Step-stress Life-testing under Tampered Random Variable Modeling for Weibull Distribution in Presence of Competing Risk Data

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Abstract

In literature, analysis of competing risk data based on different accelerated life testing (ALT) modelings has been considered by many authors. However, as per our knowledge, analysis of the Tampered Random Variable (TRV) modeling in presence of competing risk data is not studied yet in the literature. In this paper, we considered the TRV modeling for a simple step-stress life-testing (SSLT) where failures are observed due to more than one cause of failure. The lifetime of the experimental units at each stress level follows Weibull distributions with the same shape parameter and different scale parameters. In modeling of step-stress data by using TRV modeling we introduced different tempering co-efficient for different causes of failures. The maximum likelihood estimates of the model parameters and the tempering co-efficients are obtained by using SSLT data. The associated asymptotic confidence intervals

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for all the unknown parameters are obtained using Type-II censoring scheme. Further, we consider the Bayesian inference of the unknown model parameters based on fairly general prior distributions. An extensive simulation study is performed to examine the method of inference developed here. Then, we illustrate the proposed methodology with a real data set. We also provide an optimality criterion to determine the optimal stress change time. Finally, the sensitivity analysis is performed.

Keywords: Baseline lifetime, Tampered Random Variable Modeling, Tampering coefficients, Competing Risk, Maximum Likelihood estimation, Bayesian Analysis

1 Introduction

Due to the continuous development and technological improvement of the manufacturing industry, most of the industrial products, are highly reliable with complex structure, are commonly appearing in our daily life. This difficulty is overcome by accelerated life testing (ALT), wherein the experimental products are subjected to higher stress levels than the normal operating condition in order to cause rapid failures. The factor that affect the lifetime of the products are called stress factor. For example, temperature, voltage, humidity could be the stress factors for an electronic equipment such as electric bulb, fan, laptop, toaster etc. Due to this increased stress-factor in ALT experiment, one can obtained valuable information about the product reliability within short experimental time.

The ALT experiment can be performed either a constant high stress level from the beginning of the test or different stress factor can be put in different time interval. A special class of the ALT which allows the experimenter to gradually increase the stress levels, during the experiment, at some pre-fixed time points known as step-stress life testing (SSLT). In such a life-testing experiment, a simple SSLT is a special case of a SSLT when it involves only two stress levels s_1, s_2 and one pre-fixed stress changing time point, say τ. Obviously we can generalized this idea from simple SSLT to a multiple SSLT experiment by considering more than two stress levels and stress changing time points accordingly. The lifetime distribution under the initial normal stress level is termed as the baseline lifetime distribution.

To relate the different lifetime distributions under the different stress levels, there are three different types of modeling assumptions. One of these is the Cumulative Exposure (CE) modeling in which unknown constraints are introduced so that the two distributions at any two successive stress levels coincide at the corresponding change time point to ensure continuity. See, for example, Sedyakin (1966), Nelson (1980). Another approach, known as Tampered Failure Rate (TFR) modeling, scales up the failure rates at the successive stress levels. See Bhattacharyya and Soejoeti (1989) and Madi (1993) among others. The third approach, known as Tampered Random Variable (TRV) modeling, scales down the remaining lifetime at the successive stress levels. See Goel (1971), DeGroot and Goel (1979) for details. Sultana and Dewanji (2021) shown the relationship between the TRV model with the other two models, namely, CE and TFR, under the multiple step-stress framework. Also, the authors mentioned that TRV modeling equivalent with CEM and TFR if and only if the baseline lifetime is exponential and the marginal distributions of each stress levels follows scale parametric family, respectively. Therefore, one can note that these three models are coincides when the baseline distribution is exponential. The main advantage of the TRV modeling approach is that it can be readily generalized for multiple step-stress over the other two modelings. Further, it is also useful to model discrete and multivariate life time which are not straightforward for other two modelings. There are many works in the literature based on competing risk scenario using CEM and TFR modeling. However, as per our knowledge, there are no work based on TRV modeling in presence of competing risk data. In this paper, we consider the TRV modeling when the data are coming in competing risk pattern and we are taken Weibull lifetime distribution for the baseline lifetime.

In this paper, we consider a simple SSLT model for complete sample when only two different stress levels, say, s_1, s_2 and only two causes of failures say, Cause-I and Cause-II are present. Here, it is assume that when the pre fixed time point τ occurred then the stress level changes from s_1 to s_2.

In this problem, the failure time distribution for T_1 and T_2 at the stress level s_1 and for $cause-j$ is assumed to be a Weibull distribution with the common shape parameter α and the different scale parameters θ_j for $j = 1, 2$, respectively. Then the tampered random variable (TRV) model for T_1 and T_2 can be written as

$$\tilde{T}_1 = \begin{cases} T_1, & \text{if } 0 < t \leq \tau \\ \tau + \beta(T_2 - \tau), & \text{if } t > \tau, \end{cases} \tag{1}$$

and

$$\tilde{T}_2 = \begin{cases} T_2, & \text{if } 0 < t \leq \tau \\ \tau + \beta(T_2 - \tau), & \text{if } t > \tau, \end{cases} \tag{2}$$

where τ is the stress changing time, and $0 < \beta < 1$ is the acceleration factor.

The rest of the paper is organized as follows. In Section 2, the model is described along with the corresponding likelihood for Type-II censored data and we have shown the existence and uniqueness of the MLEs in graphical representation. Section 3 describes the associated asymptotic confidence intervals for all the unknown parameters are obtained using simple SSLT TRV modeling. The Bayesian inference, of the unknown model parameters based on fairly general prior distributions, is also considered in Section 4. Section 5, we provide some likelihood-ratio tests to evaluate the equivalence of the shape parameters α_1, α_2 and the importance of acceleration factor β among two different Weibull competing risks model under Type-II censoring scheme. while Section 6, presents some simulation studies to investigate the finite sample properties of the MLEs. We illustrate the proposed methods through the analysis of a real life data sets in Section 7. In section 8, we obtain an optimal stress changing time point τ by using different optimality criteria, while Section 9 ends with some concluding remarks.

2 Model Description and Likelihood Function

Let n identical units are placed on a life test under initial stress level s_1. Then the successive failure times and the corresponding risk factor are recorded. When the time τ, prefixed, occurred the stress level is changed from s_1 to s_2 and the life test continues until a pre specified $r(\leq n)$ number of failures are observed. When r is taken to be n then a complete data set of failure observations would result for this simple SSLT (i.e., no censoring). Suppose each unit fails by one of two fatal risk factors and the time-to-failure by each competing risk has an independent Weibull distribution with same shape parameter α, which follows the TRV model. Let θ_j be the location parameter by the risk factor j for $j = 1, 2$. Then the cumulative distribution function (CDF) of the lifetime T_j due to the risk j for $j = 1, 2$ is given by

$$F_j(t) = F_j(t; \alpha, \theta_j) = \begin{cases} 1 - e^{-\theta_j t^{\alpha}}, & \text{if } 0 < t \leq \tau \\ 1 - e^{-\theta_j (\tau + \frac{t-\tau}{\beta})^{\alpha}}, & \text{if } t > \tau, \end{cases} \tag{3}$$

and the corresponding density function (PDF) of T_j is given by

$$f_j(t) = f_j(t; \alpha, \theta_j, \theta_{2j}) = \begin{cases} \alpha\, \theta_j\, t^{\alpha-1}\, e^{-\theta_j t^{\alpha}}, & \text{if } 0 < t \leq \tau \\ \frac{\alpha\theta_j}{\beta}(\tau + \frac{t-\tau}{\beta})^{\alpha-1} e^{-\theta_j\, (\tau + \frac{t-\tau}{\beta})^{\alpha}}, & \text{if } t > \tau, \end{cases} \tag{4}$$

for $j = 1, 2$. Since we observe $T = min\{T_1, T_2\}$, only the smaller of T_1 and T_2, which denote the overall failure time of a test unit. Then its CDF and PDF are radily obtained as

$$F(t) = F(t; \alpha, \theta) = 1 - (1 - F_1(t))(1 - F_2(t)) = \begin{cases} 1 - e^{-(\theta_1+\theta_2)t^{\alpha}}, & \text{if } 0 < t \leq \tau \\ 1 - e^{-(\theta_1+\theta_2)\left(\tau + \frac{t-\tau}{\beta}\right)^{\alpha}}, & \text{if } t > \tau, \end{cases} \tag{5}$$

and

$$f(t) = f(t;\alpha,\theta) = \begin{cases} \alpha\,(\theta_1+\theta_2)\,t^{\alpha-1}\,e^{-(\theta_1+\theta_2)t^{\alpha}}, & \text{if } 0 < t \le \tau \\ \frac{\alpha(\theta_1+\theta_2)}{\beta}\left(\tau+\frac{t-\tau}{\beta}\right)^{\alpha-1} e^{-(\theta_1+\theta_2)\left(\tau+\frac{t-\tau}{\beta}\right)^{\alpha}}, & \text{if } t > \tau, \end{cases} \tag{6}$$

respectively, where $\theta = (\theta_1, \theta_2)$. Furthermore, let C denote the indicator for the cause of failure. Then under our assumptions, the joint PDF of (T, C) is given by

$$f_{T,C}(t,j) = f_j(t)(1-F_k(t)) = \begin{cases} \alpha\,\theta_j\,t^{\alpha-1}\,e^{-(\theta_1+\theta_2)t^{\alpha}}, & \text{if } 0 < t \le \tau \\ \frac{\alpha\theta_j}{\beta}\left(\tau+\frac{t-\tau}{\beta}\right)^{\alpha-1} e^{-(\theta_1+\theta_2)\left(\tau+\frac{t-\tau}{\beta}\right)^{\alpha}}, & \text{if } t > \tau, \end{cases} \tag{7}$$

for $j, k = 1, 2$, $j \neq k$.

Let us now define

N_{1j} =the number of units that fail before τ due to the risk factor j,

N_{2j} =the number of units that fail after τ due to the risk factor j,

for $j = 1, 2$. If we let $\hat{N}_1$ denote the total number of failures before τ and $\hat{N}_2$ the total number of failures after τ, then according to the testing scheme we have $\hat{N}_1 = N_{11} + N_{12}$, $\hat{N}_2 = N_{21} + N_{22}$ with $\hat{N}_1 + \hat{N}_2 = r \le n$, with the life testing scheme described above, the following failure times will be then observed

$$\{0 < t_{1:n} < t_{2:n} < \cdots < t_{\hat{n}_1:n} < \tau \le t_{\hat{n}_1+1:n} < \cdots < t_{r:n}\},$$

where $\hat{n}_1$ denotes the observed value of $\hat{N}_1$. For notational simplicity, let us denote $\mathbf{N} = (\mathbf{N}_1, \mathbf{N}_2)$. For $\mathbf{N}_i = (N_{i1}, N_{i2})$ for $i = 1, 2$, let n denote the observed integer vector of $\mathbf{N}$.

Since each failure times is also accompanied by the corresponding cause of failure, let $\mathbf{c} = (c_1, c_2, \cdots c_r)$ be the observed sequence of causes of failure corresponding to the observed failure time $\mathbf{t} = (t_{1:n}, t_{2:n}, \cdots, t_{r:n})$. We also denote the relative risk on a test to the risk factor j by

$$\pi_j = \Pr[C = j] = \frac{\theta_j}{\theta_1 + \theta_2}, \qquad j = 1, 2.$$

Then under the assumption of the TRV model, we formulate the likelihood of $\eta = (\theta_1, \theta_2, \alpha, \beta)$ based on this Type-II censored data as

$$\begin{aligned} L(\eta) = L(\eta|(\mathbf{t}, \mathbf{c})) &= \frac{n!}{(n-r)!} \left(\prod_{i=1}^{\hat{n_1}} f_{T,C}(t_{i:n}, c_i) \right) \left(\prod_{i=\hat{n_1}+1}^{r} f_{T,C}(t_{i:n}, c_i) \right) [1 - F(t_{r:n})]^{n-r} \\ &= \frac{n!}{(n-r)!} \alpha^r \theta_1^{n_{11}+n_{21}} \theta_2^{n_{12}+n_{22}} \beta^{-(r-\hat{n}_1)} \prod_{i=1}^{\hat{n_1}} t_i^{\alpha-1} e^{-(\theta_1+\theta_2) t_i^\alpha} \times \\ &\prod_{i=\hat{n_1}+1}^{r} \left(\tau + \frac{t_i - \tau}{\beta} \right)^{\alpha-1} e^{-(\theta_1+\theta_2)\left(\tau + \frac{t_i-\tau}{\beta}\right)^\alpha} \times e^{-(\theta_1+\theta_2)(n-r)\left(\tau + \frac{t_{r:n}-\tau}{\beta}\right)^\alpha} \end{aligned}$$

The log-likelihood function can be written as

$$\begin{aligned} l(\eta) &= r \log \alpha + (n_{11} + n_{21}) \log \theta_1 + (n_{12} + n_{22}) \log \theta_2 - (r - \hat{n}_1) \log \beta \\ &+ (\alpha - 1) \left[\sum_{i=1}^{\hat{n}_1} \log t_i + \sum_{i=\hat{n}_1+1}^{r} \log \left(\tau + \frac{t_i - \tau}{\beta} \right) \right] - (\theta_1 + \theta_2) \sum_{i=1}^{\hat{n}_1} t_i^\alpha \\ &- (\theta_1 + \theta_2) \left[\sum_{i=\hat{n}_1+1}^{r} \left(\tau + \frac{t_i - \tau}{\beta} \right)^\alpha + (n - r) \left(\tau + \frac{t_{r:n} - \tau}{\beta} \right)^\alpha \right] \end{aligned} \tag{8}$$

The MLEs of the unknown parameters can be obtained by maximizing (8) with respect to the unknown parameters. For known α and β, the MLEs of θ_1 and θ_2 are given by

$$\hat{\theta}_1 = \frac{n_{11} + n_{21}}{D_1(\alpha, \beta)}, \quad \hat{\theta}_2 = \frac{n_{12} + n_{22}}{D_1(\alpha, \beta)},$$

Here, $r = \hat{n}_1 + \hat{n}_2 = n_{11} + n_{12} + n_{21} + n_{22}$ and

$$D_1(\alpha, \beta) = \sum_{i=1}^{\hat{n}_1} t_i^{\alpha} + \sum_{i=\hat{n}_1+1}^{r} \left(\tau + \frac{t_i - \tau}{\beta}\right)^{\alpha} + (n-r)\left(\tau + \frac{t_{r:n} - \tau}{\beta}\right)^{\alpha}.$$

Note that MLEs exist only for $n_{ij} > 0\ i, j = 1, 2$. In case of unknown α and β, the explicit form of MLEs of α and β do not exist. Let us consider the profile log-likelihood of α and β without the additive constant

$$\begin{aligned} l_1(\alpha, \beta) &= r \log \alpha - r \log D_1(\alpha, \beta) - (r - \hat{n}_1) \log \beta \\ &\quad + (\alpha - 1)\left[\sum_{i=1}^{\hat{n}_1} \log t_i + \sum_{i=\hat{n}_1+1}^{r} \log\left(\tau + \frac{t_i - \tau}{\beta}\right)\right] \end{aligned} \tag{9}$$

The MLEs of α and β, denoted by $\hat{\alpha}$ and $\hat{\beta}$ can be obtained with simultaneously solving two non-linear equations (see Appendix) which can be obtained by differentiating the profile log-likelihood given in (9) with respect to α and β and equating them to zero. The behavior of the log-likelihood function (9) is noteworthy to illustrate in terms of a detailed overview to have an idea of its uniqueness and existence. For this purpose, in Figures 1 and 2 the appropriate behavior pattern is illustrated by surface and contour plots. A simulated data is used by considering $n = 50$, $m = 45$ and two different sets of parameter values as $(\theta_1, \theta_2, \alpha, \beta, \tau) = (1.25, 1.25, 2.25, 2.00, 0.50, 0.60)$ as Set 1 and $(\theta_1, \theta_2, \alpha, \beta, \tau) = (0.50, 0.75, 0.50, 0.25, 1.00)$ as Set 2. It is seen that the log-likelihood function follows a quasi-tunnel shape. This confirms that the MLEs are unique.

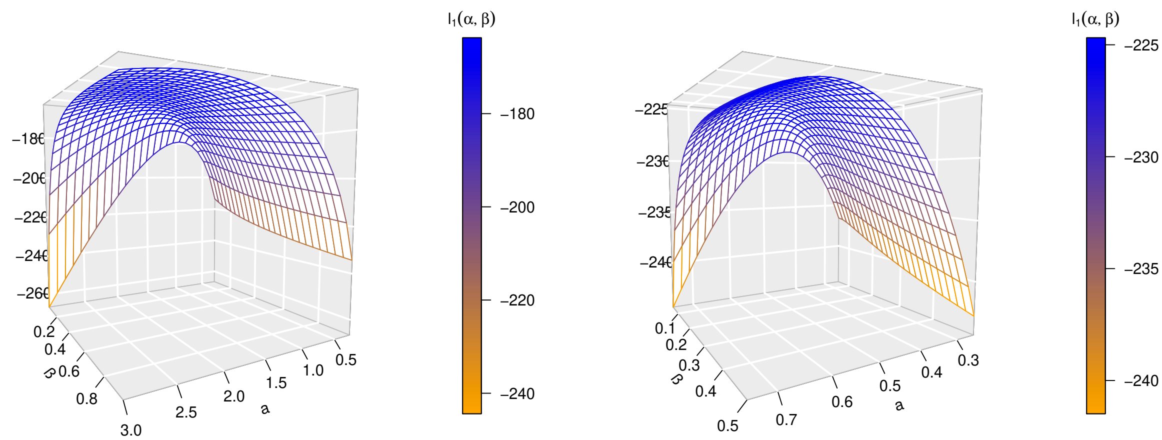

Figure (1) Surface plots for the profile log-likelihood function of α and β based on the simulated data $(n = 50, r = 45)$ under Set 1 (left) and Set 2 (right).

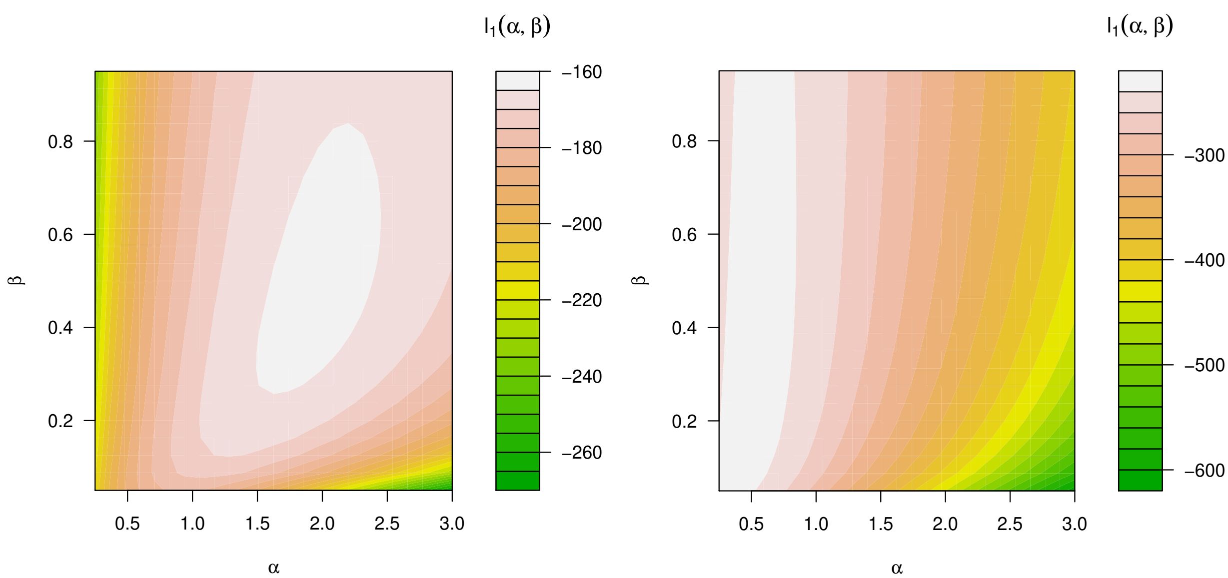

Figure (2) Contour plots for the profile log-likelihood function of α and β based on the simulated data $(n = 50, r = 45)$ under Set 1 (left) and Set 2 (right).

3 Asymptotic Confidence Interval

In this section we construct the asymptotic confidence intervals (ACI) of the unknown parameter by using the observed Fisher information matrix and then using asymptotic normality results of the

MLEs. Suppose the parameter vector $\eta = (\theta_1, \theta_2, \alpha, \beta)^T$. The observed Fisher information matrix is given by

$$F = (f_{ij}) = \left(-\frac{\partial^2 l}{\partial \eta_i \partial \eta_j} \right) \tag{10}$$

The elements of the Fisher information matrix are given in the Appendix. The asymptotic distribution of $\hat{\eta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\alpha}, \hat{\beta})^T$ is given by $\hat{\eta} - \eta \sim N_4(0, F^{-1})$. Therefore, $100(1-\delta)\%$ ACI of η_i is given by $\eta_i \pm Z_{1-\frac{p}{2}} \sqrt{v_{ii}}$ where v_{ii} is the $(i,i)^{th}$ element of the matrix F^{-1} which is given in Appendix.

In some cases, the approximate confidence intervals do not always guarantee to fall inside the unit interval $(0,1)$ for η and sometimes it can provide a higher upper or a negative lower bound. In these cases, $\sigma^2_{\ln \hat{\eta}} = \sigma_{\hat{\eta}} / \hat{\eta}^2$ and $\frac{\ln \hat{\eta} - \ln \eta}{\sqrt{\hat{\eta}^{-2} \sigma_{\hat{\eta}}}} \sim N(0,1)$ can be obtained by Taylor series expansion and the delta method. Thus, the $100(1-\gamma)\%$ asymptotic confidence interval of η can be obtained as follows

$$\left(\exp\left[\ln \hat{\eta} - Z_{\frac{\gamma}{2}} \hat{\eta}^{-1} \sigma_{\hat{\eta}} \right], \exp\left[\ln \hat{\eta} + Z_{\frac{\gamma}{2}} \hat{\eta}^{-1} \sigma_{\hat{\eta}} \right] \right) \tag{11}$$

where Z_γ denotes the 100γth percentile of the standard normal distribution $N(0,1)$.

4 Bayesian Inference

In this section, we handle the Bayesian inference method for the estimations of the unknown parameters $\eta = (\theta_1, \theta_2, \alpha, \beta)$. We first assume independent gamma priors for $\theta_1, \theta_2, \alpha$ and uniform prior as a flat prior for β. That is, θ_1, θ_2 and α have $GA(a_1, b_1)$, $GA(a_2, b_2)$ and $GA(a_3, b_3)$ priors with non-negative hyperparameters $a_1, b_1, a_2, b_2, a_3, b_3 > 0$ and β follows uniform prior as $\pi(\beta) = 1$, $0 < \beta < 1$. Thus, assumed joint prior density of independent parameters can be given as

$$
\begin{aligned}
\pi(\eta) &= \pi(\theta_1)\pi(\theta_2)\pi(\alpha)\pi(\beta) \\
&\propto \theta_1^{a_1-1}e^{-b_1\theta_1}\theta_2^{a_2-1}e^{-b_2\theta_2}\alpha^{a_3-1}e^{-b_3\alpha}, \qquad \theta_1>0,\theta_2>0,\alpha>0,0<\beta<1
\end{aligned} \tag{12}
$$

The joint posterior density function of parameters can be obtained by using the observed censored samples and the prior distributions of the parameters as

$$
\begin{aligned}
L(\mathbf{t},\mathbf{c},\eta) &\propto L(\eta|\mathbf{t},\mathbf{c})\pi(\theta_1)\pi(\theta_2)\pi(\alpha)\pi(\beta) \\
&\times \alpha^{r+a_3-1}\theta_1^{n_{11}+n_{21}+a_1-1}\theta_2^{n_{12}+n_{22}+a_2-1}\beta^{-(r-\hat{n}_1)}e^{-\alpha b_3} \\
&\times e^{(\alpha-1)\left[\sum_{i=1}^{\hat{n}_1}\log(t_i)+\sum_{i=\hat{n}_1+1}^{r}\log\left(\tau+\frac{t_i-\tau}{\beta}\right)\right]}e^{-\theta_1\left(b_1+D_1(\alpha,\beta)\right)}e^{-\theta_2\left(b_2+D_1(\alpha,\beta)\right)}
\end{aligned} \tag{13}
$$

Thus, the posterior distribution of the parameters θ_1, θ_2, α and β can be obtained by simplifying the Equation (13) as follows

$$
\begin{aligned}
\pi(\theta_1|\alpha,\beta,\mathbf{t},\mathbf{c}) &\propto \mathrm{GA}(n_{11}+n_{21}+a_1, D_1(\alpha,\beta)+b_1) \\
\pi(\theta_2|\alpha,\beta,\mathbf{t},\mathbf{c}) &\propto \mathrm{GA}(n_{12}+n_{22}+a_2, D_1(\alpha,\beta)+b_2)
\end{aligned}
$$

and

$$
\pi(\alpha|\theta_1,\theta_2,\beta,\mathbf{t},\mathbf{c}) \propto \alpha^{r+a_3-1}e^{-\alpha\left[b_3-\sum_{i=1}^{\hat{n}_1}\log(t_i)-\sum_{i=\hat{n}_1+1}^{r}\log\left(\tau+\frac{t_i-\tau}{\beta}\right)\right]}e^{-(\theta_1+\theta_2)D_1(\alpha,\beta)}
$$

$$
\begin{aligned}
\pi(\beta|\theta_1,\theta_2,\alpha,\mathbf{t},\mathbf{c}) &\propto \beta^{-(r-\hat{n}_1)}e^{(\alpha-1)\sum_{i=\hat{n}_1+1}^{r}\log\left(\tau+\frac{t_i-\tau}{\beta}\right)} \\
&\times e^{-(\theta_1+\theta_2)\left[\sum_{i=\hat{n}_1+1}^{r}\left(\tau+\frac{t_i-\tau}{\beta}\right)^{\alpha}+(n-r)\left(\tau+\frac{t_{r:n}-\tau}{\beta}\right)^{\alpha}\right]}
\end{aligned}
$$

It is clearly seen that the samples for θ_i, $i=1,2$ can be generated using the posterior gamma distributions. Since, $D_1(\alpha,\beta)\geq 0$ and the hyperparameters are non-negative values, the posterior distributions have proper gamma densities. On the other hand, the density in $\pi(\alpha|\theta_1,\theta_2,\beta,\mathbf{t},\mathbf{c})$ and $\pi(\beta|\theta_1,\theta_2,\alpha,\mathbf{t},\mathbf{c})$ can not be reduced analytically to well-known distributions and therefore it is not possible to sample directly by standard methods. It is observed that the density plots of the conditional posterior densities of α and β are like to Gaussian distribution (see Figure 3). In this case, we propose Metropolis-Hasting (M-H) sampling in Gibbs algorithm with normal proposal distribution as suggested by Tierney Tierney (1994).

The algorithm for Gibbs sampling with the M-H method can be described as follows:

Step 1: Start by using the initial values of $(\theta_1^{(0)}, \theta_2^{(0)}, \alpha^{(0)}, \beta^{(0)})$

Step 2: Set $t = 1$

Step 3: Generate $\alpha^{(t)}$ from $GA\big(n_{11} + n_{21} + a_1, D_1(\alpha^{(t-1)}, \beta^{(t-1)}) + b_1\big)$.

Step 4: Generate $\lambda^{(t)}$ from $GA\big(n_{12} + n_{22} + a_2, D_1(\alpha^{(t-1)}, \beta^{(t-1)}) + b_2)$.

Step 5: Generate $\alpha^{(t)}$ from $\pi(\alpha|\eta)$ by using the M-H algorithm with normal proposal $N(\alpha^{(t-1)}, \sigma^2_{\hat{\alpha}})$.

- Let $v = \alpha^{(t-1)}$ and generate w from the proposal as $w = N(\alpha^{(t-1)}, \sigma_{\hat{\alpha}})$.
- Let $p(v, w) = \min\Big\{1, \frac{\pi(w|\theta_1^{(t)}, \theta_2^{(t)}, \beta^{(t-1)}) N(v, \sigma_{\hat{\beta}})}{\pi(v|\theta_1^{(t)}, \theta_2^{(t)}, \beta^{(t-1)}) N(w, \sigma^2_{\hat{\alpha}})}\Big\}$
- Generate U from $U(0, 1)$, then accept proposal if $U \le p(v, w)$ and set $\alpha^{(t)} = w$ or otherwise $\alpha^{(t)} = v$

Step 6: Compute $\beta^{(t)}$ from $\pi(\beta|\eta)$ by using the M-H algorithm similarly to Step 5 at $\theta_1^{(t)}, \theta_2^{(t)}, \alpha^{(t)}$.

Step 7: Set $t = t + 1$.

Step 8: Repeat Step 3 to Step 7, for N times.

Then, the approximate posterior mean of η under the squared error $(\hat{\eta}^B_{MC})$ can be derived as

$$\hat{\eta}^B_{MC} = \frac{1}{N - B} \sum_{t=B+1}^{N} \eta^{(t)} \tag{14}$$

where B is the burn-in period. Then, the $100(1-\gamma)\%$ HPD credible interval can be constructed by using the method proposed by Chen and Shao Chen and Shao (1999) as

$$\left(\hat{\eta}^{B}_{MC[\frac{\gamma}{2}N]}, \hat{\eta}^{B}_{MC[(1-\frac{\gamma}{2})N]}\right)$$

where $[\frac{\gamma}{2}N]$ and $[(1-\frac{\gamma}{2})N]$ are the smallest integers less than or equal to $\frac{\gamma}{2}N$ and $(1-\frac{\gamma}{2})N$, respectively.

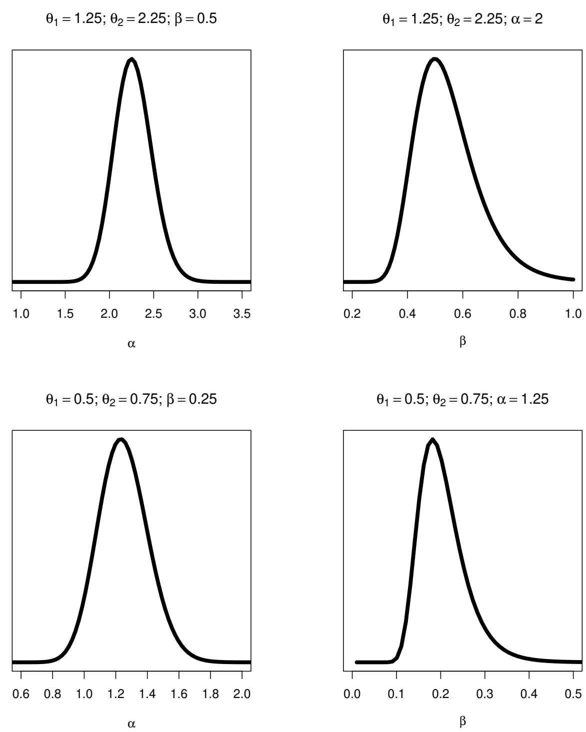

Figure (3) Density plots for the posterior distributions of α and β.

5 Some Testing Problems

In this section, we provide some likelihood-ratio tests to evaluate the equivalence of the shape parameters α_1, α_2 and the importance of acceleration factor β among two different tampered random variables from the Weibull competing risks model under Type-II censoring scheme. It is clearly seen from Eqs. (1-2), the data is not exposed to an acceleration in the case of $\beta = 1$. Therefore, experimenters may need to test if the data has an acceleration factor. In this way, the following hypothesis testings are proposed as

Test I $H_0 : \alpha_1 = \alpha_2 = \alpha$ versus $H_1 : \alpha_1 \neq \alpha_2$

Test II $H_0 : \beta_1 = 1$ versus $H_1 : \beta \neq 1$

Under large sample size n, the likelihood-ratio statistics for Test I

$$-2\{\ell(\acute{\theta}_1, \acute{\theta}_2, \acute{\alpha}, \acute{\beta}) - \ell(\hat{\theta}_1, \hat{\theta}_2, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta})\} \sim \chi^2_1$$

where $\ell(\theta_1, \theta_2, \alpha, \beta)$ is the log-likelihood function at (8) when parameters are common. Thus, the asymptotic distribution of $-2\{\ell(\acute{\theta}_1, \acute{\theta}_2, \acute{\alpha}, \acute{\beta}) - \ell(\hat{\theta}_1, \hat{\theta}_2, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta})\} \sim \chi^2_1$ can be used to construct the likelihood ratio test for Test I, and reject H_0 under this case if $-2\{\ell(\acute{\theta}_1, \acute{\theta}_2, \acute{\alpha}, \acute{\beta}) - \ell(\hat{\theta}_1, \hat{\theta}_2, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta})\} \sim \chi^2_1. > \xi^*$ where ξ^* is such that $P(\chi^2_1 > \xi^*)$ is equal to the size of the test. Similarly, the following likelihood-ratio statistics

$$-2\{\ell(\acute{\theta}_1, \acute{\theta}_2, \acute{\alpha}, \acute{\beta}) - \ell(\hat{\theta}_1, \hat{\theta}_2, \hat{\alpha}, \hat{\beta} = 1)\} \sim \chi^2_1$$

can be used to construct the likelihood ratio test for Test II. The rejection of the null hypothesis indicates that the corresponding data is not exposed to any acceleration factor.

6 Simulations

In this section, we provide different simulation schemes for the purpose of illustrative examples of the theoretical findings. Initially, we generate accelerated Weibull samples by using the inverse transformation method. For this purpose, we use the following quantile function obtained from

Equation 3

$$F_j^{-1}(u;\alpha,\theta_j) = \begin{cases} \big[-\frac{1}{\theta_j}\ln(1-u)\big]^{1/\alpha}, & \text{if } 0 < t \le \tau_j \\ \beta\Big\{\big[-\frac{1}{\theta_j}\ln(1-u)\big]^{1/\alpha} - \tau_j\Big\} + \tau_j, & \text{if } t > \tau_j, \end{cases} \tag{15}$$

where U denotes random sample from the uniform distribution and $\tau_j = 1 - e^{-\theta_j \tau^{\alpha}}$. Thus, we generate random samples with sample sizes 25, 35 and 50 by using Equation 15. Further, in the Type-II censoring framework, we used two different pre-specified number of failures in each sample sizes. Therefore, we choose $r = 18$ and $r = 22$ for $n = 25$, $r = 28$ and $r = 32$ for $n = 35$ and $r = 40$ and $r = 45$ for $n = 50$, respectively. We consider two different sets of the actual parameter values. In the first scheme, we take $(\theta_1, \theta_2, \alpha, \beta) = (1.25, 2.25, 2.00, 0.50)$ and considered two different stress change points as $\tau = 0.50$ and $\tau = 0.60$. Then, we consider a more accelerated sample by taking smaller acceleration factor as $(\theta_1, \theta_2, \alpha, \beta) = (0.50, 0.75, 0.50, 0.25)$ with $\tau = 0.75$ and $\tau = 1.00$. In both schemes, we determine stress change points by considering the ranges of the generated samples. These ranges always change depend on the different selection of the actual parameter values. We used R Team et al. (2021) software in the computational processes. In MLE computations, we used **"L-BFGS-B"** method in **"optim"** function to optimize the profile log-likelihood function given in Equation **??** within restricted $0 < \beta < 1$ area. We determine the significance level as 0.05 for approximate confidence intervals. Then, we repeated simulations 2000 times. In Bayesian procedure, we used informative hyper-parameter values by considering the mean of the gamma priors as $\left(\frac{a_i}{b_i}\right)$ for $i = 1, 2, 3$. Thus, we determined (a_i, b_i) as $(1, 25, 1.00)$, $(2, 25, 1.00)$ and $(2, 00, 1.00)$ for the first scheme and $(0, 50, 1.00)$, $(0, 75, 1.00)$ and $(0, 50, 1.00)$ for the second scheme. It is seen that, means of the gamma priors provide actual values of the parameters using these hyper-parameters. Then, we run MCMC algorithm 3500 times in each iteration of 2000 replication. We discard the first 500 values in burn-in period. Since Markov chains naturally generate autocorrelated samples, we prefer to use thinning procedure and we take every third variate as uncorrelated samples from the remaining sample after the burn-in process. Thus, we obtain 1000 uncorrelated Markov chains and we repeat this process 2000 times.

We present bias values and mean squared errors (MSE) of the point estimates and average lengths (AL) with corresponding coverage probabilities (CP) of the approximate confidence intervals. All results of these simulation schemes are presented in Tables 1, 2, 3 and 4. In the general view, we observe consistent results in all cases. It is observed that the biases, MSEs and ALs are getting smaller in parallel to the increasing sample sizes. The CPs are mostly obtained very close to their actual value 0.95. In the overall case, the informative Bayes estimation method provides better results than MLE. The difference between the two estimators are decreasing when the sample size increases. This outcome show the superiority of Bayes estimation in small samples. Especially, confidence intervals based on the HPD method are obtained quite smaller than the ACI and also it provides as good CPs as delta method. When we change the pre-fixed number of failures or stress change time, we can obtain similar performances in all cases. These consistent performances prove the efficiency and productivity of the theoretical findings.

7 Numerical Example

In this section, we used a real data set which was originally analyzed by Hoel (1972) and used by various authors such as Kundu et al. (2003), Pareek et al. (2009), Sarhan et al. (2010), Cramer and Schmiedt (2011). In this data study, Hoel Hoel (1972) tested a male mice exposed to radiation dose of 300 roentgens at age of 5-6 weeks. We handle this lifetime data under two causes of failure; reticulum cell sarcom as a first cause (1) and other causes as a second cause (0) of failure. This data used by Kundu et al. Kundu et al. (2003) under the assumption that the lifetime distributions of the individual causes are independent and exponentially distributed random variables. Further, Pareek et al. (2009) studied same data for Weibull progressively censored Weibull competing risks data. Thus, we handle this data and we also scaled data by dividing 1000 for computational sim-

Table (1) Bias and MSEs of the estimations with the ALs and CPs of the corresponding approximate confidence intervals for $\theta_1 = 1.25$, $\theta_2 = 2.25$, $\alpha = 2.00$, $\beta = 0.50$ and $\tau = 0.50$.

n	r	$\hat{\eta}$	Bias		MSE		AL		CP	
			MLE	Bayes	MLE	Bayes	ACI	HPD	ACI	HPD
25	18	$\hat{\theta}_1$	0.19608	0.03050	0.98310	0.18380	4.53402	2.15596	0.96650	0.96850
		$\hat{\theta}_2$	0.34136	0.06401	2.51521	0.36986	7.05864	3.29280	0.97200	0.98050
		$\hat{\alpha}$	0.13363	0.04754	0.32109	0.10545	2.11303	1.55490	0.95000	0.97700
		$\hat{\beta}$	0.02362	0.05644	0.07324	0.02467	1.67212	0.71105	0.94150	0.95900
	22	$\hat{\theta}_1$	0.23636	0.01149	1.09038	0.20158	4.20923	2.10904	0.96450	0.96600
		$\hat{\theta}_2$	0.41212	0.01875	2.24528	0.41193	6.61109	3.27826	0.96600	0.98100
		$\hat{\alpha}$	0.10205	0.08374	0.26262	0.10052	2.02843	1.50845	0.95350	0.96950
		$\hat{\beta}$	0.02033	0.03405	0.05763	0.01875	1.24763	0.66602	0.97600	0.97050
35	28	$\hat{\theta}_1$	0.18401	0.01034	0.56131	0.18020	3.21350	1.94833	0.95500	0.96250
		$\hat{\theta}_2$	0.34965	0.02781	1.57761	0.40243	5.13346	3.04707	0.96000	0.97950
		$\hat{\alpha}$	0.10128	0.04137	0.19075	0.08117	1.70171	1.34347	0.95050	0.97950
		$\hat{\beta}$	0.00781	0.03067	0.04782	0.01755	1.13013	0.65519	0.97000	0.97250
	32	$\hat{\theta}_1$	0.18850	0.00133	0.63895	0.17893	3.14558	1.89904	0.95450	0.96550
		$\hat{\theta}_2$	0.33022	0.00356	1.49208	0.37314	5.00356	2.98937	0.95800	0.97750
		$\hat{\alpha}$	0.08418	0.06202	0.19107	0.08469	1.67593	1.32930	0.94400	0.96700
		$\hat{\beta}$	0.01983	0.02662	0.04332	0.01634	0.95979	0.61589	0.97500	0.97350
50	40	$\hat{\theta}_1$	0.14260	0.01857	0.37370	0.15384	2.45927	1.72403	0.95100	0.95850
		$\hat{\theta}_2$	0.26887	0.04143	1.06489	0.37628	3.92040	2.71789	0.94850	0.97200
		$\hat{\alpha}$	0.07798	0.02810	0.13782	0.06981	1.39470	1.16897	0.93800	0.96900
		$\hat{\beta}$	0.01832	0.01567	0.04075	0.01685	0.91544	0.60704	0.95400	0.96350
	45	$\hat{\theta}_1$	0.12677	0.00840	0.33628	0.13802	2.36475	1.68328	0.95950	0.96450
		$\hat{\theta}_2$	0.22741	0.01273	0.92321	0.33510	3.79959	2.67467	0.95150	0.97050
		$\hat{\alpha}$	0.06260	0.04178	0.12285	0.06410	1.38020	1.16461	0.95450	0.97050
		$\hat{\beta}$	0.02146	0.01011	0.03134	0.01410	0.78685	0.57668	0.96300	0.97150

Table (2) Bias and MSEs of the estimations with the ALs and CPs of the corresponding approximate confidence intervals for $\theta_1 = 1.25$, $\theta_2 = 2.25$, $\alpha = 2.00$, $\beta = 0.50$ and $\tau = 0.60$.

n	r	$\hat{\eta}$	Bias		MSE		AL		CP	
			MLE	Bayes	MLE	Bayes	ACI	HPD	ACI	HPD
25	18	$\hat{\theta}_1$	0.20624	0.19555	0.25317	0.15691	2.44243	1.75131	0.96500	0.94150
		$\hat{\theta}_2$	0.35727	0.34431	0.58800	0.32767	3.63109	2.63153	0.96100	0.96200
		$\hat{\alpha}$	0.01012	0.04849	0.20022	0.11663	1.83331	1.54043	0.95950	0.96900
		$\hat{\beta}$	0.12639	0.09873	0.08717	0.03475	1.83390	0.73658	0.86900	0.93397
	22	$\hat{\theta}_1$	0.13308	0.00679	0.48471	0.19237	3.03493	1.96930	0.95900	0.96500
		$\hat{\theta}_2$	0.25366	0.02474	1.14271	0.38329	4.61246	2.98881	0.96950	0.97850
		$\hat{\alpha}$	0.11539	0.01808	0.21489	0.10113	1.78381	1.45664	0.95200	0.97550
		$\hat{\beta}$	0.01340	0.04724	0.06654	0.02296	1.52356	0.70218	0.94700	0.95898
35	28	$\hat{\theta}_1$	0.01155	0.05135	0.22741	0.14000	2.21180	1.68546	0.97450	0.96700
		$\hat{\theta}_2$	0.01966	0.09101	0.47594	0.26086	3.32985	2.54950	0.97450	0.97350
		$\hat{\alpha}$	0.03112	0.03915	0.13735	0.08912	1.46170	1.26539	0.95000	0.95800
		$\hat{\beta}$	0.04403	0.05324	0.06553	0.02481	1.53095	0.70661	0.92900	0.95500
	32	$\hat{\theta}_1$	0.11502	0.02097	0.35190	0.17351	2.35772	1.73889	0.95400	0.96700
		$\hat{\theta}_2$	0.20171	0.03686	0.77461	0.34416	3.59167	2.64650	0.96150	0.97350
		$\hat{\alpha}$	0.06048	0.03765	0.13948	0.08258	1.44326	1.24162	0.95150	0.96400
		$\hat{\beta}$	0.00043	0.02995	0.04804	0.01924	1.10535	0.65491	0.95850	0.96900
50	40	$\hat{\theta}_1$	0.01389	0.02011	0.15892	0.11050	1.81185	1.49186	0.96900	0.96198
		$\hat{\theta}_2$	0.02130	0.03830	0.33716	0.21870	2.73484	2.25900	0.97150	0.96798
		$\hat{\alpha}$	0.02861	0.02503	0.08596	0.06130	1.20044	1.07786	0.96050	0.96598
		$\hat{\beta}$	0.02560	0.03794	0.05703	0.02247	1.34999	0.68239	0.93147	0.95894
	45	$\hat{\theta}_1$	0.08429	0.02083	0.22440	0.13570	1.86892	1.51151	0.95450	0.95400
		$\hat{\theta}_2$	0.14581	0.03387	0.51189	0.28785	2.86021	2.31766	0.94600	0.95950
		$\hat{\alpha}$	0.05772	0.01385	0.09846	0.06550	1.20324	1.08135	0.94500	0.96200
		$\hat{\beta}$	0.01130	0.01234	0.03795	0.01661	0.93050	0.62097	0.95800	0.96700

Table (3) Bias and MSEs of the estimations with the ALs and CPs of the corresponding approximate confidence intervals for $\theta_1 = 0.50$, $\theta_2 = 0.75$, $\alpha = 0.50$, $\beta = 0.25$ and $\tau = 0.75$.

n	r	$\hat{\eta}$	Bias		MSE		AL		CP	
			MLE	Bayes	MLE	Bayes	ACI	HPD	ACI	HPD
25	18	$\hat{\theta}_1$	0.04186	0.04056	0.02723	0.02365	0.77350	0.66231	0.97700	0.94600
		$\hat{\theta}_2$	0.07640	0.07336	0.03637	0.03174	0.93893	0.82716	0.97850	0.96100
		$\hat{\alpha}$	0.02305	0.00118	0.01547	0.01262	0.48247	0.45454	0.95350	0.95950
		$\hat{\beta}$	0.00142	0.04487	0.04605	0.02420	1.45131	0.68798	0.93150	0.95298
	22	$\hat{\theta}_1$	0.02590	0.02205	0.03757	0.03138	0.82363	0.70883	0.95900	0.95350
		$\hat{\theta}_2$	0.03614	0.03062	0.05948	0.04879	1.02499	0.90200	0.95050	0.95900
		$\hat{\alpha}$	0.03886	0.01622	0.01597	0.01232	0.46212	0.43108	0.93200	0.95550
		$\hat{\beta}$	0.05357	0.06260	0.05304	0.02621	1.25758	0.66364	0.96150	0.96400
35	28	$\hat{\theta}_1$	0.00898	0.00413	0.02358	0.02082	0.68012	0.60576	0.96650	0.96050
		$\hat{\theta}_2$	0.01431	0.00757	0.03628	0.03199	0.84583	0.76897	0.96950	0.96200
		$\hat{\alpha}$	0.02297	0.00486	0.01022	0.00859	0.38485	0.36640	0.94550	0.95600
		$\hat{\beta}$	0.02667	0.05201	0.03776	0.02308	1.05636	0.64007	0.95100	0.95550
	32	$\hat{\theta}_1$	0.01862	0.01332	0.02892	0.02498	0.66509	0.59600	0.94150	0.94100
		$\hat{\theta}_2$	0.03956	0.03149	0.04671	0.03984	0.84440	0.76810	0.94700	0.95050
		$\hat{\alpha}$	0.02587	0.00847	0.01000	0.00816	0.37379	0.35331	0.94550	0.95650
		$\hat{\beta}$	0.05066	0.05390	0.04214	0.02378	0.92393	0.60346	0.94550	0.95450
50	40	$\hat{\theta}_1$	0.00992	0.00430	0.01749	0.01604	0.55804	0.51378	0.96150	0.95650
		$\hat{\theta}_2$	0.00780	0.00053	0.02873	0.02629	0.69373	0.64702	0.95850	0.95750
		$\hat{\alpha}$	0.01891	0.00445	0.00681	0.00596	0.31832	0.30717	0.94400	0.95250
		$\hat{\beta}$	0.02953	0.04856	0.02959	0.02059	0.81283	0.58849	0.95300	0.95800
	45	$\hat{\theta}_1$	0.01487	0.00824	0.01805	0.01605	0.54531	0.50300	0.95750	0.96050
		$\hat{\theta}_2$	0.02947	0.01913	0.03214	0.02811	0.69226	0.64620	0.93850	0.94500
		$\hat{\alpha}$	0.02071	0.00585	0.00695	0.00591	0.31070	0.29857	0.94000	0.94850
		$\hat{\beta}$	0.03585	0.04214	0.02583	0.01802	0.69779	0.54856	0.95500	0.96200

Table (4) Bias and MSEs of the estimations with the ALs and CPs of the corresponding approximate confidence intervals for $\theta_1 = 0.50$, $\theta_2 = 0.75$, $\alpha = 0.50$, $\beta = 0.25$ and $\tau = 1.00$.

n	r	$\hat{\eta}$	Bias		MSE		AL		CP	
			MLE	Bayes	MLE	Bayes	ACI	HPD	ACI	HPD
25	18	$\hat{\theta}_1$	0.08674	0.08117	0.02512	0.02226	0.68354	0.59391	0.97150	0.93250
		$\hat{\theta}_2$	0.13162	0.12324	0.03916	0.03446	0.82890	0.74348	0.96900	0.93850
		$\hat{\alpha}$	0.00655	0.01511	0.01197	0.01055	0.45679	0.43464	0.96600	0.95200
		$\hat{\beta}$	0.03066	0.03283	0.04028	0.02261	1.43047	0.68984	0.91050	0.93947
	22	$\hat{\theta}_1$	0.00582	0.00411	0.02815	0.02421	0.77275	0.67275	0.97100	0.95800
		$\hat{\theta}_2$	0.03238	0.02873	0.04845	0.04102	0.96659	0.86314	0.95900	0.96700
		$\hat{\alpha}$	0.03316	0.01230	0.01401	0.01129	0.43842	0.41411	0.94500	0.95750
		$\hat{\beta}$	0.04562	0.06594	0.05137	0.02632	1.34931	0.68313	0.95950	0.96750
35	28	$\hat{\theta}_1$	0.01775	0.01874	0.01961	0.01795	0.62483	0.56416	0.97250	0.95398
		$\hat{\theta}_2$	0.02797	0.02984	0.02750	0.02495	0.76802	0.70629	0.97500	0.96398
		$\hat{\alpha}$	0.01604	0.00011	0.00884	0.00776	0.36797	0.35267	0.95200	0.95148
		$\hat{\beta}$	0.00951	0.04678	0.03937	0.02339	1.16060	0.65138	0.94297	0.95598
	32	$\hat{\theta}_1$	0.01597	0.01183	0.02507	0.02216	0.63792	0.57653	0.95050	0.94450
		$\hat{\theta}_2$	0.02428	0.01836	0.03978	0.03500	0.79420	0.72949	0.95000	0.94850
		$\hat{\alpha}$	0.02560	0.00925	0.00888	0.00742	0.36050	0.34330	0.94600	0.95500
		$\hat{\beta}$	0.04656	0.05818	0.03869	0.02332	0.97161	0.62429	0.96550	0.96750
50	40	$\hat{\theta}_1$	0.00116	0.00403	0.01413	0.01329	0.52871	0.49007	0.97000	0.96600
		$\hat{\theta}_2$	0.01126	0.01527	0.02139	0.02021	0.64867	0.61000	0.97000	0.95550
		$\hat{\alpha}$	0.01244	0.00049	0.00639	0.00584	0.30161	0.29245	0.94000	0.94200
		$\hat{\beta}$	0.01469	0.04990	0.03217	0.02182	0.94693	0.62649	0.94250	0.95400
	45	$\hat{\theta}_1$	0.00979	0.00470	0.01690	0.01542	0.52326	0.48578	0.95300	0.95000
		$\hat{\theta}_2$	0.02361	0.01562	0.02900	0.02618	0.65732	0.61889	0.93950	0.95000
		$\hat{\alpha}$	0.01713	0.00352	0.00612	0.00537	0.29675	0.28673	0.94500	0.95250
		$\hat{\beta}$	0.03038	0.04348	0.02602	0.01893	0.72400	0.56314	0.95400	0.95750

plicity. We consider to pre-determined number of failures item as $r = 65$ and $r = 70$ from $n = 77$ complete data. Also, two different stress change times are handled as $\tau = 0.50$ and $\tau = 0.65$. For illustration, we present the competing risk data under $\tau = 0.65$ and $r = 70$ as in the following

$\mathbf{t_i} < \tau$	(0.040; 0), (0.042; 0), (0.051; 0), (0.062; 0), (0.163; 0), (0.179; 0), (0.206; 0), (0.222; 0), (0.228; 0), (0.249; 0), (0.252; 0), (0.282; 0), (0.317; 1), (0.318; 1), (0.324; 0), (0.333; 0), (0.341; 0), (0.366; 0), (0.385; 0), (0.399; 1), (0.407; 0), (0.420; 0), (0.431; 0), (0.441; 0), (0.461; 0), (0.462; 0), (0.482; 0), (0.495; 1), (0.517; 0), (0.517; 0), (0.522; 0), (0.524; 0), (0.525; 1), (0.536; 1), (0.549; 1), (0.552; 1), (0.554; 1), (0.557; 1), (0.558; 1), (0.564; 0), (0.567, 0), (0.571, 1), (0.586; 1), (0.586; 0), (0.594; 1), (0.596; 1), (0.605; 1), (0.612; 1), (0.619; 0), (0.620; 0), (0.621; 1), (0.628; 1), (0.631; 1), (0.636; 1), (0.643; 1), (0.647; 1), (0.647; 0), (0.648; 1), (0.649; 1)
$\mathbf{t_i} \geq \tau$	(0.651; 0), (0.65; 0), (0.661; 1), (0.66; 1), (0.666; 1), (0.67; 1), (0.686; 0), (0.69; 1), (0.697; 1), (0.700; 1), (0.70; 1)

where the first values in the parenthesis denote the lifetime t_i and the second values denote the indicator function C_i. $C_i = 1$ denotes the failure by risk I and and $C_i = 0$, otherwise. In this case, we obtain $N_{11} = 25$, $N_{12} = 34$, $N_{21} = 8$ and $N_{22} = 3$.

We use informative hyperparameters by using the MLE estimates of the parameters and determined hyperparameters as $a_i = \hat{\eta}_i$ and $b_i = 1$ for $i = 1, 2, 3$. We run MCMC 100 000 times and we take every 10th variate in thinning procedure. As initial values of the Markov chain, we prefer MLEs since they provide good estimates. Then, we obtain a Markov chain with 10 000 uncorrelated samples. We obtain the MLE and the Bayesian point estimates with their corresponding approximate confidence intervals ACIs and HPDs. The results are reported in Tables 5, 6.

Table (5) Point estimates and %95 intervals of numerical example for $r = 65$.

τ	$\hat{\eta}$	MLE	Bayes	ACI	Length	HPD	Length
0.50	$\hat{\theta}_1$	0.64491	0.62790	(0.34949; 1.19005)	0.84056	(0.32895; 1.07686)	0.74791
	$\hat{\theta}_2$	0.80058	0.77869	(0.44323; 1.44604)	1.00280	(0.42984; 1.29838)	0.86854
	$\hat{\alpha}$	1.74985	1.68475	(1.27988; 2.39241)	1.11253	(1.20997; 2.23639)	1.02642
	$\hat{\beta}$	0.29631	0.28795	(0.16410; 0.53504)	0.37094	(0.14908; 0.48793)	0.33885
0.65	$\hat{\theta}_1$	1.57772	1.55834	(0.97384; 2.55607)	1.58223	(0.95630; 2.37286)	1.41656
	$\hat{\theta}_2$	1.95855	1.92727	(1.24258; 3.08706)	1.84448	(1.21732; 2.88999)	1.67266
	$\hat{\alpha}$	2.45196	2.40835	(1.94237; 3.09524)	1.15287	(1.89854; 2.96590)	1.06737
	$\hat{\beta}$	0.25268	0.29006	(0.10984; 0.58127)	0.47143	(0.11516; 0.65210)	0.53693

Table (6) Point estimates and %95 intervals of numerical example for $r = 70$.

τ	$\hat{\eta}$	MLE	Bayes	ACI	Length	HPD	Length
0.50	$\hat{\theta}_1$	0.69421	0.67154	(0.38044; 1.26676)	0.88633	(0.36031; 1.15218)	0.79187
	$\hat{\theta}_2$	0.77835	0.75370	(0.43108; 1.40540)	0.97432	(0.40955; 1.27436)	0.86481
	$\hat{\alpha}$	1.78753	1.71949	(1.31833; 2.42371)	1.10538	(1.24067; 2.27166)	1.03099
	$\hat{\beta}$	0.29389	0.28322	(0.16497; 0.52356)	0.35859	(0.14789; 0.47339)	0.32550
0.65	$\hat{\theta}_1$	1.66785	1.63724	(1.04244; 2.66847)	1.62603	(1.02110; 2.45259)	1.43149
	$\hat{\theta}_2$	1.87001	1.83744	(1.18479; 2.95155)	1.76676	(1.16794; 2.75793)	1.58999
	$\hat{\alpha}$	2.45292	2.40077	(1.94337; 3.09609)	1.15272	(1.88941; 2.95814)	1.06873
	$\hat{\beta}$	0.33457	0.35407	(0.17671; 0.63348)	0.45677	(0.17583; 0.66787)	0.49204

We observe that the acceleration factor is remarkably small (≈ 0.30) for this data. Therefore, data acceleration speed is expected to be strong. This means early stress change time causes larger observation values on stress level s_2. Since the mean of the Weibull distribution is proportional to $\theta^{1/\alpha}$, we obtain smaller estimates in the case of early stress change time. In both cases, we obtain smaller estimation values in the case of $\tau = 0.50$. In parallel to retardation on stress change time we observe smaller mean and correspondingly higher parameter estimations. We obtain very close estimations in both methods. On the other hand, similar to the simulation studies, Bayesian HPD intervals have smaller lengths than ACIs. However, ACIs are obtained with smaller lengths for the acceleration factor estimates in this example. The reason of that can be explained by the superiority of the MLEs in large samples. Generally, we obtain consistent and efficient results in this numerical example.

We also checked the convergence of the Markov Chains. For this purpose, we first draw trace plots that show the values of the parameters against the iteration number at each iteration. The trace plots for the parameters which are given in Figure 4 show that Markov Chains for all parameters fluctuate around their centers with similar variations. Further, we see from the Figures 5, the density plots seem in symmetric and unimodal shapes. The unimodal peaks in these plots determine the mode of the posterior distributions of the parameters and they are the values with the most support from the data and the chosen priors. Additionally, we use the running mean (ergodic average) plots as another diagnostic tool. An ergodic average plot draws the mean of sampled values up to iteration t, and the precision of an ergodic average depends on the autocorrelations of the chain. We see from the Fig. 6 they stabilize with increasing iteration number which means that the chains have achieved stationarity. All plots to check the convergence in a Markov Chain can be drawn using library **mcmcplots** Curtis et al. (2018) in **R**Team et al. (2021) software.

Moreover, we test the equivalence of the shape parameters of tho competing risk variables α_1 and α_2 by using the hypothesis testing proposed in Section 5. Since the value of the likelihood ratio statistics and the associated p-value are 1.8461 and 0.1742 for hypothesis $\alpha_1 = \alpha_2 = \alpha$ with %5 level of significance, the results imply that the null hypothesis cannot be rejected. Therefore, it is recommended that the shape parameters of two Weibull samples are equal for this numerical example data. On the other hand, we test the hypothesis that the data is exposed to any acceleration factor ($\beta \neq 1$). The likelihood ratio statistics and the associated p-value are 415.9661 and ≈ 0 and this result implies that the null hypothesis is rejected. Consequently, an acceleration factor can be claimed for this numerical example. The inference values of β are obtained as ≈ 0.30 in Tables 5-6 as a higher acceleration level and this observation support our test results.

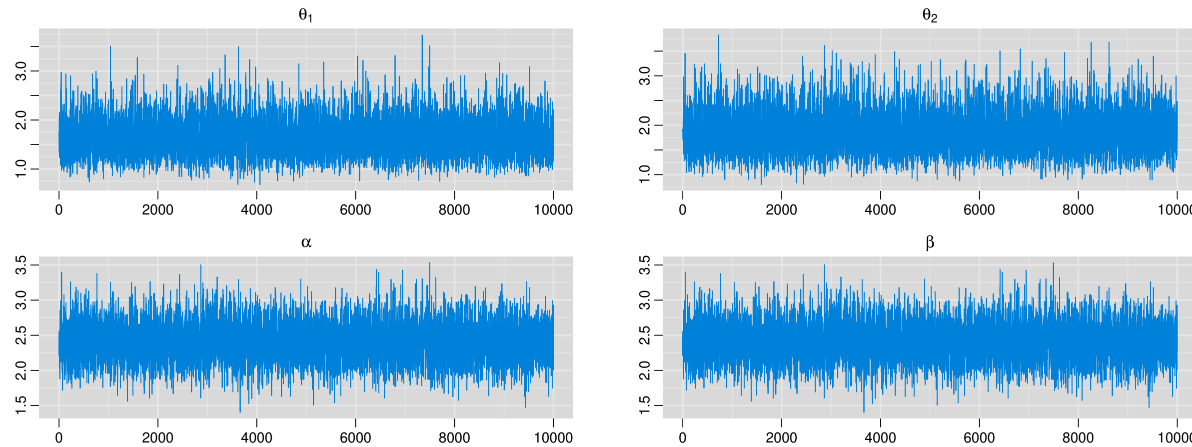

Figure (4) Trace plots of the posterior distributions of the parameters.

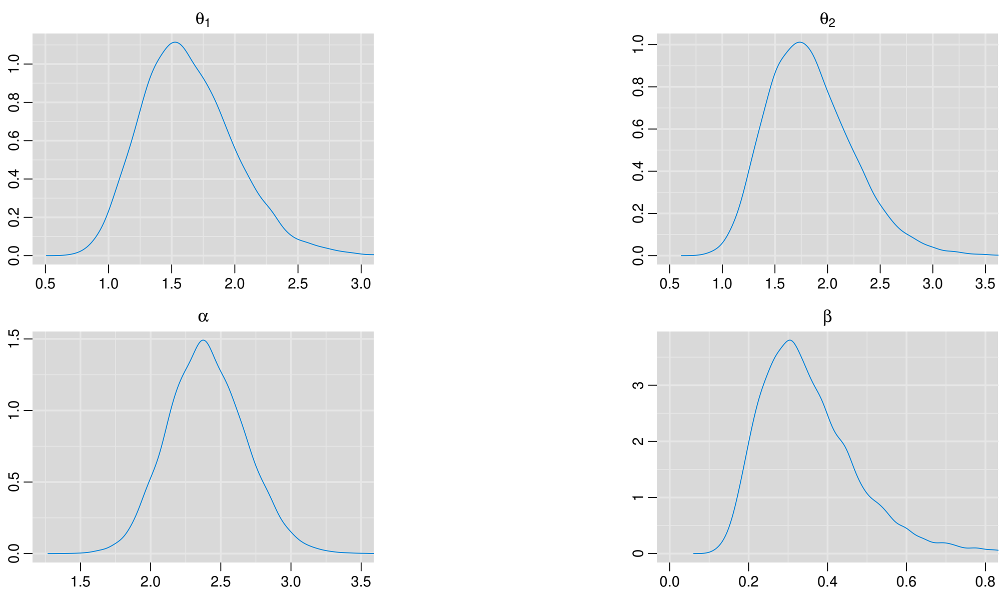

Figure (5) Density plots of the posterior distributions of the parameters.

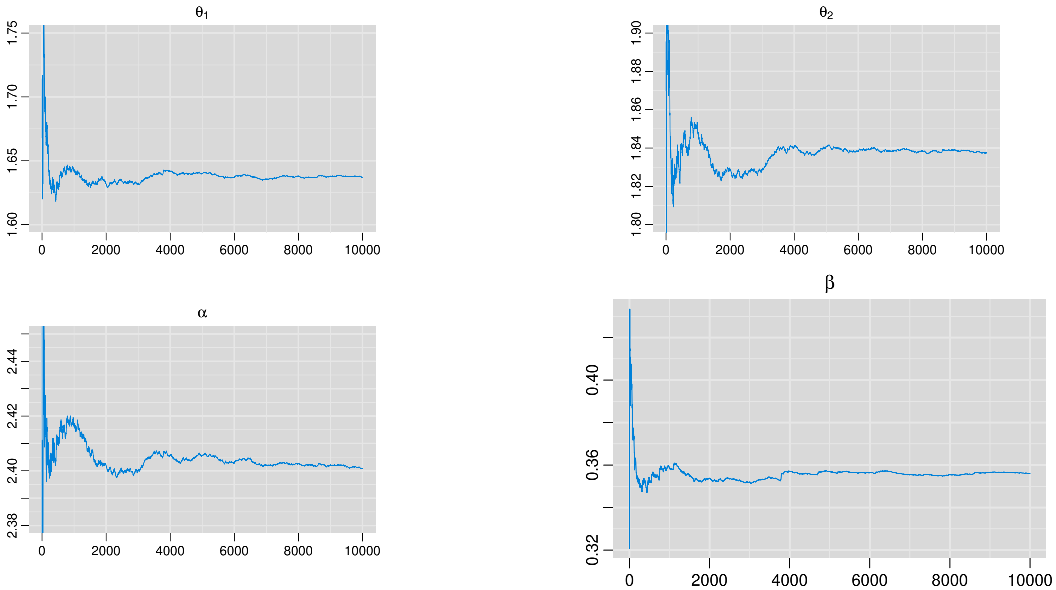

Figure (6) Running mean plots of the posterior distributions of the parameters.

8 Optimal Stress Change Time

In this section, we provide an optimal plan based on the asymptotic variances of the maximum likelihood estimators of the parameters. The asymptotic variances of the parameters can be obtained by the diagonals of the inverse of Fisher information matrix. In this section we used sum of the coefficient of variations (SVC) as optimal function instead of the sum of the variances of the parameters, as given and used by Samanta et al. (2018, 2019). Samanta et al. (2018) proposed this method to determine an optimal plan by minimizing the expected value of the SVC. Because, the sum of the variances may be dominated by the variance of any particular parameter in the case of the parameter values being in a different scale. That's why, we use the expected value of the SVC, by maximizing $E(\Phi(\tau))$, where

$$\Phi(\tau) = \frac{\sqrt{F_{11}^{-1}}}{\hat{\theta}_1} + \frac{\sqrt{F_{22}^{-1}}}{\hat{\theta}_2} + \frac{\sqrt{F_{33}^{-1}}}{\hat{\alpha}} + \frac{\sqrt{F_{44}^{-1}}}{\hat{\beta}_1}$$

However, the closed forms of the posterior variances of the parameters cannot be obtained exactly. Therefore, Samanta et al. (2018) suggest to use the following Gibbs sampling technique for computation.

Step 1: For given τ, n, r and parameter values, generate the samples T_1, T_2 and $T = min\{T_1, T_2\}$.

Step 2: Compute the objective function $\Phi(\tau)$.

Step 3: Repeat Step 1 to Step 2, for N times and get $\Phi^1(\tau), \Phi^2(\tau), \cdots, \Phi^N(\tau)$.

Step 4: Calculate the median of the objective functions and denote them by $\Phi^m(\tau)$.

Step 5: Repeat Step 1 to Step 4 for all possible values of τ.

Step 6: Determine optimal τ for which $\Phi^m(\tau)$ be the minimum.

Optimal stress change time τ values, denoted by τ^* are computed for given n, r and η_i for $i = 1, \cdots, 4$ and presented in Table 7 and Figures 7,8.

It is observed from Table 7 that optimal stress change time τ^* are obtained between 0.550 and

Table (7) Optimal stress change time τ for different sample sizes and parameter values.

	$\eta = (1.25, 2.25, 2.00, 0.50)$		$\eta = (0.50, 0.75, 0.50, 0.25)$	
(n, r)	τ^*	$\Phi(\tau^*)$	τ^*	$\Phi(\tau^*)$
(25,18)	0.600	1.93676	0.650	1.81532
(25,22)	0.650	1.67365	0.875	1.63499
(35,28)	0.600	1.52492	0.700	1.45528
(35,32)	0.650	1.36714	1.075	1.36021
(50,40)	0.550	1.27960	0.725	1.21499
(50,45)	0.625	1.15906	0.975	1.14400

0.650 for the first parameter set. Since the range of the generated data set is not very large, the range of the τ^* is not very different in the first case. On the other hand, the second parameter set causes a larger range in the generated sample. Therefore, the range of the optimal stress change time larger in this case and obtained between 0.650 and 1.075. We see that the stress change times we used in simulations are quite close to the optimal stress change times. Consequently, the consistent and well-performed simulation results are based on correctly determining the stress change time.

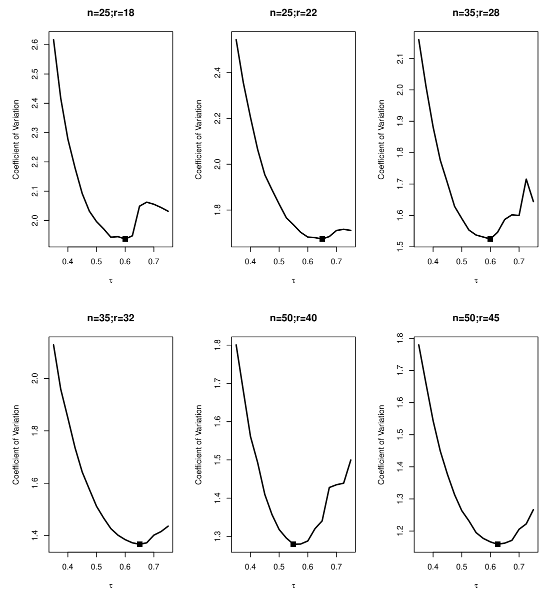

Figure (7) Plots of the SVC against different values of τ for $\theta_1 = 1.25$, $\theta_2 = 2.25$, $\alpha = 2.00$, $\beta = 0.50$.

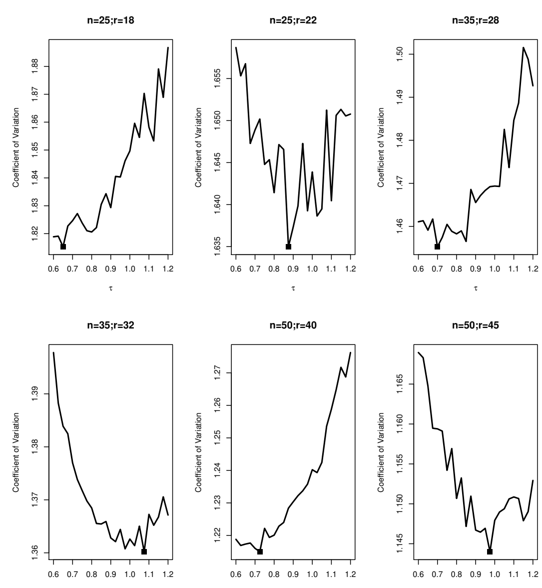

Figure (8) Plots of the SVC against different values of τ for $\theta_1 = 0.50$, $\theta_2 = 0.75$, $\alpha = 0.50$, $\beta = 0.25$.

8.1 Sensitivity Analysis

In optimal plan, incorrect choices of the acceleration factor and the model parameters α, θ_1, and θ_2 bring about poor estimates for the sum of the coefficient of variations. Therefore, we need to test the sensitivity of the expected value of the SVC given in 8. For this reason, we can construct a sensitivity analysis by following the main idea as considering the difference between incorrect and correct estimates of the parameters on SVC. In this section, we propose a sensitivity analysis following the framework proposed by Tseng et al. (2009) and Srivastava and Mittal (2010). It is expected that actual parameter values may depart from the true values in most cases. This sensitivity analysis let us to study the effects of incorrect pre-estimates of θ_1, θ_2, α and β in terms of the relative sum of the variation coefficient (RSVC) as

$$RSVC = \left| \frac{\Phi(\tau^*) - \Phi(\tau^\diamond)}{\Phi(\tau^*)} \right| \times 100$$

where $\Phi(\tau^*)$ is the SVC of the plans obtained by the correct actual values and $\tau^\diamond$ is the SVC for the scheme obtained by incorrectly specified values. We observe that the increase in the RSVC is small if the incorrect parameter values are not too far from the true values, as shown in Tables 8 and 9. When the values of the θ_1, θ_2, α and β are not far removed from their true values, the difference in the RSVC is not significantly large. Consequently, the optimal stress change point choices are not sensitive to misspecification of the model parameters as long as they are not far deviated from their true values. This sensitivity shows us the optimal plan we proposed in Section 8 is reliable for our model.

9 Concluding Remark

In the literature as one can see competing risk data based on simple SSLT modeling has been used widely using CE and TFR modeling. There has not been any inferential work for simple SSLT

Table (8) Sensivity analysis under incorrect pre-estimates of the parameters in the case of $(n, r) = (50, 40)$ and $\eta = (1.25, 2.25, 2.00, 0.50)$ where $\tau^* = 0.550$ and $\Phi(\tau^*) = 1.27960$.

Deviation	$\theta_1^\diamond$	$\theta_2^\diamond$	$\alpha^\diamond$	$\beta^\diamond$	$\tau^\diamond$	$\Phi(\tau^\diamond)$	RSVC
$(-1)\%$	1.2375	2.2275	1.9800	0.4950	0.5750	1.2721	0.0059
$(+1)\%$	1.2625	2.2725	2.0200	0.5050	0.5750	1.2728	0.0053
$(-2)\%$	1.2250	2.2050	1.9600	0.4900	0.5500	1.2686	0.0086
$(+2)\%$	1.2750	2.2950	2.0400	0.5100	0.5500	1.2838	0.0033
$(-3)\%$	1.2125	2.1825	1.9400	0.4850	0.5750	1.2700	0.0075
$(+3)\%$	1.2875	2.3175	2.0600	0.5150	0.5750	1.2794	0.0001
$(-4)\%$	1.2000	2.1600	1.9200	0.4800	0.5750	1.2681	0.0090
$(+4)\%$	1.3000	2.3400	2.0800	0.5200	0.5500	1.2808	0.0009
$(-5)\%$	1.1875	2.1375	1.9000	0.4750	0.5750	1.2649	0.0115
$(+5)\%$	1.3125	2.3625	2.1000	0.5250	0.5750	1.2841	0.0035

Table (9) Sensivity analysis under incorrect pre-estimates of the parameters in the case of $(n, r) = (50, 40)$ and $\eta = (0.50, 0.75, 0.50, 0.25)$ where $\tau^* = 0.725$ and $\Phi(\tau^*) = 1.21499$.

Deviation	$\theta_1^\diamond$	$\theta_2^\diamond$	$\alpha^\diamond$	$\beta^\diamond$	$\tau^\diamond$	$\Phi(\tau^\diamond)$	RSVC
$(-1)\%$	0.4950	0.7425	0.4950	0.2475	0.6250	1.2159	0.0498
$(+1)\%$	0.5050	0.7575	0.5050	0.2525	0.6500	1.2155	0.0501
$(-2)\%$	0.4900	0.7350	0.4900	0.2450	0.6500	1.2161	0.0497
$(+2)\%$	0.5100	0.7650	0.5100	0.2550	0.7000	1.2134	0.0518
$(-3)\%$	0.4850	0.7275	0.4850	0.2425	0.6500	1.2155	0.0501
$(+3)\%$	0.5150	0.7725	0.5150	0.2575	0.6250	1.2128	0.0522
$(-4)\%$	0.4800	0.7200	0.4800	0.2400	0.7500	1.2185	0.0478
$(+4)\%$	0.5200	0.7800	0.5200	0.2600	0.6500	1.2125	0.0525
$(-5)\%$	0.4750	0.7125	0.4750	0.2375	0.7250	1.2162	0.0495
$(+5)\%$	0.5250	0.7875	0.5250	0.2625	0.6250	1.2139	0.0514

experiments under the TRV modeling approach based on competing risk data to the best of our knowledge. The present paper fills that void by using TRV modeling on competing risk SSLT data. In our work, we consider competing risk set-up in SSLT using the TRV modeling and the parameter estimation based on Weibull distribution with same shape and different scale parameters. In this work, we develop maximum likelihood method to estimate the model parameters for the simple SSLT TRV modeling based on Type-II censored data. The asymptotic confidence intervals are obtained based on the maximum likelihood estimations. We also obtained Bayesian estimations by using conjugate Gamma priors with the corresponding highest posterior density credible intervals. Even though both methods provide satisfactory performances the Bayesian estimates have superiority over the maximum likelihood estimates. We exemplified simulation results with an example data set. Then, we propose a optimal stress change time plan following the idea of Samanta et al. (2018) and we observed that the stress change times we used in simulations are very close to the optimal ones. In final part, we provide a sensitivity analysis to study the effects of incorrect pre-estimates. Consequently, we observed the fact that the optimal stress change point choices are not sensitive to misspecification of the model parameters as long as they are not far deviated from their true values. We can generalized this idea for multiple step-stress TRV modeling as proposed by Sultana and Dewanji (2021).

Appendix

The MLEs of α and β can be obtained by solving the following equations, respectively.

$$\frac{r}{\alpha} + \sum_{i=1}^{\hat{n}_1} \log t_i + \sum_{i=\hat{n}_1+1}^{r} \log\left(\tau + \frac{t_i - \tau}{\beta}\right) - r\frac{D_1^{'(\alpha)}(\alpha,\beta)}{D_1(\alpha,\beta)} = 0$$

$$\hat{n}_1 - r - (\alpha - 1)\sum_{i=\hat{n}_1+1}^{r} \frac{(t_i - \tau)}{t_i + \tau(\beta - 1)} + \frac{r\alpha D_1^{'(\beta)}(\alpha,\beta)}{\beta D_1(\alpha,\beta)} = 0$$

where

$$D_1^{'(\alpha)}(\alpha,\beta) = \sum_{i=\hat{n}_1+1}^{r} (t_i-\tau)\left(\tau + \frac{t_i-\tau}{\beta}\right)^{\alpha-1} + (n-r)(t_{r:n}-\tau)\left(\tau + \frac{t_{r:n}-\tau}{\beta}\right)^{\alpha-1}$$

The elements of the observed Fisher information matrix is obtained as follows

$$f_{11} = \frac{n_{11}+n_{21}}{\theta_1^2}, \quad f_{22} = \frac{n_{12}+n_{22}}{\theta_2^2}, \quad f_{12} = f_{21} = 0, \quad f_{13} = f_{31} = f_{23} = f_{32} = D_1^{'(\alpha)}(\alpha,\beta)$$

$$f_{14} = f_{41} = f_{24} = f_{42} = -\frac{\alpha}{\beta^2} D_3(\alpha,\beta), \quad f_{33} = \frac{r}{\alpha^2} + (\theta_1+\theta_2) D_1^{''(\alpha)}(\alpha,\beta)$$

$$f_{34} = f_{43} = \frac{1}{\beta} \sum_{i=\hat{n}_1+1}^{r} \frac{t_i-\tau}{\beta\tau + t_i - \tau} - \frac{\theta_1+\theta_2}{\beta^2}\left[D_3(\alpha,\beta) + \alpha D_3^{'(\alpha)}(\alpha,\beta)\right]$$

$$f_{44} = -\frac{(r-\hat{n}_1)}{\beta^2} - (\alpha-1)\sum_{i=\hat{n}_1+1}^{r} \frac{(t_i-\tau)(2\beta\tau + t_i - \tau)}{[\beta^2\tau + \beta(t_i-\tau)]^2} + \frac{\alpha(\theta_1+\theta_2)}{\beta^3}\left[2D_3(\alpha,\beta) + \frac{(\alpha-1)D_3^{'(\beta)}(\alpha,\beta)}{\beta}\right]$$

where

$$
\begin{aligned}
D_3(\alpha,\beta) &= \sum_{i=\hat{n}_1+1}^{r}(t_i-\tau)\left(\tau+\frac{t_i-\tau}{\beta}\right)^{\alpha-1} + (n-r)(t_{r:n}-\tau)\left(\tau+\frac{t_{r:n}-\tau}{\beta}\right)^{\alpha-1} \\
D_1^{'(\alpha)}(\alpha,\beta) &= \sum_{i=1}^{\hat{n}_1} t_i^{\alpha}\ln t_i + \sum_{i=\hat{n}_1+1}^{r}\left(\tau+\frac{t_i-\tau}{\beta}\right)^{\alpha}\ln\left(\tau+\frac{t_i-\tau}{\beta}\right) \\
&\quad + (n-r)\left(\tau+\frac{t_{r:n}-\tau}{\beta}\right)^{\alpha}\ln\left(\tau+\frac{t_{r:n}-\tau}{\beta}\right) \\
D_1^{''(\alpha)}(\alpha,\beta) &= \sum_{i=1}^{\hat{n}_1} t_i^{\alpha}\ln^2 t_i + \sum_{i=\hat{n}_1+1}^{r}\left(\tau+\frac{t_i-\tau}{\beta}\right)^{\alpha}\ln^2\left(\tau+\frac{t_i-\tau}{\beta}\right) \\
&\quad + (n-r)\left(\tau+\frac{t_{r:n}-\tau}{\beta}\right)^{\alpha}\ln^2\left(\tau+\frac{t_{r:n}-\tau}{\beta}\right) \\
D_3^{'(\alpha)}(\alpha,\beta) &= \sum_{i=\hat{n}_1+1}^{r}(t_i-\tau)\left(\tau+\frac{t_i-\tau}{\beta}\right)^{\alpha-1}\ln\left(\tau+\frac{t_i-\tau}{\beta}\right) \\
&\quad + (n-r)(t_{r:n}-\tau)\left(\tau+\frac{t_{r:n}-\tau}{\beta}\right)^{\alpha-1}\ln\left(\tau+\frac{t_{r:n}-\tau}{\beta}\right) \\
D_3^{'(\beta)}(\alpha,\beta) &= \sum_{i=\hat{n}_1+1}^{r}(t_i-\tau)^2\left(\tau+\frac{t_i-\tau}{\beta}\right)^{\alpha-2} + (n-r)(t_{r:n}-\tau)^2\left(\tau+\frac{t_{r:n}-\tau}{\beta}\right)^{\alpha-2}
\end{aligned}
$$

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