BAYESIAN ACCEPTANCE SAMPLING PLAN FOR SIMPLE STEP-STRESS MODEL

Deepak Prajapati [1] and Debasis Kundu[2]

Abstract

This paper presents a Bayesian acceptance sampling plan for simple step-stress life testing model, when the stress changing time is random, and the experimental units are subject to Type-II censoring. It is assumed that the lifetime distribution of the experimental item is exponential at each stress level with different scale parameters. We first present the Bayes decision function under a specified loss function. Based on the Bayes decision function, the optimal Bayesian sampling plan is determined by minimizing the Bayes risk. We also present some optimal Bayesian sampling plan under specific cases. We further provide order restricted optimal Bayesian sampling plan also. The Bayesian sampling plans presented in this paper are quite beneficial when the experimental items are highly reliable, which is quite common in present days.

KEYWORDS AND PHRASES: Bayesian Sampling Plan, Bayes Risk, Bayes decision function, Random Stress-Change Time.

[1] Decision Sciences Area, Indian Institute of Management Lucknow, Lucknow-226013, India.
e-mail:deepak.prajapati@iiml.ac.in

[2] Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, India.

* Corresponding author, e-mail: kundu@iitk.ac.in

1 INTRODUCTION

Accelerated life test (ALT) has been used regularly by reliability engineers to reduce the lifetime of the experimental items or to accelerate the degradation of their performance. Such testing aims to quickly obtain data that, adequately modeled, analyzed, and yield desired information on the lifetime of the experimental units or their performances under normal operating condition. The key references on ALT model are by Nelson (1980) and Bagdonavicius and Nikulin (2002). The step-stress test is a special case of the ALT, where stress changes during the experiment in a regular interval. Depending on the type of the experimental items, several stress factors like temperature, voltage, etc. can be used during the experiment. Generally, the test starts at a specified low stress and the stress is repeatedly increased till the experiment lasts. A simple step-stress accelerated life test (SSALT) uses only two stress levels. An extensive amount of work has been done in analyzing data obtained from a simple step-stress experiment, see for example the monograph by Kundu and Ganguly (2017) in this respect.

In statistical quality control, acceptance sampling is involved with inspecting one or more samples drawn randomly from the batch and make decision based on the observed sample. An acceptance sampling plan consists of rules or procedures for making decisions on the batch of finished products. From the consumer's point of view, a product must perform the required function without failure for the desired period. The lifetime of a product is, therefore, one of the most important quality characteristics. From the perspective of the economy, the decision-theoretic approach is more scientific and realistic because the acceptance sampling plan is determined by making an optimal decision based on some economic considerations, such as maximizing the return or minimizing the loss. Under the decision-theoretic approach, the Bayesian sampling plan is quite appealing since it minimizes the Bayes risks among a class of acceptance sampling plans. Much work has been done for the design of the Bayesian sampling plans. Some of the research work along this approach are by Lam (1990, 1994), Lam and Choy (1995), Lin *et al.* (2002), Chen *et al.* (2004), Lin *et al.* (2008, 2010), Liang and Yang (2013), Tsai *et al.* (2014), Yang *et al.* (2017) and the references cited therein.

Most of the studies on the design of Bayesian sampling plans differ mainly in the assumption of the lifetime distribution of the experimental units or in the censoring scheme. So far, the testing conditions has not been considered to design the Bayesian sampling plan. In industrial experiments, tested products are often highly reliable, with significant mean times to failure under the normal operating conditions. Therefore, in conventional life testing experiments under either Type-I or Type-II censoring, it is almost impossible to obtain any failure or wait for a long time for a sufficient number of failures. Introduction of SSALTs in any life testing experiment may overcome such difficulty. SSALTs are used in many circumstances to obtain information instantly on the lifetime distribution of products. Therefore, it is very essential to obtained the Bayesian sampling plan under the SSALT condition. In this paper, we present the Bayesian sampling plan based on SSALT in which stress change time is random, and the test is subject to Type-II censoring.

The SSALT is a special case of ALT that allows for different stress at various intermediate stages of the experiment. Here random sample of n identical units are placed on life test under an initial stress level of s_1. At fixed times $\tau_1,\ \tau_2,\ \ldots,\ \tau_m$, the stress levels are increased to s_2, s_3, $\ldots$, s_{m+1}, respectively. Unfortunately, in the conventional step-stress models as described above, the parameters associated with the failure time distributions at the different stress levels are not always estimable. Therefore, we consider the following step-stress model where the the parameters associated with different parameters are always estimable. In this model the stress changes when a pre-specified number of failures takes place, which is random. This model was first considered by Xiong and Milliken (1999) and by Xiong *et al.* (2006). The random stress changing time step-stress model with m levels can be described as follows: For the given sample size n, prefixed $n_1,\ldots,n_m$, such that $n_1+\ldots+n_m < n$. Let us denote $x_{(1)} < x_{(2)} < \ldots$ as the ordered failure times. Suppose n units are placed in a life testing experiment and they are subjected to a stress level s_1. At the time of n_1-th failure $x_{(n_1)}$, the stress level is changed to s_2, then at the time of (n_1+n_2)-th failure the stress level is changed to s_3 and so on. Finally at the time of $(n_1+\ldots+n_m)$-th failure, the stress level is changed to s_m and the test continues until a total of $(n_1+\ldots+n_m)$ failures are observed.

In general different lifetime distributions are assumed at different stress levels. Therefore, we need a model to connect the distributions under different stress levels. The most popular model in this regard is the cumulative exposure model (CEM) introduced by Nelson (1990). The CEM relates the lifetime distributions of the units at one stress level to the next stress level. The model assumes that the residual life of the experimental units depends only on cumulative exposures the units have experienced, with no memory of how this exposure was accumulated. Several works have been done to analyze the data obtained from a SSALT experiment assuming CEM assumption, see for example Balakrishnan *et al.* (2007), Balakrishnan *et al.* (2009), Mitra *et al.* (2013), and Samanta and Kundu (2018). So far the Bayesian sampling plan under the simple step-stress model with random stress-change time has been not considered in the literature. This paper considers the design of the Bayesian sampling plans based on simple step-stress under random stress- change time when the lifetime is exponentially distributed. The optimal Bayesian sampling plan is obtained which satisfy the Bayes risk minimization criteria under given well-defined loss function among all sampling plans. We further consider order restricted optimal Bayesian sampling plan. It is assumed that the mean life at the second stress level is smaller than the first stress level. We provide some optimal Bayesian sampling plans under this order restriction also.

The rest of the paper is organized as follows. In Section 2, we provide the model description and well-defined loss function to obtain the Bayesian sampling plan. Bayes decision function and Bayes risk is obtained for well-defined loss function in Section 3. In Section 4, we present the Bayes decision function and Bayes risk for the quadratic loss function. The order restricted optimal Bayesian sampling plan has been considered in Section 5. Numerical results are presented in Section 6. Finally we conclude the paper in section 7. All the derivations and proofs are given in Appendix.

2 Model Description and Loss Function

Consider a batch of products from the production line. The acceptance sampling plan is performed to decide whether to accept (to sell) or to reject (not to sell) the batch of products. Whether to accept or reject the batch is based on the quality of the product of the batch. Due to the high-reliability products are designed, the sampled items are put on random stress changing time SSALT, and failed items are not replaced. For the sample size n, we prefixed n_1 and n_2, such that $n_1+n_2<n$. Suppose n identical items are put on a life test with an initial stress level s_1. At the time of n_1-th failure, the stress level is changed to s_2, and the test continues until a total of n_1+n_2 failures is observed. Assume that the lifetimes of the experimental units at stress levels s_1 and s_2 are independently distributed random variables having exponential distribution with scale parameters λ_1 and λ_2. The corresponding probability density function (PDF) and cumulative distribution function (CDF) are given, respectively, by

$$f_j(x|\lambda_j)=\lambda_j e^{-\lambda_j x}, \quad x>0, \lambda_j>0, j=1,2$$

and

$$F_j(x|\lambda_j)=1-e^{-\lambda_j x}, \quad x>0, \lambda_j>0, j=1,2.$$

Let $x_{(1)}<x_{(2)}<\ldots<x_{(n_1)}<x_{(n_1+1)}<\ldots<x_{(n_1+n_2)}$ denote the ordered failure times. We further assume that the data satisfy the CEM assumption. Under the Assumptions of the CEM, the likelihood function of the observed data is

$$L(\lambda_1,\lambda_2|data)=\frac{n!}{(n-n_1-n_2)!}\lambda_1^{n_1}\lambda_2^{n_2}e^{-\lambda_1 T_1}e^{-\lambda_2 T_2} \tag{1}$$

where

$$T_1=\sum_{i=1}^{n_1} x_{(i)}+(n-n_1)x_{(n_1)}$$

$$T_2=\sum_{i=n_1+1}^{n_1+n_2}(x_{(i)}-x_{(n_1)})+(n-n_1-n_2)(x_{(n_1+n_2)}-x_{(n_1)}),$$

see for example Kundu and Balakrishnan (2009). From the likelihood function (1), it is clear that (T_1,T_2) is a complete sufficient statistic. The maximum likelihood estimators (MLEs) of mean lifetime $\theta_1=1/\lambda_1$ and $\theta_2=1/\lambda_2$

$$\widehat{\theta}_1=\frac{T_1}{n_1} \quad \text{and} \quad \widehat{\theta}_2=\frac{T_2}{n_2}. \tag{2}$$

Kundu and Balakrishnan (2009) studied the joint distribution of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ and proved the following theorem.

Theorem 2.1. *$\widehat{\theta}_1$ and $\widehat{\theta}_2$ are distributed as $G(n_1, n_1\lambda_1)$ and $G(n_2, n_2\lambda_2)$, respectively, and they are independent. Here, $G(a,b)$ denotes the gamma distribution with mean a/b and variance a/b^2.*

Proof. See Kundu and Balakrishnan (2009). ∎

From theorem 2.1, it is clear that $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are unbiased estimators of θ_1 and θ_2, respectively. Since they are functions of the complete sufficient statistics, they are uniformly minimum variance unbiased estimator.

2.1 Prior and Loss Function

In this section, for developing the Bayesian sampling plan, first we assume the prior on parameters λ_1 and λ_2 and then define the loss function. Suppose the prior for λ_1 and λ_2 are $G(a_1, b_1)$ and $G(a_2, b_2)$, respectively. The joint posterior PDF of λ_1 and λ_2 is

$$\pi(\lambda_1, \lambda_2|\mathbf{X} = \mathbf{x}) = \frac{(t_1 + b_1)^{a_1+n_1}(t_2 + b_2)^{a_2+n_2}}{\Gamma(n_1 + a_1)\Gamma(n_2 + a_2)} \lambda_1^{n_1+a_1-1} \lambda_2^{n_2+a_2-1} e^{-\lambda_1(t_1+b_1)} e^{-\lambda_2(t_2+b_2)}.$$

Therefore, $\lambda_1|\mathbf{x} \sim G(a_1 + n_1, b_1 + t_1)$ and $\lambda_2|\mathbf{x} \sim G(a_2 + n_2, b_2 + t_2)$ and they are independent. Since T_1 and T_2 is sufficient statistic of λ_1 and λ_2 and the joint PDF $\pi(\lambda_1, \lambda_2|\mathbf{x})$ depends on $\mathbf{x}$ only through $T_1 = t_1$ and $T_2 = t_2$. Hence we may also denote the joint PDF by $\pi(\lambda_1, \lambda_2|t_1, t_2)$ and the marginal posterior PDF by $\pi(\lambda_1|t_1, t_2)$ and $\pi(\lambda_2|t_1, t_2)$.

Let δ denote an action on this acceptance sampling. When $\delta = 1$, the batch is accepted, and when $\delta = 0$, the batch is rejected. Let C_r be the loss of rejecting the batch, and let $g(\lambda_1, \lambda_2)$ be the loss of accepting the batch and be positive and increasing in λ_1(the reciprocal of θ_1) and in λ_1 (the reciprocal of θ_2) for $\lambda_1, \lambda_2 > 0$. Since variability in the data may lead to the wrong decision therefore we have a costs C_{v_1} and C_{v_2} on the variance $V(\widehat{\theta}_1)$ and $V(\widehat{\theta}_2)$, respectively. Also, C_s denotes the cost per item inspected and let $\tau = X_{(n_1+n_2)}$ be the duration of the experiment then C_τ is the cost per unit time. When the life test terminates, the un-failed components can be reused and thus, have salvage value r_s, where $C_s > r_s \geq 0$. The loss for taking an action δ and choosing the sampling parameter (n, n_1, n_2) is defined as

$$L(\delta(\mathbf{x}), \lambda_1, \lambda_2) = \begin{cases} nC_s - (n - n_1 - n_2)r_s + V(\widehat{\theta}_1)C_{v_1} + V(\widehat{\theta}_2)C_{v_2} + \tau C_\tau + g(\lambda_1, \lambda_2), & \delta(\mathbf{x}) = 1 \\ nC_s - (n - n_1 - n_2)r_s + V(\widehat{\theta}_1)C_{v_1} + V(\widehat{\theta}_2)C_{v_2} + \tau C_\tau + C_r, & \delta(\mathbf{x}) = 0. \end{cases} \tag{3}$$

The rejection cost C_r is fixed because the products are discarded or return, so the related cost is fixed. The acceptance cost $g(\lambda_1, \lambda_2)$ of the batch occurs when the some products are bad, but due to no evidence found in the samples we will accept it which means we assume that the mean lifetime of the product at stress level s_1 and s_2, is high. Therefore, if the batch is accepted, the items will be sold in the market. The loss (or gain) will depend on the customer's expectation or mean lifetime of the product. Therefore, acceptance cost $g(\lambda_1, \lambda_2)$ is depending on mean lifetime of the product at stress level s_1 and s_2 through λ_1 and λ_2. It may be mentioned that in the cost function (3) we have used $V(\widehat{\theta}_1)$ and $V(\widehat{\theta}_2)$ instead of the corresponding mean squared errors (MSEs) as both are unbiased estimators, hence, they are same.

The acceptance sampling plan under this set up is given by (n, n_1, n_2, δ), where n is the sample size, n_1 is the number of failures at stress level s_1, n_2 is the number of failures at the stress level s_2, and δ is the decision on the batch. Our aim is to determine the optimal Bayesian sampling plan, viz., $(n_B, n_{1B}, n_{2B}, \delta_B)$ by obtaining the Bayes decision function δ_B so that the Bayes risk is minimized under the given loss function Eq. (3) over all such sampling plans.

3 Bayes decision function and Bayes risk

In this section, we obtain the Bayes decision function and Bayes risk of the Bayesian sampling plan for the loss function defined in Eq. (3). In order to derive the Bayes decision function, for fixed n, n_1, and n_2, we considered the Bayes risk

$$\begin{aligned} R^\delta(n, n_1, n_2) &= nC_s - (n - n_1 - n_2)r_s + E[V(\widehat{\theta}_1)]C_{v_1} + E[V(\widehat{\theta}_2)]C_{v_2} + E[\tau]C_\tau \\ &\quad + E(g(\lambda_1, \lambda_2)) + E[(C_r - g(\lambda_1, \lambda_2))(1 - \delta(\mathbf{X}))] \\ &= nC_s - (n - n_1 - n_2)r_s + E[V(\widehat{\theta}_1)]C_{v_1} + E[V(\widehat{\theta}_2)]C_{v_2} + E[\tau]C_\tau \\ &\quad + E(g(\lambda_1, \lambda_2)) + E_{\mathbf{X}}[(1 - \delta(\mathbf{X}))(C_r - \phi_\pi(T_1, T_2))], \end{aligned} \tag{4}$$

where $\phi_\pi(T_1,T_2)=E_{(\lambda_1,\lambda_2)|\mathbf{X}}[g(\lambda_1,\lambda_2)]$. Then for a fixed n, n_1, n_2, and for given $T_1=t_1$ and $T_2=t_2$, the Bayes risk is minimize when $\delta(\mathbf{x})=1$, if $C_r-\phi_\pi(T_1,T_2)\geq 0$, for more details see Prajapati *et al.* (2020). Therefore, for a fixed n, n_1, n_2, and for given $T_1=t_1$ and $T_2=t_2$, the Bayes decision function is given by

$$\delta_B(t_1,t_2|n,n_1,n_2)=\begin{cases}1, & \phi_\pi(t_1,t_2)\leq C_r\\ 0, & \text{otherwise.}\end{cases} \tag{5}$$

Now goal is to find the optimal Bayesian sampling plan, say $(n_B,n_{1B},n_{2B},\delta_B)$, which minimizes the Bayes risk $R^{\delta_B}(n,n_1,n_2)$ among all sampling plans of type (n,n_1,n_2,δ_B). In the following, first, we present an alternative form of the Bayes decision function. To find the alternative form of the Bayes decision function we study the monotonicity of the function $\phi_\pi(t_1,t_2)$ w. r. t. t_1 and t_2 in the next section.

3.1 Alternative Form of Bayes Decision Function

In this section, we present the alternative form of Bayes decision function and corresponding Bayes risk. For a fixed n,n_1, and n_2, the Bayes decision function (5) depend on the $\phi_\pi(t_1,t_2)$ through t_1 and t_2. The function $\phi_\pi(t_1,t_2)$ is the posterior expectation of $g(\lambda_1,\lambda_2)$ which is given by

$$\begin{aligned}\phi_\pi(t_1,t_2) &= \int\int g(\lambda_1,\lambda_2)\pi(\lambda_1,\lambda_2|t_1,t_2)d\lambda_2 d\lambda_1\\ &= \int\left(\int g(\lambda_1,\lambda_2)\pi(\lambda_2|t_1,t_2)d\lambda_2\right)\pi(\lambda_1|t_1,t_2)d\lambda_1,\end{aligned} \tag{6}$$

where $g(\lambda_1,\lambda_2)$ is positive, non-constant, and increasing in λ_1 and λ_2, and the posterior PDFs $\pi(\lambda_1|t_1,t_2)$ and $\pi(\lambda_2|t_1,t_2)$ has the following properties:

Lemma 3.1. *Let t_1, t_1^* are two realization of T_1 and t_2, t_2^* are two realization of T_2, then*

(1) If $t_1^*>t_1$ and for a fixed t_2, $\dfrac{\pi(\lambda_1|t_1^*,t_2)}{\pi(\lambda_1|t_1,t_2)}$ is non-increasing in λ_1, provided $\pi(\lambda_1|t_1,t_2)\neq 0$.

(2) If $t_2^*>t_2$, and for a fixed t_1, $\dfrac{\pi(\lambda_2|t_1,t_2^*)}{\pi(\lambda_2|t_1,t_2)}$ is non-increasing in λ_2, provided $\pi(\lambda_2|t_1,t_2)\neq 0$.

Proof. (1) Since $\lambda_1|(t_1,t_2) \sim G(a_1+n_1, b_1+t_1)$, therefore density $\pi(\lambda_1|t_1,t_2)$ does not depend on t_2. Then for a fixed t_2, if $t_1^* > t_1$ the ratio

$$\frac{\pi(\lambda_1|t_1^*,t_2)}{\pi(\lambda_1|t_1,t_2)} = \frac{(b_1+t_1^*)^{n_1+a_1}}{(b_1+t_1)^{n_1+a_1}} e^{-\lambda_1(t_1^*-t_1)}$$

Hence $\dfrac{\pi(\lambda_1|t_1^*,t_2)}{\pi(\lambda_1|t_1,t_2)}$ is non-increasing in λ_1.

(2) Similarly, we can show that if $t_2^* > t_2$, and for a fixed t_1, $\dfrac{\pi(\lambda_2|t_1,t_2^*)}{\pi(\lambda_2|t_1,t_2)}$ is non-increasing in λ_2. ∎

Let us define

$$\psi_\pi(\lambda_1,t_1,t_2) = \int g(\lambda_1,\lambda_2)\pi(\lambda_2|t_1,t_2)d\lambda_2. \tag{7}$$

Then by Lemma 3.1, $\psi_\pi(\lambda_1,t_1,t_2)$ has the following desired properties:

Lemma 3.2.

(1) For given $\lambda_1 > 0$, t_2 and $t_1 \neq t_1^*$, $\psi_\pi(\lambda_1,t_1,t_2) = \psi_\pi(\lambda_1,t_1^*,t_2)$.

(2) For given $\lambda_1 > 0$ and t_1, $\psi_\pi(\lambda_1,t_1,t_2)$ is decreasing in t_2.

(3) For given t_1 and t_2, $\psi_\pi(\lambda_1,t_1,t_2)$ is increasing in λ_1.

Proof. See Appendix. ∎

From Eq. (6) and (7), the posterior expectation of $g(\lambda_1,\lambda_2)$ can be written as

$$\phi_\pi(t_1,t_2) = E_{\lambda_2|t_1,t_2}[\psi_\pi(\lambda_1,t_1,t_2)],$$

then by Lemma 3.1 and Lemma 3.2 one can prove the following theorem.

Theorem 3.1.

(1) The $\phi_\pi(t_1,t_2)$ is strictly decreasing function of t_1 and t_2.

(2) The $\delta_B(t_1,t_2|n,n_1,n_2)$ is strictly increasing function of t_1 and t_2.

Proof. See Appendix. ∎

From Theorem 3.1 we know that for a fixed $t_1 > 0$, $\phi_\pi(t_1,t_2)$ is strictly decreasing function of t_2. Therefore, for a fixed n, n_1, n_2, and $t_1 > 0$, let use define a set $S_{t_1} = \{t_2 : \phi_\pi(t_1,t_2) \leq C_r\}$, and

$$\xi(n_1,n_2,t_1) = \begin{cases} \inf\{S_{t_1}\}, & \text{if } S_{t_1} \text{ is non-empty} \\ \infty, & \text{otherwise.} \end{cases} \tag{8}$$

Then the Bayes decision function can be written as

$$\delta_B(T_1,T_2|n,n_1,n_2) = \begin{cases} 1, & T_1 \geq 0, T_2 \geq \xi(n_1,n_2,T_1) \\ 0, & \text{otherwise} \end{cases}$$

and let $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are MLE of mean lifetime θ_1 and θ_2 given in Eq.2 under the stress level s_1 and s_2, respectively, then the Bayes decision function can written as

$$\delta_B(T_1,T_2|n,n_1,n_2) = \begin{cases} 1, & \widehat{\theta}_1 \geq 0, \widehat{\theta}_2 \geq \frac{\xi(n_1,n_2,n_1\widehat{\theta}_1)}{n_2} \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

From Theorem 2.1 we know the joint distribution of $\widehat{\theta}_1$ and $\widehat{\theta}_2$, using that we can obtain risk of Bayes decision function (9) quite easily.

3.2 Derivation of Bayes risk

In this section, the Bayes risk is obtained for the loss function given in Eq. (3). Using the Bayes decision function given in Eq. (9), the Bayes risk can be written as

$$\begin{aligned} R^{\delta_B}(n,n_1,n_2) &= nC_s - (n-n_1-n_2)r_s + E[V(\widehat{\theta}_1)]C_{v_1} + E[V(\widehat{\theta}_2)]C_{v_2} + E[\tau]C_\tau + C_r \\ &\quad + E_{(\lambda_1,\lambda_2)}\left[(g(\lambda_1,\lambda_2)-C_r)P\left(\widehat{\theta}_1 > 0, \widehat{\theta}_2 > \frac{\xi(n_1,n_2,n_1\widehat{\theta}_1)}{n_2}\right)\right]. \end{aligned}$$

Since $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are $G(n_1,n_1\lambda_1)$ and $G(n_2,n_2\lambda_2)$, respectively, and they are independent. Therefore,

$$R^{\delta_B}(n,n_1,n_2) = nC_s - (n-n_1-n_2)r_s + E[V(\widehat{\theta}_1)]C_{v_1} + E[V(\widehat{\theta}_2)]C_{v_2} + E[\tau]C_\tau + C_r$$

$$+\frac{b_1^{a_1} b_2^{a_2}}{\Gamma(n_1)\Gamma(n_2)\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty (g(\lambda_1,\lambda_2) - C_r)$$
$$\times \int_0^\infty \int_{\xi(n_1,n_2,x)}^\infty \lambda_1^{n_1+a_1-1} \lambda_2^{n_2+a_2-1} x^{n_1-1} y^{n_2-1} e^{-\lambda_1(x+b_1)} e^{-\lambda_2(y+b_2)}\, dy\, dx\, d\lambda_1\, d\lambda_2$$

Then the optimal Bayesian sampling plan $(n_B, n_{1B}, n_{2B}, \delta_B)$ is a sampling plan, such that

$$R^{\delta_B}(n_B, n_{1B}, n_{2B}) = \inf_{(n,n_1,n_2)\in C} R^{\delta_B}(n, n_1, n_2)$$

where $C = \{(n, n_1, n_2) : n_1 + n_2 < n\}$. Now we will present an algorithm for searching the optimal Bayesian sampling plan. From Eq. (4), we find that the Bayes risk $R^\delta(n, n_1, n_2)$ is composed of the risk due to the sampling parameters (n, n_1, n_2), and the risk due to the decision function δ. For each fixed (n, n_1, n_2), Bayes risk $R^\delta(n, n_1, n_2)$ is a function of δ. Also in the loss function, we have costs on total time of the experiment and on sample size. Therefore, optimal values of n, n_1, and n_2, are automatically bounded above.

Algorithm A

Algorithm for finding $(n_B, n_{1B}, n_{2B}, \delta_B)$

1. For each fixed parameter (n, n_1, n_2), derive the alternative form of Bayes decision function δ_B to minimize the Bayes risks $R^\delta(n, n_1, n_2)$ among the class of all decision functions.

2. For each fixed parameter (n, n_1), find an integer $n_{2B}(n, n_1)$, $0 \le n_{2B}(n, n_1) < n - n_1$ such that

$$R^{\delta_B}(n, n_1, n_{2B}(n, n_1)) = \min_{0 \le n_2 < n-n_1} R^{\delta_B}(n, n_1, n_2).$$

3. For each fixed integer n, find an integer $n_{1B}(n) < n$ such that

$$R^{\delta_B}(n, n_{1B}(n), n_{2B}(n, n_{1B}(n))) = \min_{n_1 < n} R^{\delta_B}(n, n_1, n_{2B}(n, n_1)).$$

4. Find an integer $n_B \ge 0$ such that

$$R^{\delta_B}(n_B, n_{1B}, n_{2B}) = \min_{n \ge 0} R^{\delta_B}(n, n_{1B}(n), n_{2B}(n, n_{1B}(n))),$$

   where $n_{1B} = n_{1B}(n_B)$, $n_{2B} = n_{2B}(n_B, n_{1B}(n_B))$, and $\delta_B = \delta_B(.|n_B, n_{1B}(n_B), n_{2B}(n_B, n_{1B}(n_B)))$. Therefore, $(n_B, n_{1B}, n_{2B}, \delta_B)$ is the optimal Bayesian sampling plan.

Theorem 3.2. *Let n_B, n_{1B} and n_{2B} be the optimal values of n, n_1 and n_2, respectively. Then,*

$$n_B \le \min\left\{\frac{E_{\lambda_1,\lambda_2}[g(\lambda_1,\lambda_2)]}{C_s - r_s}, \frac{C_r}{C_s - r_s}\right\},$$

$0 \le n_{1B} < n_B$, *and* $0 \le n_{2B} < n_B$.

Proof. See Appendix. ∎

4 Bayes decision function and Bayes risk for quadratic loss function

The form of the acceptance cost is difficult to define only we know that $g(\lambda_1,\lambda_2)$ is positive and increasing function of λ_1 and λ_2. Any function of interest can be approximated by a polynomial. A polynomial loss function has been widely used as an approximation of an exact cost function when a batch is accepted. See Lam (1990, 1994) and Lam and Choy (1995) for details. In this section, a quadratic acceptance cost $g(\lambda_1,\lambda_2) = c_0 + c_1\lambda_1 + c_2\lambda_1^2 + c_3\lambda_2 + c_4\lambda_2^2 + c_5\lambda_1\lambda_2$, is applied, where $c_i > 0$ for all $i = 0,1,\ldots,5$. It is clear that if the mean lifetime of the items θ_1 in stress level s_1 and θ_2 in stress level s_2 increases the $g(\lambda_1,\lambda_2)$ decreases. The posterior expectation of $g(\lambda_1,\lambda_2)$ is given by

$$\begin{aligned}\phi_\pi(t_1,t_2) &= E_{\lambda_1,\lambda_2|t_1,t_2}[c_0 + c_1\lambda_1 + c_2\lambda_1^2 + c_3\lambda_2 + c_4\lambda_2^2 + c_5\lambda_1\lambda_2]\\ &= c_0 + c_1\frac{n_1+a_1}{t_1+b_1} + c_2\frac{(n_1+a_1)(n_1+a_1+1)}{(t_1+b_1)^2} + c_3\frac{n_2+a_2}{t_2+b_2}\\ &\qquad + c_4\frac{(n_2+a_2)(n_2+a_2+1)}{(t_2+b_2)^2} + c_5\frac{(n_1+a_1)(n_2+a_2)}{(t_1+b_1)(t_2+b_2)}.\end{aligned}$$

So to find the closed form of Bayes decision function we need to obtain the set of $t_1 \ge 0$ and $t_2 \ge 0$, such that

$$\begin{aligned}c_0 + c_1\frac{n_1+a_1}{t_1+b_1} + c_2\frac{(n_1+a_1)(n_1+a_1+1)}{(t_1+b_1)^2} + c_3\frac{n_2+a_2}{t_2+b_2}&\\ + c_4\frac{(n_2+a_2)(n_2+a_2+1)}{(t_2+b_2)^2} + c_5\frac{(n_1+a_1)(n_2+a_2)}{(t_1+b_1)(t_2+b_2)} \le C_r& \qquad (10)\end{aligned}$$

which is equivalent to find set of $t_1 \geq 0$ and $t_2 \geq 0$, such that

$$\begin{aligned} h(t_1,t_2) &= (C_r - c_0) - c_1 \frac{n_1+a_1}{t_1+b_1} - c_2 \frac{(n_1+a_1)(n_1+a_1+1)}{(t_1+b_1)^2} - c_3 \frac{n_2+a_2}{t_2+b_2} \\ &\quad - c_4 \frac{(n_2+a_2)(n_2+a_2+1)}{(t_2+b_2)^2} - c_5 \frac{(n_1+a_1)(n_2+a_2)}{(t_1+b_1)(t_2+b_2)} \geq 0, \end{aligned}$$

for which we have to solve the equation $h(t_1,t_2) = 0$. To do this we fixed $t_1 \geq 0$ and written

$$h(t_1,t_2) = A(n_1,t_1)(t_2+b_2)^2 - B(n_2,t_1)(t_2+b_2) - C(n_2,t_1) = 0, \tag{11}$$

where

$$\begin{aligned} A(n_1,t_1) &= (C_r - c_0)(t_1+b_1)^2 - c_1(n_1+a_1)(t_1+b_1) - c_2(n_1+a_1)(n_1+a_1+1) \\ B(n_2,t_1) &= c_3(n_2+a_2)(t_1+b_1)^2 + c_5(n_1+a_1)(n_2+a_2)(t_1+b_1) \\ C(n_2,t_1) &= c_4(n_2+a_2)(n_2+a_2+1)(t_1+b_1)^2, \end{aligned}$$

now solving the Eq. (11) w.r.t. t_2 we will get

$$\xi(n_1,n_2,t_1) = \frac{B(n_2,t_1) + \sqrt{B(n_2,t_1)^2 + 4A(n_1,t_1)C(n_2,t_1)}}{2A(n_1,t_1)} - b_2.$$

provided $A(n_1,t_1) \geq 0$. Therefore, the set of $t_1 \geq 0$ and $t_2 \geq 0$ which satisfy the Eq. 10 is equivalent to the set $S = \{(t_1,t_2) : t_1 \geq 0, t_2 \geq \xi(n_1,n_2,t_1)\}$.

The Bayes risk for the quadratic loss function is given by

$$\begin{aligned} R(n,n_1,n_2) &= nC_s - (n-n_1-n_2)r_s + E[V(\widehat{\theta}_1)]C_{v_1} + E[V(\widehat{\theta}_2)]C_{v_2} + E[\tau]C_\tau + C_r \\ &\quad + \sum_{l=0}^{5} \frac{\Gamma(n_1+a_1+p_l)\Gamma(a_2+q_l)C_l}{b_1^{p_l} b_2^{q_l} \Gamma(n_1)\Gamma(a_1)\Gamma(a_2)} \int_0^1 u^{n_1-1}(1-u)^{a_1+p_l-1}[1 - I_{S(u)}(n_2,a_2+q_l)]du \end{aligned}$$

where $S(u) = \frac{\xi(n_1,n_2,ub_1/(1-u))}{b_2+\xi(n_1,n_2,ub_1/(1-u))}$, $I_x(a,b)$ is distribution function of beta random variable,

$$\begin{aligned} E[V(\widehat{\theta}_1)] &= \frac{b_1^2}{n_1(a_1-1)(a_1-2)} \\ E[V(\widehat{\theta}_2)] &= \frac{b_2^2}{n_2(a_2-1)(a_2-2)} \\ E[\tau] &= \frac{n!}{(n-n_1-n_2)!(n_1-1)!(n_2-1)!} \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} \frac{\binom{n_1-1}{i}\binom{n_2-1}{j}(-1)^{i+j}}{(j+n-n_1-n_2+1)(i+n-n_1+1)} \end{aligned}$$

$$\times\left[\frac{b_2}{(a_2-1)(j+n-n_1-n_2+1)}+\frac{b_1}{(a_1-1)(i+n-n_1+1)}\right]$$

and $(p_0,q_0)=(0,0)$, $(p_1,q_1)=(1,0)$, $(p_2,q_2)=(2,0)$, $(p_3,q_3)=(0,1)$, $(p_4,q_4)=(0,2)$ and $(p_5,q_5)=(1,1)$. The derivation of the Bayes risk is given in the Appendix.

5 ORDER RESTRICTED BAYESIAN SAMPLING PLAN

In this section, we assume that the mean lifetime under the stress level s_2 is smaller than the mean lifetime at the stress level s_1. If the lifetimes of the experimental units at stress levels s_1 and s_2 are independently distributed having exponential distribution with scale parameters λ_1 and λ_2, then $\lambda_1<\lambda_2$. First, it is assumed that $\lambda=\lambda_1<\lambda_2=k\lambda$, where $k>1$, and it is known. Based on the CEM assumption, the likelihood function can be written as

$$L(\lambda|data)=\frac{n!}{(n-n_1-n_2)!}k^{n_2}\lambda^{n_1+n_2}e^{-\lambda Y(T_1,T_2)};\quad\forall\ 1\le n_1+n_2<n,\ \ n_1>0,n_2>0, \tag{12}$$

where $Y(T_1,T_2)=T_1+kT_2$. From the likelihood function (12), the MLE of $\theta=1/\lambda$ is

$$\widehat{\theta}=\frac{Y(T_1,T_2)}{n_1+n_2};\ \ \forall\ \ 1\le n_1+n_2<n,\ n_1>0,n_2>0. \tag{13}$$

From Theorem 2.1, the exact distribution of $\widehat{\theta}$ can be easily obtained, and it is clear that $\widehat{\theta}$ is an unbiased estimator of θ. The following loss function has been used here:

$$L(\delta(\mathbf{x}),\lambda)=\begin{cases} nC_s-(n-n_1-n_2)r_s+V(\widehat{\theta})C_v+\tau C_\tau+g(\lambda), & \delta(\mathbf{x})=1\\ nC_s-(n-n_1-n_2)r_s+V(\widehat{\theta})C_v+\tau C_\tau+C_r, & \delta(\mathbf{x})=0, \end{cases} \tag{14}$$

where $g(\lambda)=a_0+a_2\lambda+a_2\lambda^2$. The prior distribution of λ is given by

$$\pi(\lambda|a,b)=\frac{\lambda}{\Gamma(a)}\lambda^{a-1}e^{-\lambda b},\ \ \lambda>0,\ a>0,\ b>0. \tag{15}$$

Therefore, for the given loss function (14) and the prior distribution (15) the Bayes risk is given by

$$\begin{aligned} R^{\delta}(n,n_1,n_2) &= nC_s-(n-n_1-n_2)r_s+E[V(\widehat{\theta})]C_v+E[\tau]C_\tau\\ &\qquad +E(g(\lambda))+E[(C_r-g(\lambda))(1-\delta(\mathbf{X}))] \end{aligned}$$

$$= \quad nC_s - (n - n_1 - n_2)r_s + E[V(\widehat{\theta})]C_v + E[\tau]C_\tau$$
$$+E(g(\lambda)) + E_{\mathbf{X}}[(1-\delta(\mathbf{X}))(C_r - \phi_\pi(Y(T_1,T_2)))] \qquad (16)$$

where $\phi_\pi(Y(T_1,T_2)) = E_{\lambda|\mathbf{X}}[g(\lambda)]$. Therefore, for a fixed n, n_1, n_2, and for given $T_1 = t_1$ and $T_2 = t_2$, the Bayes decision function is given by

$$\delta_B(t_1,t_2|n,n_1,n_2) = \begin{cases} 1, & \phi_\pi(y(t_1,t_2)) \le C_r \\ 0, & \text{otherwise.} \end{cases}$$

Following discussions analogous to that of Lemmas 2.1 and 2.2 of Liang and Yang (2013), one can show that for a fixed n, n_1, n_2, the $\phi_\pi(y(t_1,t_2))$ is strictly decreasing in $y(t_1,t_2)$ for $y(t_1,t_2) > 0$. Let $S = \{y > 0 | \phi_\pi(y) \le C_r\}$. Define

$$\xi(t_1,t_2) = \begin{cases} \inf\{S\}, & \text{if } S \text{ is non-empty} \\ \infty, & \text{otherwise.} \end{cases}$$

Then the Bayes decision function can be written as

$$\delta_B(T_1,T_2|n,n_1,n_2) = \begin{cases} 1, & Y(T_1,T_2) = T_1 + kT_2 > \xi(n_1,n_2) \\ 0, & \text{otherwise} \end{cases} \qquad (17)$$

which can be written as

$$\delta_B(T_1,T_2|n,n_1,n_2) = \begin{cases} 1, & \widehat{\theta} > \frac{\xi(n_1,n_2)}{n_1+n_2} \\ 0, & \text{otherwise.} \end{cases} \qquad (18)$$

If $g(\lambda) = a_0 + a_1\lambda + a_2\lambda^2$ then the bound $\xi(n_1,n_2)$ is obtained by obtaining the set S which collect all values of $y > 0$ such that

$$a_0 + a_1\frac{(n_1+n_2+a)}{(y+b)} + a_2\frac{(n_1+n_2+a)(n_1+n_2+a+1)}{(y+b)^2} \le C_r$$

which is equivalent to

$$y+b \ge \frac{a_1(n_1+n_2+a) + \sqrt{a_1^2(n_1+n_2+a)^2 + 4(C_r-a_0)a_2(n_1+n_2+a)(n_1+n_2+a+1)}}{2(C_r-a_0)}.$$

Therefore,

$$\xi(n_1,n_2)=\max\left\{\frac{a_1(n_1+n_2+a)+\sqrt{a_1^2(n_1+n_2+a)^2+4(C_r-a_0)a_2(n_1+n_2+a)(n_1+n_2+a+1)}}{2(C_r-a_0)}-b,0\right\},$$

then the Bayes risk is given by

$$\begin{aligned}
R^{\delta_B}(n,n_1,n_2) &= nC_s-(n-n_1-n_2)r_s+E[V(\widehat{\theta})]C_v+E[\tau]C_\tau+a_0+a_1\frac{a}{b}+a_2\frac{a(a+1)}{b^2}\\
&\quad+\sum_{l=0}^{2}\frac{b^a}{\Gamma a}C_l\int_0^\infty\lambda^{a+l-1}e^{-\lambda b}P\big(\widehat{\theta}<\xi(n_1,n_2)/n_1+n_2\big)d\lambda\\
&= nC_s-(n-n_1-n_2)r_s+E[V(\widehat{\theta})]C_v+E[\tau]C_\tau+a_0+a_1\frac{a}{b}+a_2\frac{a(a+1)}{b^2}\\
&\quad+\sum_{l=0}^{2}\frac{b^a(n_1+n_1)^{n_1+n_2}}{\Gamma a\Gamma(n_1+n_2)}C_l\int_0^\infty\int_0^{\frac{\xi(n_1,n_2)}{n_1+n_2}}\lambda^{n_1+n_2+a+l-1}x^{n_1+n_2-1}e^{-\lambda(b+(n_1+n_2)x)}dx\,d\lambda\\
&= nC_s-(n-n_1-n_2)r_s+E[V(\widehat{\theta})]C_v+E[\tau]C_\tau+a_0+a_1\frac{a}{b}+a_2\frac{a(a+1)}{b^2}\\
&\qquad+\sum_{l=0}^{2}\frac{\Gamma(a+l)}{b^l\Gamma a}C_lI_S(n_1+n_2,a+l) \qquad (19)
\end{aligned}$$

where $S=\frac{\xi(n_1,n_2)/b}{1+\xi(n_1,n_2)/b}$, $I_x(a,b)$ is distribution function of beta random variable and

$$\begin{aligned}
E[V(\widehat{\theta})] &= \frac{b^2}{(n_1+n_2)(a-1)(a-2)}\\
E[\tau] &= \frac{n!}{(n-n_1-n_2)!(n_1-1)!(n_2-1)!}\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}\frac{\binom{n_1-1}{i}\binom{n_2-1}{j}(-1)^{i+j}}{(j+n-n_1-n_2+1)(i+n-n_1+1)}\\
&\quad\times\frac{b}{a-1}\left[\frac{1}{k(j+n-n_1-n_2+1)}+\frac{1}{(i+n-n_1+1)}\right]\\
C_l &= \begin{cases} C_r-a_0 & \text{if } l=0\\ -a_l & \text{if } l=1,2.\end{cases}
\end{aligned}$$

Now to obtained the optimal Bayesian sampling plan $(n_B,n_{1B},n_{2B},\delta_B)$ we have to minimize Bayes risk (19), which can be done using Algorithm A given in Section 3.2.

Now we will discuss the case when $\lambda=\lambda_1<\lambda_2=k\lambda$, where $k>1$, is not known. In this case the closed form of the Bayes decision function cannot be obtained. Keeping in mind the order restriction on the scale parameters $(\lambda_1<\lambda_2)$, the following prior assumption on (λ_1,λ_2) is assumed;

$$\pi(\lambda_1,\lambda_2) = \mathscr{C}\frac{\Gamma(a_1+a_2)}{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)}b_0^{a_0}\lambda^{a_0-a_1-a_2}e^{-b_0\lambda}(\lambda_1^{a_1-1}\lambda_2^{a_2-1}+\lambda_1^{a_2-1}\lambda_2^{a_1-1});\ \ 0<\lambda_1<\lambda_2<\infty,$$

where $\lambda = \lambda_1+\lambda_2$ and $\mathscr{C}$ is the normalizing constant. This is the ordered Beta-Gamma PDF and we will denote it by $OBG(a_0,b_0,a_1,a_2)$. It is quite straightforward to generate a sample from an ordered Beta-Gamma distribution. At first, we generate a sample from a Beta-Gamma distribution and then by sorting, we obtain a random sample from an ordered Beta-Gamma distribution. Next, for the loss function (3) and for Bayes decision function (5), the Bayes risk can be written as

$$\begin{aligned} R^{\delta}(n,n_1,n_2) &= nC_s-(n-n_1-n_2)r_s+E[V(\widehat{\theta}_1)]C_{v_1}+E[V(\widehat{\theta}_2)]C_{v_2}+E[\tau]C_\tau \\ &\quad +C_r+E_{\lambda_1,\lambda_2}[(g(\lambda_1,\lambda_2)-C_r)P(\phi_\pi(T_1,T_2)<C_r)] \\ &= nC_s-(n-n_1-n_2)r_s+E[V(\widehat{\theta}_1)]C_{v_1}+E[V(\widehat{\theta}_2)]C_{v_2}+E[\tau]C_\tau \\ &\quad +C_r+R_1(\delta_B|n,n_1,n_2). \end{aligned} \tag{20}$$

The exact expression of Bayes risk (20) cannot be obtained as $E[V(\widehat{\theta}_1)]$, $E[V(\widehat{\theta}_2)]$, $E(\tau)$, and $R_1(\delta_B|n,n_1,n_2)$ are not in closed forms. Therefore, we have to go for the simulation approach. We will use the algorithm to obtain the Bayes risk.

Algorithm B

Step 1 Generate $(\lambda_1,\ \lambda_2)$ from $\pi(\lambda_1,\lambda_2)$.

Step 2 Given n, n_1 and n_2 and given a generated (λ_1,λ_2), generate the data $\mathbf{X}$ under step-stress from exponential population, $Exp(\lambda_1)$ and $Exp(\lambda_2)$, compute (T_1,T_2) M times, say $\{(T_1^1,T_2^1), \ldots,(T_1^M,T_2^M)\}$.

Step 3 To obtain probability $P(\phi_\pi(T_1,T_2)<C_r)$, compute $(\phi_1,\ldots,\phi_M)$, where $\phi_i = \phi_\pi(T_1^i,T_2^i)$ given in (6), and probability can be approximated as $\dfrac{\sum_{i=1}^M \mathrm{I}(\phi_i<C_r)}{M}$, where $\mathrm{I}(A)$ is indicator function on set A, if A occurs then $\mathrm{I}(A)=1$, otherwise $\mathrm{I}(A)=0$.

Step 4 Use $(g(\lambda_1,\lambda_2)-C_r)\dfrac{\sum_{i=1}^M \mathrm{I}(\phi_i<C_r)}{M}$ to yield the simulated value of $R_1(\delta_B|n,n_1,n_2)$ and also simulate $E(\tau)$ by $\dfrac{\sum_{i=1}^M X^i_{(n_1+n_2)}}{M}$.

Step 5 Repeat Steps 1-4, N times. Denote the simulated values $R_1(\delta_B|n,n_1,n_2)$ by R_l and $E(\tau)$ by T_l, $l=1,2,\ldots,N$.

Step 6 Compute the $\frac{\sum_{l=1}^{N} R_l}{N}$ and $\frac{\sum_{l=1}^{N} T_l}{N}$, to obtain the estimated $R_1(\delta_B|n,n_1,n_2)$ and $E(\tau)$, respectively. Also simulate $E[V(\widehat{\theta}_1)]$ by $\frac{1}{Nn_1}\sum_{l=1}^{N}\frac{1}{\lambda_1^l}$ and $E[V(\widehat{\theta}_2)]$ by $\frac{1}{Nn_2}\sum_{l=1}^{N}\frac{1}{\lambda_2^l}$.

Using this simulation approach we can obtain the simulated Bayes risk and by Algorithm A the optimal Bayesian sampling plan $(n_B, n_{1B}, n_{2B}, \delta_B)$ can be obtained.

6 Numerical Studies

In this section, we present the optimal Bayesian sampling plan under SSALT with random stress-change time and for illustrative purpose, we consider the following set of hyper parameters, costs and coefficients, $a_1 = 2.5,\ b_1 = 2.5,\ a_2 = 2.2,\ b_2 = 1.0,\ c_0 = 2,\ c_1 = 3,\ c_2 = 2,\ c_3 = 6,\ c_4 = 8,\ c_5 = 2\ C_s = 1.0,\ r_s = 0.5,\ C_\tau = 1.5, C_{v_1} = 0.5, C_{v_2} = 0.1$, and $C_r = 75$ in the loss function. In Tables 1-6 the optimal Bayesian sampling plan and the optimal Bayes risk are obtained by varying the one hyperparameter, cost or coefficient one at a time and keeping others fixed. In Table 1, we have obtained the optimal Bayesian sampling plan by varying the hyperparameter or the inspection cost C_s one at a time and keeping others fixed. We can see the optimal Bayesian sampling plan is quite sensitive to the hyperparameter and when the inspection cost C_s increases the optimal value of n decreases. Therefore, if the inspection cost is high then we need more sample to get on the correct decision. In Table 2, when the cost on time C_τ increases optimal Bayes risk is increasing and the optimal value of n_1 decreases which indicate that we are going to observe less number of failures in the experiment.

Table 1: Minimum Bayes risks and their optimal Bayesian sampling plans for different values of a_1, b_1, a_2, b_2 and C_s when $c_0 = 2,\ c_1 = 3,\ c_2 = 2,\ c_3 = 6,\ c_4 = 8,\ c_5 = 2\ C_s = 1.0,\ r_s = 0.5,\ C_\tau = 1.5, C_{v_1} = 0.5, C_{v_2} = 0.1$, and $C_r = 75$

a_1	b_1	a_2	b_2	C_s	R_B	n_B	n_{1B}	n_{2B}	a_1	b_1	a_2	b_2	C_s	R_B	n_B	n_{1B}	n_{2B}
2.5	2.5	2.2	1.0		68.8141	10	2	6					0.7	65.6293	12	2	7
2.5	3.0	2.2	1.0		68.9119	10	2	6					0.8	66.7858	11	2	6
2.5	2.5	2.5	1.0	1.0	73.0308	8	2	5	2.5	2.5	2.2	1.0	1.0	68.8141	10	2	6
2.5	2.5	2.2	0.8		73.4249	8	2	5					1.5	72.7426	7	2	4
2.8	2.5	2.2	1.0		68.2306	7	1	5					1.7	74.1426	7	2	4
2.2	2.5	2.5	1.1		74.0471	11	3	6					1.8	74.8073	6	1	4

Table 2: Minimum Bayes risks and their optimal Bayesian sampling plans for different values of C_τ and r_s when $a_1 = 2.5,\ b_1 = 2.5,\ a_2 = 2.2,\ b_2 = 1.0,\ c_0 = 2,\ c_1 = 3,\ c_2 = 2,\ c_3 = 6,\ c_4 = 8, c_5 = 2$ $C_s = 1.0, C_{v_1} = 0.5, C_{v_2} = 0.1$, and $C_r = 75$

C_τ	r_s	R_B	n_B	n_{1B}	n_{2B}	r_s	C_τ	R_B	n_B	n_{1B}	n_{2B}
0.1		66.4360	9	2	6	0.1		69.2453	9	2	6
0.3		66.7803	9	2	6	0.3		69.0453	9	2	6
0.5	0.5	67.1244	9	2	6	0.5	1.5	68.8141	10	2	6
0.8		67.6407	9	2	6	0.6		68.6141	10	2	6
1.5		68.8141	10	2	6	0.7		68.3858	11	2	6
3.0		70.6341	10	2	5	0.8		68.0566	12	2	6

The salvage value r_s increases the optimal value of the n increases but optimal values of n_1 and n_2 are fixed. It indicate that sample size increases but the number of failure is fixed, i.e., if salvage value is high then more number of items does not fail in the experiments to increase the profit. In Table 3, we vary the rejection cost C_r and we observed that as C_r increases the optimal Bayes risk is increasing. The optimal value of the n and n_2 are increasing which indicate that we

Table 3: Minimum Bayes risks and their optimal Bayesian sampling plans for different values of C_r and c_0 when $a_1 = 2.5,\ b_1 = 2.5,\ a_2 = 2.2,\ b_2 = 1.0,\ c_1 = 3,\ c_2 = 2,\ c_3 = 6,\ c_4 = 8,\ c_5 = 2$ $C_s = 1.0,\ r_s = 0.5$, $C_{v_1} = 0.5$, $C_{v_2} = 0.1$, and $C_\tau = 1.5$

C_r	c_0	R_B	n_B	n_{1B}	n_{2B}	c_0	C_r	R_B	n_B	n_{1B}	n_{2B}
72		67.2805	9	2	5	0.1		67.6229	10	2	6
75		68.8141	10	2	6	0.3		68.8141	10	2	6
80	2.0	71.1599	10	2	6	1.0	75	68.2927	10	2	6
85		73.3926	10	2	6	2.0		68.8141	10	2	6
90		75.4666	11	2	7	3.0		69.3140	9	2	5
95		77.3911	11	2	7	6.0		70.7564	9	2	5

cannot afford to reject the batch of the item if rejection cost is high. Also in Table 3 we vary c_0 keeping others fixed, for small value of c_0 there is no effect on the optimal Bayesian sampling plan but when the c_0 is high, it has an effect on the optimal Bayesian sampling plan. In Tables

Table 4: Minimum Bayes risks and their optimal Bayesian sampling plans for different values of c_1 and c_2 when $a_1 = 2.5,\ b_1 = 2.5,\ a_2 = 2.2,\ b_2 = 1.0,\ c_3 = 6,\ c_4 = 8,\ c_5 = 2\ C_s = 1.0,\ r_s = 0.5$, $C_{v_1} = 0.5$, $C_{v_2} = 0.1$, and $C_\tau = 1.5$

c_1	c_2	R_B	n_B	n_{1B}	n_{2B}	c_2	c_1	R_B	n_B	n_{1B}	n_{2B}
1.0		67.7821	10	2	6	0.5		67.7399	10	2	6
1.5		68.0423	10	2	6	0.8		67.9572	10	2	6
2.0	2.0	68.3009	10	2	6	1.0	3.0	68.1015	10	2	6
2.5		68.5582	10	2	6	1.5		68.4595	10	2	6
3.0		68.8141	10	2	6	2.0		68.8141	10	2	6
4.0		69.3052	9	2	5	2.2		68.9549	10	2	6

4-5, we vary the coefficient c_1, c_2, c_3 and c_4 of the acceptance cost. We observed that there is no effect on the optimal Bayesian sampling plan for small values of c_1 or c_2 but the optimal Bayes

Table 5: Minimum Bayes risks and their optimal Bayesian sampling plans for different values of c_3 and c_4 when $a_1 = 2.5,\ b_1 = 2.5,\ a_2 = 2.2,\ b_2 = 1.0,\ c_1 = 3,\ c_2 = 2,\ c_5 = 2\ C_s = 1.0,\ r_s = 0.5$, $C_{v_1} = 0.5$, $C_{v_2} = 0.1$, and $C_\tau = 1.5$

c_3	c_4	R^{δ_B}	n_B	n_{1B}	n_{2B}	c_4	c_3	R^{δ_B}	n_B	n_{1B}	n_{2B}
1.0		65.0208	10	2	6	6.5		66.6984	10	2	6
3.0		66.6189	10	2	6	7.0		67.4593	10	2	6
4.0	8.0	67.3769	10	2	6	8.0	6.0	68.8141	10	2	6
6.0		68.8141	10	2	6	10.0		70.9339	9	2	5
10.0		71.2633	9	2	5	12.0		72.5756	9	2	5
15.0		73.8339	9	2	5	15.0		74.4637	9	2	5

risk increases as c_1 or c_2 increases. Therefore, the optimal Bayesian sampling plan is robust for small value of c_1 and c_2. Wherein when c_3 or c_4 vary the optimal Bayesian sampling plan changes and the optimal Bayes is increasing as c_3 or c_4 increases. It is also observed that the coefficients c_3 and c_4 have more effect on optimal Bayesian sampling plan compare to the coefficients c_1 and c_2. Therefore, coefficient of acceptance cost under stress level s_2 effect the optimal Bayesian sampling plan more in comparison of acceptance cost under the stress level s_1. In Table 6 we vary the cost on the variance C_{v_1} and C_{v_2} and obtained the optimal Bayesian sampling plan and optimal Bayes risk. We observed that as C_{v_1} increases optimal value of n and n_1 increases but the optimal value on n_2 is fixed. Which indicate that more number of failures are observed in stress level s_1 if the cost on variance of the mean lifetime θ_1 is high. Similarly, when the C_{v_2} increases the optimal value of n and n_2 increases but the optimal value on n_1 is fixed. Which indicate that more number of failures are observed in stress level s_2 if the cost on variance of the mean lifetime θ_2 is high. Cost C_{v_1} has effect the optimal Bayesian sampling plan more in comparison to cost C_{v_2}, which indicate that we should choose high cost on variance of the θ_1 to insure the less variability in optimal decision. In Table 7, we vary the coefficient c_5 to obtain the optimal Bayesian sampling plan. As c_5 increases the optimal value of n and n_2 decreases and the optimal Bayes risk increases. Which indicate that the coefficient c_5 has a significant effect on the optimal Bayesian sampling plan.

Table 6: Minimum Bayes risks and their optimal Bayesian sampling plans for different values of C_{v_1} and C_{v_2} when $a_1 = 2.5,\ b_1 = 2.5,\ a_2 = 2.2,\ b_2 = 1.0,\ c_1 = 3,\ c_2 = 2,\ c_3 = 6,\ c_4 = 8,\ c_5 = 2$ $C_s = 1.0,\ r_s = 0.5$, and $C_\tau = 1.5$

C_{v_1}	C_{v_2}	R^{δ_B}	n_B	n_{1B}	n_{2B}	C_{v_2}	C_{v_1}	R^{δ_B}	n_B	n_{1B}	n_{2B}
0.1		66.3282	7	1	5	0.1		68.8141	10	2	6
0.3		67.9807	10	2	6	0.4		69.0224	10	2	6
0.5	0.1	68.8141	10	2	6	0.8	0.5	69.3002	10	2	6
0.8		70.0641	10	2	6	1.5		69.7863	10	2	6
1.0		70.7197	11	3	6	2.5		70.4807	10	2	6
2.0		73.3045	12	4	6	5.5		72.2769	11	2	7

Table 7: Minimum Bayes risks and their optimal Bayesian sampling plans for different values of c_5 when $a_1 = 2.5,\ b_1 = 2.5,\ a_2 = 2.2,\ b_2 = 1.0,\ c_0 = 2,\ c_1 = 3,\ c_2 = 2,\ c_3 = 6,\ c_4 = 8,$ $C_s = 1.0,\ r_s = 0.5, C_{v_1} = 0.5, C_{v_2} = 0.1$, and $C_\tau = 1.5$

c_5	R^{δ_B}	n_B	n_{1B}	n_{2B}
1.0	68.1267	10	2	6
2.0	68.8141	10	2	6
3.0	69.4476	9	2	5
4.0	70.0455	9	2	5
5.0	70.6183	9	2	5
7.0	71.6924	9	2	5

6.1 Order Restricted Case

In this section, we present the order restricted optimal Bayesian sampling plan under SSALT with random stress change time and for illustration purpose, we considered the following set of hyper-parameters, costs and coefficients, $a = 2.5,\ b = 0.8, C_s = 0.5,\ r_s = 0.3,\ C_r = 75,\ C_v = 3.0,\ a_0 =$

6, $a_1 = 3$, $a_2 = 8$ in the loss function. In Table 8 we considered the values of $k = 2,4$ and obtained the optimal Bayesian sampling plan by varying the hyperparameters a and b for fixed costs and coefficients. It is observed that as k increases from 2 to 4 for fixed a and b the Bayes risk of optimal Bayesian sampling plan decreases. For example, when $a = 2.5$ and 0.8 then for $k = 2$ the Bayes risk R^{δ_B} is 64.6218 and for $k = 4$ the Bayes risk R^{δ_B} is 64.2119. In Tables 9 and 10 the optimal Bayesian sampling plan is obtained for $a = 2.5$ and $b = 0.8$ and varying the costs and coefficients with $k = 2,4$. In this case also as the value of k increases, the Bayes risk decreases. In all the cases, more failures are observed under the stress level s_2 than the stress level s_1, which indicates that step-stress plays an important role in deciding on the batch of items when the mean lifetime of the items are high. It may be seen that when k is known $n_{1B} = 1$ in all the cases. It is not surprising. Since k is known the information on λ can be obtained from both the stress levels. On the other choosing $n_{1B} = 1$, the lower bound, the experimental time becomes minimum. Since there is a cost associated with the experimental time, the optimal Bayesian sampling plan always chooses $n_{1B} = 1$.

Table 8: Minimum Bayes risks and their optimal Bayesian sampling plans for different values of a and b when $a_0 = 6$, $a_1 = 3$, $a_2 = 8$, $C_s = 0.5$, $r_s = 0.3$, $C_r = 75$, $C_v = 3.0$, $C_\tau = 2.0$, and $C_\tau = 2.0$

k	a	b	R^{δ_B}	n_B	n_{1B}	n_{2B}	k	a	b	R^{δ_B}	n_B	n_{1B}	n_{2B}
2.0	3.5	0.8	73.8107	5	1	3	4.0	3.5	0.8	73.6289	6	1	4
	3.0	0.8	70.2559	7	1	5		3.0	0.8	69.9659	7	1	5
	2.5	0.8	64.6218	9	1	6		2.5	0.8	64.2119	8	1	6
	2.5	1.0	58.3371	9	1	6		2.5	1.0	57.8746	8	1	6
	2.5	1.2	52.2415	9	1	6		2.5	1.2	51.7266	8	1	6
	4.0	2.0	51.7188	8	1	5		4.0	2.0	51.2977	7	1	5

Table 9: Minimum Bayes risks and optimal Bayesian sampling plan when C_s, r_s, C_τ, C_v, a_0, a_1, a_2 vary with hyper parameters $a = 2.5$, $b = 0.8$ and $k = 2$.

k	C_s	C_τ	C_v	C_r	r_s	a_0	a_1	a_2	R^{δ_B}	n_B	n_{1B}	n_{2B}
2.0	0.5								64.6218	9	1	6
	1.0	2.0	3.0	75	0.3	6.0	3.0	8.0	67.8278	5	1	3
	1.5								70.2262	4	1	2
		0.5							63.8995	8	1	6
	0.5	1.0	3.0	75	0.3	6.0	3.0	8.0	64.1452	8	1	6
		2.0							64.6218	9	1	6
			1.0						64.3742	8	1	5
	0.5	2.0	3.0	75	0.3	6.0	3.0	8.0	64.6218	9	1	6
			9.0						65.3409	10	1	7
				50					48.3556	7	1	4
	0.5	2.0	3.0	75	0.3	6.0	3.0	8.0	64.6218	9	1	6
				100					77.3313	10	1	7
					0.1				64.8367	8	1	6
	0.5	2.0	3.0	75	0.3	6.0	3.0	8.0	64.6218	9	1	6
					0.4				64.3915	10	1	6
						2.0			62.8638	9	1	6
	0.5	2.0	3.0	75	0.3	6.0	3.0	8.0	64.6218	9	1	6
						8.0			65.4715	9	1	6
							3.0		64.6218	9	1	6
	0.5	2.0	3.0	75	0.3	6.0	6.0	8.0	66.6239	8	1	5
							8.0		67.7821	8	1	5
								4.0	54.8751	9	1	6
	0.5	2.0	3.0	75	0.3	6.0	3.0	8.0	64.6218	9	1	6
								10.0	67.1597	8	1	5

Table 10: Minimum Bayes risks and optimal Bayesian sampling plan when C_s, r_s, C_τ, C_v, a_0, a_1, a_2 vary with hyper parameters $a = 2.5$, $b = 0.8$, and $k = 4$.

k	C_s	C_τ	C_v	C_r	r_s	a_0	a_1	a_2	R^{δ_B}	n_B	n_{1B}	n_{2B}
4.0	0.5								64.2119	8	1	6
	1.0	2.0	3.0	75	0.3	6.0	3.0	8.0	67.5255	6	1	4
	1.5								70.0039	4	1	2
		0.5							63.7933	8	1	6
	0.5	1.0	3.0	75	0.3	6.0	3.0	8.0	63.9328	8	1	6
		2.0							64.2119	8	1	6
			1.0						63.9681	8	1	6
	0.5	2.0	3.0	75	0.3	6.0	3.0	8.0	64.2119	8	1	6
			9.0						64.9021	9	1	7
				50					48.0165	6	1	4
	0.5	2.0	3.0	75	0.3	6.0	3.0	8.0	64.2119	8	1	6
				100					76.8723	10	1	8
					0.1				64.4119	8	1	6
	0.5	2.0	3.0	75	0.3	6.0	3.0	8.0	64.2119	8	1	6
					0.4				64.0971	9	1	6
						2.0			62.4538	8	1	6
	0.5	2.0	3.0	75	0.3	6.0	3.0	8.0	64.2119	8	1	6
						8.0			65.0616	8	1	6
							3.0		64.2119	8	1	6
	0.5	2.0	3.0	75	0.3	6.0	6.0	8.0	66.2239	8	1	6
							8.0		67.4050	7	1	5
								4.0	54.4651	8	1	6
	0.5	2.0	3.0	75	0.3	6.0	3.0	8.0	64.2119	8	1	6
								10.0	66.7502	8	1	6

One natural question is how to choose the prior and also the associated hyperpapemeters. Since we have assumed that the lifetime distribution is exponential, then the choice of the gamma prior is quite natural as it provides the conjugate prior. Extensive work has been done in this respect. On the other hand the choice of the hyperparameters is not very straight forward. It is recommended that one should use some prior or expert's knowledge in this respect. We suggest if we have any such information we should use it, otherwise we may perform some pilot study to get some prior knowledge about the hyperparameters, and that can be used to find the optimal sampling plans.

7 Conclusion

This paper studies the problem of designing Bayesian sampling plan based on SSALT in which the stress changing time is random, and the data are subject to Type-II censoring. We first derive the conventional Bayesian sampling plan by constructing the Bayes decision function. The monotonicity of the Bayes decision function δ_B is investigated. This monotonicity enables us to generate alternative form of the Bayes decision function. It is shown that the Bayes decision function minimizes the Bayes risk of the optimal Bayesian sampling plan. Numerical results indicate that the optimal Bayesian sampling plan is sensitive to the choice of the hyperparameters. The cost C_s, C_τ, C_r, C_{v_1}, and C_{v_2} have significant effect on the optimal Bayesian sampling plan as expected. When the coefficient of acceptance cost is small then they do not have significant effect on the optimal Bayesian sampling plan. In this paper we have mainly dealt with exponential lifetime distribution. It will be interesting to generalize it for other lifetime distributions also. More work is needed in that direction.

Acknowledgements:

The authors would like to thank the Associate Editor and two unknown reviewers for their constructive suggestions which have helped to improve the earlier version of the paper significantly.

Appendix

Proof of Lemma 3.2

Proof. (1) Note that for a fixed t_2 and $t_1 \neq t_1^*$, $\pi(\lambda_2|t_1,t_2) = \pi(\lambda_2|t_1^*,t_2)$, which implies that $\psi_\pi(\lambda_1,t_1,t_2) = \psi_\pi(\lambda_1,t_1^*,t_2)$.

(2) For a fixed t_1, and $t_2^* > t_2$,

$$\psi_\pi(\lambda_1,t_1,t_2^*) - \psi_\pi(\lambda_1,t_1,t_2) = \int g(\lambda_1,\lambda_2)[\pi(\lambda_2|t_1,t_2^*) - \pi(\lambda_2|t_1,t_2)]d\lambda_2$$

Define a set E and F such that

$$E = \{\lambda_2 : \frac{\pi(\lambda_2|t_1,t_2^*)}{\pi(\lambda_2|t_1,t_2)} > 1\}$$

$$F = \{\lambda_2 : \frac{\pi(\lambda_2|t_1,t_2^*)}{\pi(\lambda_2|t_1,t_2)} \leq 1\}$$

Then by Lemma 3.1, if $a = \sup_E\{g(\lambda_1,\lambda_2)\}$ and $b = \inf_F\{g(\lambda_1,\lambda_2)\}$, then $b - a > 0$ and

$$\begin{aligned}
\psi_\pi(\lambda_1,t_1,t_2^*) - \psi_\pi(\lambda_1,t_1,t_2) &= \int_E g(\lambda_1,\lambda_2)[\pi(\lambda_2|t_1,t_2^*) - \pi(\lambda_2|t_1,t_2)]d\lambda_2 \\
&\qquad + \int_F g(\lambda_1,\lambda_2)[\pi(\lambda_2|t_1,t_2^*) - \pi(\lambda_2|t_1,t_2)]d\lambda_2 \\
&\leq a\int_E [\pi(\lambda_2|t_1,t_2^*) - \pi(\lambda_2|t_1,t_2)]d\lambda_2 \\
&\qquad + b\int_F [\pi(\lambda_2|t_1,t_2^*) - \pi(\lambda_2|t_1,t_2)]d\lambda_2 \\
&= -(b-a)\int_E [\pi(\lambda_2|t_1,t_2^*) - \pi(\lambda_2|t_1,t_2)]d\lambda_2 < 0
\end{aligned}$$

This implies that

$$\psi_\pi(\lambda_1,t_1,t_2^*) < \psi_\pi(\lambda_1,t_1,t_2) \tag{21}$$

Hence for a fixed t_1 and $\lambda_1 > 0$, $\psi_\pi(\lambda_1,t_1,t_2)$ is strictly decreasing in t_2. ∎

PROOF OF THEOREM 3.1

Proof. Let $t_1^* > t_1$ and $t_2^* > t_2$, then consider

$$\phi_\pi(t_1^*,t_2^*) - \phi_\pi(t_1,t_2) = \int \psi_\pi(\lambda_1,t_1^*,t_2^*)\pi(\lambda_1|t_1^*,t_2^*)d\lambda_1 - \int \psi_\pi(\lambda_1,t_1,t_2)\pi(\lambda_1|t_1,t_2)d\lambda_1$$

By Lemma 3.2, for a fixed t_2^* and $t_1 \neq t_1^*$, $\psi_\pi(\lambda_1,t_1^*,t_2^*) = \psi_\pi(\lambda_1,t_1,t_2^*)$ and then for a fixed t_1, $\psi_\pi(\lambda_1,t_1,t_2)$ is decreasing in t_2.

$$\begin{aligned}
\phi_\pi(t_1^*,t_2^*) - \phi_\pi(t_1,t_2) &= \int \psi_\pi(\lambda_1,t_1,t_2^*)\pi(\lambda_1|t_1^*,t_2^*)d\lambda_1 - \int \psi_\pi(\lambda_1,t_1,t_2)\pi(\lambda_1|t_1,t_2)d\lambda_1 \\
&\leq \int \psi_\pi(\lambda_1,t_1,t_2)\pi(\lambda_1|t_1^*,t_2^*)d\lambda_1 - \int \psi_\pi(\lambda_1,t_1,t_2)\pi(\lambda_1|t_1,t_2)d\lambda_1 \\
&= \int \psi_\pi(\lambda_1,t_1,t_2)[\pi(\lambda_1|t_1^*,t_2^*) - \pi(\lambda_1|t_1,t_2)]d\lambda_1 \\
&= \int \psi_\pi(\lambda_1,t_1,t_2)[\pi(\lambda_1|t_1^*,t_2) - \pi(\lambda_1|t_1,t_2)]d\lambda_1,
\end{aligned}$$

last equality is due to the fixed t_1^* and $t_2^* \neq t_2$, $\pi(\lambda_1|t_1^*,t_2^*) = \pi(\lambda_1|t_1^*,t_2)$. Let us define a set A and B such that

$$A = \{\lambda_1 : \frac{\pi(\lambda_1|t_1^*,t_2)}{\pi(\lambda_1|t_1,t_2)} > 1\}$$

$$B = \{\lambda_1 : \frac{\pi(\lambda_1|t_1^*,t_2)}{\pi(\lambda_1|t_1,t_2)} \leq 1\}$$

Then by Lemma 3.1, if $u = \sup_A\{\psi_\pi(\lambda_1,t_1,t_2)\}$ and $v = \inf_B\{\psi_\pi(\lambda_1,t_1,t_2)\}$, then $v - u > 0$ and

$$\begin{aligned}
\phi_\pi(t_1^*,t_2^*) - \phi_\pi(t_1,t_2) &\leq \int \psi_\pi(\lambda_1,t_1,t_2)[\pi(\lambda_1|t_1^*,t_2) - \pi(\lambda_1|t_1,t_2)]d\lambda_1 \\
&= \int_A \psi_\pi(\lambda_1,t_1,t_2)[\pi(\lambda_1|t_1^*,t_2) - \pi(\lambda_1|t_1,t_2)]d\lambda_1 \\
&\qquad + \int_B \psi_\pi(\lambda_1,t_1,t_2)[\pi(\lambda_1|t_1^*,t_2) - \pi(\lambda_1|t_1,t_2)]d\lambda_1 \\
&\leq u\int_A [\pi(\lambda_1|t_1^*,t_2) - \pi(\lambda_1|t_1,t_2)]d\lambda_1 \\
&\qquad + v\int_B [\pi(\lambda_1|t_1^*,t_2) - \pi(\lambda_1|t_1,t_2)]d\lambda_1 \\
&= -(v-u)\int_A [\pi(\lambda_1|t_1^*,t_2) - \pi(\lambda_1|t_1,t_2)]d\lambda_1 < 0
\end{aligned}$$

which implies that

$$\phi_\pi(t_1^*, t_2^*) < \phi_\pi(t_1, t_2) \tag{22}$$

Hence $\phi_\pi(t_1, t_2)$ is strictly decreasing in t_1 and t_2. ∎

EXPECTED DURATION TIME OF THE EXPERIMENT

$$E(X_{n_1+n_2}|\lambda_1, \lambda_2) = \int_0^\infty y f_{X_{n_1+n_2}}(y|\lambda_1, \lambda_2)dy \tag{23}$$

where

$$\begin{aligned}
f_{X_{n_1+n_2}}(y|\lambda_1,\lambda_2) &= \frac{n!\lambda_1^{n_1}\lambda_2^{n_2}}{(n-n_1-n_2)!}\int_0^y \left[\int_0^{x_{n_1}}\cdots\int_0^{x_2} e^{-\lambda_1[\sum_{i=1}^{n_1}x_i+(n-n_1)x_{n_1}]}dx_1\ldots dx_{n_1-1}\right] \\
&\times \left[\int_{x_{n_1}}^{y}\cdots\int_{x_{n_1}}^{x_{n_1+2}} e^{-\lambda_2[\sum_{i=n_1+1}^{n_1+n_2-1}(x_i-x_{n_1})+(n-n_1-n_2+1)(y-x_{n_1})]}dx_{n_1+1}\ldots dx_{n_1+n_2-1}\right]dx_{n_1} \\
&= \frac{n!\lambda_1^{n_1}\lambda_2^{n_2}}{(n-n_1-n_2)!}\int_0^y \frac{\left[1-e^{-\lambda_1 x_{n_1}}\right]^{n_1-1}}{(n_1-1)!\lambda_1^{n_1-1}}\frac{\left[1-e^{-\lambda_2(y-x_{n_1})}\right]^{n_2-1}}{(n_2-1)!\lambda_2^{n_2-1}}e^{-\lambda_1(n-n_1+1)x_{n_1}} \\
&\qquad \times e^{-\lambda_2(n-n_1-n_2+1)(y-x_{n_1})}dx_{n_1} \\
&= \frac{n!\lambda_1\lambda_2}{(n-n_1-n_2)!(n_1-1)!(n_2-1)!}\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}\binom{n_1-1}{i}\binom{n_2-1}{j}(-1)^{i+j} \\
&\times \int_0^y e^{-i\lambda_1 x_{n_1}}e^{-j\lambda_2(y-x_{n_1})}e^{-\lambda_1(n-n_1+1)x_{n_1}}e^{-\lambda_2(n-n_1-n_2+1)(y-x_{n_1})}dx_{n_1} \\
&= \frac{n!\lambda_1\lambda_2}{(n-n_1-n_2)!(n_1-1)!(n_2-1)!}\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}\binom{n_1-1}{i}\binom{n_2-1}{j}(-1)^{i+j} \\
&\times e^{-(j+n-n_1-n_2+1)\lambda_2 y}\int_0^y e^{-[(i+n-n_1+1)\lambda_1-(j+n-n_1-n_2+1)\lambda_2]x_{n_1}}dx_{n_1} \\
&= \frac{n!\lambda_1\lambda_2}{(n-n_1-n_2)!(n_1-1)!(n_2-1)!}\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}\binom{n_1-1}{i}\binom{n_2-1}{j}(-1)^{i+j} \\
&\times \frac{1-e^{-[(i+n-n_1+1)\lambda_1-(j+n-n_1-n_2+1)\lambda_2]y}}{[(i+n-n_1+1)\lambda_1-(j+n-n_1-n_2+1)\lambda_2]}e^{-(j+n-n_1-n_2+1)\lambda_2 y} \\
&= \frac{n!\lambda_1\lambda_2}{(n-n_1-n_2)!(n_1-1)!(n_2-1)!}\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}\binom{n_1-1}{i}\binom{n_2-1}{j}(-1)^{i+j} \\
&\qquad \times \frac{e^{-(j+n-n_1-n_2+1)\lambda_2 y}-e^{-(i+n-n_1+1)\lambda_1 y}}{[(i+n-n_1+1)\lambda_1-(j+n-n_1-n_2+1)\lambda_2]}
\end{aligned}$$

Then the expectation is

$$
\begin{aligned}
E(X_{n_1+n_2}|\lambda_1,\lambda_2) &= \frac{n!\lambda_1\lambda_2}{(n-n_1-n_2)!(n_1-1)!(n_2-1)!}\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}\frac{\binom{n_1-1}{i}\binom{n_2-1}{j}(-1)^{i+j}}{[(i+n-n_1+1)\lambda_1-(j+n-n_1-n_2+1)\lambda_2]} \\
&\quad\times\left[\frac{1}{\lambda_2^2(j+n-n_1-n_2+1)^2}-\frac{1}{\lambda_1^2(i+n-n_1+1)^2}\right] \\
&= \frac{n!\lambda_1\lambda_2}{(n-n_1-n_2)!(n_1-1)!(n_2-1)!}\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}\frac{\binom{n_1-1}{i}\binom{n_2-1}{j}(-1)^{i+j}}{[(i+n-n_1+1)\lambda_1-(j+n-n_1-n_2+1)\lambda_2]} \\
&\quad\times\left[\frac{\lambda_1^2(i+n-n_1+1)^2-\lambda_2^2(j+n-n_1-n_2+1)^2}{\lambda_2^2(j+n-n_1-n_2+1)^2\lambda_1^2(i+n-n_1+1)^2}\right] \\
&= \frac{n!}{(n-n_1-n_2)!(n_1-1)!(n_2-1)!}\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}\binom{n_1-1}{i}\binom{n_2-1}{j}(-1)^{i+j} \\
&\quad\times\left[\frac{\lambda_1(i+n-n_1+1)+\lambda_2(j+n-n_1-n_2+1)}{\lambda_2(j+n-n_1-n_2+1)^2\lambda_1(i+n-n_1+1)^2}\right] \\
&= \frac{n!}{(n-n_1-n_2)!(n_1-1)!(n_2-1)!}\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}\frac{\binom{n_1-1}{i}\binom{n_2-1}{j}(-1)^{i+j}}{(j+n-n_1-n_2+1)(i+n-n_1+1)} \\
&\quad\times\left[\frac{1}{\lambda_2(j+n-n_1-n_2+1)}+\frac{1}{\lambda_1(i+n-n_1+1)}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
E(X_{n_1+n_2}) &= \frac{n!}{(n-n_1-n_2)!(n_1-1)!(n_2-1)!}\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}\frac{\binom{n_1-1}{i}\binom{n_2-1}{j}(-1)^{i+j}}{(j+n-n_1-n_2+1)(i+n-n_1+1)} \\
&\times \int_0^\infty\int_0^\infty\left[\frac{1}{\lambda_2(j+n-n_1-n_2+1)}+\frac{1}{\lambda_1(i+n-n_1+1)}\right]\frac{b_1^{a_1}b_2^{a_2}}{\Gamma a_1\Gamma a_2}\lambda_1^{a_1-1}\lambda_2^{a_2-1}e^{-\lambda_1 b_1-\lambda_2 b_2}d\lambda_1 d\lambda_2 \\
&= \frac{n!}{(n-n_1-n_2)!(n_1-1)!(n_2-1)!}\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}\frac{\binom{n_1-1}{i}\binom{n_2-1}{j}(-1)^{i+j}}{(j+n-n_1-n_2+1)(i+n-n_1+1)} \\
&\quad\times\left[\frac{b_2}{(a_2-1)(j+n-n_1-n_2+1)}+\frac{b_1}{(a_1-1)(i+n-n_1+1)}\right]
\end{aligned}
$$

BAYES RISK DERIVATION

The Bayes risk is obtained for the loss function given in Eq. (3) with the acceptance cost $g(\lambda_1,\lambda_2)=c_0+c_1\lambda_1+c_2\lambda_1^2+c_3\lambda_2+c_4\lambda_2^2+c_5\lambda_1\lambda_2$, and the Bayes decision function given in Eq. (9),

$$
\begin{aligned}
R^{\delta_B}(n,n_1,n_2) &= nC_s-(n-n_1-n_2)r_s+E[V(\widehat{\theta}_1)]C_{v_1}+E[V(\widehat{\theta}_2)]C_{v_2}+E[\tau]C_\tau+C_r \\
&\qquad +E_{(\lambda_1,\lambda_2)}\Big[(g(\lambda_1,\lambda_2)-C_r)P(T_1>0,T_2>\xi(n_1,n_2,T_1))\Big]
\end{aligned}
$$

$$\begin{aligned}
&= nC_s-(n-n_1-n_2)r_s+E[V(\widehat{\theta}_1)]C_{v_1}+E[V(\widehat{\theta}_2)]C_{v_2}+E[\tau]C_\tau+C_r\\
&\qquad +\sum_{l=0}^{5}C_lE_{(\lambda_1,\lambda_2)}\left[\lambda_1^{p_l}\lambda_2^{q_l}P\Big(\widehat{\theta}_1>0,\widehat{\theta}_2>\frac{\xi(n_1,n_2,T_1)}{n_2}\Big)\right]\\
&= nC_s-(n-n_1-n_2)r_s+E[V(\widehat{\theta}_1)]C_{v_1}+E[V(\widehat{\theta}_2)]C_{v_2}+E[\tau]C_\tau+C_r\\
&+ \sum_{l=0}^{5}C_l\frac{b_1^{a_1}b_2^{a_2}}{\Gamma a_1\Gamma a_2}\int_0^\infty\int_0^\infty \lambda_1^{a_1+p_l-1}\lambda_2^{a_2+p_l-1}e^{-\lambda_1 b_1}e^{-\lambda_2 b_2}P\Big(\widehat{\theta}_1>0,\widehat{\theta}_2>\tfrac{\xi(n_1,n_2,T_1)}{n_2}\Big)\,d\lambda_1\,d\lambda_2\\
&= nC_s-(n-n_1-n_2)r_s+E[V(\widehat{\theta}_1)]C_{v_1}+E[V(\widehat{\theta}_2)]C_{v_2}+E[\tau]C_\tau+C_r\\
&+\sum_{l=0}^{5}C_l\frac{b_1^{a_1}b_2^{a_2}}{\Gamma n_1\Gamma n_2\Gamma a_1\Gamma a_2}\int_0^\infty\int_0^\infty\int_0^\infty\int_{\xi(n_1,n_2,x)}^\infty \lambda_1^{n_1+a_1+p_l-1}\lambda_2^{n_2+a_2+q_l-1}\\
&\qquad\times x^{n_1-1}y^{n_2-1}e^{-\lambda_1(x+b_1)}e^{-\lambda_2(y+b_2)}\,dy\,dx\,d\lambda_1\,d\lambda_2\\
&= nC_s-(n-n_1-n_2)r_s+E[V(\widehat{\theta}_1)]C_{v_1}+E[V(\widehat{\theta}_2)]C_{v_2}+E[\tau]C_\tau+C_r\\
&+\sum_{l=0}^{5}\frac{\Gamma(n_1+a_1+p_l)\Gamma(n_2+a_2+q_l)b_1^{a_1}b_2^{a_2}C_l}{\Gamma n_1\Gamma n_2\Gamma a_1\Gamma a_2}\\
&\qquad\times\int_0^\infty\int_{\xi(n_1,n_2,x)}^\infty\frac{x^{n_1-1}y^{n_2-1}}{(x+b_1)^{n_1+a_1+p_l}(y+b_2)^{n_2+a_2+q_l}}\,dy\,dx\\
&= nC_s-(n-n_1-n_2)r_s+E[V(\widehat{\theta}_1)]C_{v_1}+E[V(\widehat{\theta}_2)]C_{v_2}+E[\tau]C_\tau+C_r\\
&+\sum_{l=0}^{5}\frac{\Gamma(n_1+a_1+p_l)\Gamma(n_2+a_2+q_l)C_l}{\Gamma n_1\Gamma n_2\Gamma a_1\Gamma a_2 b_1^{n_1+p_l}b_2^{n_2+q_l}}\\
&\qquad\times\int_0^\infty\int_{\xi(n_1,n_2,x)}^\infty\frac{x^{n_1-1}y^{n_2-1}}{(1+x/b_1)^{n_1+a_1+p_l}(1+y/b_2)^{n_2+a_2+q_l}}\,dy\,dx
\end{aligned}$$

Let $x/b_1=u/(1-u)$ and $y/b_2=v/(1-v)$, then the Bayes risk can be written as

$$\begin{aligned}
R^{\delta_B}(n,n_1,n_2) &= nC_s-(n-n_1-n_2)r_s+E[V(\widehat{\theta}_1)]C_{v_1}+E[V(\widehat{\theta}_2)]C_{v_2}+E[\tau]C_\tau+C_r\\
&+\sum_{l=0}^{5}\frac{\Gamma(n_1+a_1+p_l)\Gamma(n_2+a_2+q_l)C_l}{b_1^{p_l}b_2^{q_l}\Gamma n_1\Gamma n_2\Gamma a_1\Gamma a_2}\\
&\times\int_0^1 u^{n_1-1}(1-u)^{a_1+p_l-1}\int_{\frac{\xi(n_1,n_2,ub_1/(1-u))}{b_2+\xi(n_1,n_2,ub_1/(1-u))}}^1 v^{n_2-1}(1-v)^{a_2+q_l-1}dvdu\\
&= nC_s-(n-n_1-n_2)r_s+E[V(\widehat{\theta}_1)]C_{v_1}+E[V(\widehat{\theta}_2)]C_{v_2}+E[\tau]C_\tau+C_r\\
&+\sum_{l=0}^{5}\frac{\Gamma(n_1+a_1+p_l)\Gamma(a_2+q_l)C_l}{b_1^{p_l}b_2^{q_l}\Gamma n_1\Gamma a_1\Gamma a_2}\int_0^1 u^{n_1-1}(1-u)^{a_1+p_l-1}[1-I_{S(u)}(n_2,a_2+q_l)]du
\end{aligned}$$

where $S(u)=\frac{\xi(n_1,n_2,ub_1/(1-u))}{b_2+\xi(n_1,n_2,ub_1/(1-u))}$, $I_x(a,b)$ is distribution function of beta random variable and

$$E[V(\widehat{\theta}_1)] = \frac{b_1^2}{n_1(a_1-1)(a_1-2)}$$

$$
\begin{aligned}
E[V(\widehat{\theta}_2)] &= \frac{b_2^2}{n_2(a_2-1)(a_2-2)} \\
E[\tau] &= \frac{n!}{(n-n_1-n_2)!(n_1-1)!(n_2-1)!}\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}\frac{\binom{n_1-1}{i}\binom{n_2-1}{j}(-1)^{i+j}}{(j+n-n_1-n_2+1)(i+n-n_1+1)} \\
&\quad \times\left[\frac{b_2}{(a_2-1)(j+n-n_1-n_2+1)}+\frac{b_1}{(a_1-1)(i+n-n_1+1)}\right],
\end{aligned}
$$

where $E[V(\widehat{\theta}_1)]$, $E[V(\widehat{\theta}_1)]$ exists for $a_1 > 2$ and $a_2 > 2$ and $E[\tau]$ exist when $a_1 > 1$ and $a_2 > 1$.

Proof of Theorem 3.2

Let $(n_B, n_{1B}, n_{2B}, \delta_B)$ be the optimal Bayesian sampling plan, then minimum Bayes risk is given by $R^{\delta_B}(n_B, n_{1B}, n_{2B})$. From Eq. 4 we know that $E[V(\widehat{\theta}_1)] \geq 0$, $E[V(\widehat{\theta}_2)] \geq 0$, $E[\tau] \geq 0$, and risk of δ_B, which is $E(g(\lambda_1,\lambda_2)) + E_{\mathbf{X}}[(1-\delta_B(\mathbf{X}))(C_r - \phi_\pi(T_1,T_2))] \geq 0$ therefore it is clear that

$$
R^{\delta_B}(n_B, n_{1B}, n_{2B}) \geq n_B(C_s - r_s). \tag{24}
$$

provided $C_s > r_s \geq 0$. Let $(0,0,0,\delta_B = 0)$ denote the sampling plan, when we reject the batch without sampling with Bayes risk $R^{\delta_B=0}(0,0,0) = C_r$ and $(0,0,0,\delta_B = 1)$ denote the sampling plan when we accept the batch without sampling with Bayes risk $R^{\delta_B=1}(0,0,0) = E_{\lambda_1,\lambda_2}[g(\lambda_1,\lambda_2)]$. Then the Bayes risk of optimal Bayesian sampling plan:

$$
R^{\delta_B}(n_B, n_{1B}, n_{2B}) \leq \min\left\{E_{\lambda_1,\lambda_2}[g(\lambda_1,\lambda_2)], C_r\right\}. \tag{25}
$$

Now from (24) and (25), it follows that

$$
n_B(C_s - r_s) \leq \min\left\{E_{\lambda_1,\lambda_2}[g(\lambda_1,\lambda_2)], C_r\right\},
$$

which implies that

$$
n_B \leq \min\left\{\frac{E_{\lambda_1,\lambda_2}[g(\lambda_1,\lambda_2)]}{C_s - r_s}, \frac{C_r}{C_s - r_s}\right\},
$$

and since $n_{1B} + n_{2B} < n_B$ which implies that $0 \leq n_{1B} < n_B$ and $0 \leq n_{2B} < n_B$.

References

Bagdonavicius, V. B. and Nikulin, M., “Accelerated life models: modeling and statistical analysis ”, *Chapman and Hall CRC Press, Boca Raton, Florida, 2002*.

Balakrishnan, N. and Kundu, D., “Hybrid censoring: models, inferential results and applications”, *Computational Statistics and Data Analysis, vol. 57, 166-209, 2013*.

Balakrishnan, N., Kundu, D., Ng, H. K. T., and Kannan, N., “Point and interval estimation for a simple step-stress model with Type-II censoring”, *Journal of Quality Technology, vol. 9, 35–47, 2007*.

Balakrishnan, N., Beutner, E., and Kateri, M., “Order restricted inference for exponential step-stress models”, *IEEE Transactions on Reliability, vol. 58, 132–142, 2009*.

Bhattacharya, R., Pradhan, B., and Dewanji, A., “Optimum life testing plans in presence of hybrid censoring: a cost function approach”, *Applied Stochastic Models in Business and Industry, vol. 30, 519-528, 2014*.

Chen, J., Chou, W., Wu, H., and Zhou, H., “Designing acceptance sampling schemes for life testing with mixed censoring”, *Naval Research Logistics, vol. 51, 597–612, 2004*.

Chen, L. S., Yang, M. C., and Liang T., “Bayesian sampling plans for exponential distributions with interval censored samples”, *Naval Research Logistics, vol. 62, 604–616, 2015*.

Chen, L. S., Yang, M. C., and Liang T., “Curtailed Bayesian sampling plans for exponential distributions based on Type-II censored samples”, *Journal of Statistical Computation and Simulation, vol. 87, 1160-1178, 2017*.

Ganguly, A., Kundu, D. and Mitra, S., “Bayesian analysis of a simple step-stress model under Weibull lifetimes”, *IEEE Transactions on Reliability, vol. 64(1), 473-485, 2014.*.

Herstein, I. N., “Topics in Algebra”, *John Wiley & Sons, New York, 1975*.

Kundu, D. and Ganguly, A., “Analysis of Step-Stress Model: Existing Methods and Recent Developments ”, *Elsevier/Academic Press, Singapore, 2017*.

Kundu, D. and Balakrishnan, N., “Point and Interval Estimation for a Simple Step-Stress Model with Random Stress-Change Time”, *Journal of Probability and Statistical Science, vol. 7, 113-126, 2009*.

Lam, Y., “An optimal single variable sampling plan with censoring”, *The Statistician, vol. 39, 53–66, 1990*.

Lam, Y., “Bayesian variable sampling plans for the exponential distribution with Type-I censoring”, *The Annals of Statistics, vol. 22, 696–711, 1994*.

Lam, Y. and Choy, S. T. B., “Bayesian variable sampling plans for the exponential distribution with uniformly distributed random censoring”, *Journal of Statistical Planning and Inference, vol. 47, 277–293, 1995*.

Liang, T. and Yang, M. C., “Optimal Bayesian sampling plans for exponential distributions based on hybrid censored samples”, *Journal of Statistical Computation and Simulation, vol. 83, 922–940, 2013*.

Yang, M. C., Chen, L. S., and Liang, T., “Optimal Bayesian variable sampling plans for exponential distributions based on modified type-II hybrid censored samples”, *Communications in Statistics-Simulation and Computation, vol. 46, 4722-4744 , 2017*.

Lin, C. T., Huang, Y., and Balakrishnan, N., “Exact Bayesian variable sampling plans for the exponential distribution based on Type-I and Type-II hybrid censored samples”, *Communications in Statistics Simulation and Computation, vol. 37, 1101–1116, 2008*.

Lin, C. T., Huang, Y., and Balakrishnan, N., “Corrections on Exact Bayesian variable sampling plans for the exponential distribution based on type-I and type-II hybrid censored samples”, *Communications in Statistics: Simulation and Computation, vol. 39, 1499–1505, 2010*.

Lin, Y., Liang, T., and Huang, W., “Bayesian sampling plans for exponential distribution based on type-I censoring data”, *Annals of the Institute of Statistical Mathematics, vol. 54, 100–113, 2002*.

Mitra, S., Ganguly, A., Samanta, D., and Kundu, D., “On simple step-stress model for two-parameter exponential distribution.”, *Statistical Methodology, vol. 15, 95–114, 2013*.

Nelson, W. B., “Accelerated life testing: step-stress models and data analysis ”, *IEEE Transactions on Reliability, vol. 29, 103–108, 1980*.

Nelson, W., “Accelerated Testing: Statistical Models, Test Plans, and Data Analysis, ”, *John Wiley & Sons, New York, USA*.

Prajapati, D., Mitra, S. and Kundu, D., “Bayesian sampling plan for the exponential distribution with generalized Type-II hybrid censoring scheme ”, *Communications in Statistics-Simulation and Computation, 1-32,* `https://doi.org/10.1080/03610918.2020.1861293`.

Samanta, D. and Kundu, D., "Order restricted inference of a multiple step-stress model ", *Computational Statistics and Data Analysis, vol. 117, 62–75, 2018*.

Tsai, T. R., Chiang, J. Y., Liang, T., and Yang, M. C., "Efficient Bayesian sampling plans for exponential distributions with type-I censored samples", *Journal of Statistical Computation and Simulation, vol. 84, 964–981, 2014*.

Xiong, C. and Milliken, G. A., "Step-stress life-testing with random stress-change times for exponential data", *IEEE Transactions on Reliability, vol. 48, 141-148, 1999*.

Xiong, C., Zhu, K., and Ji, M., "Analysis of a simple step-stress life test with a random stress-change time", *IEEE Transactions on Reliability, vol. 55, 67-74, 2006*.