Exponentiated Pareto), respectively. Suppose that M_i is the true model and M_j is false. For fixed $\boldsymbol{\theta}_i$, let $\boldsymbol{\theta}_j^*$ be the value of $\boldsymbol{\theta}_j$ that minimize the expected negative likelihood $E_{M_i}\{-l_j(\boldsymbol{\theta}_j|\boldsymbol{t},\, \boldsymbol{\delta})\}$ with respect to M_i i.e.,

$$\boldsymbol{\theta}_j^* = \text{argmin}_{\boldsymbol{\theta}_j}\big[E_{M_i}\{-l_j(\boldsymbol{\theta}_j|\boldsymbol{t},\, \boldsymbol{\delta})\}\big]. \tag{4.1}$$

Here, $\boldsymbol{\theta}_j^*$ is called the asymptotic value of $\boldsymbol{\theta}_j$ and the best parameter setting under model M_i with respect to the true model. The incorrect MLEs in this context are called quasi-MLEs (QMLEs). Let $\hat{\boldsymbol{\theta}}_j$ be the QMLE when model M_j is fitted to the data. Therefore, when the true model is M_i, by Theorem 3.2 in White [21] the asymptotic distribution of $\hat{\boldsymbol{\theta}}_j$ is

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_j - \boldsymbol{\theta}_j) \xrightarrow{d} Normal(0, C(\boldsymbol{\theta}_i, \boldsymbol{\theta}_j^*)) \tag{4.2}$$

where $C(\boldsymbol{\theta}_i, \boldsymbol{\theta}_j^*)$ is the variance-covariance matrix.

The p-th quantile $t_p(\boldsymbol{\theta})$ turns out often to be a reliable characteristic of the failure time distribution. In this context, one can plausibly check the effect of the misspecification based on the p-th quantile. The p-th quantile for the proposed dependent competing risks mixture model (with S causes of failure) under Generalized Exponential distribution is given by

$$t_p(\boldsymbol{\theta}_{GE}) = \sum_{i=1}^{S} \phi_i\big[-\frac{1}{\lambda_i}\log\big(1-(1-(1-p)^{\frac{1}{k}})^{\frac{1}{\alpha_i}}\big)\big].$$

The p-th quantile for the proposed dependent competing risks mixture model (with S causes of failure) under Exponentiated Pareto distribution is given by

$$t_p(\boldsymbol{\theta}_{EP}) = \sum_{i=1}^{S} \phi_i \big[((1-(1-(1-p)^{\frac{1}{k}})^{\frac{1}{\alpha_i}})^{-\frac{1}{\lambda_i}}) - 1\big].$$

Let $t_{p1}(\boldsymbol{\theta}_1)$ and $t_{p2}(\boldsymbol{\theta}_2)$ be the p-th quantiles under models M_1 (Generalized Exponential) and M_2 (Exponentiated Pareto), respectively. Suppose that M_i is the true model and M_j is false. The asymptotic relative bias (RB) of the p-th quantile under model mis-specification is then given by

$$RB = \frac{ABias(t_{pj}(\hat{\boldsymbol{\theta}}_j)|\boldsymbol{\theta}_i)}{t_{pi}(\boldsymbol{\theta}_i)} = \frac{t_{pj}(\boldsymbol{\theta}_j^*) - t_{pi}(\boldsymbol{\theta}_i)}{t_{pi}(\boldsymbol{\theta}_i)}, \tag{4.3}$$

where the p-th quantile of model M_j is given by $t_{pj}(\boldsymbol{\theta}_j)$. The asymptotic relative variance (RV) of the p-th quantile under the model misspecification is given by

$$RV = \frac{AMSE(t_p(\hat{\boldsymbol{\theta}}_j)|\boldsymbol{\theta}_i)}{AMSE(t_p(\hat{\boldsymbol{\theta}}_i)|\boldsymbol{\theta}_i)} \tag{4.4}$$

5 Simulation Results, Misspecification Study and Expected Test Time

In this Section, an extensive simulation study is first carried out to investigate the performance of the proposed estimation methodology and for misspecification study when the failure time distribution belongs to the Lehmann family of distributions. We consider two important members of the Lehmann family -the Generalized Exponential (GE) and Exponentiated Pareto (EP)distributions introduced in the earlier sections for illustration purpose. Next we explore the expected test time (ETT) to failure of the life test under the proposed experimental set up.

5.1 Simulation Results

Associated with each of the two failure time distributions considered, we assume that a unit may fail owing to two competing causes. Hence, in this set up, the associated model parameters are $\phi, \alpha_1, \lambda_1, \alpha_2$ and λ_2. We choose different combinations of (k, n, m) and various PFFCSs including left censoring (Scheme I), right censoring (Scheme II) and middle censoring (Scheme III). In particular, we take $S = 2$, $k = 1, 2, 3$, and for each of these choices of k, we take $(n, m) = (40, 30),\ (40, 40),\ (60, 50),$ and $(60, 60)$. The studied schemes are as follows:

$$
\begin{aligned}
\text{Scheme I} \quad &: \quad R_1 = n - m,\ R_i = 0 \ \text{ for } \ i \neq 1 \\
\text{Scheme II} \quad &: \quad R_m = n - m,\ R_i = 0 \ \text{ for } \ i \neq m \\
\text{Scheme III} \quad &: \quad R_{\frac{m+1}{2}} = n - m,\ R_i = 0 \ \text{ for } \ i \neq \frac{m+1}{2} \ \text{ if } m \text{ is odd,} \\
&\quad\;\; R_{\frac{m}{2}} = n - m,\ R_i = 0 \ \text{ for } \ i \neq \frac{m}{2} \ \text{ if } m \text{ is even.}
\end{aligned}
$$

Now, corresponding to each of the GE and EP distributions, we consider the following two sets of parameter values for simulation purposes:

$$
\begin{aligned}
\textbf{GE Case} \quad &: \quad \phi = 0.35,\ \alpha_1 = 3.00,\ \alpha_2 = 4.00,\ \lambda_1 = 1.50 \ \text{ and } \ \lambda_2 = 1.75. \\
&\quad\;\; \phi = 0.40,\ \alpha_1 = 4.75,\ \alpha_2 = 2.25,\ \lambda_1 = 1.75 \ \text{ and } \ \lambda_2 = 1.20. \\
\textbf{EP Case} \quad &: \quad \phi = 0.35,\ \alpha_1 = 3.00,\ \alpha_2 = 4.00,\ \lambda_1 = 1.50 \ \text{ and } \ \lambda_2 = 1.75. \\
&\quad\;\; \phi = 0.40,\ \alpha_1 = 4.75,\ \alpha_2 = 2.25,\ \lambda_1 = 1.75 \ \text{ and } \ \lambda_2 = 1.20.
\end{aligned}
$$

At first, progressive first-failure-censored competing risks data are generated with these parameter values following Algorithm 2 discussed in Section 3. All the results are obtained based on 1000 Monte Carlo simulations. MLEs of all the model parameters are computed based on the proposed EM algorithm. The codes used to get the results are available on a GitHub repository at `https://github.com/deepak-stats/FMM_DCR_PFFCS_EM.git`. To initiate the proposed EM algorithm in a simulation study , one may select the true values of the model parameters as the initial guesses. However, the problem becomes non-trivial in analyzing a real life data and it will be discussed in detail in the Section 6.1.1. The average estimates (AEs) of the MLEs and the corresponding mean squared errors (MSEs) are obtained within brackets to assess the performance of the MLEs. For interval estimation of the model parameters, we resort to use the asymptotic CIs. The results are provided for both GE distribution (see Table 1, 3 and Table 5, 7) and EP distribution (see Table 2, 4 and Table 6, 8) for two sets of parameter values. Some of the observations are evident from the results in Tables 1-8. It is evident from the tables that Scheme-I and Scheme-III perform better than Scheme-II in terms of MSEs. The coverage probabilities (CPs) and average

Table 1: AEs and their MSEs (within brackets) of MLEs of ϕ, α_1, α_2, λ_1 and λ_2 for different censoring schemes (Generalized Exponential case, $\phi = 0.35$, $\alpha_1 = 3.0$, $\lambda_1 = 1.5$, $\alpha_2 = 4.0$ and $\lambda_2 = 1.75$).

k	n	m	Scheme	$\hat{\phi}$	$\hat{\alpha}_1$	$\hat{\lambda}_1$	$\hat{\alpha}_2$	$\hat{\lambda}_2$
1	40	30	I	0.3462 (0.0078)	3.6920 (1.7009)	1.7080 (0.2809)	4.1214 (0.8597)	1.7827 (0.1017)
			II	0.3628 (0.0108)	3.8294 (2.0787)	1.7649 (0.5295)	4.2617 (0.8143)	1.8505 (0.1336)
			III	0.3494 (0.0080)	3.6220 (1.6110)	1.6907 (0.2856)	4.1310 (0.8329)	1.7784 (0.1054)
			*	0.3469 (0.0058)	3.5468 (1.4296)	1.6582 (0.1796)	4.1227 (0.8059)	1.7707 (0.0810)
2			I	0.3562 (0.0110)	3.6788 (1.7074)	1.7837 (0.5502)	4.1389 (0.7937)	1.8035 (0.1480)
			II	0.3388 (0.0188)	3.8574 (2.1258)	2.0135 (1.1215)	4.2228 (0.8309)	1.8386 (0.2057)
			III	0.3468 (0.0125)	3.6427 (1.6335)	1.7829 (0.5437)	4.2065 (0.7420)	1.8353 (0.1463)
			*	0.3495 (0.0086)	3.4818 (1.3698)	1.6917 (0.3461)	4.1976 (0.7286)	1.8251 (0.1195)
3			I	0.3456 (0.0161)	3.6688 (1.6732)	1.8837 (0.8626)	4.1966 (0.7214)	1.8445 (0.1742)
			II	0.3102 (0.0254)	3.8860 (2.1774)	2.2361 (1.7385)	4.2464 (0.7364)	1.8268 (0.2116)
			III	0.3460 (0.0185)	3.5508 (1.5672)	1.8095 (0.7688)	4.2669 (0.7202)	1.8805 (0.1917)
			*	0.3532 (0.0134)	3.5272 (1.3781)	1.7501 (0.5322)	4.2453 (0.6859)	1.8652 (0.1517)
1	60	50	I	0.3538 (0.0044)	3.4507 (1.1301)	1.5990 (0.1023)	4.0921 (0.7190)	1.7594 (0.0737)
			II	0.3658 (0.0058)	3.5392 (1.4587)	1.6113 (0.2271)	4.2120 (0.7833)	1.8294 (0.1043)
			III	0.3533 (0.0046)	3.4337 (1.1704)	1.6015 (0.1036)	4.1436 (0.7135)	1.7743 (0.0751)
			*	0.3492 (0.0038)	3.3857 (1.0450)	1.5924 (0.0966)	4.0836 (0.7183)	1.7602 (0.0669)
2			I	0.3524 (0.0068)	3.4308 (1.1774)	1.6463 (0.2593)	4.1547 (0.6731)	1.8087 (0.1103)
			II	0.3525 (0.0116)	3.5824 (1.5231)	1.7731 (0.5230)	4.2008 (0.6947)	1.8244 (0.1509)
			III	0.3517 (0.0077)	3.4192 (1.2128)	1.6438 (0.2479)	4.1754 (0.6367)	1.8250 (0.1087)
			*	0.3582 (0.0060)	3.4287 (1.0996)	1.6137 (0.1840)	4.1687 (0.6622)	1.8141 (0.1004)
3			I	0.3535 (0.0107)	3.4043 (1.1304)	1.6681 (0.3702)	4.1829 (0.6319)	1.8412 (0.1406)
			II	0.3286 (0.0184)	3.6524 (1.5605)	1.9575 (0.9259)	4.1619 (0.6285)	1.8013 (0.1605)
			III	0.3504 (0.0117)	3.4317 (1.1354)	1.6928 (0.3748)	4.1797 (0.6108)	1.8407 (0.1448)
			*	0.3571 (0.0098)	3.3976 (1.0684)	1.6434 (0.2867)	4.2092 (0.6274)	1.8464 (0.1355)

lengths (ALs) are obtained. For fixed k, when we increase the (n, m) with a bit of high $\frac{m}{n}$

ratio, the performances of all the parameter estimates get better in terms of MSEs and ALs. We have also considered the $n = m$ case (marked with $*$) to see the results corresponding to the first failure censoring scheme. It is to note here that the results are entirely satisfactory in every instance and the trend remains the same in the two different sets of parameter values considered for both the distributions. The results of the first failure censoring scheme are the best, as expected in every model. It is worthwhile to mention that lower the mean failure time of the distribution, the higher will be the corresponding mixing weight.

Table 2: AEs and their MSEs (within brackets) of MLEs of ϕ, α_1, α_2, λ_1 and λ_2 for different censoring schemes (Exponentiated Pareto case, $\phi = 0.35$, $\alpha_1 = 3.0$, $\lambda_1 = 1.5$, $\alpha_2 = 4.0$ and $\lambda_2 = 1.75$).

k	n	m	Scheme	$\hat{\phi}$	$\hat{\alpha}_1$	$\hat{\lambda}_1$	$\hat{\alpha}_2$	$\hat{\lambda}_2$
1	40	30	I	0.3472 (0.0075)	3.6015 (1.6260)	1.6824 (0.2614)	4.1183 (0.8603)	1.7682 (0.0968)
			II	0.3657 (0.0117)	3.7726 (2.0563)	1.7547 (0.5200)	4.2652 (0.8714)	1.8537 (0.1435)
			III	0.3534 (0.0085)	3.6066 (1.6292)	1.6801 (0.2629)	4.1085 (0.8430)	1.7650 (0.1095)
			*	0.3506 (0.0059)	3.5588 (1.4201)	1.6566 (0.1912)	4.0763 (0.8450)	1.7532 (0.0912)
2			I	0.3539 (0.0108)	3.6659 (1.6885)	1.7366 (0.4147)	4.1756 (0.7827)	1.8259 (0.1408)
			II	0.3414 (0.0200)	3.8820 (2.1084)	2.0364 (1.1425)	4.2100 (0.7600)	1.8268 (0.1815)
			III	0.3507 (0.0134)	3.5297 (1.4713)	1.7440 (0.4848)	4.2052 (0.7752)	1.8302 (0.1536)
			*	0.3581 (0.0088)	3.5470 (1.4573)	1.6640 (0.2870)	4.1789 (0.7293)	1.8182 (0.1212)
3			I	0.3466 (0.0161)	3.6907 (1.7499)	1.8476 (0.7626)	4.2033 (0.7031)	1.8524 (0.1742)
			II	0.3142 (0.0259)	3.8613 (2.1420)	2.1962 (1.6429)	4.2681 (0.7554)	1.8470 (0.2194)
			III	0.3457 (0.0190)	3.5739 (1.5461)	1.8458 (0.8362)	4.2639 (0.7077)	1.8837 (0.1885)
			*	0.3498 (0.0131)	3.5392 (1.4308)	1.7544 (0.4805)	4.1967 (0.6811)	1.8379 (0.1509)
1	60	50	I	0.3511 (0.0043)	3.4396 (1.1901)	1.6075 (0.1154)	4.1051 (0.7881)	1.7568 (0.0773)
			II	0.3633 (0.0058)	3.4711 (1.3151)	1.6146 (0.2306)	4.1584 (0.7613)	1.8070 (0.0963)
			III	0.3502 (0.0050)	3.4484 (1.2513)	1.6087 (0.1222)	4.1337 (0.6707)	1.7780 (0.0752)
			*	0.3524 (0.0040)	3.4525 (1.1157)	1.5974 (0.0987)	4.0995 (0.6895)	1.7650 (0.0662)
2			I	0.3533 (0.0075)	3.4718 (1.1567)	1.6711 (0.2737)	4.1263 (0.6277)	1.7961 (0.1044)
			II	0.3557 (0.0132)	3.6419 (1.5587)	1.8114 (0.6189)	4.2491 (0.6409)	1.8531 (0.1366)
			III	0.3540 (0.0073)	3.3803 (1.1046)	1.6149 (0.2088)	4.1834 (0.6670)	1.8219 (0.1169)
			*	0.3555 (0.0061)	3.3405 (1.0463)	1.6163 (0.2090)	4.1664 (0.6106)	1.8073 (0.0923)
3			I	0.3516 (0.0111)	3.4485 (1.1879)	1.6851 (0.3568)	4.2041 (0.6402)	1.8523 (0.1428)
			II	0.3222 (0.0173)	3.6735 (1.5847)	1.9696 (0.9370)	4.1728 (0.5955)	1.8098 (0.1516)
			III	0.3533 (0.0120)	3.4017 (1.2055)	1.6729 (0.4138)	4.2156 (0.6028)	1.8601 (0.1457)
			*	0.3529 (0.0100)	3.3441 (1.0189)	1.6495 (0.3354)	4.2298 (0.6158)	1.8572 (0.1306)

Table 3: CPs and ALs (within brackets) linked with 95% CIs MLEs of ϕ, α_1, α_2, λ_1 and λ_2 for different censoring schemes (Generalized Exponential case, $\phi = 0.35$, $\alpha_1 = 3.0$, $\lambda_1 = 1.5$, $\alpha_2 = 4.0$ and $\lambda_2 = 1.75$).

k	n	m	Scheme	$\hat{\phi}$	$\hat{\alpha}_1$	$\hat{\lambda}_1$	$\hat{\alpha}_2$	$\hat{\lambda}_2$
1	40	30	I	0.9430 (0.3418)	0.9680 (4.8034)	0.9940 (1.9987)	0.8970 (4.7970)	0.8710 (1.3467)
			II	0.9590 (0.4060)	0.9550 (5.0483)	0.9970 (3.3399)	0.9130 (4.7683)	0.8760 (1.6839)
			III	0.9420 (0.3399)	0.9570 (4.7844)	0.9950 (2.1606)	0.9230 (4.8216)	0.8850 (1.4045)
			*	0.9610 (0.2957)	0.9840 (4.7514)	0.9980 (1.7454)	0.9260 (4.6790)	0.9070 (1.2339)
2			I	0.9530 (0.4115)	0.9750 (4.8914)	0.9970 (3.0544)	0.9350 (4.6907)	0.9040 (1.6567)
			II	0.9230 (0.6730)	0.9710 (4.9929)	0.9978 (4.5392)	0.9410 (4.7572)	0.9060 (2.0097)
			III	0.9350 (0.4467)	0.9770 (4.8333)	0.9960 (3.2692)	0.9580 (4.7384)	0.9220 (1.7201)
			*	0.9470 (0.3679)	0.9780 (4.8425)	0.9910 (2.6938)	0.9520 (4.5396)	0.9150 (1.5480)
3			I	0.9080 (0.5338)	0.9770 (4.8710)	0.9970 (3.9143)	0.9620 (4.5715)	0.9250 (1.8078)
			II	0.9100 (0.9092)	0.9580 (4.8984)	0.9998 (5.1997)	0.9600 (4.7347)	0.9190 (2.2613)
			III	0.9190 (0.5881)	0.9760 (4.8790)	0.9980 (4.0550)	0.9540 (4.5459)	0.9210 (1.8810)
			*	0.8990 (0.4876)	0.9750 (4.7877)	0.9870 (3.4150)	0.9600 (4.4237)	0.9290 (1.7481)
1	60	50	I	0.9550 (0.2626)	0.9880 (4.4763)	0.9970 (1.5462)	0.9400 (4.5478)	0.9310 (1.1439)
			II	0.9630 (0.3009)	0.9880 (4.8734)	0.9980 (2.2962)	0.9360 (4.5678)	0.9220 (1.4563)
			III	0.9400 (0.2639)	0.9900 (4.4339)	0.9970 (1.5892)	0.9460 (4.5312)	0.9300 (1.1689)
			*	0.9540 (0.2398)	0.9830 (4.3551)	0.9950 (1.4045)	0.9330 (4.4257)	0.9280 (1.0677)
2			I	0.9480 (0.3374)	0.9850 (4.4057)	0.9920 (2.3039)	0.9520 (4.3559)	0.9360 (1.4698)
			II	0.9070 (0.5043)	0.9860 (4.8365)	0.9990 (3.4471)	0.9590 (4.3316)	0.9190 (1.7872)
			III	0.9320 (0.3559)	0.9820 (4.3671)	0.9920 (2.3968)	0.9500 (4.3579)	0.9330 (1.5088)
			*	0.9350 (0.3103)	0.9680 (4.1878)	0.9780 (2.0822)	0.9520 (4.3145)	0.9310 (1.3969)
3			I	0.9160 (0.4595)	0.9700 (4.4605)	0.9830 (2.8929)	0.9690 (4.3192)	0.9400 (1.7220)
			II	0.8680 (0.6593)	0.9800 (4.8448)	0.9940 (4.3260)	0.9640 (4.1151)	0.9180 (1.8785)
			III	0.9070 (0.4928)	0.9790 (4.4123)	0.9870 (2.9974)	0.9740 (4.3186)	0.9490 (1.7728)
			*	0.9260 (0.4297)	0.9700 (4.2866)	0.9560 (2.5938)	0.9680 (4.2714)	0.9380 (1.6586)

Table 4: CPs and ALs (within brackets) linked with 95% CIs MLEs of ϕ, α_1, α_2, λ_1 and λ_2 for different censoring schemes (Exponentiated Pareto case, $\phi = 0.35$, $\alpha_1 = 3.0$, $\lambda_1 = 1.5$, $\alpha_2 = 4.0$ and $\lambda_2 = 1.75$).

k	n	m	Scheme	$\hat{\phi}$	$\hat{\alpha}_1$	$\hat{\lambda}_1$	$\hat{\alpha}_2$	$\hat{\lambda}_2$
1	40	30	I	0.9260 (0.3429)	0.9600 (4.3764)	0.9960 (1.9490)	0.8660 (4.3864)	0.8190 (1.3041)
			II	0.9640 (0.4051)	0.9800 (5.0701)	0.9990 (3.3280)	0.9150 (4.8407)	0.8800 (1.6880)
			III	0.9260 (0.3368)	0.9700 (4.8400)	0.9950 (2.1847)	0.9370 (4.9599)	0.9120 (1.4268)
			*	0.9660 (0.2928)	0.9740 (4.7349)	0.9900 (1.7464)	0.9480 (5.0700)	0.9310 (1.2666)
2			I	0.9480 (0.4090)	0.9670 (4.8623)	0.9860 (2.9781)	0.9480 (4.8907)	0.9260 (1.6933)
			II	0.9220 (0.6609)	0.9700 (5.0074)	0.9990 (4.5726)	0.9480 (4.8482)	0.9180 (2.0231)
			III	0.9390 (0.4454)	0.9640 (4.8188)	0.9810 (3.1690)	0.9540 (4.9272)	0.9180 (1.7588)
			*	0.9490 (0.3696)	0.9600 (4.6904)	0.9870 (2.4904)	0.9650 (4.9729)	0.9450 (1.6351)
3			I	0.9140 (0.5375)	0.9740 (4.8876)	0.9850 (3.7362)	0.9660 4.9331)	0.9370 (1.9281)
			II	0.9280 (0.8988)	0.9570 (4.8752)	0.9989 (5.1174)	0.9530 (4.9793)	0.9280 (2.2527)
			III	0.9130 (0.5864)	0.9730 (4.8632)	0.9840 (3.8005)	0.9690 (4.8881)	0.9290 (2.0055)
			*	0.9360 (0.4913)	0.9680 (4.7308)	0.9750 (3.0751)	0.9710 (4.8990)	0.9380 (1.8843)
1	60	50	I	0.9610 (0.2634)	0.9890 (4.4798)	0.9950 (1.5029)	0.9410 (4.8226)	0.9410 (1.1606)
			II	0.9610 (0.3002)	0.9770 (4.7288)	0.9850 (2.1790)	0.9480 (4.9425)	0.9290 (1.4890)
			III	0.9480 (0.2643)	0.9800 (4.4416)	0.9940 (1.5495)	0.9590 (4.7981)	0.9550 (1.1921)
			*	0.9540 (0.2415)	0.9820 (4.3016)	0.9950 (1.3592)	0.9400 (4.5283)	0.9490 (1.0717)
2			I	0.9450 (0.3397)	0.9700 (4.3021)	0.9770 (2.0832)	0.9670 (4.7824)	0.9440 (1.5652)
			II	0.9180 (0.5128)	0.9850 (4.9059)	0.9940 (3.1823)	0.9630 (4.6786)	0.9350 (1.8694)
			III	0.9280 (0.3585)	0.9600 (4.3042)	0.9720 (2.1781)	0.9750 (4.7819)	0.9580 (1.6103)
			*	0.9420 (0.3146)	0.9740 (4.2804)	0.9780 (1.9293)	0.9640 (4.5305)	0.9560 (1.4562)
3			I	0.9410 (0.4651)	0.9630 (4.2749)	0.9610 (2.4801)	0.9760 (4.7872)	0.9560 (1.8632)
			II	0.8930 (0.6715)	0.9880 (4.9678)	0.9970 (4.0127)	0.9760 (4.5404)	0.9440 (2.0290)
			III	0.9090 (0.4839)	0.9560 (4.2831)	0.9420 (2.5663)	0.9780 (4.8317)	0.9630 (1.9196)
			*	0.9200 (0.4347)	0.9590 (4.0973)	0.9580 (2.3482)	0.9650 (4.5512)	0.9540 (1.7726)

5.2 Misspecification Study

The effects of model misspecification on the system lifetime is an interesting research problem. When it comes to misspecification analysis, the results stated in Section 4 are based on the infinite sample property of model misspecification. In real-life applications, sample size is always finite. Monte Carlo simulation experiment is hence used to investigate the effect of misspecification. To check the misspecification performance, we have considered the p-th quantile with the use of the relative bias (RB) and relative variance (RV) measures,

Table 5: AEs and their MSEs (within brackets) of MLEs of ϕ, α_1, α_2, λ_1 and λ_2 for different censoring schemes (Generalized Exponential case, $\phi = 0.40$, $\alpha_1 = 4.75$, $\lambda_1 = 2.25$, $\alpha_2 = 1.75$ and $\lambda_2 = 1.20$).

k	n	m	Scheme	$\hat{\phi}$	$\hat{\alpha}_1$	$\hat{\lambda}_1$	$\hat{\alpha}_2$	$\hat{\lambda}_2$
1	40	30	I	0.3960 (0.0077)	4.6231 (0.3735)	2.3223 (0.1690)	1.9507 (0.2349)	1.3170 (0.0938)
			II	0.3480 (0.0099)	4.8012 (0.4638)	2.6139 (0.4656)	1.9962 (0.3339)	1.3080 (0.1499)
			III	0.4007 (0.0077)	4.6464 (0.3592)	2.3186 (0.1574)	1.9565 (0.2218)	1.3297 (0.1077)
			*	0.4044 (0.0057)	4.6528 (0.2306)	2.2560 (0.0873)	1.9555 (0.2011)	1.3010 (0.0797)
2			I	0.3863 (0.0066)	4.7223 (0.2547)	2.4541 (0.2522)	1.9751 (0.1889)	1.2830 (0.1075)
			II	0.4040 (0.0087)	4.4816 (0.5049)	2.2275 (0.4635)	1.9504 (0.2701)	1.3423 (0.2505)
			III	0.3982 (0.0072)	4.7652 (0.2441)	2.4800 (0.2913)	1.9472 (0.1649)	1.2948 (0.1276)
			*	0.3788 (0.0056)	4.7545 (0.2286)	2.4421 (0.1999)	1.9399 (0.1037)	1.2837 (0.0930)
3			I	0.4030 (0.0037)	4.6782 (0.3175)	2.3844 (0.2488)	1.9552 (0.1719)	1.2872 (0.1181)
			II	0.4362 (0.0032)	4.5943 (0.5298)	2.2391 (0.4373)	1.8446 (0.2029)	1.2828 (0.3003)
			III	0.4140 (0.0039)	4.7263 (0.2721)	2.4437 (0.2785)	1.8963 (0.1256)	1.2666 (0.1228)
			*	0.3971 (0.0041)	4.6610 (0.3319)	2.3829 (0.2047)	1.8945 (0.0700)	1.2602 (0.0759)
1	60	50	I	0.4002 (0.0051)	4.6297 (0.3004)	2.2629 (0.0785)	1.9373 (0.1977)	1.2967 (0.0702)
			II	0.3653 (0.0055)	4.8649 (0.3400)	2.5289 (0.2329)	1.9267 (0.2846)	1.2418 (0.1729)
			III	0.4042 (0.0045)	4.6330 (0.3109)	2.2627 (0.0748)	1.8976 (0.1769)	1.2821 (0.0719)
			*	0.4000 (0.0037)	4.5903 (0.2003)	2.2595 (0.0762)	1.8846 (0.1585)	1.2672 (0.0557)
2			I	0.3816 (0.0040)	4.7422 (0.2193)	2.4312 (0.1702)	1.9300 (0.1441)	1.2586 (0.0791)
			II	0.3849 (0.0058)	4.5436 (0.4565)	2.3098 (0.2873)	1.9117 (0.2231)	1.2912 (0.1997)
			III	0.3838 (0.0050)	4.7982 (0.1667)	2.4822 (0.2082)	1.8766 (0.1132)	1.2388 (0.0771)
			*	0.3782 (0.0042)	4.7635 (0.1937)	2.4360 (0.1487)	1.9008 (0.1193)	1.2447 (0.0698)
3			I	0.3959 (0.0027)	4.6699 (0.2737)	2.3967 (0.1048)	1.8997 (0.1184)	1.2482 (0.0853)
			II	0.4222 (0.0045)	4.5674 (0.3938)	2.2393 (0.2288)	1.8413 (0.1828)	1.2531 (0.1724)
			III	0.4078 (0.0028)	4.7361 (0.2265)	2.4212 (0.1077)	1.8422 (0.0854)	1.2263 (0.0816)
			*	0.3942 (0.0028)	4.7231 (0.2194)	2.4044 (0.1037)	1.8552 (0.0903)	1.2260 (0.0677)

for more details see [20]. We have used the parameter values as $\phi = 0.35$, $\alpha_1 = 3.0$, $\alpha_2 = 4.0$, $\lambda_1 = 1.50$ and $\lambda_2 = 1.75$ to generate 5000 data points by Monte Carlo simulation from the GE (EP) distribution. Next, from the simulated data, the empirical RBs and RVs for the p-th-quantile is obtained in Tables 9 and 10. It is observed that when the true model is GE and wrongly fitted by EP, empirical RBs and RVs are smaller in estimating the p-th quantile compare to when the true model is EP and wrongly fitted by GE.

Table 6: AEs and their MSEs (within brackets) of MLEs of ϕ, α_1, α_2, λ_1 and λ_2 for different censoring schemes (Exponentiated Pareto case, $\phi = 0.40$, $\alpha_1 = 4.75$, $\lambda_1 = 2.25$, $\alpha_2 = 1.75$ and $\lambda_2 = 1.20$).

k	n	m	Scheme	$\hat{\phi}$	$\hat{\alpha}_1$	$\hat{\lambda}_1$	$\hat{\alpha}_2$	$\hat{\lambda}_2$
1	40	30	I	0.4000 (0.0083)	4.5373 (0.5530)	2.2731 (0.2300)	2.0263 (0.4387)	1.3222 (0.1520)
			II	0.3548 (0.0100)	4.8000 (0.6904)	2.5940 (0.4877)	2.0286 (0.5731)	1.2814 (0.2179)
			III	0.4048 (0.0073)	4.5809 (0.4739)	2.2693 (0.2159)	1.9975 (0.4082)	1.3223 (0.1609)
			*	0.4016 (0.0060)	4.5114 (0.5363)	2.2256 (0.1390)	1.9647 (0.3789)	1.2858 (0.1136)
2			I	0.3789 (0.0078)	4.7566 (0.2347)	2.4908 (0.3214)	1.9585 (0.1888)	1.2910 (0.1155)
			II	0.4062 (0.0096)	4.7268 (0.5238)	2.4311 (0.4517)	1.9985 (0.2547)	1.4095 (0.2428)
			III	0.3888 (0.0066)	4.7783 (0.2374)	2.5208 (0.3568)	1.9536 (0.1663)	1.3209 (0.1478)
			*	0.3788 (0.0049)	4.7698 (0.2208)	2.4609 (0.2267)	1.9381 (0.1563)	1.2917 (0.1079)
3			I	0.3979 (0.0038)	4.6307 (0.3831)	2.3914 (0.1671)	1.9650 (0.1729)	1.3032 (0.1265)
			II	0.4364 (0.0064)	4.6160 (0.4883)	2.2474 (0.3385)	1.8614 (0.2134)	1.2821 (0.2028)
			III	0.4131 (0.0038)	4.6873 (0.3030)	2.4300 (0.1748)	1.8968 (0.1250)	1.2531 (0.1159)
			*	0.3980 (0.0040)	4.7265 (0.2433)	2.4160 (0.1502)	1.9113 (0.1243)	1.2754 (0.1044)
1	60	50	I	0.4025 (0.0045)	4.6327 (0.3052)	2.2591 (0.0842)	1.9148 (0.1910)	1.2821 (0.0675)
			II	0.3592 (0.0079)	4.8696 (0.4552)	2.5383 (0.2328)	1.9180 (0.2783)	1.2468 (0.1712)
			III	0.4001 (0.0047)	4.6325 (0.3338)	2.2715 (0.0801)	1.8887 (0.1746)	1.2805 (0.0711)
			*	0.4024 (0.0040)	4.6305 (0.3158)	2.2567 (0.0644)	1.8848 (0.1686)	1.2623 (0.0545)
2			I	0.3809 (0.0044)	4.7643 (0.2101)	2.4397 (0.1786)	1.9330 (0.1434)	1.2526 (0.0758)
			II	0.3858 (0.0066)	4.5971 (0.3928)	2.3285 (0.2902)	1.9178 (0.1939)	1.2781 (0.1903)
			III	0.3879 (0.0043)	4.7978 (0.1768)	2.4521 (0.1823)	1.8840 (0.1162)	1.2456 (0.0762)
			*	0.3753 (0.0035)	4.7557 (0.1980)	2.4484 (0.1570)	1.8885 (0.1184)	1.2370 (0.0658)
3			I	0.3966 (0.0027)	4.6753 (0.2767)	2.3888 (0.1102)	1.8897 (0.1128)	1.2323 (0.0759)
			II	0.4224 (0.0040)	4.5606 (0.4159)	2.2318 (0.2281)	1.8448 (0.1795)	1.2502 (0.1722)
			III	0.4084 (0.0028)	4.7534 (0.2094)	2.4190 (0.1048)	1.8584 (0.0894)	1.2257 (0.0756)
			*	0.3952 (0.0027)	4.7484 (0.2088)	2.4057 (0.0990)	1.8558 (0.0872)	1.2172 (0.0658)

5.3 Expected Test Time

Since the exact expression for the expected test time (ETT) i.e. $\mathrm{E}(T^{\boldsymbol{R}}_{m:m:n:k})$ is difficult to tract in this case, we have computed the empirical ETTs. Corresponding to every choice of $(k, n, m, \boldsymbol{R})$, we have generated $M = 15000$ samples. The empirical ETT is then given by $\dfrac{1}{M}\displaystyle\sum_{i=1}^{M} t^{\boldsymbol{R}(i)}_{m:m:n:k}$, where $t^{\boldsymbol{R}(i)}_{m:m:n:k}$ is the $m-$ th failure in the $i-$ th sample drawn. All the results are available in Table 11. Some of the observations are evident from the results. For fixed n and m, as k increases, ETT decreases for both the GE and EP distributions (see [16] in this respect for similar behaviour). Again, for complete sample case, ETT is larger compared to

Table 7: CPs and ALs (within brackets) linked with 95% CIs MLEs of ϕ, α_1, α_2, λ_1 and λ_2 for different censoring schemes (Generalized Exponential case, $\phi = 0.40$, $\alpha_1 = 4.75$, $\lambda_1 = 2.25$, $\alpha_2 = 1.75$ and $\lambda_2 = 1.20$).

k	n	m	Scheme	$\hat{\phi}$	$\hat{\alpha}_1$	$\hat{\lambda}_1$	$\hat{\alpha}_2$	$\hat{\lambda}_2$
1	40	30	I	0.9590 (0.3457)	0.9880 (8.8239)	0.9970 (2.4324)	0.9710 (3.0138)	0.8990 (1.3705)
			II	0.9140 (0.3696)	0.9860 (10.4834)	0.9930 (3.7559)	0.9830 (3.2561)	0.9060 (1.7378)
			III	0.9470 (0.3510)	0.9900 (8.7631)	0.9970 (2.5800)	0.9710 (2.8861)	0.9050 (1.4351)
			*	0.9550 (0.3001)	0.9920 (7.9292)	0.9900 (2.1247)	0.9720 (2.5472)	0.9190 (1.1722)
2			I	0.9280 (0.3985)	0.9920 (9.1142)	0.9940 (3.4909)	0.9790 (2.5594)	0.9180 (1.7358)
			II	0.8630 (0.6807)	0.9910 (10.9811)	0.9920 (5.5087)	0.9820 (2.5909)	0.9360 (2.6703)
			III	0.9360 (0.4297)	0.9880 (8.8486)	0.9970 (3.6300)	0.9870 (2.3776)	0.9310 (1.7757)
			*	0.9310 (0.3502)	0.9950 (8.1812)	0.9930 (3.0099)	0.9800 (2.2667)	0.9100 (1.4922)
3			I	0.9180 (0.5120)	0.9920 (9.1047)	0.9960 (4.1405)	0.9770 (2.2084)	0.9050 (1.9396)
			II	0.8910 (0.9596)	0.9910 (11.1161)	0.9990 (6.8342)	0.9760 (2.0211)	0.9110 (2.8500)
			III	0.9080 (0.5560)	0.9980 (8.9882)	0.9980 (4.3989)	0.9800 (2.0201)	0.8880 (1.9758)
			*	0.9200 (0.4504)	0.9930 (8.1118)	0.9930 (3.6082)	0.9840 (2.0123)	0.9120 (1.7713)
1	60	50	I	0.9440 (0.2686)	0.9820 (6.9767)	0.9900 (1.8982)	0.9640 (2.0778)	0.9110 (1.0231)
			II	0.9060 (0.2664)	0.9930 (8.0504)	0.9950 (2.5599)	0.9480 (2.1487)	0.8470 (1.1260)
			III	0.9470 (0.2707)	0.9840 (6.8804)	0.9930 (1.9199)	0.9670 (2.0924)	0.9140 (1.0687)
			*	0.9370 (0.2460)	0.9810 (6.4886)	0.9960 (1.7145)	0.9540 (1.9093)	0.9000 (0.9327)
2			I	0.9220 (0.3123)	0.9920 (7.2467)	0.9970 (2.7169)	0.9790 (1.8282)	0.9170 (1.2689)
			II	0.8450 (0.4475)	0.9860 (8.4375)	0.9890 (3.8547)	0.9830 (1.9723)	0.9050 (1.7760)
			III	0.9410 (0.3304)	0.9950 (7.0634)	0.9990 (2.7167)	0.9760 (1.7259)	0.9000 (1.2876)
			*	0.9090 (0.2841)	0.9950 (6.6935)	0.9970 (2.4615)	0.9750 (1.6256)	0.8910 (1.1211)
3			I	0.9160 (0.4136)	0.9930 (7.2442)	0.9950 (3.2663)	0.9800 (1.7002)	0.9000 (1.5337)
			II	0.8990 (0.7062	0.9940 (8.4759)	0.9970 (4.7414)	0.9710 (1.6758)	0.8940 (2.2369)
			III	0.9030 (0.4320)	0.9970 (7.1548)	0.9950 (3.3220)	0.9800 (1.5969)	0.8750 (1.5340)
			*	0.9060 (0.3788)	0.9940 (6.7772)	0.9970 (2.9932)	0.9790 (1.5556)	0.8820 (1.3890)

other schemes as expected.

6 Real data analysis

In this Section, we consider a real data example from a warranty claims database company to investigate the performance of the proposed method.

Table 8: CPs and ALs (within brackets) linked with 95% CIs MLEs of ϕ, α_1, α_2, λ_1 and λ_2 for different censoring schemes (Exponentiated Pareto case, $\phi = 0.40$, $\alpha_1 = 4.75$, $\lambda_1 = 2.25$, $\alpha_2 = 1.75$ and $\lambda_2 = 1.20$).

k	n	m	Scheme	$\hat{\phi}$	$\hat{\alpha}_1$	$\hat{\lambda}_1$	$\hat{\alpha}_2$	$\hat{\lambda}_2$
1	40	30	I	0.9400 (0.3447)	0.9780 (8.8182)	0.9930 (2.4585)	0.9690 (2.9455)	0.9210 (1.3678)
			II	0.9160 (0.3733)	0.9840 (10.4529)	0.9870 (3.6502)	0.9810 (3.4038)	0.9100 (1.7903)
			III	0.9270 (0.3507)	0.9870 (8.7655)	0.9930 (2.5523)	0.9730 (2.8055)	0.9240 (1.4347)
			*	0.9540 (0.2991)	0.9870 (7.9551)	0.9950 (2.1424)	0.9590 (2.5261)	0.9040 (1.1683)
2			I	0.9200 (0.3959)	0.9910 (8.9684)	0.9960 (3.4013)	0.9790 (2.5722)	0.9060 (1.7028)
			II	0.8730 (0.6739)	0.9930 (10.8018)	0.9930 (5.3745)	0.9840 (2.6034)	0.9170 (2.6625)
			III	0.9110 (0.4270)	0.9910 (8.7468)	0.9940 (3.5264)	0.9780 (2.5595)	0.9050 (1.8601)
			*	0.9240 (0.3471)	0.9930 (8.1092)	0.9950 (3.0111)	0.9880 (2.2499)	0.9150 (1.4773)
3			I	0.9040 (0.5040)	0.9930 (9.0143)	0.9960 (4.1071)	0.9790 (2.2691)	0.9060 (1.9980)
			II	0.8810 (0.9485)	0.9930 (11.0638)	1.0000 (6.8806)	0.9740 (2.0102)	0.9210 (2.7914)
			III	0.9150 (0.5537)	0.9940 (8.9058)	0.9960 (4.3773)	0.9860 (2.0464)	0.9170 (2.0126)
			*	0.9170 (0.4571)	0.9960 (8.1433)	0.9980 (3.6040)	0.9790 (1.9925)	0.9030 (1.7662)
1	60	50	I	0.9320 (0.2690)	0.9910 (6.9781)	0.9970 (1.8954)	0.9670 (2.1680)	0.8900 (1.0336)
			II	0.9140 (0.2669)	0.9950 (8.0466)	0.9950 (2.5543)	0.9550 (2.1501)	0.8450 (1.1334)
			III	0.9540 (0.2713)	0.9870 (6.9191)	0.9910 (1.9212)	0.9720 (2.0428)	0.9150 (1.0638)
			*	0.9270 (0.2453)	0.9930 (6.5127)	0.9970 (1.7360)	0.9590 (1.9416)	0.9010 (0.9357)
2			I	0.9220 (0.3113)	0.9920 (7.1806)	0.9940 (2.6691)	0.9770 (1.8244)	0.9080 (1.2420)
			II	0.8380 (0.4529)	0.9830 (8.3392)	0.9860 (3.8034)	0.9790 (2.0497)	0.8690 (1.8221)
			III	0.9350 (0.3319)	0.9980 (7.0866)	0.9980 (2.7143)	0.9810 (1.7633)	0.9030 (1.3001)
			*	0.9110 (0.2822)	0.9980 (6.7244)	0.9970 (2.4689)	0.9660 (1.6310)	0.8830 (1.1202)
3			I	0.9160 (0.4156)	0.9930 (7.2703)	0.9930 (3.2305)	0.9760 (1.6895)	0.9020 (1.5311)
			II	0.8710 (0.6998)	0.9900 (8.5087)	0.9970 (4.7791)	0.9810 (1.6682)	0.8890 (2.2340)
			III	0.9180 (0.4303)	0.9950 (7.1575)	0.9980 (3.3541)	0.9750 (1.5655)	0.8700 (1.5262)
			*	0.9030 (0.3721)	0.9950 (6.7153)	0.9970 (2.9574)	0.9730 (1.5233)	0.8770 (1.3503)

6.1 Automotive Warranty Claims Data

For illustration purposes, a real life data set obtained from a warranty claims database company is considered (see [17] for the original data). In this particular example, mileages to repeated failures of 189 vehicles of a certain make are recorded along with the associated causes of failure. These data actually focus on early failures, from production to a maximum mileage of 3000. Although the original data is obtained at the level of individual prototypes, we carry out its analysis following [17] at the superposed fleet level to make it consistent with the theme of our proposed estimation methodology. To enable reasonable inference, fourteen

Table 9: Misspecification Analysis, $p = 0.5$, $\phi = 0.35$, $\alpha_1 = 3.0$, $\alpha_2 = 4.0$, $\lambda_1 = 1.5$ and $\lambda_2 = 1.75$

k	n	m	True Model	t_p	Fitted Model	Scheme	$\hat{t}_p$	MSE	RB	RV
1	40	30				I	0.9328	0.0231	-0.1124	1.7602
			Generalized Exponential	1.0510	Exponentiated Pareto	II	1.0198	0.0100	-0.0296	0.8130
						III	0.9922	0.0124	-0.0559	1.0698
						*	0.9361	0.0192	-0.1093	1.9417
2						I	0.7029	0.0076	-0.0611	1.3245
				0.7487		II	0.7520	0.0042	0.0044	0.6456
						III	0.7292	0.0041	-0.0260	0.8102
						*	0.7028	0.0045	-0.0612	1.1250
3						I	0.6045	0.0027	-0.0324	0.6923
				0.6247		II	0.6436	0.0034	0.0301	0.7555
						III	0.6183	0.0026	-0.0103	0.6969
						*	0.6046	0.0021	-0.0322	0.8235
1	60	50				I	0.9896	0.0103	-0.0583	1.1976
				1.0510		II	1.0214	0.0072	-0.0281	0.9501
						III	1.0062	0.0089	-0.0425	1.1710
						*	0.9869	0.0098	-0.0609	1.5076
2						I	0.7213	0.0033	-0.0366	0.9494
				0.7487		II	0.7392	0.0026	-0.0126	0.7647
						III	0.7292	0.0027	-0.0260	1.0384
						*	0.7207	0.0029	-0.0374	1.0740
3						I	0.6101	0.0018	-0.0235	0.8181
				0.6247		II	0.6248	0.0016	0.0001	0.5521
						III	0.6131	0.0019	-0.0186	0.9213
						*	0.6084	0.0018	-0.0262	0.9782

different labor codes (LC) for failures were combined into three broad groups of failure causes FM1–FM3. Clearly, the causes of failure are not independent (see [17] for more details) and we can actually use our proposed FMM based estimation method to deal with this dependent competing risks data. The mileage records (t_i) and the indicator denoting the cause of failure - δ_i (1 for FM1, 2 for FM2, 3 for FM3) have been recorded for all the events. There are 372 events in all at the fleet level, all of which are censored at 2972 miles (the largest mileage below 3000). Out of these, 99, 118, and 155 failures are attributable to causes FM1, FM2, and FM3, respectively.

Firstly, we opt to carry out the data analysis with two failure causes - (FM1 and FM3)

Table 10: Misspecification Analysis, $p = 0.5$, $\phi = 0.35$, $\alpha_1 = 3.0$, $\alpha_2 = 4.0$, $\lambda_1 = 1.5$ and $\lambda_2 = 1.75$

k	n	m	True Model	t_p	Fitted Model	Scheme	$\hat{t}_p$	MSE	RB	RV
1	40	30				I	2.5209	0.7570	0.3548	5.8765
			Exponentiated Pareto	1.8606	Generalized Exponential	II	2.2146	0.3144	0.1902	2.2697
						III	2.3837	0.5397	0.2810	4.8130
						*	2.4581	0.5863	0.3210	6.6162
2						I	1.3425	0.1001	0.2045	3.1300
				1.1146		II	1.3030	0.0818	0.1690	2.2002
						III	1.3054	0.0808	0.1711	3.1474
						*	1.3128	0.0750	0.1778	3.3940
3						I	1.0285	0.0480	0.1845	3.0034
				0.8682		II	1.0143	0.0470	0.1682	1.8331
						III	1.0117	0.0427	0.1652	3.0913
						*	1.0119	0.0376	0.1654	3.0659
1	60	50				I	2.5134	0.6342	0.3508	8.2379
				1.8606		II	2.2714	0.3054	0.2207	3.7101
						III	2.4490	0.5280	0.3162	7.7960
						*	2.4926	0.5638	0.3396	9.0463
2						I	1.3220	0.0717	0.1860	4.3533
				1.1146		II	1.3292	0.0784	0.1925	3.3242
						III	1.3162	0.0711	0.1808	4.5883
						*	1.3030	0.0586	0.1690	4.3876
3						I	1.0090	0.0328	0.1621	3.4617
				0.8682		II	1.0175	0.0392	0.1719	2.8190
						III	1.0059	0.0323	0.1585	3.3249
						*	0.9995	0.0284	0.1510	3.8763

and consider these as the two dependent causes of failure. Next, all the three causes of failure are taken into consideration for analysis purpose. The proposed model can thus easily be extended to analyze dependent competing risks data with multiple causes of failure. Analysis of the automotive warranty claims data is carried out after the original data are divided by 3000. It is not going to affect the conclusions of the study since this transformation is only a scale transformation (see also Section 6.1 of [22] for similar transformation in this regard). For illustration purposes, we consider two important members of the Lehmann family-GE and EP distributions.

Table 11: Expected total times for different censoring schemes ($\phi = 0.35$, $\alpha_1 = 3.0$, $\lambda_1 = 1.5$, $\alpha_2 = 4.0$ and $\lambda_2 = 1.75$).

k	n	m	Scheme	ETT(GE)	ETT(EP)
1	40	30	I	0.00029	0.00089
			II	0.00012	0.00025
			III	0.00016	0.00032
			*	0.00034	0.00107
2			I	0.00012	0.00029
			II	0.00007	0.00011
			III	0.00018	0.00026
			*	0.00013	0.00032
3			I	0.00007	0.00020
			II	0.00047	0.00010
			III	0.00009	0.00021
			*	0.00011	0.00029
1	60	50	I	0.00032	0.00260
			II	0.00012	0.00033
			III	0.00020	0.00333
			*	0.00032	0.00499
2			I	0.00009	0.00037
			II	0.00008	0.00014
			III	0.00012	0.00029
			*	0.00012	0.00030
3			I	0.00009	0.00022
			II	0.00006	0.00012
			III	0.00012	0.00017
			*	0.00009	0.00022

6.1.1 Case when number causes of failure is two

It is to note that 99 and 155 failures are attributable to causes FM1 and FM3, respectively in the original data set. We first consider these 254 failures from the the data set given in Table 1 of the supplementary file in [17] and these data points are then randomly grouped into 127 sets, each set containing 2 elements. Now applying the PFFCS on these sets with $n = 127$ and $k = 2$, we consider $m = 113$ failures (competing risk data) listed in Table 12.

Using the FMM representation and the proposed EM algorithm, the MLEs are computed for all the model parameters. All data and codes used to get the results are available on a GitHub repository at `https://github.com/deepak-stats/FMM_DCR_PFFCS_EM.git`. The selection of the initial parameters in an EM algorithm is a non-trivial problem in real life data analysis. While implementing the EM algorithm for the GE model in this case, we adopt some points: (i) Out of these 113 failures, 57 failures are attributable to cause FM1. An initial guess of ϕ is then taken as $\phi^{(0)} = \dfrac{57}{113} = 0.50442$. (ii) Note that 57 failures and 56 failures are attributable to causes FM1 and FM3 respectively. We assume the first set of failures (57 failures) as a set of complete data coming from GE(α_1, λ_1) and the second set of failures (56 failures) as a set of complete data coming from GE(α_2, λ_2). The initial guesses are then taken as the respective MLEs. Hence, $\alpha_1^{(0)} = 0.66508$, $\lambda_1^{(0)} = 4.73546$, $\alpha_2^{(0)} = 1.12315$, $\lambda_2^{(0)} = 4.75083$. Similarly, we can select the initial parameters while implementing the EM algorithm for the EP model. To check the goodness of fit of both the GE and EP distributions, we have computed the Kolmogorov-Smirnov (K-S) distances between the fitted and the empirical CDFs and the associated p-values. The Akaike information criterion (AIC) value is often used as a model selection criteria and is given by $2q - 2l(\boldsymbol{\theta}|\boldsymbol{t},\ \boldsymbol{\delta})$, where q is the number of unknown parameters in the model and $l(\boldsymbol{\theta}|\boldsymbol{t},\ \boldsymbol{\delta})$ is the associated log-likelihood function. Lower the AIC value, better is the model. In this context, MLEs of all the model parameters, 95% asymptotic CIs, the associated AIC values, K-S distances and the associated p-values are provided in Table 13. The K-S distance and the associated p-value reveals that both the models fit the data reasonably well. However, based on the K-S distance, it can be seen that the GE model provides a better fit compared to the EP model. Again from the AIC values, it is clear that the GE model will be preferred on EP model since it has the smallest AIC value. Although we could not prove theoretically the unimodality of the profile log-likelihood function, plots of the profile log-likelihood function w.r.t. one parameter by fixing the other parameters as MLEs. Figure 2 and Figure 3 provide the plots of the profile log-likelihood functions w.r.t $\alpha_1, \lambda_1, \alpha_2$ and λ_2 for the GE and EP distributions respectively. It is evident that in all the cases, the profile log-likelihood functions are unimodal. The plot of the empirical v/s the fitted CDFs is shown in Figure 4.

Table 12: The PFFCS dependent competing risks data with two causes of failure

Mileage	Cause	R_i	Mileage	Cause	R_i	Mileage	Cause	R_i	Mileage	Cause	R_i
1.21	1	1	61.65	1	0	250	3	0	543	3	0
2.23	1	1	62.37	1	0	259	1	0	559	3	0
2.3	3	0	62.71	1	0	288	3	0	562	3	0
2.49	1	1	74	3	0	294	3	0	569	1	0
3.23	1	0	77.2	1	0	300	3	0	590.55	1	0
3.64	1	0	77.22	1	0	313.45	3	0	604.09	3	0
3.76	1	0	97	3	0	313.65	3	0	620.45	1	0
5.27	1	0	98	1	0	325	3	0	620.8	1	0
5.34	1	1	109.27	1	0	329	1	0	620.82	1	0
5.47	1	1	120	3	0	331	1	0	645	3	0
6.23	1	0	123	3	1	367	1	0	654.26	3	0
7	3	1	125.06	3	1	374	3	0	654.27	1	0
10.21	3	0	125.31	3	1	378	3	0	665	1	0
10.74	3	1	127	3	0	386	1	0	677	3	0
10.82	3	1	129.43	1	0	395	3	0	696	3	0
11.55	1	0	129.99	1	0	396.44	1	0	709	3	0
11.64	1	0	138	3	0	396.76	1	0	753	3	0
12.08	1	0	153.26	1	0	397.59	1	0	756	3	0
12.76	1	0	154	1	0	397.99	1	0	760.03	3	0
13.15	3	0	155	1	0	400	1	0	760.65	3	0
13.74	1	1	156	3	0	436	3	0	776	3	0
21.31	1	0	157	3	0	447	1	0	777	1	0
24	1	0	170	1	0	461.97	3	0	801	3	0
28	1	0	174	1	0	470	3	0	827.49	3	0
30	1	0	196.81	3	0	484	3	0	827.55	3	0
38	3	0	218	3	0	503	3	0	840.17	3	0
40.35	3	0	223.85	1	0	515	3	0			
40.72	1	1	233	1	0	520	1	0			
40.92	3	1	248	1	0	525.62	3	0			

Table 13: Estimation of the model parameters associated with GE and EP models.

Case	Parameter	MLE	Asymptotic CI	AIC	K-S distance	p-value
GE model	ϕ	0.4188	(0.4122, 0.5966)			
	α_1	0.6185	(0.5261, 0.7689)			
	λ_1	4.5679	(3.6596, 4.9089)	9.3789	0.2210	0.8084
	α_2	1.1098	(0.8652, 1.2858)			
	λ_2	4.1463	(3.1800, 5.1720)			
EP model	ϕ	0.4205	(0.4122, 0.5966)			
	α_1	0.6357	(0.4200, 0.9100)			
	λ_1	5.0141	(2.0847, 7.3862)	13.7062	0.2648	0.7476
	α_2	1.1579	(0.8385, 1.4077)			
	λ_2	4.7244	(3.0997, 6.4019)			

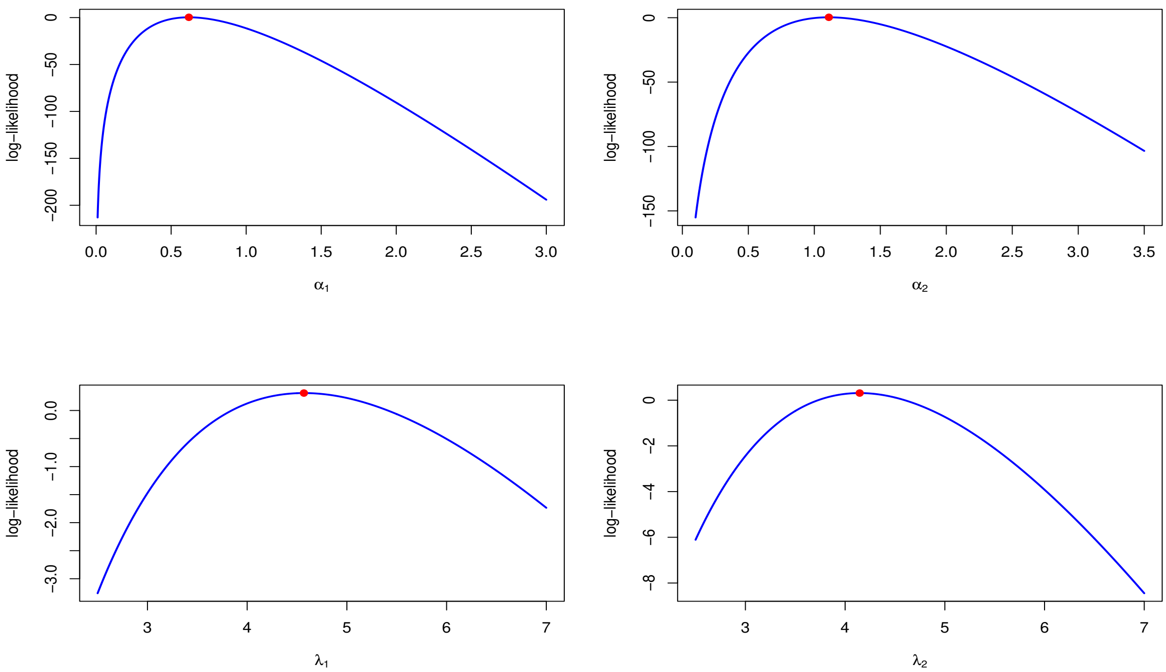

Figure 2: Plots of the profile log-likelihood functions associated with GE distribution.

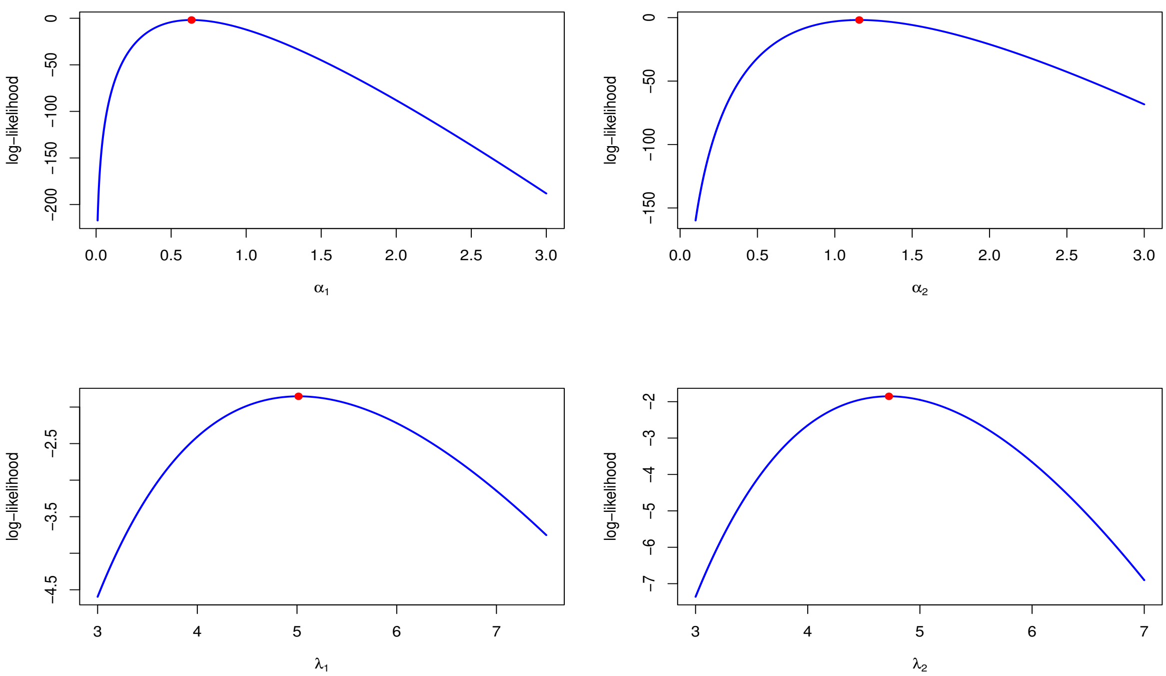

Figure 3: Plots of the profile log-likelihood functions associated with EP distribution.

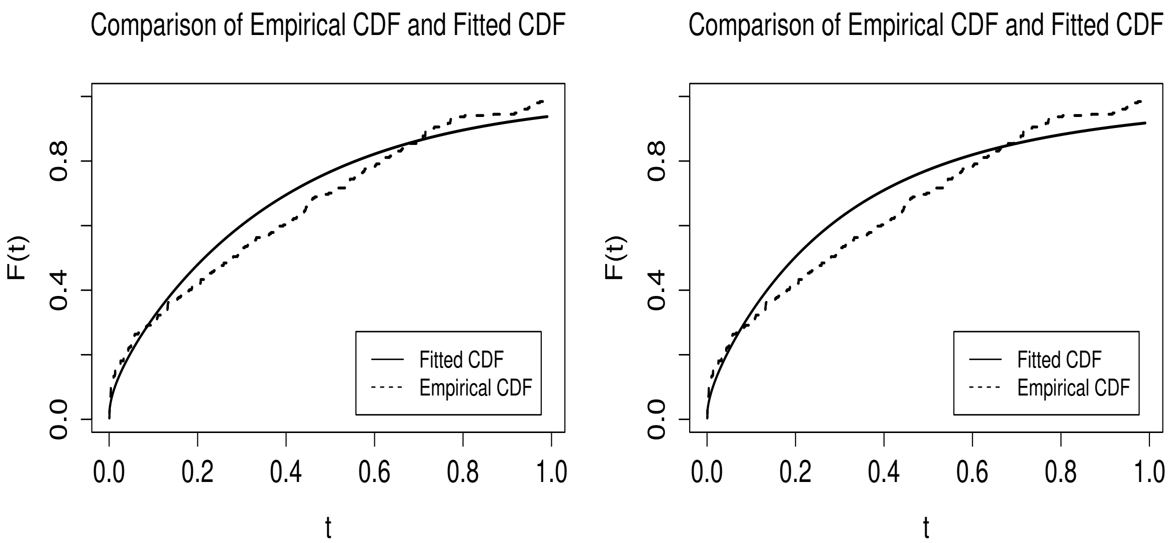

Figure 4: Fitted and Empirical CDF associated with GE and EP distribution for two causes.

6.1.2 Case when number of causes of failure is three

In the original data set (see Table 1 in the supplementary file of [17]), 99, 118, and 155 failures are attributable to causes FM1, FM2, and FM3, respectively. To illustrate the proposed model with three causes, these 372 data points are randomly grouped into $n = 186$ sets, each set containing $k = 2$ elements listed in ascending order. Now, from these data, we choose $m = 166$ and obtain the progressive first-failure censored competing risks data provided in Table 14. Employing the FMM representation coupled with the proposed EM algorithm, the MLEs are evaluated for all the model parameters. The selection of the initial parameters in the proposed EM algorithm is done using the same method discussed in the previous case when the number of causes of failures is two. In order to check the goodness of fit of both the GE and EP distributions, we have computed the K-S distances between the fitted and the empirical CDFs and the associated p-values. In this context, MLEs of all the model parameters, 95% asymptotic CIs, the associated AIC values, K-S distances and the associated p-values are provided in Table 15. From the K-S distance and the associated p-value, it seems quite clear that both the models fit the data reasonably well. However, based on the K-S distance, it can be observed that the GE model provides a better fit compared to the EP model. Comparing the AIC values, it is evident that the GE model will be preferred upon EP model as it has the smallest AIC value. Plots of the profile log-likelihood function w.r.t. one parameter (by fixing the other parameters as MLEs) are provided in Figure 5 and Figure 6 for the GE and EP distributions respectively. It is evident that in all the cases, the profile log-likelihood functions are unimodal. The plot of the empirical v/s the fitted CDFs is shown in Figure 7.

Table 14: The PFFCS dependent competing risks data with three causes of failure

Milage	Cause	R_i	Milage	Cause	R_i	Milage	Cause	R_i	Milage	Cause	R_i	Milage	Cause	R_i	Milage	Cause	R_i
1.21	1	1	12.08	1	0	77.22	1	0	196.07	2	0	397.99	1	0	590.55	1	0
1.48	2	0	12.76	1	0	81	2	0	196.21	2	0	400	1	0	590.98	2	0
1.53	2	0	13.15	3	0	97	3	0	196.81	3	0	408	2	0	604.09	3	0
1.66	2	0	13.74	1	0	98	1	0	210	2	0	416	2	0	604.39	2	0
1.81	2	1	14.18	1	0	102	2	0	218	3	1	436	3	0	604.54	2	0
2.3	3	0	14.59	2	0	109.27	1	0	223.85	1	0	447	1	0	604.82	2	0
2.41	1	0	16.42	2	0	109.55	2	0	223.87	2	0	461.68	2	0	620.45	1	0
2.49	1	0	16.43	2	0	117	2	0	237	2	1	461.97	3	0	620.8	1	0
2.75	1	1	21.19	2	1	120	3	0	248	1	1	470	3	0	620.82	1	0
3.61	2	0	21.56	2	0	121	2	0	250	3	0	484	3	0	645	3	0
3.64	1	0	24	1	0	123	3	0	259	1	0	490	2	0	654.26	3	0
3.76	1	1	28	1	1	124	2	0	288	3	0	503	3	0	654.27	1	0
5.27	1	1	38	3	0	125.06	3	0	293	2	0	509.3	2	0	665	1	0
5.37	1	0	40.05	2	0	125.31	3	0	294	3	0	509.49	2	0	677	3	0
5.47	1	0	40.35	3	1	127	3	1	300	3	0	514	2	0	696	3	0
5.51	1	1	40.92	3	0	129.99	1	1	309	2	0	515	3	0	709	3	0
6.39	2	0	41	1	0	136	2	0	313.45	3	0	519	2	0	753	3	0
7	3	0	47.1	2	0	138	3	0	313.65	3	0	520	1	0	756	3	0
8.28	2	0	47.66	2	0	149.07	1	0	325	3	0	525.62	3	0	760.03	3	0
8.62	1	0	48.41	2	0	149.07	1	0	329	1	0	526	2	0	760.65	3	0
10.21	3	0	48.93	2	0	153.08	1	0	331	1	0	532	2	0	760.7	2	0
10.26	2	1	54	2	0	153.26	1	0	367	1	0	535	2	0			
10.74	3	0	59	2	1	154	1	1	374	3	0	543	3	0			
10.78	2	0	61.59	2	0	156	3	0	378	3	0	558.1	2	0			
10.82	3	0	61.65	1	0	157	3	0	386	1	0	558.79	2	0			
11.19	1	0	62.37	1	0	170	1	0	395	3	0	559	3	0			
11.46	1	0	62.71	1	0	174	1	1	396.44	1	0	562	3	0			
11.55	1	0	74	3	0	182	2	0	396.76	1	0	567	2	0			
11.64	1	1	76	2	1	189	2	0	397.59	1	0	569	1	0			

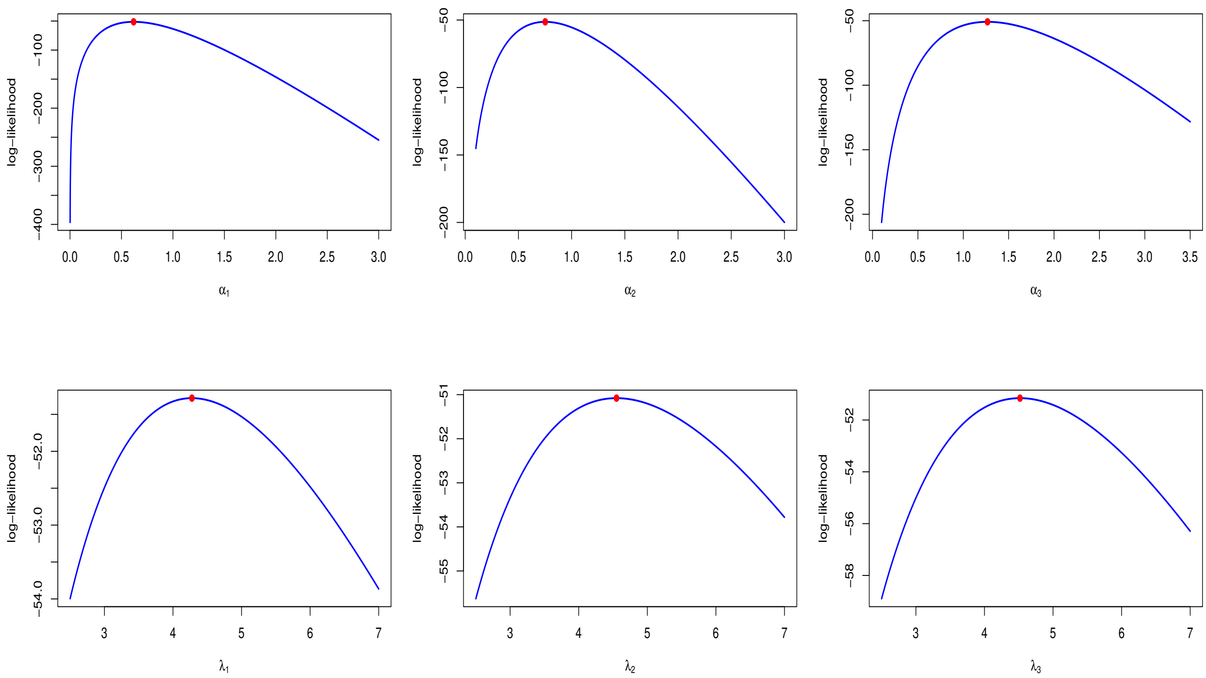

Figure 5: Plots of the profile log-likelihood functions associated with GE distribution.

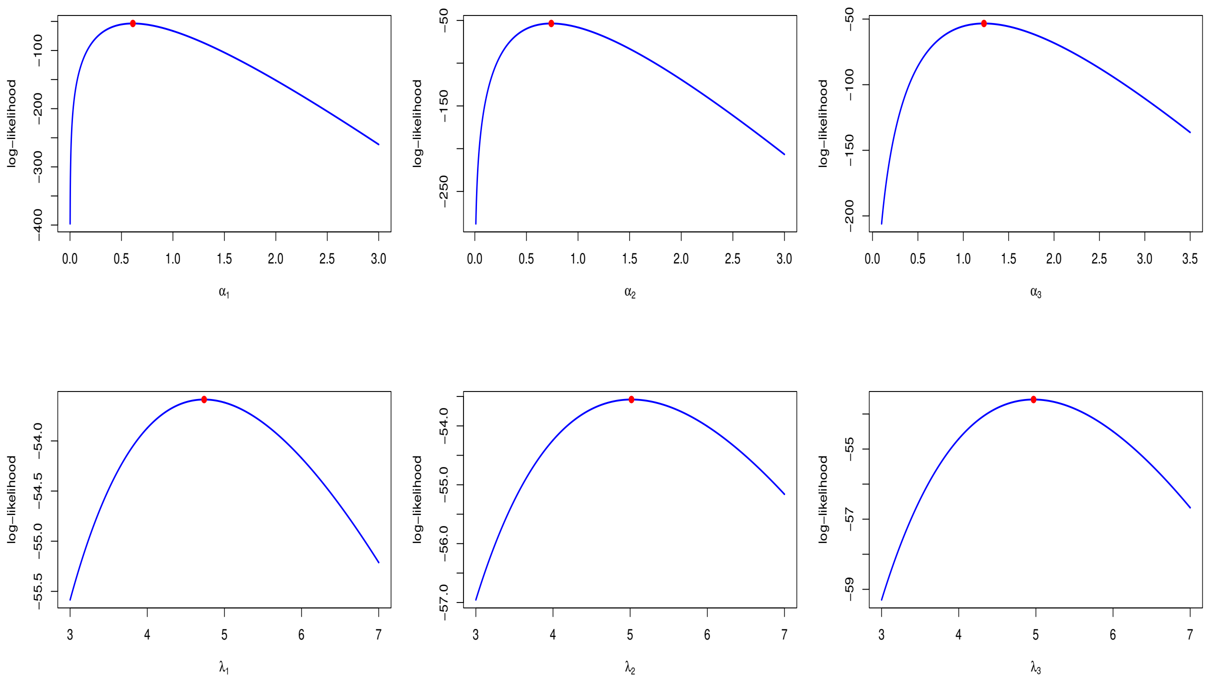

Figure 6: Plots of the profile log-likelihood functions associated with EP distribution.

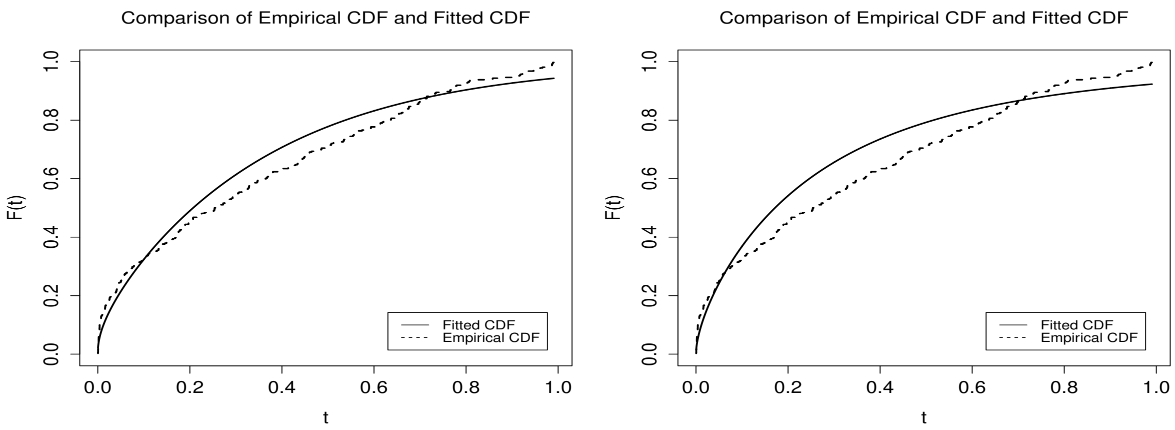

Figure 7: Fitted and Empirical CDF associated with GE and EP distribution for three causes.

Table 15: Estimation of the model parameters associated with GE and EP models.

Case	Parameter	MLE	Asymptotic CI	AIC	K-S distance	p-value
GE model	ϕ_1	0.2840	(0.2711, 0.4156)			
	ϕ_2	0.3376	(0.2768, 0.4219)			
	α_1	0.5685	(0.5275, 0.7621)			
	λ_1	4.7705	(3.9051, 5.1294)			
	α_2	0.7044	(0.5993, 0.8687)	112.4786	0.0853	0.9866
	λ_2	4.1894	(3.5725, 5.8416)			
	α_3	1.1056	(0.8659, 1.3166)			
	λ_3	4.4402	(4.9279, 5.3786)			
EP model	ϕ_1	0.3005	(0.2711, 0.4156)			
	ϕ_2	0.3201	(0.2768, 0.4219)			
	α_1	0.6110	(0.3918, 0.8978)			
	λ_1	4.7363	(1.7256, 7.3089)			
	α_2	0.7314	(0.4750, 0.9929)	117.1747	0.1186	0.9680
	λ_2	5.2750	(2.2491, 7.1650)			
	α_3	1.1479	(0.7905, 1.3920)			
	λ_3	4.9697	(4.8525, 5.4540)			

7 Conclusion

In this paper, we have proposed a FMM approach to analyze the dependent competing risks data under the PFFCS when the cause-specific failure time distributions are members of the flexible Lehmann family of distributions. The flexibility of the proposed FMM also stems from the fact that in our case, both the parameters of the cause-specific failure time distributions are allowed to vary. It is observed that MLEs of any of the model parameters cannot be obtained explicitly on maximization of the log-likelihood function. However, the point estimation methodology of the model parameters can easily be carried out using the EM-based maximum likelihood principle. The dimension of the optimization problem is greatly reduced and the complexity of the problem no longer depends on the number of causes. Extensive simulation studies are carried out under different scenarios to check the performance of the proposed FMM approach. For the illustration purposes, we consider the two important members of the Lehmann family- the GE and EP distributions. Additionally, to check the model validation, we study the misspecification analysis between the distributions using the pth-quantile. Finally, analysis of an automotive warranty claims data with multiple causes is provided. The proposed FMM can be easily be generalized to other important families of distributions. One can well and easily extend this model to deal with covariates particularly in survival analysis. Another interesting problem will be to find the optimal grouping of the proposed framework. Work is in progress along this direction, and we hope to report our findings in our future results.

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Statements and Declarations

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8 Appendix : Elements of Fisher Information Matrix

8.1 General Case

Suppose $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_{3S-1}) = (\phi_1, \phi_2, \ldots, \phi_{S-1}, \alpha_1, \alpha_2, \ldots, \alpha_S, \lambda_1, \lambda_2, \ldots, \lambda_S)$, and for $i, j = 1, 2, \ldots, 3S-1$, $I(\boldsymbol{\theta}) = \left(\left(\dfrac{\partial^2 l(\boldsymbol{\theta}|\boldsymbol{t},\, \boldsymbol{\delta})}{\partial\theta_i \partial\theta_j}\right)\right)$ be the observed Fisher information matrix. Note that here $l = l(\boldsymbol{\theta}|\boldsymbol{t},\, \boldsymbol{\delta})$ is the log-likelihood function as defined in (2.5). For $j, k = 1, 2, \ldots, S-1$, $j \neq k$, and $x,\ y = 1, 2, \ldots, S$, $x \neq y$, the elements of the observed Fisher information matrix can then be expressed as follows:

$$\frac{\partial^2 l}{\partial \phi_x^2} = -\frac{m_x}{\phi_x^2} - \frac{m_S}{(1-\sum_{i=1}^{S-1}\phi_i)^2} - \sum_{i=1}^{m} I_x(i)\frac{\{k(R_i+1)-1\}\{1-[G_0(t_i;\lambda_x)]^{\alpha_x}\}^2}{[\sum_{j=1}^{S}\phi_j\{1-[G_0(t_i;\lambda_j)]^{\alpha_j}\}]^2},$$

$$\begin{aligned}\frac{\partial^2 l}{\partial \phi_x \partial \phi_y} &= \frac{\partial^2 l}{\partial \phi_y \partial \phi_x} = -\frac{m_S}{(1-\sum_{i=1}^{S-1}\phi_i)^2}\\ &-\sum_{i=1}^{m} I_x(i)\frac{\{k(R_i+1)-1\}\{1-[G_0(t_i;\lambda_x)]^{\alpha_x}\}\{1-[G_0(t_i;\lambda_y)]^{\alpha_y}\}}{[\sum_{j=1}^{S}\phi_j\{1-[G_0(t_i;\lambda_j)]^{\alpha_j}\}]^2},\end{aligned}$$

$$\frac{\partial^2 l}{\partial \phi_x \partial \alpha_x} = \frac{\partial^2 l}{\partial \alpha_x \partial \phi_x} = \sum_{i=1}^{m} I_x(i)(K(R_i+1)-1)\frac{\partial l}{\partial \alpha_x}\left[\frac{1-[G(t_i;\lambda_x)]^{\alpha_x}}{\sum_{j=1}^{S}\phi_i(1-[G(t_i;\lambda_j)]^{\alpha_j})}\right]$$

$$\frac{\partial^2 l}{\partial \phi_y \partial \alpha_x} = \frac{\partial^2 l}{\partial \alpha_x \partial \phi_y} = \sum_{i=1}^{m} I_y(i)(K(R_i+1)-1)\frac{\partial l}{\partial \alpha_x}\left[\frac{1-[G(t_i;\lambda_y)]^{\alpha_y}}{\sum_{j=1}^{S}\phi_i(1-[G(t_i;\lambda_j)]^{\alpha_j})}\right]$$

$$\frac{\partial^2 l}{\partial \phi_x \partial \lambda_x} = \frac{\partial^2 l}{\partial \lambda_x \partial \phi_x} = \sum_{i=1}^{m} I_x(i)(K(R_i+1)-1)\frac{\partial l}{\partial \lambda_x}\left[\frac{1-[G(t_i;\lambda_x)]^{\alpha_x}}{\sum_{j=1}^{S}\phi_i(1-[G(t_i;\lambda_j)]^{\alpha_j})}\right]$$

$$\frac{\partial^2 l}{\partial \phi_y \partial \lambda_x} = \frac{\partial^2 l}{\partial \lambda_x \partial \phi_y} = \sum_{i=1}^{m} I_y(i)(K(R_i+1)-1)\frac{\partial l}{\partial \lambda_x}\left[\frac{1-[G(t_i;\lambda_y)]^{\alpha_y}}{\sum_{j=1}^{S}\phi_i(1-[G(t_i;\lambda_j)]^{\alpha_j})}\right]$$

$$\begin{aligned}\frac{\partial^2 l}{\partial \lambda_x^2} &= \sum_{i=1}^{m} I_x(i)\Bigg\{\frac{\partial}{\partial \lambda_x}\Big[\frac{g_0^{'}(t_i;\lambda_x)}{g_0(t_i;\lambda_x)}\Big] + (\alpha_x-1)\frac{\partial}{\partial \lambda_x}\Big[\frac{G_0^{'}(t_i;\lambda_x)}{G_0(t_i;\lambda_x)}\Big]\\ &-\{k(R_i+1)-1\}\phi_x\alpha_x\frac{\partial}{\partial \lambda_x}\left[\frac{[G_0(t_i;\lambda_x)]^{\alpha_x-1}G_0^{'}(t_i;\lambda_x)}{\sum_{j=1}^{S}\phi_j\{1-[G_0(t_i;\lambda_j)]^{\alpha_j}\}}\right]\Bigg\},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial \alpha_x \partial \lambda_x} &= \frac{\partial^2 l}{\partial \lambda_x \partial \alpha_x} = \sum_{i=1}^{m} I_x(i)\Bigg\{\frac{G_0^{'}(t_i;\lambda_x)}{G_0(t_i;\lambda_x)}\\ &-\{k(R_i+1)-1\}\phi_x G_0^{'}(t_i;\lambda_x)\frac{\partial}{\partial \alpha_x}\left[\frac{\alpha_x[G_0(t_i;\lambda_x)]^{\alpha_x-1}}{\sum_{j=1}^{S}\phi_j\{1-[G_0(t_i;\lambda_j)]^{\alpha_j}\}}\right]\Bigg\},\end{aligned}$$

$$\frac{\partial^2 l}{\partial\alpha_y\partial\lambda_x} = \frac{\partial^2 l}{\partial\lambda_x\partial\alpha_y} = -\sum_{i=1}^{m} I_x(i)\Bigg\{\{k(R_i+1)-1\}\phi_x G_0^{'}(t_i;\lambda_x)\alpha_x[G_0(t_i;\lambda_x)]^{\alpha_x-1} \times\frac{\partial}{\partial\alpha_y}\left[\frac{1}{\sum\limits_{j=1}^{S}\phi_j\{1-[G_0(t_i;\lambda_j)]^{\alpha_j}\}}\right]\Bigg\},$$

$$\frac{\partial^2 l}{\partial\lambda_y\partial\lambda_x} = \frac{\partial^2 l}{\partial\lambda_x\partial\lambda_y} = -\sum_{i=1}^{m} I_x(i)\Bigg\{\{k(R_i+1)-1\}\phi_x G_0^{'}(t_i;\lambda_x)\alpha_x[G_0(t_i;\lambda_x)]^{\alpha_x-1} \times\frac{\partial}{\partial\lambda_y}\left[\frac{1}{\sum\limits_{j=1}^{S}\phi_j\{1-[G_0(t_i;\lambda_j)]^{\alpha_j}\}}\right]\Bigg\},$$

$$\frac{\partial^2 l}{\partial\alpha_x^2} = -\frac{m_x}{\alpha_x^2} - \sum_{i=1}^{m} I_x(i)\Bigg\{\{k(R_i+1)-1\}\phi_x \ln G_0(t_i;\lambda_x)\frac{\partial}{\partial\alpha_x}\left[\frac{[G_0(t_i;\lambda_x)]^{\alpha_x}}{\sum\limits_{j=1}^{S}\phi_j\{1-[G_0(t_i;\lambda_j)]^{\alpha_j}\}}\right]\Bigg\},$$

$$\frac{\partial^2 l}{\partial\alpha_x\alpha_y} = \frac{\partial^2 l}{\partial\alpha_y\alpha_x} = -\sum_{i=1}^{m} I_x(i)\Bigg\{\{k(R_i+1)-1\}\phi_x[G_0(t_i;\lambda_x)]^{\alpha_x}\ln G_0(t_i;\lambda_x) \times\frac{\partial}{\partial\alpha_y}\left[\frac{1}{\sum\limits_{j=1}^{S}\phi_j\{1-[G_0(t_i;\lambda_j)]^{\alpha_j}\}}\right]\Bigg\},$$

where

$$g_0^{'}(t_i;\lambda_x) = \frac{\partial}{\partial\lambda_x}g_0(t_i;\lambda_x),\;\; G_0^{'}(t_i;\lambda_x) = \frac{\partial}{\partial\lambda_x}G_0(t_i;\lambda_x).$$