ARTICLE TEMPLATE

Estimation of the Stress-Strength Parameter under Two-Sample Balanced Progressive Censoring Scheme

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ABSTRACT
In this paper, we obtain the stress-strength reliability estimation under balanced joint Type-II progressive censoring (BJPC) scheme for independent samples from two different populations. We simultaneously place two independent samples where the experimental units follow Weibull distributions with common shape parameter β and different scale parameters α, λ, respectively for both populations. The maximum likelihood estimators of the unknown parameters are derived. The asymptotic confidence intervals are also constructed. Further, the Bayesian inference associated with any parametric function of the model parameters is considered using Lindley's approximation method. We also obtain the Bayes estimators and associated credible intervals of the unknown parameters using Gibbs sampling method. Extensive simulations are performed to see the effectiveness of the proposed estimation methods. Further, we derive the optimal censoring scheme in the Bayesian framework by using the variable neighborhood search (VNS) method proposed by Bhattacharya et. al [8]. In addition, the likelihood ratio test is provided for testing the equivalence of different shape parameters. Further, some simulation schemes are provided to compare the performances of the estimations under the jointly censored samples versus two separate censored samples.

KEYWORDS
Bayesian Inference; Joint Censoring Scheme; Maximum likelihood; Optimal censoring Scheme; Stress-strength Reliability; Weibull Distribution

1. Introduction

The stress-strength (SS) reliability model has been widely investigated in reliability and statistics theory. This model was first introduced by Birnbaum [9] and developed by Birnbaum and McCarty [10]. The reliability of the stress–strength model is defined as $\theta = P(X > Y)$, where X and Y are two independent random variables. Let random variables X and Y be the amount of strength of the system component and the amount of stress on a system component, respectively. Therefore, if the stress exceeds the

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strength ($i.e. X \leq Y$), the system component will fail and the probability of not failing of the system component (i.e. the parameter of θ) is considered a measure of system reliability.

The SS reliability model has wide applications in different areas of scientific studies such as engineering, physics, quality control, clinical trial and medicine. For example; fatigue failure of aircraft structures, static fatigue of ceramic components, deterioration of rocket motors, aging of concrete pressure vessels and etc. In a clinical experiment, let Y be the response of the control group and X be the response of the treatment group. In such an experiment, $P(X > Y)$ is a measure of the effectiveness of the treatment. Let X denote the maximum chamber pressure generated by the ignition of a solid propellant and Y be the strength of the rocket chamber. Then, $P(X > Y)$ is a measurement of a successful firing on the engine [21]. In engineering problems, numerous examples of the usage of the SS models can be found in the literature. For instance, Bhasin and Grimstad [6] handled the SS reliability model for the assessment of tunnel stability. Quin and Wedira [38] handled the SS model for the determination of safety factors for pressure vessel design. Li et al. [26] considered a discrete SS model for maintenance service outsourcing. Recently, Zhou et al. [48] used the SS model the creep of basalt fiber–reinforced alkali-activated slag concrete. For more examples and different applications the readers are referred to the monograph by Kotz et al. [23]

In the literature, stress-strength reliability takes part in many studies. Based on the different probability distributions, many authors studied the stress-strength reliability under the complete samples. For examples of recent studies; Pak et al. [36] studied the stress-strength reliability problem based on Weibull record data in the presence of inter-record times, Çetinkaya and Genç [13] studied the SS reliability under the two-sided power distribution. Sharma et al. [42] studied it based on the inverse Lindley distribution with application to head and neck cancer data. Kundu and Gupta [24] handled it under the generalized exponential distribution and for more examples, one can see the references cited therein. Further, Turkkan et al. [45] handled the stress-strength model in the multivariate case, Singh [43] considered multivariate normal populations under the stress-strength model, Cramer [17] studied it based on the Weinman multivariate exponential samples Rezaeai [40] studied it for the multivariate SGPII distribution.

Due to substantial improvement in science and technology in recent times, in any life testing experiment, it may not be possible to observe the failure times of all the experimental units. In real life, due to substantial improvements in science and technology, we can not get the adequate number of failure times of some units which are put on the life testing experiment. Thus, experimenters deal with the censored data sets. The most basic censoring schemes available in the literature are Type-I, Type-II and Hybrid censoring schemes. But in all of these censoring schemes, the experimenter can not remove any experimental unit during the experiment. Whereas, the progressive censoring scheme allows the withdrawal of experimental units during the experiment. Also, the experimenter can use the removed items for further studies. Due to this removal flexibility of the items during the test, the progressive censoring scheme has become very popular in literature.

The SS reliability studies are handled by various authors under censored samples. The following references can be seen as examples of SS reliability studies under different censoring schemes. Maurya and Tripathi [30] studied multi-component SS reliability for Burr XII distribution under progressive censoring. Çetinkaya [12] handled multi-component SS reliability with generalized progressive hybrid censored Weibull components. Asadi and Panahi [3] studied the stress-strength model under censored data and its evaluation for coating processes, Arshad et al. [2] studied the stress-strength reliability model with p-type non-identical multicomponent from the proportional reversed hazard rate family based on records, Nagy and Hassan [35] handled multicomponent stress-strength model for the generalized inverted exponential model based on ranked set sampling, Saini et al. [41] studied SS model under the generalized Maxwell failure distribution under progressive first failure censoring. Kohansal et al. [22] studied Type-II hybrid progressive censored samples for a Kumaraswamy distribution. Bai et al. [5] handled SS reliability estimation for finite mixture distributions under progressively interval censoring.

On the other hand, in parallel to development in reliability studies, comparative lifetime experiments are needed to study to save experimental costs and accelerate the execution of the experiments. Hence, recent studies lead for the jointly censoring schemes since the experimenters may need to decide under conducting comparative life tests of experimental units. Therefore, a joint censoring scheme is quite useful in conducting comparative lifetime test of products coming from different units within the same facility. For instance, some products can be produced by two different production lines within the same factory and they can be needed to be placed simultaneously on a life-testing experiment due to some constraints such as cost or time. For two samples, Rasouli and Balakrishnan [39] introduced the joint progressive Type-II censoring. For example, comparative lifetimes of two groups in insulating fluids subjected to high voltage stress, comparative lifetimes of the air conditioning system of a fleet of 13 Boeing 720 jet airplanes (See Rasouli and Balakrishnan [39]), comparative lifetimes of the two fabric weaving in the same factory and etc. Further, Mondal and Kundu [31] proposed balanced joint Progressive Type-II censoring (BJPC) scheme for exponential populations. The author have compared the estimators based on their proposed joint censoring with the joint progressive Type-II censoring. In general, the joint censoring scheme provides more efficient estimators with smaller experimental time. In particular, we have compared the BJPC scheme with the single sample Progressive Type-II censoring. We observed that the BJPC scheme provides more efficient estimators than progressive Type-II censoring scheme with lesser experimental time. However, we may need to know the reliability of stress-strength problems when the components are jointly censored. One may need jointly censored stress and strength variables. For instance, let's consider the same factory has simultaneously two fabrics weaving in production and we want to calculate the probability that the production performances of one fabric exceed the performance of the other fabric. In spite of the fact that some inferential studies based on the joint censoring schemes take part in the literature, the first study of the stress-strength model under jointly Type-II censoring scheme was being studied by Çetinkaya [12]. This study deals with the jointly censored component strengths under Type-II censoring. However, a stress-strength model has never been studied when both X and Y are exposed to a jointly censoring scheme. Further in the literature, there is no work related to optimal censoring schemes for stress-strength reliability estimation. However, we have obtained the optimal censoring scheme in the Bayesian framework with the help of some standard neighborhood search algorithm. The expected running time of the experiment we have calculated and we have chosen the objective function as the weighted average of Variance of the estimator of stress-strength reliability parameter and total time of the test. This optimality will provide an optimal progressive censoring scheme which will be useful for the experimenter to finish the test in an affordable time. Finally, we also make a comparison under the jointly censored samples and without joint censored sample i.e., single censored sample based on progressive censoring scheme for the performances of the estimation of stress-strength reliability parameter. We have shown that estimator of stress-strength reliability parameter under joint censoring performing better than the single censoring scheme with respect to its mean squared error.

In this study, it is assumed that both X and Y are simultaneously exposed to the balanced jointly progressive Type-II censoring (BJPC) scheme defined by Mondal and Kundu [31]. In Sections 2, we first obtained the maximum likelihood estimation of $\theta = P(X > Y)$ with the corresponding asymptotic and bootstrap confidence intervals. Then, we derived Bayesian estimations by using the Lindley's approximation in Section 3.1 and Monte Carlo Markov Chain by using the Metropolis-Hasting algorithm in Section 3.2. We have discussed selecting the optimal censoring plans based on the squared error loss function depending on the precision of the estimators. To derive the optimal censoring scheme in the Bayesian framework, we used the variable neighborhood search (VNS) method as considered in Bhattacharya et al. [7] in the section 4. Then, we evaluate the performances of the estimation procedures with simulation studies and a numerical example in Sections 5-6. Finally, in Section 10, we compared the estimation performances for $P(X > Y)$ under jointly censored samples versus separately single censored samples by using the classical progressive Type-II censoring scheme.

2. Model Description and Classical Inference

A random variable T has a Weibull distribution with the scale parameter γ and the shape parameter β if its density function is given as in the following

$$f(t;\gamma,\beta) = \frac{\beta}{\gamma}\left(\frac{t}{\gamma}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\gamma}\right)^{\beta}\right], \qquad t>0, \gamma>0, \beta>0.$$

For the sake of the simplicity in the inferential process, using the transformation $\delta = \gamma^{-\beta} > 0$, the probability density function (PDF) and the cumulative distribution function (CDF) of the Weibull distribution can respectively be rewritten as

$$f(t;\delta,\beta) = \delta\beta t^{\beta-1}\exp\left(-\delta t^{\beta}\right) \qquad \text{and} \qquad F(t;\delta,\beta) = 1-\exp\left(-\delta t^{\beta}\right). \tag{1}$$

Note that WE(λ,β) denote the Weibull distribution with scale and shape parameters λ and β, respectively.

Now, if there is a change in the stress that will affect the scale parameter directly but not the shape parameter of the lifetime distribution. In literature, many authors considered the same shape and different scale parameters for Weibull distribution based on different kinds of step-stress life testing and joint censoring schemes. One can see [19], [37], [28], [33], [32] assume the lifetime of the stress and strength follows Weibull distribution with the same shape parameter and different scale parameters. Suppose $X \sim$ WE(α,β) and $Y \sim$ WE(λ,β), also assume that they are independently distributed. Then, the stress-strength reliability can be obtained as

$$\theta = P(Y < X) = \int_0^{\infty} f_X\left(t;\alpha,\beta\right)\bar{F}_y\left(t;\lambda,\beta\right)dt = \frac{\lambda}{\alpha+\lambda}, \tag{2}$$

where, $\bar{F}(\cdot;\cdot)$ is the survival function of Weibull distribution. Now we choose n items from one population which have lifetime distributions $(X_1, X_2, \cdots, X_n) = \mathbf{X}$, say, Samp-I, are they are independent and identically distributed (i.i.d.) WE(α, β). Similarly, we draw a random sample of size m, $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_m)$,say, Samp-II, from another population and they are i.i.d. WE(λ, β). Let us suppose both $\mathbf{X}$ and $\mathbf{Y}$ are independent and simultaneously exposed to the Balanced Joint Progressive Censoring (BJPC) scheme proposed by Mondal and Kundu [31]. Now we want to estimate θ, based on the observed BJPC sample. Such a model can be constructed as follows.

We first pre-determine $k < \min(m, n)$, the total number of failures to be observed. We pre-fixed the removals as $R_1, R_2, \cdots, R_{k-1}$ such that $\sum_{i=1}^{k-1}(1 + R_i) < \min(m, n)$. Now, when the first failure takes place at the time point W_1 and suppose it comes from Samp-I, then R_1 units are chosen at random from the remaining $m-1$ surviving units. At the same time $R_1 + 1$ units are chosen at random from the remaining n surviving units of Sample-II. Similarly, if the second failure is coming from the Samp-II at the time point W_2, $R_2 + 1$ surviving units are withdrawn from the remaining $m - R_1 - 1$ units from the Samp-I and R_2 surviving units withdrawn from the remaining $n - R_1 - 2$ units from the Samp-II randomly at W_2, and we continue the test till the k^{th} failure occur. Finally, at the time of the k^{th} failure, W_k, it may be either from Samp-I or Samp-II, the experiment will be stop.

Therefore we obtained the set of failure of the lifetime units from both the samples as $\mathbf{W} = (W_1, W_2, \cdots, W_k)$ and $\mathbf{Z} = (Z_1, Z_2, \cdots, Z_k)$ where Z_i for $i = 1, 2, \cdots, k$ takes two values either 1 or 0 depending on whether the failures W_i's are from either $\mathbf{X}$ or $\mathbf{Y}$, respectively. The number of failures of $\mathbf{X}$ among W_i can be denoted as $k_1 = \sum_{i=1}^{k} Z_i$ and $k_2 = \sum_{i=1}^{k}(1 - Z_i)$ denoted the number of failures from $\mathbf{Y}$ such that $k_1 + k_2 = r$. The schematic presentations of this censoring plan can be seen in Figure 1 and Figure 2.

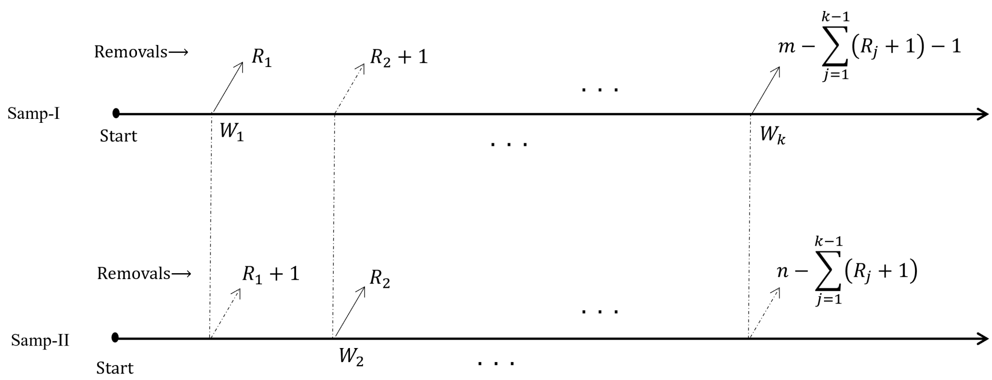

Figure 1. Schematic presentation of Case-I: k^{th} failure comes from the Samp-I.

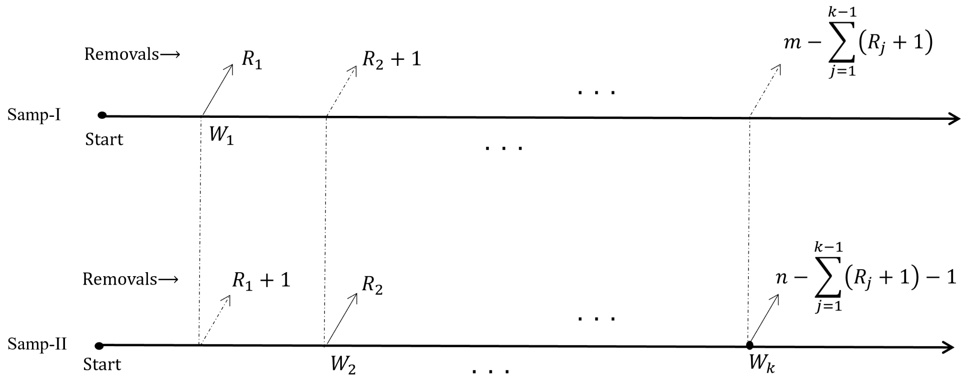

Figure 2. Schematic presentation of Case-I: k^{th} failure comes from the Samp-II.

Thus, the joint density of the observed sample **(w,z)** can be written as:

$$L\left(\mathbf{w},\mathbf{z}|\alpha,\lambda,\beta\right) = C_1 \times \beta^k \alpha^{k_1} \lambda^{k_2} \times \prod_{i=1}^{k} w_i^{\beta-1} e^{-\alpha A_1(\beta)-\lambda A_2(\beta)}, \tag{3}$$

where

$$C_1 = \prod_{i=1}^{k} \left[\left(m - \sum_{j=1}^{i-1} \left(R_j+1\right) \right) z_i + \left(n - \sum_{j=1}^{i-1} \left(R_j+1\right) \right) (1-z_i) \right],$$

$$A_1(\beta) = \sum_{i=1}^{k-1} \left(R_i+1\right) w_i^{\beta} + \left(m - \sum_{i=1}^{k-1} \left(R_i+1\right) \right) w_k^{\beta},$$

$$A_2(\beta) = \sum_{i=1}^{k-1} \left(R_i+1\right) w_i^{\beta} + \left(n - \sum_{i=1}^{k-1} \left(R_i+1\right) \right) w_k^{\beta},$$

and the corresponding log-likelihood function can be written as

$$\ell\left(\mathbf{w},\mathbf{z}|\alpha,\lambda,\beta\right) \propto k\log(\beta)+k_1\log(\alpha)+k_2\log(\lambda)+(\beta-1)\sum_{i=1}^{k}\log(w_i)-\alpha A_1(\beta)-\lambda A_2(\beta). \tag{4}$$

Thus, the normal equations can be obtained by taking partial derivatives of the log-likelihood function with respect to the parameters and equating them to zero as given in the following

$$\begin{aligned}
\frac{\partial \ell}{\partial \alpha} &= \frac{k_1}{\alpha} - A_1(\beta) = 0, \\
\frac{\partial \ell}{\partial \lambda} &= \frac{k_2}{\lambda} - A_2(\beta) = 0, \\
\frac{\partial \ell}{\partial \beta} &= \frac{k}{\beta} + \sum_{i=1}^{k} \log(w_i) - \alpha A_1'(\beta) - \lambda A_2'(\beta) = 0,
\end{aligned} \tag{5}$$

where $A_1'(\beta)$ and $A_2'(\beta)$ are respectively, the first order derivatives of $A_1(\beta)$ and $A_2(\beta)$ with respect to β and they are given by

$$A_1'(\beta) = \sum_{i=1}^{k-1} (R_i + 1)\, w_i^{\beta} \log(w_i) + \left(m - \sum_{i=1}^{k-1} (R_i + 1) \right) w_k^{\beta} \log(w_k),$$

and

$$A_2'(\beta) = \sum_{i=1}^{k-1} (R_i + 1)\, w_i^{\beta} \log(w_i) + \left(n - \sum_{i=1}^{k-1} (R_i + 1) \right) w_k^{\beta} \log(w_k).$$

Thus, the maximum likelihood estimations (MLEs) of α and λ are given by

$$\hat{\alpha} = \frac{k_1}{A_1(\hat{\beta})} \qquad \text{and} \qquad \hat{\lambda} = \frac{k_2}{A_2(\hat{\beta})}, \tag{6}$$

where $\hat{\beta}$ is the MLE of β and it can be obtained with the solution of the following nonlinear equation

$$\frac{k}{\beta} + \sum_{i=1}^{k} \log(w_i) - k_1 \frac{A_1'(\beta)}{A_1(\beta)} - k_2 \frac{A_2'(\beta)}{A_2(\beta)} = 0. \tag{7}$$

The existence and uniqueness of the MLEs are given by Mondal and Kundu [34]. Thus, the invariance property of the MLE helps us to obtain MLE of θ, denoted by $\hat{\theta}$ as replacing the parameters in Equation (2) with their estimates to be found by using the quantities given above. That is the MLE of the stress-strength reliability parameter θ can be obtained as

$$\hat{\theta} = \frac{\hat{\lambda}}{\hat{\alpha} + \hat{\lambda}}. \tag{8}$$

2.1. Asymptotic Confidence Interval

In this subsection, we construct asymptotic confidence interval (ACI) for the stress-strength reliability parameter θ as an approximate confidence interval. We first obtain the inverse of the observed Fisher information matrix, for $\psi = (\psi_1, \psi_2, \psi_3) = (\alpha, \lambda, \beta)$, as

$$I^{-1}(\psi) = -\begin{pmatrix} \frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} & \frac{\partial^2 \ell}{\partial \alpha \partial \beta} \\ \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} & \frac{\partial^2 \ell}{\partial \lambda^2} & \frac{\partial^2 \ell}{\partial \lambda \partial \beta} \\ \frac{\partial^2 \ell}{\partial \beta \partial \alpha} & \frac{\partial^2 \ell}{\partial \beta \partial \lambda} & \frac{\partial^2 \ell}{\partial \beta^2} \end{pmatrix}^{-1} = \begin{pmatrix} \mathrm{Var}(\hat{\alpha}) & \mathrm{Cov}(\hat{\alpha}, \hat{\lambda}) & \mathrm{Cov}(\hat{\alpha}, \hat{\beta}) \\ & \mathrm{Var}(\hat{\lambda}) & \mathrm{Cov}(\hat{\lambda}, \hat{\beta}) \\ & & \mathrm{Var}(\hat{\beta}) \end{pmatrix}.$$

It should be noted that the expected values in the Fisher information matrix cannot be obtained explicitly since the distribution of the MLEs under the jointly censored samples can not be obtained explicitly. Therefore, we use the elements of the observed Fisher information matrix which is obtained by the second partial derivatives of the log-likelihood function (4) with respect to α, λ, and β. The corresponding derivatives terms, of the Fisher information matrix, are given in the Appendix-A to omit the trivial details.
It is known that the maximum likelihood estimators have asymptotic normality properties. Therefore, the MLE of θ, that is $\hat{\theta}$, is asymptotically normal distributed with mean θ and variance

$$\sigma^2_{\hat{\theta}} = \sum_{j=1}^{3} \sum_{i=1}^{3} \frac{\partial \theta}{\partial \psi_i} \frac{\partial \theta}{\partial \psi_j} I^{-1}_{ij}(\psi).$$

Then,

$$\sigma^2_{\hat{\theta}} = \left(\frac{\partial \theta}{\partial \alpha} \right)^2 I^{-1}_{11} + \left(\frac{\partial \theta}{\partial \lambda} \right)^2 I^{-1}_{22} + 2 \left(\frac{\partial \theta}{\partial \alpha} \right) \left(\frac{\partial \theta}{\partial \lambda} \right) I^{-1}_{12}, \tag{9}$$

where

$$\frac{\partial \theta}{\partial \alpha} = -\frac{\lambda}{(\alpha + \lambda)^2} \qquad \text{and} \qquad \frac{\partial \theta}{\partial \lambda} = \frac{\alpha}{(\alpha + \lambda)^2}. \tag{10}$$

Thus, the $100(1-\omega)\%$ asymptotic confidence interval of θ can be constructed by

$$\theta \in \left(\hat{\theta} \pm z_{\omega/2} \hat{\sigma}_{\theta} \right),$$

where z_ω is $100\omega^{th}$ percentile of standard normal distribution and also $\hat{\sigma}_\theta$ is the value of θ at the MLEs of the parameters.

2.2. Bootstrap Confidence Intervals

In this subsection, the bootstrap confidence intervals (BCI) of the unknown parameters are obtained in addition to ACI. It is clearly known that ACI is not well-performed in the case of small sample sizes. Therefore, we obtain parametric percentile bootstrap method (boot-p) and studentized bootstrap method (boot-t) to obtain approximate confidence intervals for $\hat{\theta}$ estimation. The corresponding algorithms can be described as follows.

Bootstrap-p Method:

Step 1: Compute $\hat{\alpha}$, $\hat{\lambda}$, and $\hat{\beta}$ by using the sample $W = (W_1, W_2, \cdots, W_k)$.

Step 2: Generate bootstrap sample $W^* = (W_1^*, W_2^*, \cdots, W_k^*)$ based on the $WE(\hat{\alpha}, \hat{\beta})$, and $WE(\hat{\lambda}, \hat{\beta})$, then compute the bootstrap estimate of θ and denote by $\hat{\theta}^*$

Step 3: Repeat Step 2, NBOOT times.

Step 4: Let $G(z) = P(\hat{\theta}^* \leq z)$, be the cumulative distribution of $\hat{\theta}^*$. Define $\hat{\theta}_{Boot-p} = G^{-1}(z)$ for a given z. Then, the approximate $100(1-\omega)\%$ confidence interval of θ is given by

$$\left(\hat{\theta}_{Boot-p}\left(\frac{\omega}{2}\right), \hat{\theta}_{Boot-p}\left(1-\frac{\omega}{2}\right)\right).$$

Bootstrap-t Method:

Step 1: Compute $\hat{\alpha}$, $\hat{\lambda}$, and $\hat{\beta}$ by using the sample $W = (W_1, W_2, \cdots, W_k)$.

Step 2: Generate bootstrap sample $W^* = (W_1^*, W_2^*, \cdots, W_k^*)$ based on the $WE(\hat{\alpha}, \hat{\beta})$, and $WE(\hat{\lambda}, \hat{\beta})$, then compute the bootstrap estimate of θ and denote by $\hat{\theta}^*$ and also calculate $t^* = (\hat{\theta}^* - \hat{\theta})/\sigma_{\hat{\theta}^*}$. Compute $\sigma_{\hat{\theta}^*}$ by using (9).

Step 3: Repeat step 2, NBOOT times.

Step 4: Let $H(z) = P(t^* \leq z)$ be the cdf t^*. From the obtained t^* values, the apprioximate $100(1-\omega)\%$ confidence interval of θ can be obtained as follows. For a given z, define $\hat{\theta}_{Boot-t} = \hat{\theta} + H^{-1}(z)\sigma_{\hat{\theta}^*}$. Thus, the approximate boot-t $100(1-\omega)\%$ confidence interval of θ is given by

$$\left(\hat{\theta}_{Boot-t}\left(\frac{\omega}{2}\right), \hat{\theta}_{Boot-t}\left(1-\frac{\omega}{2}\right)\right).$$

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2.3. Testing Problem

Throughout this study, inferential processes have been discussed under equal shape β but unequal scale parameters α, λ. It should be noted that the equivalence of the shape parameters of the strength and stress variables has an important and interesting issue in practice. For this reason, one may be interested in testing whether the shape parameters of the strength and stress variables are the same. For example; Ma et al. [29] studied the same problem for the inverted exponential Rayleigh distribution. Similarly, Wang et al. [46] handled resilience and truncated parameters of a general family of truncated distributions. Recently, Çetinkaya [11] tested the equality of the shape parameters for the generalized progressive hybrid censored Weibull populations. Hence, the likelihood ratio testing is proposed to compare the strength and stress shape parameters.

Let us suppose that the strength and stress variables follow the Weibull distributions with shape parameters β_1, β_2 and the following hypothesis testing problem is investigated as

$$H_0 : \beta_1 = \beta_2 = \beta \qquad \text{versus} \qquad H_1 : \text{at least one} \quad \beta_i \neq \beta \quad i = 1, 2.$$

In the case of large sample size, the likelihood ratio statistic

$$\triangle = -2\{\ell_1(\hat{\beta}_1, \hat{\beta}_2) - \ell_2(\hat{\beta})\} \sim \chi_1^2$$

Here, ℓ_1 and ℓ_2 are the log-likelihood functions given in the Equation (4) under arbitrary $(\hat{\beta}_1, \hat{\beta}_2)$ and common parameter $(\check{\beta})$ cases, respectively. They are obtained as given in the following

$$\ell_1(\hat{\beta}_1, \hat{\beta}_2) \propto k_1 \log(\beta_1) + k_2 \log(\beta_2) + k_1 \log(\alpha) + k_2 \log(\lambda) + (\beta_1 - 1) \sum_{i=1}^{k} z_i \log(w_i)$$
$$+ (\beta_2 - 1) \sum_{i=1}^{k} (1 - z_i) \log(w_i) - \alpha A_1(\beta_1) - \lambda A_2(\beta_2).$$

and

$$\ell_2(\hat{\beta}) \propto k \log(\hat{\beta}) + k_1 \log(\alpha) + k_2 \log(\lambda) + (\hat{\beta} - 1) \sum_{i=1}^{k} \log(w_i) - \alpha A_1(\hat{\beta}) - \lambda A_2(\hat{\beta}).$$

The likelihood ratio test can be constructed using the asymptotic distribution of $\triangle$ when $\triangle > C^*$, where C^* is such that $P(\chi_1^2 > C^*)$ size of the test.

On the other hand, one natural question can be asked what should be done if the null hypothesis is rejected. This question seeks inference of R in the case of $\beta_1 \neq \beta_2$. Thus, we should assume that $X \sim \mathrm{WE}(\alpha, \beta_1)$ and $Y \sim \mathrm{WE}(\lambda, \beta_2)$, also suppose that they are independently distributed. In this case, the stress-strength reliability can not be obtained in closed form as being in Equation (2) and obtained as in the following.

$$\begin{aligned} \theta = P(Y < X) &= \int_0^\infty f_X\left(t; \alpha, \beta_1\right) \bar{F}_y\left(t; \lambda, \beta_2\right) dt \\ &= 1 - \alpha\beta_1 \int_0^\infty t^{\beta_1 - 1} \exp\left(-\alpha t^{\beta_1} - \lambda t^{\beta_2}\right) dt \end{aligned} \tag{11}$$

In this case, many steps of the inferential studies of this stress-strength model will face some struggles.

3. Bayesian Inference

In this subsection, we derive an estimation of θ and its corresponding confidence interval based on the Bayesian estimation procedure. The first step in the Bayesian method is determining prior distributions for the unknown parameters that summarize subjective prior beliefs about the parameters. This stage plays an important role in Bayesian inference. When all the parameters are unknown, then the joint conjugate prior is difficult to obtain. Further, Jeffrey's prior can not be defined in our problem since all the elements of the corresponding expected Fisher information matrix do not have close forms. Arnold and Press [1] highlighted that it is not obvious to claim that any prior is better than the others. They also suggested electing to restrict attention to a given flexible family of prior distributions. Thus, one of the distribution families which seems the best fitting for our problem can be chosen. The gamma distribution is versatile for adjusting different shapes of the density function. It has a log-concave density function in the interval $(0, \infty)$. Jeffery's prior can be obtained as a special case of the gamma prior. Due to the importance of the gamma distributions, we have chosen gamma priors to obtain Bayes estimates. For this purpose, we first assume that the unknown parameters α, λ, and β follow independent gamma priors such that $\alpha \sim GA(a_1, b_1)$, $\lambda \sim GA(a_2, b_2)$, and $\beta \sim GA(a_3, b_3)$ with density functions are given as in the following

$$\pi(\psi_i) \propto \psi_i^{a_i-1} e^{-b_i \psi_i}, \quad a_i, \ b_i > 0, \quad \text{for} \quad i = 1, 2, 3,$$

where a_i and b_i are called the hyper-parameters.

In the case of non-informative priors, we prefer to use very small and non-negative values of the hyper-parameters, i.e. $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0.0001$, as suggested by Congdon [15] which are almost like Jeffrey's priors, but they are proper, inversely. This value of the hyper-parameters is used in many studies as non-informative priors. In the case of informative hyper-parameters, it is suggested to use hyper-parameters providing the actual values of the parameters. In gamma priors, since $\mathrm{E}(\psi_i) = \frac{a_i}{b_i}$, an average value of the actual parameters can be obtained by choosing available a_i and b_i. Since the actual values of the parameters are unknown in real data studies, a_i and b_i can be chosen as providing MLEs of the parameters.

Then, the joint posterior distribution for α, λ, and β is given by

$$L(\alpha, \lambda, \beta, \mathbf{w}, \mathbf{z}) = \mathbf{L}(\mathbf{w}, \mathbf{z} | \alpha, \lambda, \beta) \pi(\alpha) \pi(\lambda) \pi(\beta),$$

and the joint posterior density of α, λ, and β given data is obtained by

$$L(\alpha, \lambda, \beta | \mathbf{w}, \mathbf{z}) = \frac{L(\mathbf{w}, \mathbf{z} | \alpha, \lambda, \beta) \pi(\alpha) \pi(\lambda) \pi(\beta)}{\int_0^\infty \int_0^\infty \int_0^\infty L(\mathbf{w}, \mathbf{z} | \alpha, \lambda, \beta) \pi(\alpha) \pi(\lambda) \pi(\beta) d\alpha \ d\lambda \ d\beta}. \tag{12}$$

Then, the Bayes estimate of θ, denoted by θ^B, under the squared error loss function, can be obtained as the mean of the posterior function in equation (12) as given in the following

$$E(\theta | \mathbf{w}, \mathbf{z}) = \int_0^\infty \int_0^\infty \int_0^\infty \theta \ L(\alpha, \lambda, \beta | \mathbf{w}, \mathbf{z}) d\alpha \ d\lambda \ d\beta. \tag{13}$$

It could be useful to explain that different loss functions can be used in this stage. There are some asymmetrical loss functions such as Linex, general entropy and so on. However, we prefer to use symmetric loss function to provide equal over and underestimates to the actual value of θ. We considered the squared error loss function for this reason.
Cleary, it is not possible to obtain an explicit expression for the posterior mean of θ in equation (13). For this reason, Lindley's approximation [27] to obtain the Bayes estimate of the parameter θ. Note that Lindley's method is not useful to obtain the credible inverval. Therefore, we also use Markov Chain Monte Carlo (MCMC) method to obtain the Bayes estimate and corresponding credible interval of the unknown parameter θ.

3.1. Lindley's Approximation

In this subsection, we derive the Bayes estimation of θ by using the method to obtain approximate Bayesian estimations for a function of parameters described by Lindley [27]. This approximation method saves computation time for the Bayesian method and we can also obtain mean and variance of θ explicitly for the optimum censoring plan which will be given in Section 4. Let $U(\psi) = U(\alpha, \lambda, \beta)$ be a function of parameters, then under the squared error loss function, the Bayes estimator of $U(\psi)$ can be obtained by using the Lindley's approximation formulas for case with three parameters as given in the following

$$E[U(\psi)|data] = \frac{\int U(\psi)e^{\ell(\psi)+\rho(\psi)}d\psi}{\int e^{\ell(\psi)+\rho(\psi)}d\psi}, \tag{14}$$

where $\ell(\psi)$ is the log-likelihood function of ψ and $\rho(\psi)$ is the natural logarithm of the joint prior density of ψ. Lindley's approximation for $E[U(\psi)|data]$ is given as

$$E[U(\psi)|data] \approx U(\psi) + \frac{1}{2}\sum_i\sum_j(u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_l \ell_{ijk}\sigma_{ij}\sigma_{kl}u_l|_{\psi=\hat{\psi}}, \tag{15}$$

where ψ is defined in Section 2.1, $\hat{\psi}$ is the MLE of ψ, $u = U(\psi)$, $u_i = \frac{\partial u}{\partial\psi_i}$, $u_{ij} = \frac{\partial^2 u}{\partial\psi_i\partial\psi_j}$, $l_{ijk} = \frac{\partial^3 l}{\partial\psi_i\partial\psi_j\partial\psi_k}$, $\rho_j = \frac{\partial\rho}{\partial\psi_j}$ and $\sigma_{ij} = (i,j)^{th}$ element of the matrix I^{-1}, for $i,j,k = 1,2,3$, with all parameters replaced by their MLEs. Thus, Lindley's approximation gives the Bayes estimation of θ, denoted by $\hat{\theta}^B_{Ln}$. Here, from prior densities of parameters, we have

$$\rho_1 = \frac{a_1 - 1}{\alpha} - b_1, \qquad \rho_2 = \frac{a_2 - 1}{\lambda} - b_2, \qquad \text{and} \qquad \rho_3 = \frac{a_3 - 1}{\beta} - b_3,$$

then, we have the following values of ℓ_{ijk} for $i,j,k = 1,2,3$.

$$\ell_{111} = \frac{2k_1}{\alpha^3}, \qquad \ell_{222} = \frac{2k_2}{\lambda^3}, \qquad \ell_{333} = \frac{2k}{\beta^3} - \alpha A_1^{'''}(\beta) - \lambda A_2^{'''}(\beta),$$
$$\ell_{331} = -A_1^{''}(\beta), \qquad \ell_{332} = -A_2^{''}(\beta), \qquad \text{and} \qquad \ell_{112} = \ell_{113} = \ell_{221} = \ell_{223} = 0.$$

Also, u_1, u_2, and u_3 is obtained in equation (10). Here the experssion for $-A_1^{'''}(\beta)$ and $-A_2^{'''}(\beta)$ are given in the Appendix-A. Then, we have $u_{13} = u_{31} = u_{23} = u_{32} = 0$, and

$$u_{11} = \frac{2\lambda}{(\alpha+\lambda)^3}, \qquad u_{12} = \frac{\lambda - \alpha}{(\alpha+\lambda)^3}, \qquad u_{22} = \frac{-2\alpha}{(\alpha+\lambda)^3}.$$

By calculating the necessary quantities above, Lindley's approximation for the Bayes estimation of θ, denoted by $\hat{\theta}^B_{Ln}$ can be obtained as in the following

$$\hat{\theta}^B_{Ln} = \hat{\theta} + \frac{1}{2}\sum_i\sum_j(u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_l \ell_{ijk}\sigma_{ij}\sigma_{kl}u_l|_{\psi=\hat{\psi}}, \tag{16}$$

where $\hat{\theta}$ is the MLE of θ and all parameters are evaluated at $\hat{\alpha}, \hat{\lambda}, \hat{\beta}$.

It should be note that, the highest posterior density (HPD) credible interval for θ can not be obtained using Lindley's approximation due to lack of an explicit form of the PDF for θ. Therefore, MCMC method is used to derive another approximate Bayes estimation and corresponding HPD credible intervals.

3.2. MCMC Method

In this subsection, we consider the MCMC method and generate a sample by using Gibbs sampling algorithm. In this purpose, the joint posterior density function of $\hat{\alpha}, \hat{\lambda}$, and $\hat{\beta}$ is obtained as

$$L\left(\psi|\mathbf{w},\mathbf{z}\right) \propto \beta^{k+a_3-1}\alpha^{k_1+a_1-1}\lambda^{k_2+a_2-1}e^{-\beta\left[b_3-\sum_{i=1}^{k}\log(w_i)\right]}e^{-\alpha\left[A_1(\beta)+b_1\right]-\lambda\left[A_2(\beta)+b_2\right]}, \tag{17}$$

Then, the conditional posterior density functions of the parameters can be obtained easily as in the following

$$\begin{aligned}\pi(\alpha|\lambda,\beta) &\propto GA\bigg(k_1+a_1, A_1(\beta)+b_1\bigg),\\ \pi(\lambda|\alpha,\beta) &\propto GA\bigg(k_2+a_2, A_2(\beta)+b_2\bigg),\end{aligned} \tag{18}$$

and

$$\pi(\beta|\alpha,\lambda) \propto \beta^{k+a_3-1}e^{-\beta\left[b_3-\sum_{i=1}^{k}\log(w_i)\right]}e^{-\alpha A_1(\beta)-\lambda A_2(\beta)}. \tag{19}$$

It is clearly seen from (18) that the samples for α and λ can be easily obtained by using gamma densities. However, the density from $\pi(\beta|\alpha,\lambda)$ can not be reduced analytically to well known distributions and therefore it is not possible to generate sample directly by standard methods. We observed that the density plot of the conditional posterior density of β is like to Gaussian distribution as seen in Figure 3.

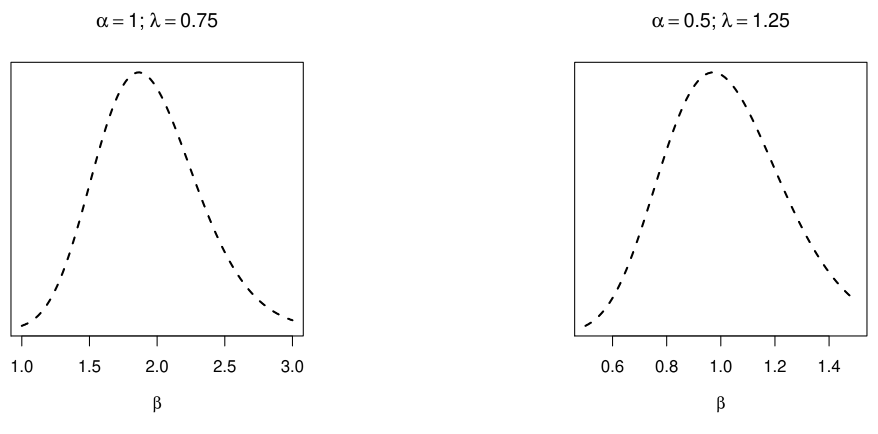

Figure 3. Plots for the posterior pdfs of β.

Hence, the Metropolis-Hasting (M-H) algorithm with normal proposal distribution as suggested by Tierney [44] can be used for the Bayesian estimation of θ. The algorithm for Gibbs sampling with M-H method can be described as follows:

Step 1: Start by using the initial values of $(\alpha^{(0)}, \lambda^{(0)}, \beta^{(0)})$
Step 2: Set $t = 1$
Step 3: Generate $\alpha^{(t)}$ from $GA\big(k_1 + a_1, A_1(\beta) + b_1\big)$.
Step 4: Generate $\lambda^{(t)}$ from $GA\big(k_2 + a_2, A_2(\beta) + b_2\big)$.
Step 5: Generate $\beta^{(t)}$ from $\pi(\beta|\alpha, \lambda)$ by using the M-H algorithm with normal proposal $N(\beta^{(t-1)}, \sigma^2_{\hat{\beta}})$.

- Let $v = \beta^{(t-1)}$ and generate w from the proposal as $w = N(\beta^{(t-1)}, \sigma_{\hat{\beta}})$.
- Let $p(v, w) = \min\left\{1, \frac{\pi(w|\alpha^{(t)}, \lambda^{(t)})N(v, \sigma_{\hat{\beta}})}{\pi(v|\alpha^{(t)}, \lambda^{(t)})N(w, \sigma^2_{\hat{\beta}})}\right\}$
- Generate U from $U(0, 1)$, then accept proposal if $U \leq p(v, w)$ and set $\beta^{(t)} = w$ or otherwise $\beta^{(t)} = v$

Step 6: Compute $\theta^{(t)}$ from (8) at $(\alpha^{(t)}, \lambda^{(t)}, \beta^{(t)})$.
Step 7: Set $t = t + 1$.
Step 8: Repeat Step 3 to Step 7, for N times.

Then, the approximate posterior mean of θ under the squared error $(\hat{\theta}^B_{MC})$ can be derived as

$$\hat{\theta}^B_{MC} = \frac{1}{N - B} \sum_{t=B+1}^{N} \theta^{(t)} \tag{20}$$

where B is the burn-in period. Then, the $100(1 - \omega)\%$ HPD credible interval can be constructed by using the method proposed by Chen and Shao [14] as

$$\left(\hat{\theta}^B_{MC[\frac{\gamma}{2}N]}, \hat{\theta}^B_{MC[\frac{\gamma}{2}N]}\right)$$

where $[\frac{\gamma}{2}N]$ and $[(1 - \frac{\gamma}{2})N]$ are the smallest integers less than or equal to $\frac{\gamma}{2}N$ and $(1 - \frac{\gamma}{2})N$, respectively.

4. Optimal Censoring

The censoring schemes are mostly predetermined in most cases. However, finding "optimum" censoring scheme is very important since it provides better performance such as lower variance, minimum cost and etc.

In this study, our aim is to obtain the optimal progressive censoring scheme $R = (R_1, R_2, \cdots, R_{k-1})$ such that $\sum_{i=1}^{k-1}(R_i + 1) < \min(m, n)$ for different choices of m, n and k based on a variance minimization criterion as considered in Bhattacharya et al. [7]. We consider a weighted posterior variance with the expected total time of the experiment. The expected running time of the experiment, denoted by $E(W_k)$, for the BJPC scheme, can be calculated by using the following Lemma.

Lemma 4.1. *If $W_1 \leq \cdots \leq W_k$ are the ordered lifetime from a BJPC, then*

$$W_i \triangleq \sum_{s=1}^{i} V_s$$

where V_s's are independent random variables such that

$$V_s \sim \operatorname{Exp}\left(\frac{1}{E_s}\right), \quad E_s = \alpha\left(m - \sum_{j=1}^{s-1}(R_j+1)\right) + \lambda\left(n - \sum_{j=1}^{s-1}(R_j+1)\right)$$

Proof. The above expression is obtained from the distribution of W_k, derived by the moment generating function technique. See the details in the Appendix-C. □

Then, the expected running time of the experiment can easily obtain by using Lemma 4.1 as

$$E\left(W_k\right) = \sum_{s=1}^{k} E\left(V_s\right) = \sum_{s=1}^{k} \frac{1}{E_s}$$

Thus, the constrained optimization problem is now formally formulated as

$$\text{Minimize } \phi(R) = \varphi_1 E\left[V\left(\theta \mid \text{ data }\right)\right] + \varphi_2 E\left(W_k\right) \tag{21}$$

such that $\sum_{i=1}^{k-1}(R_i+1) < \min(m,n)$, $0 \leq \varphi_i \leq 1$, $i = 1,2$ and $\sum_{i=}^{2} \varphi_1 = 1$. Here, $E\left[V\left(\theta \mid \text{ data }\right)\right]$ is the expected posterior variances of θ and can be calculated by using the Lindley's method which is described in the Section 3.1 as

$$\operatorname{Var}[U(\psi)|data] \approx E[V(\psi)|data] - \left\{E[U(\psi)|data]\right\}^2$$

where $V(\psi) = U^2(\psi) = \frac{\lambda^2}{(\alpha+\lambda)^2}$ and

$$E[V(\psi)|data] \approx V(\psi) + \frac{1}{2}\sum_i\sum_j(v_{ij} + 2v_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_l \ell_{ijk}\sigma_{ij}\sigma_{kl}v_l\big|_{\psi=\hat{\psi}}$$

where $v_1 = \lambda^2\Big/(\alpha+\lambda)^3$, $v_2 = 2\alpha\lambda\Big/(\alpha+\lambda)^3$ and

$$v_{11} = \frac{6\lambda^2}{(\alpha+\lambda)^4}, \qquad v_{12} = \frac{2\lambda^2 - 4\alpha\lambda}{(\alpha+\lambda)^4}, \qquad v_{22} = \frac{2\alpha^2 - 4\alpha\lambda}{(\alpha+\lambda)^4}$$

and the other expressions are the same as given in the Section 3.1.

Then, we use a VNS based algorithm given by Bhattacharya et al. [8] to find the optimum solution. An initial guess of $R_0 = (R_{01}, R_{02}, \cdots, R_{0(k-1)})$ is taken from the set of $CS(m,n,k) = \left\{R = (R_1, R_2, \cdots, R_{k-1}) : \sum_{i=1}^{k-1}(R_i+1) < \min(m,n)\right\}$ and let

denote $\ell_{\max} = \max R_{0j}, 1 \leq j \leq k-1$. for $\ell = 1, \cdots, \ell_{\max}$ construct the neighborhood $N_\ell(R_0)$ of R_0 given by $N_l(R_0) = \{R : R \in CS(n,m,k) \text{ and } \|R - R_0\| < \ell + 1\}$. Here, $\|\cdot\|$ denotes standard Euclidean norm. Thus, for fixed n, m and k with initial guess R_0 we can obtain the optimum censoring scheme by solving the constrained optimization problem 21. The computational algorithm is presented in Algorithm-I in the Appendix-B section.
For the purpose of illustration, we have computed optimum plans by considering various sample sizes and weights. We used the actual parameter values as $\alpha = 1$, $\lambda = 0.75$ and $\beta = 2$ and preferred non-informative hyperparameters such as $a_j = b_j = 0$ for $j = 1, 2, 3$. The whole results of these calculations are presented in Table 9. We observed that when we put the whole weight on φ_2, that is $\varphi_2 = 1$ i.e., our main objective is to reduce the experimental time and early removals are prefable in this case. On the other hand, late removal is preferable when we put the weight on the variance part $\varphi_1 = 1$. In the case of more weight on the expected running time of the experiment, the schemes which all removals are placed at the kth failure have mostly optimum values of the objective functions.

5. Simulation Study

In this section, we performed simulation studies to exemplify our theoretical findings under different censoring schemes. We evaluate the point estimates of θ with their average biases and mean squared errors (MSE). At the same time, we evaluated the approximate confidence intervals with their average lengths (AL) and coverage probabilities (CP). We take the actual parameter values as $\alpha = 1$, $\lambda = 0.75$, $\beta = 2$ and the corresponding $\theta = 0.4286$. In the Bayesian estimations, very small non-negative values of the hyper-parameters, i.e. $a_i = b_i = 0.0001$ for $i = 1, 2, 3$ can be considered as suggested by Congdon [15] which are almost like Jefirey's priors, but they are proper, inversely. However, the Bayesian estimates are very close to the MLEs under these non-informative priors. Therefore, we prefer more informative priors such as $a_1 = b_1 = 1$, $a_2 = 1.25; b_2 = 1.65$, and $a_3 = 2; b_3 = 1$, respectively. We consider three different sets of sample sizes (m, n) as $(20, 25), (35, 35)$, and $(50, 40)$. The corresponding censoring schemes are constructed by considering different observed failures k, and the removals R_i, $i = 1, 2, \cdots, k-1$. We determined the removals by providing fast failure, moderate failure and late failure, respectively.
We repeated the simulations 2000 times, and we used 500 bootstrap samples for each replication. We used 3500 samples of Markov chain and discarded the first 500 values as burn-in period and took every third variate in thinning procedure in the MCMC samples. Then, the algorithm is performed for 2000 replications. The significant level was taken as $\omega = 0.05$.

Table 1. Absolute biases and MSEs (within parentheses) of the estimators with average lengths and coverage probabilities (within parentheses) of their confidence intervals for $m = 20, n = 25$.

	Average Estimate			*Approximate CIs*			
Censoring Schemes	$\hat{\theta}$	$\hat{\theta}^B_{Ln}$	$\hat{\theta}^B_{MC}$	*ACI*	Boot-p	Boot-t	*HPD*
$k = 14; R = (6, 0_{(12)})$	0.0045	0.0121	0.0167	0.5061	0.4970	0.6073	0.4524
	(0.0198)	(0.0138)	(0.0148)	(89.65)	(91.20)	(88.10)	(93.80)
$k = 14; R = (0_{(6)}, 6, 0_{(6)})$	0.0030	0.0089	0.0162	0.5041	0.4950	0.6086	0.4509
	(0.0199)	(0.0139)	(0.0149)	(90.10)	(92.75)	(87.90)	(95.05)
$k = 14; R = (0_{(12)}, 6)$	0.0020	0.0008	0.0070	0.4951	0.4841	0.5830	0.4451
	(0.0182)	(0.0129)	(0.0135)	(88.55)	(91.00)	(88.15)	(94.40)
$k = 16; R = (5, 0_{(14)})$	0.0092	0.0016	0.0061	0.4782	0.4719	0.5845	0.4319
	(0.0181)	(0.0128)	(0.0135)	(88.95)	(90.25)	(85.80)	(94.10)
$k = 16; R = (0_{(7)}, 5, 0_{(7)})$	0.0085	0.0008	0.0076	0.4794	0.4742	0.5914	0.4323
	(0.0169)	(0.0120)	(0.0126)	(90.90)	(92.20)	(87.90)	(95.25)
$k = 16; R = (0_{(14)}, 5)$	0.0024	0.0061	0.0116	0.4703	0.4674	0.5571	0.4262
	(0.0158)	(0.0117)	(0.0122)	(91.85)	(93.25)	(89.40)	(95.20)

Table 2. Absolute biases and MSEs (within parentheses) of the estimators with average lengths and coverage probabilities (within parentheses) of their confidence intervals for $m = 35, n = 35$.

	Average Estimate			*Approximate CIs*			
Censoring Schemes	$\hat{\theta}$	$\hat{\theta}^B_{Ln}$	$\hat{\theta}^B_{MC}$	*ACI*	Boot-p	Boot-t	*HPD*
$k = 20; R = (5, 0_{(18)})$	0.0012	0.0003	0.0015	0.4223	0.4089	0.4643	0.3866
	(0.0123)	(0.0096)	(0.0098)	(92.75)	(93.20)	(88.95)	(94.85)
$k = 20; R = (0_{(9)}, 5, 0_{(9)})$	0.0027	0.0007	0.0030	0.4227	0.4076	0.4612	0.3874
	(0.0120)	(0.0093)	(0.0096)	(93.15)	(93.10)	(88.10)	(95.00)
$k = 20; R = (0_{(18)}, 5)$	0.0018	0.0001	0.0023	0.4218	0.4068	0.4617	0.3864
	(0.0130)	(0.0101)	(0.0104)	(92.60)	(93.00)	(87.55)	(94.45)
$k = 25; R = (8, 0_{(23)})$	0.0038	0.0037	0.0031	0.3791	0.3793	0.4202	0.3522
	(0.0101)	(0.0083)	(0.0085)	(92.40)	(92.70)	(89.65)	(94.40)
$k = 25; R = (0_{(11)}, 8, 0_{(12)})$	0.0012	0.0006	0.0013	0.3801	0.3808	0.4214	0.3533
	(0.0096)	(0.0079)	(0.0080)	(93.45)	(93.85)	(91.00)	(94.90)
$k = 25; R = (0_{(23)}, 8)$	0.0009	0.0014	0.0004	0.3800	0.3772	0.4164	0.3530
	(0.0095)	(0.0077)	(0.0079)	(93.80)	(93.60)	(91.15)	(95.25)

Table 3. Absolute biases and MSEs (within parentheses) of the estimators with average lengths and coverage probabilities (within parentheses) of their confidence intervals for $m = 50, n = 40$.

	Average Estimate			*Approximate CIs*			
Censoring Schemes	$\hat{\theta}$	$\hat{\theta}^B_{Ln}$	$\hat{\theta}^B_{MC}$	*ACI*	Boot-p	Boot-t	*HPD*
$k = 30; R = (10, 0_{(28)})$	0.0028	0.0069	0.0087	0.3829	0.3793	0.4251	0.3501
	(0.0101)	(0.0077)	(0.0080)	(93.4)	(93.7)	(92.7)	(95.75)
$k = 30; R = (0_{(14)}, 10, 0_{(14)})$	0.0010	0.0044	0.0074	0.3793	0.3758	0.4199	0.3469
	(0.0101)	(0.0077)	(0.0080)	(93.00)	(93.65)	(91.85)	(95.30)
$k = 30; R = (0_{(28)}, 10)$	0.0102	0.0037	0.0023	0.3548	0.3519	0.3878	0.3277
	(0.0089)	(0.0070)	(0.0071)	(91.70)	(91.85)	(90.15)	(94.05)
$k = 35; R = (5, 0_{(33)})$	0.0014	0.0047	0.0062	0.3516	0.3476	0.3805	0.3261
	(0.0085)	(0.0068)	(0.0070)	(93.30)	(93.75)	(91.40)	(95.25)
$k = 35; R = (0_{(17)}, 5, 0_{(16)})$	0.0106	0.0044	0.0025	0.3532	0.3514	0.3861	0.3266
	(0.0085)	(0.0066)	(0.0068)	(93.30)	(93.25)	(91.65)	(95.60)
$k = 35; R = (0_{(33)}, 5)$	0.0027	0.0050	0.0061	0.3416	0.3372	0.3662	0.3182
	(0.0080)	(0.0066)	(0.0067)	(93.65)	(93.35)	(92.35)	(95.20)

The whole results are presented in Tables 1-3. Note that, the notation in the table, for example, $(2, 4, 0_{(3)}, 2, 0)$ means that the censoring scheme employed is $(2, 4, 0, 0, 0, 2, 0)$. It is observed that the MSEs and biases decrease with increasing sample sizes as expected. The Bayes estimates have smaller MSEs, especially in Lindley's method and also in small sample sizes. In parallel to increasing sample sizes, all estimates provide closer results to each other. In terms of CPs, estimation methods perform well and mostly closer results to 0.95 are provided. Also, the ALs are getting smaller with increasing sample sizes as expected. Similar to MSE and bias results, Bayes estimates show better performances in small samples. In terms of overall performances of the approximate confidence intervals, the Bayesian MCMC method presents better than other bootstrap and ACI methods. Unlike Bayesian methods, boot-t has the worst ALs and CPs between approximate confidence intervals. On the other hand, it is observed that the increase in the number of observed failures increases the estimation performances as expected. And also, there is no significant effect of the position of removals on estimation performances. The theoretical findings perform well in all cases of the censoring schemes.

6. Numerical Example

In this section, we apply our theoretical findings to real data. The original data are reported by Badar and Priest [4]. These data sets represent the strength measured in GPa (giga-Pascals) for single carbon fibers, and impregnated 1000−carbon fiber tows. Single fibers were tested under tension at gauge lengths of $20mm$ (Data Set-I) and 10 mm (Data Set-II). Kundu and Gupta [25] analyzed these data sets by fitting the Weibull distribution after subtracting 0.75 from both these data sets and they observed that the Weibull distributions with equal shape parameters fit to both these data sets. The sample sizes of the corresponding data sets are 69 and 63 respectively. The data sets are given as

Table 4. Data Set-I (gauge lengths of 20 mm).

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958
1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179
2.224	2.240	2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382
2.382	2.426	2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554
2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726
2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585	

Table 5. Data Set-I (gauge lengths of 10 mm).

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397	2.445
2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614	2.616	2.618
2.624	2.659	2.675	2.738	2.740	2.856	2.917	2.928	2.937	2.937
2.977	2.996	3.030	3.125	3.139	3.145	3.220	3.223	3.235	3.243
3.264	3.272	3.294	3.332	3.346	3.377	3.408	3.435	3.493	3.501
3.537	3.554	3.562	3.628	3.852	3.871	3.886	3.971	4.024	4.027
4.225	4.395	5.020							

In this study, we follow Kundu and Gupta [25] and extracted 0.75 from the both data samples.Then, we consider the Data Set-II as strength and the Data Set-I as stress variables. We calculate the 95% bootstrap confidence intervals for θ based on 1000 replications for each of which 500 bootstrap intervals are used. We also perform the simulation algorithm for the MCMC method given in Subsection 3.2 with the iteration number $N = 100000$. We run the iterations of the Gibbs algorithm by using the maximum likelihood estimates of the parameters as initial values. Since the MLEs can be evaluated as good starting values, we prefer not to use burn-in operation. However, since the Gibbs algorithm naturally generates an autocorrelated Markov chain (Zhang [47]), the thinning operation is performed to reduce the autocorrelation. It is known that highly autocorrelated Markov chains show slow moves around the parameter space. In this case, much time is needed to attain the correct balance among the different regions of the parameter space. The higher the autocorrelation in the chain causes the larger variance and worse approximation (Hoff [20]). Therefore, we use thinning and determined 10 as the thinning number so as not to lose much information from the data. Thus, we obtain 10000 uncorrelated Markov chain. We consider different censoring schemes (CS) as given in the following.

Table 6. Censoring schemes for the numerical example

$CS-I$	:	$k = 40;\ R = (20, 0_{(38)})$
$CS-II$	:	$k = 40;\ R = (0_{(19)}, 20, 0_{(19)})$
$CS-III$	:	$k = 40;\ R = (0_{(38)}, 20)$
$CS-IV$	:	$k = 50;\ R = (13, 0_{(48)})$
$CS-V$	:	$k = 50;\ R = (0_{(24)}, 13, 0_{(24)})$
$CS-VI$	:	$k = 50;\ R = (0_{(48)}, 13)$

We reported estimations and the corresponding confidence intervals in Tables.

Table 7. Estimations of θ for the real data example.

	Censoring Schemes					
Estimations	CS-I	CS-II	CS-III	CS-IV	CS-V	CS-VI
$\hat{\theta}$	0.7482	0.7865	0.7742	0.7725	0.7986	0.7685
$\hat{\theta}^B_{Ln}$	0.7616	0.7949	0.7777	0.7846	0.8099	0.7741
$\hat{\theta}^B_{MC}$	0.7504	0.7881	0.7758	0.7748	0.8005	0.7698

Table 8. Approximate confidence intervals of θ for the real data example.

	Approximate Confidence Intervals			
CSs	ACI	$Boot-p$	$Boot-t$	HPD
$CS-I$	(0.6021, 0.8944) 0.2923	(0.6017, 0.9004) 0.2987	(0.6135, 0.9589) 0.3454	(0.5967, 0.8791) 0.2823
$CS-II$	(0.6494, 0.9236) 0.2743	(0.6445, 0.9072) 0.2626	(0.6611, 0.9534) 0.2923	(0.6421, 0.9044) 0.2622
$CS-III$	(0.6388, 0.9097) 0.2709	(0.6399, 0.9099) 0.2700	(0.6549, 0.9647) 0.3098	(0.6294, 0.8936) 0.2642
$CS-IV$	(0.6455, 0.8995) 0.2540	(0.6403, 0.8950) 0.2547	(0.6534, 0.9344) 0.2809	(0.6390, 0.8869) 0.2479
$CS-V$	(0.6768, 0.9205) 0.2437	(0.6748, 0.9212) 0.2463	(0.6903, 0.9746) 0.2842	(0.6735, 0.9056) 0.2321
$CS-VI$	(0.6451, 0.8918) 0.2467	(0.6395, 0.8835) 0.2440	(0.6524, 0.9178) 0.2654	(0.6380, 0.8802) 0.2422

Kundu and Gupta [25] calculated MLE and approximate MLE of θ as 0.7624 and 0.7608 respectively under complete samples. In the scope of our study, the estimations of θ and their approximate confidence intervals are given in Tables 7-8 based on the different censoring schemes given above. We observed that estimations are closer to the estimations of the complete cases under the fast and late failure cases. In the moderate failure cases, estimations slightly differ. On the other hand, approximate confidence intervals have smaller lengths in the case of more observed failures. Among the methods, the Bayesian HPD credible intervals have the smallest lengths and the boot-t intervals show the worst performances. These conclusions show consistency with the simulation results.

We also test the equivalence between the shape parameters of the strength and stress variables as $\beta_1 = \beta_2 = \beta$ by using the hypothesis testing proposed in Section 2.3. Since the value of the likelihood ratio statistics and the associated p-value are 0.7609 and 0.3830 with %5 level of significance, the results imply that the null hypothesis cannot be rejected. Consequently, it is recommended that the shape parameters of the Weibull models for the stress and strength components, β_1 and β_2, are equal for this breaking strength of carbon fibers data.

We also checked the convergence of the Markov Chains. For this purpose, we used the Markov Chains generated for the real data example. Firstly, we draw trace plots that show the values of the parameters against the iteration number at each iteration. The trace plot for the posterior distribution of θ which is given in Fig. 4 shows that the Markov Chain fluctuates around its center with similar variations. Further, we see from the Fig. 5, the density plot seems in symmetric and unimodal shape. The unimodal peak in this plot determines the mode of the posterior distribution of θ and they are the values with the most support from the data and the chosen priors. Additionally, we use the running mean (ergodic average) plot as another diagnostic tool. An ergodic average plot draws the mean of sampled values up to iteration t, and the precision of an ergodic average depends on the auto-correlations of the chain. We see from the Fig. 5 it stabilizes with increasing iteration number which means that the chain has achieved stationarity. All plots to check the convergence in a Markov Chain can be drawn using library MCMC plots (Curtis et al.[18]) in R (Core Team, [16]) software.

7. Comparative Study

In this section, we compared the estimation performances under jointly censoring versus single sample censoring schemes. For this purpose, we first handled the stress and strength variables based on the classical progressive censoring (PC) scheme and computed the estimations. Then, we obtained the results for the jointly progressive censoring (JPC) scheme under the same censoring plans and compared the results with the previous results. We used the following algorithm for this comparative study.

Step 1: Generate strength sample X from the WE(α, β) under the PC scheme.
Step 2: Generate stress sample Y from the WE(λ, β) under the PC scheme.
Step 3: Take the $Max\{X_{(k)}, Y_{(k)}\}$ and calculate $P(X > Y)$.
Step 4: Generate strength and stress samples under JPC from the samples $X \sim$ WE(α, β) and $Y \sim$ WE(λ, β).
Step 5: Calculate $W_{(k)}$ and $P(X > Y)$ based on the samples obtained from the JPC.
Step 6: Repeat Steps 1-5, M times.

Following, we calculate MSEs for the estimates based on PC scheme and named as MSE_{PC}. Then, we obtained MSEs for the estimates based on JPC and named as MSE_{JPC}. We also evaluated the end time of the experiment. Therefore, we obtained the averages of the $Max\{X_{(k)}, Y_{(k)}\}$ and $W_{(r)}$. The results are presented in Table 10. We observed that JPC scheme mostly has smaller MSE values and the running time of the experiment is shorter. We observed our results for the different cases of the sample sizes and censoring schemes. Consequently, we can strongly state that this jointly censoring scheme gives better performance. In addition to the advantages of a jointly censoring scheme, theoretical superiority leads us to use the jointly censoring scheme for any stress-strength model.

8. Conclusion

In this study, we considered the stress-strength reliability when both strength and stress variables are exposed to jointly balanced progressive censoring scheme. In this paper, we have taken the Weibull distribution of the both stress and strength variable with same shape and different scale parameters. However, one can take different shape parameter for stress and strength variables, that will be bit complicated for calculating the MLEs. We used classical and Bayesian methods for the estimation procedures and we observed consistent results. It is observed that estimation methods perform well in all cases of the removals since there is no significant effect of the position of removals on estimation performances. Further, we examined the optimum censoring plans and observed that optimal plan change depend on the selection of the weights on the variance of the reliability and expected running time. In most cases,the schemes in

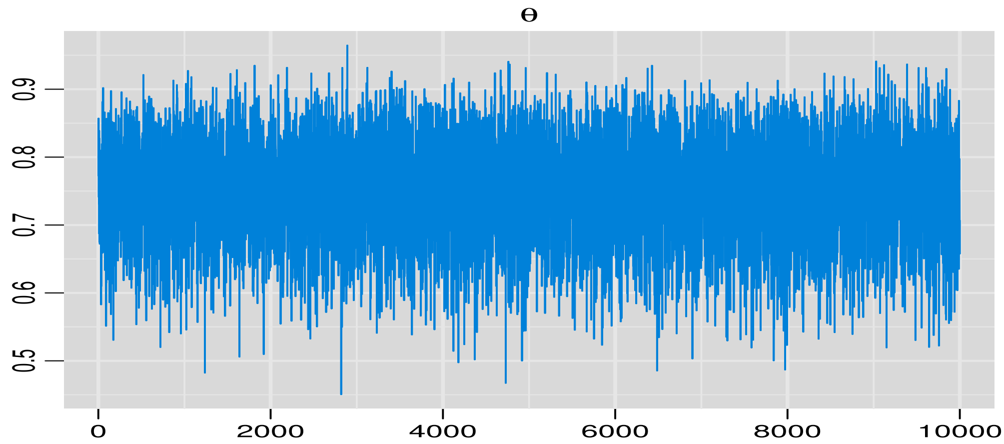

Figure 4. Trace plots of the posterior distribution of θ.

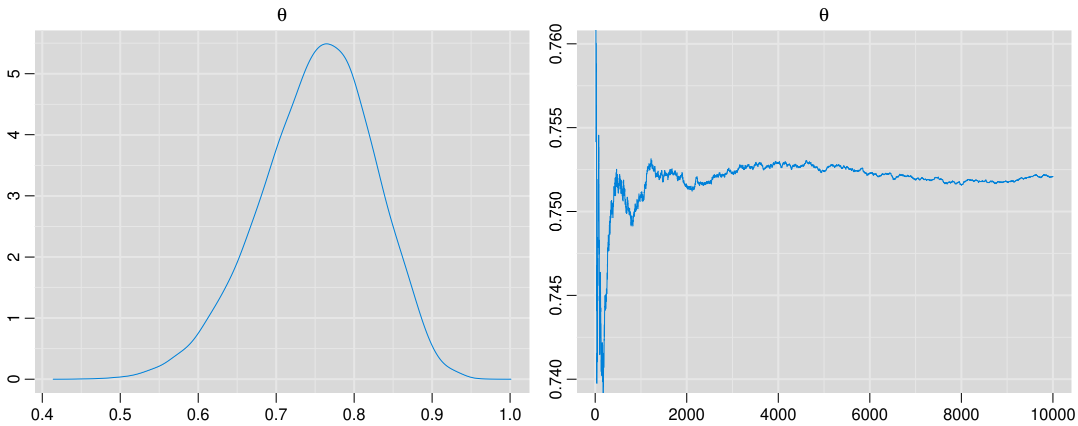

Figure 5. Density (left) and ergodic mean (right) plot of the posterior distribution of θ.

that removals occur at the end of the experiment, that is at the kth failure, gives the optimal scheme. Additionally, we presented a comparative study to examine whether the jointly censoring scheme has superiority over the censoring separately or not. This comparative study shows us that jointly censored samples contribute to better estimation performances.

Appendix-A

The second order partial derivatives of the log-likelihood function in equation (4) with respect to α, λ, and β are

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{k_1}{\alpha^2}, \qquad \frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{k_2}{\lambda^2}, \qquad \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} = 0,$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = -\frac{k}{\beta^2} - \alpha A_1^{''}(\beta) - \lambda A_2^{''}(\beta), \qquad \frac{\partial^2 \ell}{\partial \alpha \partial \beta} = -A_1^{''}(\beta), \qquad \frac{\partial^2 \ell}{\partial \lambda \partial \beta} = -A_2^{''}(\beta),$$

where

$$A_1^{''}(\beta) = \sum_{i=1}^{k-1} (R_i + 1)\, w_i^{\beta} \log^2(w_i) + \left(m - \sum_{i=1}^{k-1} (R_i + 1)\right) w_k^{\beta} \log^2(w_k),$$

and

$$A_2^{''}(\beta) = \sum_{i=1}^{k-1} (R_i + 1)\, w_i^{\beta} \log^2(w_i) + \left(n - \sum_{i=1}^{k-1} (R_i + 1)\right) w_k^{\beta} \log^2(w_k)$$

Also,

$$A_1^{'''}(\beta) = \sum_{i=1}^{k-1} (R_i + 1)\, w_i^{\beta} \log^3(w_i) + \left(m - \sum_{i=1}^{k-1} (R_i + 1)\right) w_k^{\beta} \log^3(w_k),$$

$$A_2^{'''}(\beta) = \sum_{i=1}^{k-1} (R_i + 1)\, w_i^{\beta} \log^3(w_i) + \left(n - \sum_{i=1}^{k-1} (R_i + 1)\right) w_k^{\beta} \log^3(w_k).$$

Appendix-B

Algorithm 1: Constrained optimization method using VNS based algorithm

Input: Fix n, m and k
Output: $R^{opt} = R^{new}$
Data: BJPC Weibull samples

Choose a random point R_0 from $CS(n, m, k)$
Find $E(W_k)$ by solving (4) for given n, m and k
Set $R^{int} = R_0$ as initial solution;
Suppose, $\ell_{\max} = \max R_{0j}, 1 \leq j \leq k-1$
Set $\ell = 1$
while $\ell \leq \ell_{max}$ **do**
    K_ℓ denotes the number of points in each neighborhood
for $i = 1, \cdots, K_\ell$ **do**
    Perform local search on $N_\ell(R_0)$
    Suppose R^{new} is a neighbor of R_0 in $N_\ell(R_0)$
    if $\left(\phi\left(R^{new}\right) < \phi\left(R^{int}\right)\right)$ **then**
        $R^{new} = R^{int}$
        $\ell = 1$
    else
        $\ell = \ell + 1$
$R^{opt} = R^{new}$

Appendix-C

Proof of Lemma 4.1 can be obtained by generalization to Weibull populations from the results for exponential samples by Mondal and Kundu [31] as follows. For this purpose, let define

$$Q_h = \left\{ \mathbf{z} = (z_1, \ldots, z_k) : \sum_{i=1}^{k} z_i = h \right\}; h = 0, \ldots, k$$

$$\begin{aligned}
E\left(e^{tW_j}\right) &= \sum_{h=0}^{k} \sum_{\mathbf{z}\in Q_r} E\left(e^{tW_j} \mid k_1 = h, \mathbf{Z} = \mathbf{z}\right) P\left(\mathbf{Z} = \mathbf{z} \mid k_1 = h\right) P\left(k_1 = h\right) \\
&= C \sum_{h=1}^{k-1} \sum_{z\in Q_h} \int_0^\infty \int_{w_1}^\infty \cdots \int_{w_{k-1}}^\infty \beta^k \alpha^h \lambda^{k-h} e^{tW_j} \prod_{i=1}^{k} w_i^{\beta-1} e^{-\alpha A_1(\beta) - \lambda A_2(\beta)} dw_k \ldots dw_2 dw_1 \\
&= C \sum_{h=0}^{\infty} \sum_{\mathrm{z}\in Q_h} \alpha^h \lambda^{k-h} \\
&\times \left\{a_k \left(a_k + a_{k-1}\right) \cdots \left(a_k + a_{k-1} + \cdots + a_{j+1}\right)\right. \\
&\times \left.\left(a_k + a_{k-1} + \cdots + a_{j+1} + a_j' + a_{j-1}\right)\right\}^{-1} \cdots \\
&\times \left(a_k + a_{k-1} + \cdots + a_{j+1} + a_j' + a_{j-1} + \cdots + a_1\right)^{-1} \\
&= C \sum_{h=0}^{\infty} \sum_{\mathrm{z}\in Q_h} \alpha^h \lambda^{k-h} \\
&\times \left\{a_k \left(a_k + a_{k-1}\right) \cdots \left(a_k + a_{k-1} + \cdots + a_j\right) \cdots\right. \\
&\times \left.\left(a_k + a_{k-1} + \cdots + a_j + a_{j-1} + \cdots a_1\right)\right\}^{-1} \\
&\times \frac{\left(a_k + a_{k-1} + \cdots + a_j\right) \cdots \left(a_k + a_{k-1} + \cdots + a_j + a_{j-1} + \cdots + a_1\right)}{\left(a_k + a_{k-1} + \cdots + a_j'\right) \cdots \left(a_k + a_{k-1} + \cdots + a_j' + aj - 1 + \cdots + a_1\right)} \\
&= \sum_{h=0}^{k} P\left(k_1 = h\right) \\
&\times \left\{ \frac{\left(a_k + a_{k-1} + \cdots + a_j\right) \cdots \left(a_k + a_{k-1} + \cdots + a_j + a_{j-1} + \cdots + a_1\right)}{\left(a_k + a_{k-1} + \cdots + a_j'\right) \cdots \left(a_k + a_{k-1} + \cdots + a_j' + a_{j-1} + \cdots + a_1\right)} \right\} \\
&= \left\{ \frac{\left(a_k + a_{k-1} + \cdots + a_j\right) \cdots \left(a_k + a_{k-1} + \cdots + a_j + a_{j-1} + \cdots + a_1\right)}{\left(a_k + a_{k-1} + \cdots + a_j'\right) \cdots \left(a_k + a_{k-1} + \cdots + a_j' + a_{j-1} + \cdots + a_1\right)} \right\} \\
&= \times \sum_{h=0}^{k} P\left(k_1 = h\right) \\
&= \prod_{s=1}^{j} \left(1 - \frac{t}{E_s}\right)^{-1}
\end{aligned}$$

where

$$a_j = (\alpha+\lambda)(R_j+1), \quad j=1,\ldots,k-1$$

$$a_k = \alpha\left(m-\sum_{j=1}^{k-1}(R_j+1)\right)+\lambda\left(n-\sum_{j=1}^{k-1}(R_j+1)\right); a'_j = a_j - t$$

$$E_s = \alpha\left(m-\sum_{j=1}^{s-1}(R_j+1)\right)+\lambda\left(n-\sum_{j=1}^{s-1}(R_j+1)\right)$$

Table 9. Optimal BJPC schemes under the Bayesian framework with different choice of n, m and k.

(m,n,k)	φ_1	φ_2	Optimum Scheme	Optimum Value
	0.2	0.8	$(0_{(7)})$	0.51154
	0.4	0.6	$(0_{(7)})$	0.38859
	0.5	0.5	$(0_{(7)})$	0.32985
15,12,8	0.6	0.4	$(4,0_{(6)})$	0.34214
	0.8	0.2	$(0_{(7)})$	0.14998
	0.0	1.0	$(3,0_{(6)})$	0.63163
	1.0	0.0	$(0_{(7)})$	0.03285
	0.2	0.8	$(0_{(9)})$	0.70152
	0.4	0.6	$(0_{(9)})$	0.54955
	0.5	0.5	$(0_{(9)})$	0.46206
15,12,10	0.6	0.4	$(0_{(9)})$	0.38017
	0.8	0.2	$(0_{(9)})$	0.20272
	0.0	1.0	$(2,0_{(8)})$	0.87973
	1.0	0.0	$(0_{(9)})$	0.02489
	0.2	0.8	$(0_{(9)})$	0.43264
	0.4	0.6	$(0_{(9)})$	0.34738
	0.5	0.5	$(4,0_{(8)})$	0.30163
20,14,10	0.6	0.4	$(3,1,0_{(7)})$	0.26311
	0.8	0.2	$(3,0,1,0_{(6)})$	0.52698
	0.0	1.0	$(1,1,0_{(5)},1,1)$	0.53662
	1.0	0.0	$(0_{(9)})$	0.03067

Table 10. Comparison of the performances of the estimates of $\theta = P(X > Y)$ between PC and JPC model with different choice of (n, m, k) and censoring schemes.

(n, m)	r	Schemes	(α, λ, β)	$Max(X_{(k)}, Y_{(k)})$	$W_{(k)}$	MSE_{PC}	MSE_{JPC}
(10,10)	5	$(0_{(4)}, 5)$	(2,2,3)	0.7308	0.6978	0.1682	0.0139
(10,10)	5	$(1_{(5)})$	(2,2,3)	1.0621	0.9875	0.0881	0.0280
(10,10)	5	$(2, 1, 0_{(2)}, 2)$	(2,2,3)	0.8510	0.8136	0.1314	0.0289
(20,20)	10	$(0_{(9)}, 10)$	(2,2,3)	0.7277	0.7079	0.1526	0.0005
(20,20)	10	$(1_{(10)})$	(2,2,3)	1.1265	1.0614	0.0461	0.0129
(20,20)	10	$(2_{(2)}, 1, 0_{(4)}, 1, 2_{(2)})$	(2,2,3)	0.8851	0.8498	0.0927	0.0125
(30,30)	15	$(0_{(14)}, 15)$	(2,2,3)	0.7265	0.7126	0.1448	0.0018
(30,30)	15	$(1_{(15)})$	(2,2,3)	1.1588	1.0995	0.0299	0.0087
(30,30)	15	$(2_{(3)}, 1, 0_{(3)}, 1, 0_{(3)}, 1, 2_{(3)})$	(2,2,3)	0.9152	0.8813	0.0647	0.0085
(10,10)	5	$(0_{(4)}, 5)$	(1.25,0.75,0.50)	0.9620	0.7566	0.1089	0.0011
(10,10)	5	$(1_{(5)})$	(1.25,0.75,0.50)	10.3173	7.4502	0.0468	0.0286
(10,10)	5	$(2, 1, 0_{(2)}, 2)$	(1.25,0.75,0.50)	2.4977	1.9943	0.0728	0.0264
(20,20)	10	$(0_{(9)}, 10)$	(1.25,0.75,0.50)	0.9247	0.7917	0.0763	0.0162
(20,20)	10	$(1_{(10)})$	(1.25,0.75,0.50)	13.6344	10.0731	0.0228	0.0213
(20,20)	10	$(2_{(2)}, 1, 0_{(4)}, 1, 2_{(2)})$	(1.25,0.75,0.50)	3.0413	2.4184	0.0373	0.0212
(30,30)	15	$(0_{(14)}, 15)$	(1.25,0.75,0.50)	0.8827	0.7877	0.0633	0.0078
(30,30)	15	$(1_{(15)})$	(1.25,0.75,0.50)	15.7577	11.9923	0.0155	0.0196
(30,30)	15	$(2_{(3)}, 1, 0_{(3)}, 1, 0_{(3)}, 1, 2_{(3)})$	(1.25,0.75,0.50)	3.5405	2.8645	0.0262	0.0189

Appendix-D

The following R codes can be used for computations.

```
MCN<-2000;BOOT<-500;MCMC<-3500;idx<-seq(503,3500,3)
al<-1;lm<-.75;bt<-2
a1<-0.0001;b1<-0.0001;a2<-0.0001;b2<-0.0001;a3<-0.0001;b3<-0.0001

m<-50;n<-40;k<-30;R<-c(10,rep(0,28));R0<-lm/(al+lm)

f_bt<-function(beta){
  a1<-sum((1+R)*(wa^beta))+(m-sum((R+1)))*(wk^beta)
  a2<-sum((1+R)*(wa^beta))+(n-sum((R+1)))*(wk^beta)
  a11<-sum(((1+R)*(wa^beta)*log(wa)))+(m-sum((R+1)))*(wk^beta)*log(wk)
  a22<-sum(((1+R)*(wa^beta)*log(wa)))+(n-sum((R+1)))*(wk^beta)*log(wk)
  k/beta+sum(log(w))-k1*a11/a1-k2*a22/a2
}

f_bot<-function(beta){
  a1b<-sum((1+R)*(wab^beta))+(m-sum((R+1)))*(wkb^beta)
  a2b<-sum((1+R)*(wab^beta))+(n-sum((R+1)))*(wkb^beta)
  a11b<-sum(((1+R)*(wab^beta)*log(wab)))+
    (m-sum((R+1)))*(wkb^beta)*log(wkb)
  a22b<-sum(((1+R)*(wab^beta)*log(wab)))+
    (n-sum((R+1)))*(wkb^beta)*log(wkb)
  k/beta+sum(log(wb))-k1b*a11b/a1b-k2b*a22b/a2b
}

f_bys<-function(beta){
  a1bys<-sum((1+R)*(wa^beta))+(m-sum((R+1)))*(wk^beta)
  a2bys<-sum((1+R)*(wa^beta))+(n-sum((R+1)))*(wk^beta)
  (beta^(k+a3-1))*exp(-beta*(b3-sum(log(w))))*
    exp(-alf_bys0[jj]*a1bys-lam_bys0[jj]*a2bys)
}

rlb<-rlnd<-rbys<-0;ACI<-bootp<-boott<-HPD<-matrix(0,MCN,2)

for(j in 1:MCN){

  x<-(-(log(1-runif(m)))/al)^(1/bt);y<-(-(log(1-runif(n)))/lm)^(1/bt)
  w0<-c(x,y);kg<-c(rep(1,m),rep(0,n))
  veri<-data.frame(kg,w0);verim<-veri[order(veri$w0),]
  w1<-verim[,2];z0<-verim[,1]

  dataw<-dataz<-0
  for(i in 1:(k-1)){
    dataw[i]<-w1[1];dataz[i]<-z0[1]
    qs1<-c(sample(which(z0==1),R[i]),sample(which(z0==0),R[i]+1))
    qs2<-c(sample(which(z0==1),R[i]+1),sample(which(z0==0),R[i]))
    if(dataz[i]==1){
```

```
    w1<-w1[-c(1,qs1)];z0<-z0[-c(1,qs1)]
  }else{
    w1<-w1[-c(1,qs2)];z0<-z0[-c(1,qs2)]
  }
}
dataw[k]<-w1[1];dataz[k]<-z0[1]

w<-dataw;z<-dataz;k1<-sum(z);k2<-sum(1-z);wa<-w[1:(k-1)];wk<-w[k]
bt_mle<-uniroot(f_bt,c(0.001,100))$root
alf_mle<-k1/(sum((1+R)*(wa^bt_mle))+(m-sum(R+1))*(wk^bt_mle))
lam_mle<-k2/(sum((1+R)*(wa^bt_mle))+(n-sum(R+1))*(wk^bt_mle))

if(k1==0|k2==0) next

rlb[j]<-(lam_mle)/(alf_mle+lam_mle)

AA1<-sum((1+R)*(wa^bt_mle)*((log(wa))^2))+(m-sum((R+1)))*
  (wk^bt_mle)*((log(wk))^2)
AA2<-sum((1+R)*(wa^bt_mle)*((log(wa))^2))+(n-sum((R+1)))*
  (wk^bt_mle)*((log(wk))^2)
Fsr<-matrix(0,3,3)
Fsr[1,1]<-k1/(alf_mle^2); Fsr[2,2]<-k2/(lam_mle^2)
Fsr[3,3]<-k/(bt_mle^2)+alf_mle*(AA1)+lam_mle*(AA2)
Fsr[1,3]<-Fsr[3,1]<-sum(((1+R)*(wa^bt_mle)*log(wa)))+
  (m-sum((R+1)))*(wk^bt_mle)*log(wk)
Fsr[2,3]<-Fsr[3,2]<-sum(((1+R)*(wa^bt_mle)*log(wa)))+
  (n-sum((R+1)))*(wk^bt_mle)*log(wk)
InvM<-solve(Fsr)
R1<--lam_mle/((alf_mle+lam_mle)^2);R2<-alf_mle/((alf_mle+lam_mle)^2)
Sigma<-(R1^2)*InvM[1,1]+(R2^2)*InvM[2,2]+2*R1*R2*InvM[1,2]
ACI[j,]<-c(rlb[j]-qnorm(0.975)*sqrt(Sigma),
           rlb[j]+qnorm(0.975)*sqrt(Sigma))

q1<-(a1-1)/alf_mle-b1;q2<-(a2-1)/lam_mle-b2;q3<-(a3-1)/bt_mle-b3

u1<--lam_mle/((alf_mle+lam_mle)^2);u2<-alf_mle/((alf_mle+lam_mle)^2)
u3<-0;u11<-2*lam_mle/((alf_mle+lam_mle)^3)
u12<-u21<-(lam_mle-alf_mle)/((alf_mle+lam_mle)^3)
u22<--2*alf_mle/((alf_mle+lam_mle)^3);u13<-u31<-u32<-u23<-u33<-0

A13B<-sum((1+R)*(wa^bt_mle)*((log(wa))^3))+
  (m-sum((R+1)))*(wk^bt_mle)*((log(wk))^3)
A23B<-sum((1+R)*(wa^bt_mle)*((log(wa))^3))+
  (n-sum((R+1)))*(wk^bt_mle)*((log(wk))^3)

l111<-(2*k1)/(alf_mle^3);l222<-(2*k2)/(lam_mle^3)
l333<-(2*k)/(bt_mle^3)-alf_mle*A13B-lam_mle*A23B
l331<--AA1;l332<--AA2;l112<-l113<-l221<-l223<-0

sum1<-(u11+2*u1*q1)*InvM[1,1]+(u12+2*u1*q2)*InvM[1,2]+
```

```
    (u13+2*u1*q3)*InvM[1,3]+(u21+2*u2*q1)*InvM[2,1]+
    (u22+2*u2*q2)*InvM[2,2]+(u23+2*u2*q3)*InvM[2,3]+
    (u31+2*u3*q1)*InvM[3,1]+(u32+2*u3*q2)*InvM[3,2]+
    (u33+2*u3*q3)*InvM[3,3]
  sum2<-l111*InvM[1,1]*(InvM[1,1]*u1+InvM[1,2]*u2+InvM[1,3]*u3)+
    l222*InvM[2,2]*(InvM[2,1]*u1+InvM[2,2]*u2+InvM[2,3]*u3)+
    l333*InvM[3,3]*(InvM[3,1]*u1+InvM[3,2]*u2+InvM[3,3]*u3)

  rlnd[j]<-rlb[j]+sum1/2+sum2/2;rlb_boot<-tstar<-0

  for(u in 1:BOOT){

    xb<-(-(log(1-runif(m)))/alf_mle)^(1/bt_mle)
    yb<-(-(log(1-runif(n)))/lam_mle)^(1/bt_mle)
    w0b<-c(xb,yb);kgb<-c(rep(1,m),rep(0,n))
    verib<-data.frame(kgb,w0b);verimb<-verib[order(verib$w0b),]
    w1b<-verimb[,2];z0b<-verimb[,1]

    datawb<-datazb<-0
    for(i in 1:(k-1)){
      datawb[i]<-w1b[1];datazb[i]<-z0b[1]
      qs1b<-c(sample(which(z0b==1),R[i]),sample(which(z0b==0),R[i]+1))
      qs2b<-c(sample(which(z0b==1),R[i]+1),sample(which(z0b==0),R[i]))
      if(datazb[i]==1){
        w1b<-w1b[-c(1,qs1b)];z0b<-z0b[-c(1,qs1b)]
      }else{
        w1b<-w1b[-c(1,qs2b)];z0b<-z0b[-c(1,qs2b)]
      }
    }
    datawb[k]<-w1b[1];datazb[k]<-z0b[1]

    wb<-datawb;zb<-datazb;k1b<-sum(zb);k2b<-sum(1-zb)
    wab<-wb[1:(k-1)];wkb<-wb[k]
    bt_boot<-uniroot(f_bot,c(0.001,100))$root
    alf_boot<-k1b/(sum((1+R)*(wab^bt_boot))+(m-sum(R+1))*(wkb^bt_boot))
    lam_boot<-k2b/(sum((1+R)*(wab^bt_boot))+(n-sum(R+1))*(wkb^bt_boot))
    if(k1b==0|k2b==0) next
    rlb_boot[u]<-(lam_boot)/(alf_boot+lam_boot)

    AA1b<-sum((1+R)*(wab^bt_boot)*((log(wab))^2))+
      (m-sum((R+1)))*(wkb^bt_boot)*((log(wkb))^2)
    AA2b<-sum((1+R)*(wab^bt_boot)*((log(wab))^2))+
      (n-sum((R+1)))*(wkb^bt_boot)*((log(wkb))^2)
    Fsrb<-matrix(0,3,3)
    Fsrb[1,1]<-k1b/(alf_boot^2);Fsrb[2,2]<-k2b/(lam_boot^2)
    Fsrb[3,3]<-k/(bt_boot^2)+alf_boot*(AA1b)+lam_boot*(AA2b)
    Fsrb[1,3]<-Fsrb[3,1]<-sum(((1+R)*(wab^bt_boot)*log(wab)))+
      (m-sum((R+1)))*(wkb^bt_boot)*log(wkb)
    Fsrb[2,3]<-Fsrb[3,2]<-sum(((1+R)*(wab^bt_boot)*log(wab)))+
      (n-sum((R+1)))*(wkb^bt_boot)*log(wkb)
```

```
    InvMb<-solve(Fsrb)
    R1b<--lam_boot/((alf_boot+lam_boot)^2)
    R2b<-alf_boot/((alf_boot+lam_boot)^2)
    hata<-(R1b^2)*InvMb[1,1]+(R2b^2)*InvMb[2,2]+2*R1b*R2b*InvMb[1,2]
    tstar[u]<-(rlb_boot[u]-rlb[j])/sqrt(hata)
  }

  tstar<-na.omit(tstar);rlb_boot<-na.omit(rlb_boot)
  bootp[j,]<-quantile(rlb_boot,c(0.025,0.975))
  boott[j,]<-c(rlb[j]+quantile(tstar,c(0.025))*sqrt(Sigma),
               rlb[j]+quantile(tstar,c(0.975))*sqrt(Sigma))


  alf_bys0<-alf_mle;lam_bys0<-lam_mle;bet_bys0<-bt_mle;
  Vb<-InvM[3,3];R_mcmc<-0

  for(jj in 2:MCMC){

    alf_bys0[jj]<-rgamma(1,k1+a1,b1+sum((1+R)*(wa^bet_bys0[jj-1]))+
                           (m-sum((R+1)))*(wk^bet_bys0[jj-1]))
    lam_bys0[jj]<-rgamma(1,k2+a2,b2+sum((1+R)*(wa^bet_bys0[jj-1]))+
                           (n-sum((R+1)))*(wk^bet_bys0[jj-1]))
    v0<-bet_bys0[jj-1];w0<-abs(rnorm(1,v0,Vb))
    pm<-min((f_bys(w0)*rnorm(1,v0,Vb))/(f_bys(v0)*rnorm(1,w0,Vb)),1)
    u<-runif(1)
    if(u<pm) bet_bys0[jj]<-w0 else bet_bys0[jj]<-v0
    R_mcmc[jj]<-lam_bys0[jj]/(alf_bys0[jj]+lam_bys0[jj])
  }

  R_mcmc<-na.omit(R_mcmc[idx])
  rbys[j]<-mean(R_mcmc);HPD[j,]<-quantile(R_mcmc,c(0.025,0.975))
  print(j)
}

CP_ACI<-(length(which(ACI[,1]<R0&ACI[,2]>R0))/MCN)*100
CP_bootp<-(length(which(bootp[,1]<R0&bootp[,2]>R0))/MCN)*100
CP_boott<-(length(which(boott[,1]<R0&boott[,2]>R0))/MCN)*100
CP_HPD<-(length(which(HPD[,1]<R0&HPD[,2]>R0))/MCN)*100
lng_ACI<-mean(ACI[,2])-mean(ACI[,1])
lng_bootp<-mean(bootp[,2])-mean(bootp[,1])
lng_boott<-mean(boott[,2])-mean(boott[,1])
lng_HPD<-mean(HPD[,2])-mean(HPD[,1])

rlb<-na.omit(rlb);rlnd<-na.omit(rlnd);rbys<-na.omit(rbys)

row1<-c(R0-mean(rlb),R0-mean(rlnd),R0-mean(rbys),
        lng_ACI,lng_bootp,lng_boott,lng_HPD)
row2<-c(mean((rlb-R0)^2),mean((rlnd-R0)^2),mean((rbys-R0)^2),
        CP_ACI,CP_bootp,CP_boott,CP_HPD)
Results<-data.frame(rbind(row1,row2))
```

```
round(Results,4)
```

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