Bivariate Load Sharing Model: Bayesian Inference

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Abstract

This work is motivated by a data set obtained from an experiment performed on diabetic retinopathy patients. The experiment was conducted to test whether there is any significance difference between two different treatments namely laser treatment and traditional treatment. The two treatments have been administered on two different eyes of the same patient. The two eyes of a patient are modeled as a two-component load-sharing system. In such systems, the failure of one component can increase the stress on the other, leading to earlier failure. In most of the existing literature, the initial lifetimes of the two components are assumed to be independent, which may not be a reasonable assumption in this context. In this paper it has been assumed that the lifetimes of the two components may be dependent. The model allows for a positive probability that both components fail simultaneously. If they do not fail simultaneously, the survivor's lifetime is adjusted according to the tampered failure rate assumption, resulting in a bivariate distribution with a singular part. We have developed Bayesian inference of the unknown parameters under a very general set of prior assumptions when the baseline distributions are exponential, Weibull or piecewise constant hazard functions. Bayesian hypothesis testing is conducted using Bayes factors approach. Extensive simulations have been performed to show the performances of the proposed methods. The analysis of the diabetic retinopathy data set has been conducted to show the effectiveness of the Bayesian approach.

Keywords: Load-sharing model; Bayes estimated; Importance sampling; Credible intervals; Posterior analysis

1 Introduction

The motivation of our work came when we were trying to analyze a data set based on a diabetic retinopathy study. Diabetic retinopathy is a medical condition of the eye and it causes blindness. It is one of the leading cause of blindness, often progressing without early symptoms, and its severity has been shown to increase with disease duration, patient age, hypertension, and elevated serum lipid levels. The purpose of the experiment is to test whether there is any significance difference between two treatments namely laser treatment and traditional treatment. In this experiment for each patient in one eye (chosen at random) the laser treatment and in the other eye the traditional treatment have been administered. The observation is of the form (Y_1, Y_2), where Y_1 and Y_2 denote the time to onset of blindness of the laser treated eye and the traditionally treated eye, respectively. The data set was developed by Vincent [20] and it is based on 197 diabetic patients. All possible outcomes ($Y_1 < Y_2$, $Y_1 > Y_2$, and $Y_1 = Y_2$) are represented in this experiment. The data are heavily censored in both variables, primarily due to: (i) the event not occurring before the study ended, (ii) loss to follow-up, and (iii) withdrawal from the study. Out of 197 observations only 38 observations are complete and the rest of the observations are censored either in one or in both variables. Kundu [13] first analyzed this complete data set of 38 observations based on the assumption that it is two-component load-sharing system. In this paper also we have made the same assumption. The main difference of the present paper with that of Kundu [13] is that he has only developed the classical inference based on the complete observations, where as here we have developed Bayesian inference based on complete as well as censored observations.

In a load-sharing system, when one component fails, the load on the remaining components changes. This usually means the surviving components will have shorter expected lifetimes than they would have under the original conditions. In some cases, the failure of one component might free up resources, allowing the remaining components to last longer. A model that captures these changes in the lifetimes of components in a parallel system is called a load-sharing model. The load-sharing model was first proposed by Daniels [5] in the analysis of textile data. Since then, it has been applied in many areas. In reliability studies, it has been used to model systems such as electrical generators, multi-processor computer systems, suspension bridge cables, and valves or pumps in hydraulic systems (see, for example, Suprasad et al. [19], Kvam and Pean [14] and related references). Gross et al. [10] applied the model to biological systems involving paired organs, such as eyes or kidneys. For more recent developments in load-sharing models, see Wang et al. [21], Xu et al. [22], Park et al. [15], Rychlik and Spizzichino [17] and references therein.

The bivariate load-sharing model introduced by Freund [8] has been widely used in practice. In this model, the two lifetimes are initially assumed to be independent and exponentially distributed. Once one component fails, the survivor's lifetime distribution changes to reflect the additional load. Many extensions of this model have been proposed in the literature, see for example Asha et al. [1], Franco et al. [7] and the references cited there in, but most retain the assumption of initial independence, which may not be realistic in practice.

Recent studies show that Marshall–Olkin bivariate Weibull type models are still widely used and have many new developments, including Bayesian analyses. Since our model allows dependence and also permits simultaneous failures, it has a clear Marshall-Olkin type structure, and it reduces to the usual Marshall-Olkin or bivariate prportional hazard (BPH) model for special choices of the load factors. Recent Bayesian methods for Marshall-Olkin Weibull models have also been developed in reliability problems. Zhang et al. [24] presented a Bayesian inference approach for the Marshall–Olkin Weibull model.

A more general approach was developed by Kundu [13], where the initial lifetimes to be dependent and included the possibility of simultaneous failures. If the components do not fail together, the survivor's lifetime changes according to the tampered failure rate model, see Bhattacharyya and Soejoeti [3], a natural choice in step-stress settings. A schematic diagram of the load sharing of a two component system is provided in Figure 1. This model is flexible and includes a singular component in its distribution, which earlier models lack. Methodologically, classical inference for the bivariate load-sharing model is based on maximum likelihood estimation, which does not directly incorporate prior knowledge. Bayesian inference, on the other hand, offers several advantages. In particular, results under classical inference are typically asymptotic, whereas Bayesian methods provide inference that remains valid for finite samples. For these reasons, it is both natural and necessary to extend the model of Kundu [13] to a Bayesian framework.

In this paper, we investigate the bivariate load-sharing model within a Bayesian framework. Prior distributions are specified for the model parameters, and methods are developed to obtain the Bayes estimates and credible intervals under different baseline hazard functions, including exponential, Weibull, and piecewise constant forms. Specifically, we assume a flexible Dirichlet–Gamma prior on the model parameters, as suggested by Pena and Gupta [16] for the Marshall–Olkin bivariate exponential distribution. A Gamma prior is used for the Weibull hazard rate parameter, and an independent Dirichlet prior is adopted for the piecewise constant hazard rate parameters. It is observed that the Bayes estimates of the unknown parameters cannot be obtained in closed form, since they involve multidimensional integrations. To address this, we propose the use of the importance sampling technique to compute the Bayes estimators and the associated credible intervals. Bayesian hypothesis testing is also carried out using Bayes factors, with particular emphasis on testing the equality of the marginal lifetime distributions. The performance of the proposed methods is examined through extensive simulation studies under various parameter settings and sample sizes. Finally, the diabetic retinopathy data set has been analyzed in presence of censoring to demonstrate how the proposed method can be used in practice.

The rest of the paper is organized as follows. Section 2 introduces the model and the corresponding likelihood function for the complete data set. Section 3 presents the prior

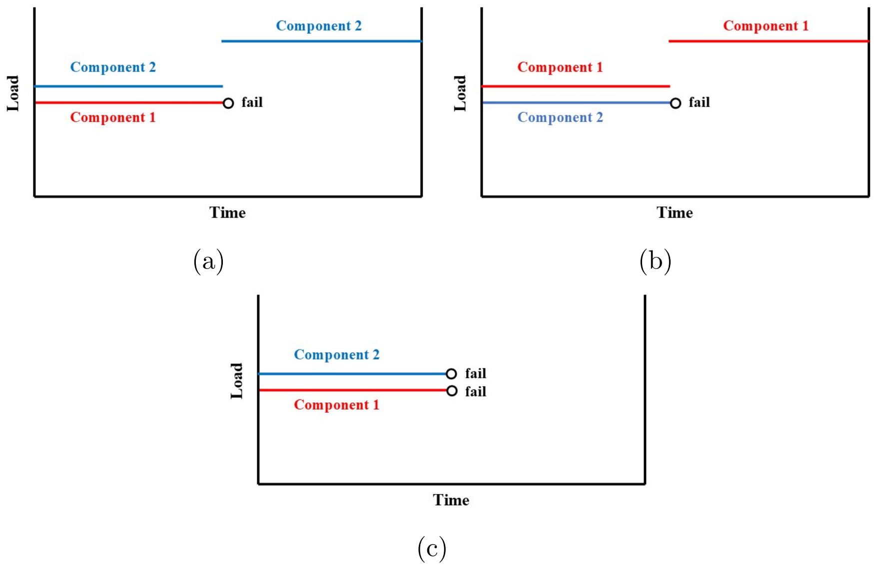

Figure 1: Schematic diagram of the load-sharing of a two component system when (a) the first component fails before the second component (b) the second component fails before the first component and (c) both the component fail at the same time.

and posterior distributions, along with the Bayes estimates, highest posterior density (HPD) intervals, and symmetric credible intervals. In Section 4, the likelihood function for complete and censored data is developed, followed by the corresponding posterior analysis. Bayesian hypothesis testing problems are addressed in Section 5. Section 6 reports the results of simulation studies, while Section 7 presents the analysis of the retinopathy data set, first under complete observations and then under the complete and censored data. Finally, Section 8 provides concluding remarks.

2 Model and Likelihood Function for Complete Data

We consider a system with two components, A and B. The system stops working only when both components fail. Let X and Y be the lifetimes of components A and B, respectively. These lifetimes are continuous random variables and may be related to each other. There are three possible cases:

Case 1: Both components A and B fail at the same time. The system stops immediately.

Case 2: Component A fails first, but component B is still working. The system keeps running because B is still active. However, the failure of A increases the load on B, so its lifetime changes from Y to Y^*. The system stops when B finally fails.

Case 3: This is similar to Case 2, but B fails first. The lifetime of A changes from X to X^* after B fails. The system stops when A finally fails.

Let (Y_1, Y_2) be the lifetimes of components A and B at the time the system fails. Then,

$$(Y_1, Y_2) = \begin{cases} (X, Y), & \text{if } Y_1 = Y_2, \\ (X, Y^*), & \text{if } Y_1 < Y_2, \\ (X^*, Y), & \text{if } Y_1 > Y_2. \end{cases} \tag{1}$$

We assume (X, Y) follows a bivariate proportional hazards model with the baseline survival function S_B and parameters $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_3 > 0$, i.e. (X, Y) has the following joint survival function

$$P(X > x, Y > y) = \begin{cases} (S_B(x))^{\lambda_1}(S_B(y))^{\lambda_2+\lambda_3} & \text{if} \quad x < y \\ (S_B(x))^{\lambda_1+\lambda_3}(S_B(y))^{\lambda_2} & \text{if} \quad x \geq y. \end{cases}$$

The joint probability density function (PDF) of (Y_1, Y_2), given by Kundu [13], is:

Case 1: When $y_1 = y_2 = y$,

$$f_{Y_1,Y_2}(y, y) = \lambda_3 \, h_B(y) \, [S_B(y)]^{\lambda_1+\lambda_2+\lambda_3}, \tag{2}$$

where h_B is the baseline hazard function.

Case 2: When $y_1 < y_2$,

$$f_{Y_1,Y_2}(y_1, y_2) = \lambda_1\lambda_5 \, h_B(y_1)h_B(y_2) \, [S_B(y_1)]^{\lambda_1+\lambda_2+\lambda_3-\lambda_5}[S_B(y_2)]^{\lambda_5}, \tag{3}$$

where $\lambda_5 = \gamma(\lambda_2 + \lambda_3)$ and $\gamma > 0$ is the load factor on B after A fails.

Case 3: When $y_1 > y_2$,

$$f_{Y_1,Y_2}(y_1, y_2) = \lambda_2\lambda_4 \, h_B(y_1)h_B(y_2) \, [S_B(y_2)]^{\lambda_1+\lambda_2+\lambda_3-\lambda_4}[S_B(y_1)]^{\lambda_4}, \tag{4}$$

where $\lambda_4 = \delta(\lambda_1 + \lambda_3)$ and $\delta > 0$ is the load factor on A after B fails.

The load-sharing parameters describe how the risk of failure of one component changes after the other component fails. In this model, δ controls how the hazard of the surviving component A is multiplied when B fails: $\delta > 1$ means the survivor is under extra load and fails faster, while $\delta < 1$ means the survivor experiences relief and fails more slowly. The parameter γ plays a similar role for B when A fails. In the diabetic retinopathy study, the two components are the patient's two eyes. If blindness occurs in one eye, the patient may rely more on the other eye in daily life, which can be interpreted as an increased functional load on the remaining eye and hence a higher risk of blindness, corresponding to $\delta > 1$ (or

$\gamma > 1$). Thus, $\delta = \gamma = 1$ represents no load-sharing effect, and the model reduces to the usual BPH (Marshall–Olkin type) model.

Next, to find the likelihood function, we take a random sample of size n,

$$S = \{(y_{11}, y_{21}), \ldots, (y_{1n}, y_{2n})\}.$$

Define the index sets

$$I_1 = \{i : y_{1i} < y_{2i}\}, \quad I_2 = \{i : y_{1i} > y_{2i}\}, \quad I_3 = \{i : y_{1i} = y_{2i} = y_i\}.$$

Let n_1, n_2, and n_3 be the number of elements in I_1, I_2, and I_3, respectively.

Using the joint density functions given in (2), (3), and (4), the likelihood function can be written as

$$l(\Theta, \alpha \,|\, S) = l_1(\Theta, \alpha \,|\, S)\; l_2(\alpha \,|\, S),$$

where $\Theta = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)^\top$ and α is the parameter of the baseline distribution. There are two situations: the baseline distribution may be known or unknown.

The terms $l_1(\Theta, \alpha \,|\, S)$ and $l_2(\alpha \,|\, S)$ are given by

$$\begin{aligned}
l_1(\Theta, \alpha \,|\, S) &= \lambda_1^{n_1}\lambda_2^{n_2}\lambda_3^{n_3}\lambda_4^{n_2}\lambda_5^{n_1} \prod_{i\in I_1}[S_B(y_{1i})]^{\lambda_1+\lambda_2+\lambda_3-\lambda_5} \prod_{i\in I_1}[S_B(y_{2i})]^{\lambda_5} \\
&\quad \times \prod_{i\in I_2}[S_B(y_{2i})]^{\lambda_1+\lambda_2+\lambda_3-\lambda_4} \prod_{i\in I_2}[S_B(y_{1i})]^{\lambda_4} \prod_{i\in I_3}[S_B(y_i)]^{\lambda_1+\lambda_2+\lambda_3} \\
&= \lambda_1^{n_1}\lambda_2^{n_2}\lambda_3^{n_3}\lambda_4^{n_2}\lambda_5^{n_1}\, e^{-\left(\lambda_1+\lambda_2+\lambda_3-\lambda_5\right)\sum_{i\in I_1} H_B(y_{1i};\alpha)} e^{-\lambda_5 \sum_{i\in I_1} H_B(y_{2i};\alpha)} \\
&\quad \times e^{-\left(\lambda_1+\lambda_2+\lambda_3-\lambda_4\right)\sum_{i\in I_2} H_B(y_{2i};\alpha)} e^{-\lambda_4 \sum_{i\in I_2} H_B(y_{1i};\alpha)} \\
&\quad \times e^{-\left(\lambda_1+\lambda_2+\lambda_3\right)\sum_{i\in I_3} H_B(y_i;\alpha)},
\end{aligned} \tag{5}$$

where H_B is the cumulative baseline hazard function, and

$$l_2(\alpha \,|\, S) = \prod_{i\in I_3} h_B(y_i;\alpha) \prod_{i\in I_1\cup I_2} \big[h_B(y_{1i};\alpha)\; h_B(y_{2i};\alpha)\big]. \tag{6}$$

3 Prior and Posterior Analysis

3.1 Prior Assumptions

In this section, we describe the prior distributions for the unknown parameters. We first specify the priors for λ_i, $i = 1, \ldots, 5$, and then, based on the baseline distribution, we choose priors for the baseline parameters.

Following Pena and Gupta [16], we assume a conjugate prior for $(\lambda_1, \lambda_2, \lambda_3)$. The conjugate prior is used mainly for convenience and stability. Under conjugacy, the posterior stays in the same family as the prior, so we can obtain the posterior by simply updating the hyper-parameters. This makes computation fast and stable, which is important here because the model has many parameters and the data can be heavily censored. Morepver, the proposed conjugate prior is a very flexible one. The joint PDF can take variety of shapes. Similar use of conjugate priors for efficient Bayesian updating in reliability problems is discussed in Xu and Wang [23].

Assume that, $\lambda = \lambda_1 + \lambda_2 + \lambda_3$ has a Gamma(a, b), $a > 0, b > 0$ prior. The probability density function of Gamma(a, b) is

$$\pi_0(\lambda|a,b) = \begin{cases} \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda} & \text{if } \lambda > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Given λ, $\left(\frac{\lambda_1}{\lambda}, \frac{\lambda_2}{\lambda}\right)$ has a Dirichlet prior, that is

$$\pi_1\left(\frac{\lambda_1}{\lambda}, \frac{\lambda_2}{\lambda}|\lambda, a_1, a_2, a_3\right) = \frac{\Gamma(a_1+a_2+a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\left(\frac{\lambda_1}{\lambda}\right)^{a_1-1}\left(\frac{\lambda_2}{\lambda}\right)^{a_2-1}\left(\frac{\lambda_3}{\lambda}\right)^{a_3-1}, \tag{8}$$

where $\lambda_3 = \lambda - \lambda_1 - \lambda_2$, and the hyperparameters a_1, a_2 and a_3 are > 0. The joint prior of λ_1, λ_2 and λ_3 are obtained as

$$\pi_2(\lambda_1, \lambda_2, \lambda_3|a, b, a_1, a_2, a_3) = \frac{\Gamma(\tilde{a})}{\Gamma(a)}(b\lambda)^{a-\tilde{a}}\prod_{i=1}^{3}\left[\frac{b^{a_i}}{\Gamma(a_i)}\lambda_i^{a_i-1}e^{-b\lambda_i}\right], \tag{9}$$

where $\tilde{a} = a_1 + a_2 + a_3$. The probability density function in (9) is known as the density function of Gamma-Dirichlet distribution.

Next, for λ_4 and λ_5, we consider the independent Gamma priors, that is

$$\pi_3(\lambda_4|a_4,b_4) = \begin{cases} \frac{b_4^{a_4}}{\Gamma(a_4)}\lambda_4^{a_4-1}e^{-b_4\lambda_4} & \text{if } \lambda_4 > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

and

$$\pi_4(\lambda_5|a_5,b_5) = \begin{cases} \frac{b_5^{a_5}}{\Gamma(a_5)}\lambda_5^{a_5-1}e^{-b_5\lambda_5} & \text{if } \lambda_5 > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

If the baseline distribution is unknown, we consider the suitable prior for the baseline parameters.

3.2 Posterior Analysis

In this section, We first look at the case when the baseline distribution is known, and then the case when it is not.

Known Baseline Distribution Here, the baseline parameter α is known. So, the likelihood depends only on $l_1(\Theta|D)$. The joint posterior distribution of $\Theta = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)^\top$ can be obtained as

$$\begin{aligned}\Pi(\Theta|D) \propto \lambda_1^{n_1+a_1-1}\lambda_2^{n_2+a_2-1}\lambda_3^{n_3+a_3-1}\lambda_4^{n_2+a_4-1}\lambda_5^{n_1+a_5-1}e^{-(S_1+b)\lambda_1} \\ e^{-(S_1+b)\lambda_2}e^{-(S_1+b)\lambda_3}e^{-(S_2+b_4)\lambda_4}e^{-(S_3+b_5)\lambda_5} \times \lambda^{a-\tilde{a}} \\ = C \times f_G(\lambda_1; n_1+a_1, S_1+b) f_G(\lambda_2; n_2+a_2, S_1+b) f_G(\lambda_3; n_3+a_3, S_1+b) \\ f_G(\lambda_4; n_2+a_4, S_2+b_4) f_G(\lambda_5; n_1+a_5, S_3+b_5) \times h(\lambda_1, \lambda_2, \lambda_3). \end{aligned} \tag{12}$$

Here, $f_G(\lambda_i; c, d) \sim \text{Gamma}(c, b)$, $S_1 = \sum_{i\in I_1} H_B(y_{1i}) + \sum_{i\in I_2} H_B(y_{2i}) + \sum_{i\in I_3} H_B(y_i)$, $S_2 = \sum_{i\in I_2} \big[H_B(y_{1i}) - H_B(y_{2i})\big]$ and $S_3 = \sum_{i\in I_1} \big[H_B(y_{2i}) - H_B(y_{1i})\big]$, $h(\lambda_1, \lambda_2, \lambda_3) = \lambda^{a-\tilde{a}}$.

Baseline distribution: exponential

We take the baseline distribution as exponential with $\alpha = 1$ and $H_B(y) = y$. Here

$$S_1 = \sum_{i\in I_1} y_{1i} + \sum_{i\in I_2} y_{2i} + \sum_{i\in I_3} y_i, \quad S_2 = \sum_{i\in I_2} \big[y_{1i} - y_{2i}\big], \quad S_3 = \sum_{i\in I_1} \big[y_{2i} - y_{1i}\big].$$

We use the importance sampling procedure to compute the Bayes estimate of $g = g(\Theta)$ and also to construct the corresponding credible intervals of g. We use the following procedure:

Step 1: Generate

$$\begin{aligned}\lambda_1 &\sim \text{Gamma}(n_1 + a_1, S_1 + b) \\ \lambda_2 &\sim \text{Gamma}(n_2 + a_2, S_1 + b) \\ \lambda_3 &\sim \text{Gamma}(n_3 + a_3, S_1 + b) \\ \lambda_4 &\sim \text{Gamma}(n_2 + a_4, S_2 + b_4) \\ \lambda_5 &\sim \text{Gamma}(n_1 + a_5, S_3 + b_5)\end{aligned}$$

Step 2: Repeat this N times to obtain $\{\lambda_{1i}, \lambda_{2i}, \lambda_{3i}, \lambda_{4i}, \lambda_{5i}; i = 1, \ldots, N\}$.

Step 3: For squared error loss, the Bayes estimate of $g(\Theta)$ can be obtained as

$$\hat{g}(\Theta) = \frac{\sum_{i=1}^{N} g_i h(\lambda_{1i}, \lambda_{2i}, \lambda_{3i})}{\sum_{i=1}^{N} h(\lambda_{1i}, \lambda_{2i}, \lambda_{3i})}, \tag{13}$$

where $g_i = g(\lambda_{1i}, \lambda_{2i}, \lambda_{3i}, \lambda_{4i}, \lambda_{5i})$.

Now to construct a $(1-\gamma)100\%$ HPD credible interval of $g(\Theta)$:

- First compute
$$w_i = \frac{h(\lambda_{1i}, \lambda_{2i}, \lambda_{3i})}{\sum_{i=1}^{N} h(\lambda_{1i}, \lambda_{2i}, \lambda_{3i})}, \tag{14}$$

- Sort the pairs $\{(g_1, w_1), \ldots, (g_N, w_N)\}$ as $\{(g_{(1)}, w_{(1)}), \ldots, (g_{(N)}, w_{(N)})\}$, where $g_{(1)} < \ldots < g_{(N)}$ and $w'_{(i)}$s are associated with $g_{(i)}$.

- For any $p \in (0, 1)$,
$$\sum_{i=1}^{N_p} w_{(i)} \leq p \leq \sum_{i=1}^{N_p+1} w_{(i)} \text{ and } \hat{g}_p = g_{(N_p)}$$

- A $(1-\gamma)100\%$ credible interval of $g(\Theta)$ can be obtained as $(\hat{g}_\delta, \hat{g}_{\delta+1-\gamma})$, for $\gamma = w_{(1)}, w_{(1)} + w_{(2)}, \ldots, \sum_{i=1}^{N_\gamma} w_{(i)}$. Hence, a HPD $(1-\gamma)100\%$ credible interval of $g(\Theta)$ can be obtained as $(\hat{g}_\delta*, \hat{g}_{\delta*+1-\gamma})$, where
$$\hat{g}_{\delta*+1-\gamma} - \hat{g}_{\delta*} \leq \hat{g}_{\delta+1-\gamma} - \hat{g}_\delta, \quad \text{for all } \delta.$$

Unknown Baseline Distribution If the baseline parameter α is unknown, we put a prior $\pi(\alpha)$ on it. The joint posterior of (Θ, α) can be written as
$$\Pi(\Theta, \alpha|D) = \Pi_1(\Theta|\alpha, D) \times h(\Theta, \alpha) \times \Pi_2(\alpha|D), \tag{15}$$
where
$$\begin{aligned}\Pi_1(\Theta|\alpha, D) \propto \quad & f_G(\lambda_1; n_1 + a_1, S_1(\alpha) + b) f_G(\lambda_2; n_2 + a_2, S_1(\alpha) + b) f_G(\lambda_3; n_3 + a_3, S_1(\alpha) + b) \\ & f_G(\lambda_4; n_2 + a_4, S_2(\alpha) + b_4) f_G(\lambda_5; n_1 + a_5, S_3(\alpha) + b_5) \end{aligned} \tag{16}$$

$$h(\Theta, \alpha) \times \Pi_2(\alpha|D) = \frac{\lambda^{a-\tilde{a}} \times l_2(\alpha|D) \times \pi(\alpha)}{(S_1(\alpha) + b)^{n+\tilde{a}} (S_2(\alpha) + b_4)^{n_2+a_4} (S_3(\alpha) + b_5)^{n_1+a_5}} \tag{17}$$

Here $S_1(\alpha) = \sum_{i \in I_1} H_B(y_{1i}; \alpha) + \sum_{i \in I_2} H_B(y_{2i}; \alpha) + \sum_{i \in I_3} H_B(y_i; \alpha)$, $S_2(\alpha) = \sum_{i \in I_2} \left[H_B(y_{1i}; \alpha) - H_B(y_{2i}; \alpha)\right]$ and $S_3(\alpha) = \sum_{i \in I_1} \left[H_B(y_{2i}; \alpha) - H_B(y_{1i}; \alpha)\right]$.

Baseline distribution: Weibull

If the baseline distribution is Weibull, we have one parameter α, and $H_B(y; \alpha) = y^\alpha$. Then
$$S_1(\alpha) = \sum_{i \in I_1} y_{1i}^\alpha + \sum_{i \in I_2} y_{2i}^\alpha + \sum_{i \in I_3} y_i^\alpha, \qquad S_2(\alpha) = \sum_{i \in I_2} \left[y_{1i}^\alpha - y_{2i}^\alpha\right], \qquad S_3(\alpha) = \sum_{i \in I_1} \left[y_{2i}^\alpha - y_{1i}^\alpha\right].$$

Here,

$$\Pi_2(\alpha|D) = \frac{\pi(\alpha) \times \alpha^{2n_1+2n_2+n_3}\Big(\prod_{i\in I_3} y_i^{\alpha-1}\Big)\Big(\prod_{i\in I_1\cup I_2} y_{1i}^{\alpha-1}y_{2i}^{\alpha-1}\Big)}{(S_1(\alpha)+b)^{n+\tilde{a}}(S_2(\alpha)+b_4)^{n_2+a_4}(S_3(\alpha)+b_5)^{n_1+a_5}}$$

and $h(\Theta) = \lambda^{a-\tilde{a}}$.

We take the prior for α as Gamma distribution: $\pi(\alpha) = f_G(\alpha; c, d)$.

Lemma 3.1. If the probability density function of $\pi(\alpha)$ is log-concave, then $\Pi_2(\alpha|D)$ is log-concave.

Proof. The proof is similar as the proof of Theorem 2 of [12], so we do not repeat it here. □

Using Lemma 1, we consider an importance sampling method to obtain the Bayes estimators and credible intervals of Θ and α as follows

Step 1: Generate α from the density $\Pi_2(\alpha|D)$ using the method proposed by [6].

Step 2: Generate

$$\begin{aligned}
\lambda_1 &\sim \text{Gamma}(n_1 + a_1, S_1(\alpha) + b)\\
\lambda_2 &\sim \text{Gamma}(n_2 + a_2, S_1(\alpha) + b)\\
\lambda_3 &\sim \text{Gamma}(n_3 + a_3, S_1(\alpha) + b)\\
\lambda_4 &\sim \text{Gamma}(n_2 + a_4, S_2(\alpha) + b_4)\\
\lambda_5 &\sim \text{Gamma}(n_1 + a_5, S_3(\alpha) + b_5)
\end{aligned}$$

Step 3: Repeat the Steps 1 - 2 to get $\{\alpha_i, \lambda_{1i}, \lambda_{2i}, \lambda_{3i}, \lambda_{4i}, \lambda_{5i}; i = 1, \ldots, N\}$.

Step 4: The Bayes estimator of $g = g(\alpha, \Theta)$ under the squared error loss function is

$$\hat{g}(\alpha, \Theta) = \frac{\sum_{i=1}^{N} g_i h(\lambda_{1i}, \lambda_{2i}, \lambda_{3i})}{\sum_{i=1}^{N} h(\lambda_{1i}, \lambda_{2i}, \lambda_{3i})}, \tag{18}$$

where $g_i = g(\alpha_i, \lambda_{1i}, \lambda_{2i}, \lambda_{3i}, \lambda_{4i}, \lambda_{5i})$.

In the same way as for the exponential case, one can also find the HPD credible intervals.

Baseline distribution: Piecewise constant hazard (PCH) In this case we do not assume any specific form of baseline distribution function. Instead the baseline hazard function $h_B(t)$ is assumed to be piecewise constant. This means it is defined by a set of constants $(c_1, \ldots, c_M)$ and cut points $0 = \tau_0 < \tau_1 < \ldots < \tau_M$. The hazard $h_B(t; \alpha)$ has the following form

$$h_B(t : \alpha) = \sum_{j=1}^{M} c_j I_{[\tau_{j-1}, \tau_j)}(t), \tag{19}$$

where c_j's are constants and $I_{[\tau_{j-1}, \tau_j)}(t)$ is the indicator function as

$$I_{[\tau_{j-1}, \tau_j)}(t) = \begin{cases} 1 & \text{if } t \in [\tau_{j-1}, \tau_j) \\ 0 & \text{otherwise.} \end{cases}$$

It may be mentioned that in case of proportional hazard class of distributions as defined above, $c_1, \ldots, c_M$ are identifiable only up to a multiplicative constant. Hence, without loss of generality it can be assumed that

$$\sum_{j=1}^{M} c_j = 1, \quad c_j > 0.$$

Thus, in this case $\alpha = (c_1, \ldots, c_M)^T$. Further, when $M = 1$, it reduces to exponential hazard function. Based on the hazard function (18), the cumulative hazard function is obtained as

$$H_B(t; \alpha) = \sum_{j=1}^{M} c_j \Big(\min\{t, \tau_j\} - \tau_{j-1} \Big) I_{[\tau_{j-1}, \tau_j)}(t). \tag{20}$$

For the PCH case, we to write y_{1i}, y_{2i} and y_i in the following way: for $y_{1i}(y_{2i}) \in I_1 \cup I_2$ as $y_{1uvw}(y_{1uvw})$, where

- $u = 1$ or 2 depending on whether $i \in I_1$ or $i \in I_2$,
- $v = m$, if $\tau_{m-1} < y_{1i}(y_{2i}) \leq \tau_m$, for $m = 1, \ldots, M$,
- $w = 1, \ldots, n_{1um}(n_{2um})$.

Here, $n_{1um}(n_{2um})$ denotes the number of observations satisfying $\{i; y_{1i}(y_{2i}) \in I_u, \tau_{m-1} < y_{1i}(y_{2i}) \leq \tau_m\}$. Similarly, we write y_i as y_{vw}, and define n_{um}. With these notations, the cumulative hazard function becomes

$$\sum_{i\in I_1} H_B(y_{1i};\alpha) = c_1D_{11} + c_2D_{12} + \ldots + c_MD_{1M} \tag{21}$$
$$\sum_{i\in I_2} H_B(y_{1i};\alpha) = c_1D_{21} + c_2D_{22} + \ldots + c_MD_{2M} \tag{22}$$
$$\sum_{i\in I_1} H_B(y_{2i};\alpha) = c_1E_{11} + c_2E_{12} + \ldots + c_ME_{1M} \tag{23}$$
$$\sum_{i\in I_2} H_B(y_{2i};\alpha) = c_1E_{21} + c_2E_{22} + \ldots + c_ME_{2M} \tag{24}$$
$$\sum_{i\in I_3} H_B(y_{i};\alpha) = c_1F_{1} + c_2F_{2} + \ldots + c_MF_{M}. \tag{25}$$

The explicit expressions of D'_{ij}s, E'_{ij}s and F'_js are described in the Appendix.

We now define

$$\begin{aligned} S_1(\alpha) &= c_1U_1 + c_2U_2 + \ldots + c_MU_M \\ S_2(\alpha) &= c_1V_1 + c_2V_2 + \ldots + c_MV_M \\ S_3(\alpha) &= c_1W_1 + c_2W_2 + \ldots + c_MW_M, \end{aligned}$$

where $u_j = D_{1j} + E_{2j} + F_j$, $V_j = D_{2j} - E_{2j}$ and $W_j = E_{1j} - D_{1j}$.

Next, we assume the prior for $\alpha = (c_1, \ldots, c_M)^\top$ is Dirichlet:

$$\pi(c_1, \ldots, c_M|\theta) = \frac{1}{B(\theta)} \prod_{j=1}^{M} c_j^{\theta_j - 1}, \tag{26}$$

here $\theta = (\theta_1, \ldots, \theta_M)$,$\theta_1 > 0, \ldots, \theta_M > 0$ and $B(\theta) = \Big[\prod_{j=1}^{M} \Gamma(\theta_j)\Big]\Big/\Gamma(\sum_{j=1}^{M} \theta_j)$ is the normalising constant.

From (16),

$$\Pi_2(\alpha) = f_D(\alpha; \theta + k),$$

where $f_D(\alpha;\theta+k) \sim$ Dirichlet$(\theta+k)$, $\theta + k = (\theta_1 + k_1, \ldots, \theta_M + k + M)$ and $k_j = \#\{j : \tau_{j-1} < y_j \leq \tau_j\} + \#\{j : \tau_{j-1} < y_{1j} \leq \tau_j\} + \#\{j : \tau_{j-1} < y_{2j} \leq \tau_j\}$, for $j = 1, \ldots, M$.

The weight function is

$$h(\Theta, \alpha) = \frac{\lambda^{a-\tilde{a}}}{(S_1(\alpha) + b)^{n+\tilde{a}}(S_2(\alpha) + b_4)^{n_2+a_4}(S_3(\alpha) + b_5)^{n_1+a_5}} \tag{27}$$

Algorithm to compute the Bayes estimates and the corresponding credible intervals

Step 1: generate $\alpha = (c_1, \ldots, c_M) \sim$ Dirichlet$(\theta + k)$.

Step 2: generate

$$\begin{aligned}
\lambda_1 &\sim \text{Gamma}(n_1 + a_1, S_1(\alpha) + b) \\
\lambda_2 &\sim \text{Gamma}(n_2 + a_2, S_1(\alpha) + b) \\
\lambda_3 &\sim \text{Gamma}(n_3 + a_3, S_1(\alpha) + b) \\
\lambda_4 &\sim \text{Gamma}(n_2 + a_4, S_2(\alpha) + b_4) \\
\lambda_5 &\sim \text{Gamma}(n_1 + a_5, S_3(\alpha) + b_5)
\end{aligned}$$

Step 3: repeat the Step 1 and Step 2 to obtain $\{(c_{1i}, \ldots, c_{Mi}, \lambda_{1i}, \lambda_{2i}, \lambda_{3i}, \lambda_{4i}, \lambda_{5i}); i = 1, \ldots, N\}$.

Step 4: The Bayes estimator of $g = g(\alpha, \Theta)$ under the squared error loss function is

$$\hat{g}(\alpha, \Theta) = \frac{\sum_{i=1}^{N} g_i h(\lambda_{1i}, \lambda_{2i}, \lambda_{3i})}{\sum_{i=1}^{N} h(\lambda_{1i}, \lambda_{2i}, \lambda_{3i})}, \tag{28}$$

where $g_i = g(c_{1i}, \ldots, c_{Mi}, \lambda_{1i}, \lambda_{2i}, \lambda_{3i}, \lambda_{4i}, \lambda_{5i})$.

The HPD intervals can be computed following the same steps as given in the case of exponential.

When the baseline is PCH, it can take any shape depending on the values of τ_j and c_j. This makes it a data-driven model, which can be regarded as a semi-parametric–parametric approach. It has been used by several researchers (e.g., Balakrishnan et al. [2]; Ibrahim et al. [11]; Samanta and Kundu [18]; Ganguly et al. [9] and the references cited therein).

4 Likelihood function and Posterior Analysis for Complete and Censored data

In this section, first we discuss the likelihood function for the complete and censored data and then find the posterior distribution. We have a complete and censored bivariate dataset where each observation belongs to one of four types:

Type 1: (y_1, y_2), here y_1 is the exact failure time of the first component and y_2 is the exact failure time of the second component. It may happen that $y_1 < y_2$, $y_1 > y_2$ or $y_1 = y_2$. Let T_1 is the set of observation of this type of data. The likelihood contribution of this data type is

– When $y_1 < y_2$,

$$\lambda_1\lambda_5\, h_B(y_1)h_B(y_2)\,[S_B(y_1)]^{\lambda_1+\lambda_2+\lambda_3-\lambda_5}[S_B(y_2)]^{\lambda_5},$$

– When $y_1 > y_2$,

$$\lambda_2\lambda_4\, h_B(y_1)h_B(y_2)\,[S_B(y_2)]^{\lambda_1+\lambda_2+\lambda_3-\lambda_4}[S_B(y_1)]^{\lambda_4},$$

– When $y_1 = y_2$,

$$\lambda_3 h_B(y)\,[S_B(y)]^{\lambda_1+\lambda_2+\lambda_3}.$$

Type 2: (y_1, y_2^*), here y_1 is the exact failure time of the first component and y_2^* denotes that the second component has been censored at y_2. In this case $y_1 \leq y_2$ and the likelihood contribution is

$$\lambda_1 h_B(y_1)[S_B(y_1)]^{\lambda_1+\lambda_2+\lambda_3-\lambda_5}[S_B(y_2)]^{\lambda_5}.$$

Type 3: (y_1^*, y_2), here y_1^* denotes that the first component has been censored at y_1 and y_2 is the exact failure time of the second component. In this case $y_1 \geq y_2$ and the likelihood contribution is

$$\lambda_2 h_B(y_2)[S_B(y_2)]^{\lambda_1+\lambda_2+\lambda_3-\lambda_4}[S_B(y_1)]^{\lambda_4}.$$

Type 4: (y_1^*, y_2^*), here both the components have been censored at y_1 and and y_2. In this case $y_1 = y_2 = y$ and the likelihood contribution is

$$\lambda_3 h_B(y)[S_B(y)]^{\lambda_1+\lambda_2+\lambda_3}.$$

We denote the index sets as

$$T_1 = \{i : (y_{1i}, y_{2i})\}, \quad T_2 = \{i : (y_{1i}, y_{2i}^*)\}, \quad T_3 = \{i : (y_{1i}^*, y_{2i})\}, \quad T_4 = \{i : (y_{1i}^*, y_{2i}^*)\}.$$

and

$$I_{11} = T_1 \cap I_1, \quad I_{12} = T_1 \cap I_2, \quad I_{13} = T_1 \cap I_3, \quad I_{21} = T_2 \cap I_1, \quad I_{32} = T_3 \cap I_2, \quad I_{43} = T_4 \cap I_3.$$

Let

$$m_1 = \#I_{11} + \#I_{21}, \quad m_2 = \#I_{12} + \#I_{32}, \quad m_3 = \#I_{13} + \#I_{43}, \quad m_4 = \#I_{12}, \quad m_5 = \#I_{11}$$

The likelihood function can be written as

$$l_c(\Theta, \alpha \,|\, S) = l_{c1}(\Theta, \alpha \,|\, S)\; l_{c2}(\alpha \,|\, S),$$

where

$$\begin{aligned}
l_{c1}(\Theta,\alpha\,|\,S) &= \lambda_1^{m_1}\lambda_2^{m_2}\lambda_3^{m_3}\lambda_4^{m_4}\lambda_5^{m_5}\prod_{i\in I_{11}\cup I_{21}}[S_B(y_{1i})]^{\lambda_1+\lambda_2+\lambda_3-\lambda_5}\prod_{i\in I_{11}\cup I_{21}}[S_B(y_{2i})]^{\lambda_5}\\
&\quad\times\prod_{i\in I_{12}\cup I_{32}}[S_B(y_{2i})]^{\lambda_1+\lambda_2+\lambda_3-\lambda_4}\prod_{i\in I_{12}\cup I_{32}}[S_B(y_{1i})]^{\lambda_4}\prod_{i\in I_{13}\cup I_{43}}[S_B(y_i)]^{\lambda_1+\lambda_2+\lambda_3}\\
&= \lambda_1^{m_1}\lambda_2^{m_2}\lambda_3^{m_3}\lambda_4^{m_4}\lambda_5^{m_5}\,e^{-\Big(\lambda_1+\lambda_2+\lambda_3-\lambda_5\Big)\sum_{i\in I_{11}\cup I_{21}}H_B(y_{1i};\alpha)}e^{-\lambda_5\sum_{i\in I_{11}\cup I_{21}}H_B(y_{2i};\alpha)}\\
&\quad\times e^{-\Big(\lambda_1+\lambda_2+\lambda_3-\lambda_4\Big)\sum_{i\in I_{12}\cup I_{32}}H_B(y_{2i};\alpha)}e^{-\lambda_4\sum_{i\in I_{12}\cup I_{32}}H_B(y_{1i};\alpha)}\\
&\quad\times e^{-\Big(\lambda_1+\lambda_2+\lambda_3\Big)\sum_{i\in I_{13}\cup I_{43}}H_B(y_i;\alpha)},
\end{aligned}\tag{29}$$

and

$$l_{c2}(\alpha\,|\,S)=\prod_{i\in I_{13}\cup I_{43}}h_B(y_i;\alpha)\prod_{i\in I_{11}\cup I_{12}\cup I_{21}\cup I_{32}}\Big[h_B(y_{1i};\alpha)\;h_B(y_{2i};\alpha)\Big].\tag{30}$$

Here, the priors for $\lambda_i, i=1,\ldots,5$, have been considered as similar as discussed in Section 3.1. So the joint posterior can be obtained as

Known Baseline Distribution

$$\begin{aligned}
\Pi_c(\Theta|D) &\propto \lambda_1^{m_1+a_1-1}\lambda_2^{m_2+a_2-1}\lambda_3^{m_3+a_3-1}\lambda_4^{m_4+a_4-1}\lambda_5^{m_5+a_5-1}e^{-(S_{c1}+b)\lambda_1}\\
&\qquad e^{-(S_{c1}+b)\lambda_2}e^{-(S_{c1}+b)\lambda_3}e^{-(S_{c2}+b_4)\lambda_4}e^{-(S_{c3}+b_5)\lambda_5}\times\lambda^{a-\tilde{a}}\\
&= C\times f_G(\lambda_1;m_1+a_1,S_{c1}+b)f_G(\lambda_2;m_2+a_2,S_{c1}+b)f_G(\lambda_3;m_3+a_3,S_{c1}+b)\\
&\qquad f_G(\lambda_4;m_4+a_4,S_{c2}+b_4)f_G(\lambda_5;m_5+a_5,S_{c3}+b_5)\times h(\lambda_1,\lambda_2,\lambda_3).
\end{aligned}\tag{31}$$

Here,

$$\begin{aligned}
S_{c1} &= \sum_{i\in I_{11}\cup I_{21}}H_B(y_{1i})+\sum_{i\in I_{12}\cup I_{32}}H_B(y_{2i})+\sum_{i\in I_{13}\cup I_{43}}H_B(y_i),\\
S_{c2} &= \sum_{i\in I_{12}\cup I_{32}}\Big[H_B(y_{1i})-H_B(y_{2i})\Big],\\
S_{c3} &= \sum_{i\in I_{11}\cup I_{21}}\Big[H_B(y_{2i})-H_B(y_{1i})\Big],\\
h(\lambda_1,\lambda_2,\lambda_3) &= \lambda^{a-\tilde{a}}.
\end{aligned}$$

and

Unknown Baseline Distribution

$$\Pi_c(\Theta,\alpha|D)=\Pi_{c1}(\Theta|\alpha,D)\times h(\Theta,\alpha)\times\Pi_{c2}(\alpha|D).\tag{32}$$

Here

$$\begin{aligned}\Pi_{c1}(\Theta|\alpha, D) \propto \quad & f_G(\lambda_1; m_1 + a_1, S_{c1}(\alpha) + b) f_G(\lambda_2; m_2 + a_2, S_{c1}(\alpha) + b) f_G(\lambda_3; m_3 + a_3, S_{c1}(\alpha) + b) \\ & f_G(\lambda_4; m_4 + a_4, S_{c2}(\alpha) + b_4) f_G(\lambda_5; m_5 + a_5, S_{c3}(\alpha) + b_5) \end{aligned} \tag{33}$$

$$h(\Theta, \alpha) \times \Pi_2(\alpha|D) = \frac{\lambda^{a-\tilde{a}} \times l_{c2}(\alpha|D) \times \pi(\alpha)}{(S_{c1}(\alpha) + b)^{m_1+m_2+m_3+\tilde{a}} (S_{c2}(\alpha) + b_4)^{m_4+a_4} (S_{c3}(\alpha) + b_5)^{m_5+a_5}} \tag{34}$$

where

$$\begin{aligned} S_{c1} &= \sum_{i \in I_{11} \cup I_{21}} H_B(y_{1i}; \alpha) + \sum_{i \in I_{12} \cup I_{32}} H_B(y_{2i}; \alpha) + \sum_{i \in I_{13} \cup I_{43}} H_B(y_i; \alpha), \\ S_{c2} &= \sum_{i \in I_{12} \cup I_{32}} \big[H_B(y_{1i}; \alpha) - H_B(y_{2i}; \alpha)\big], \\ S_{c3} &= \sum_{i \in I_{11} \cup I_{21}} \big[H_B(y_{2i}; \alpha) - H_B(y_{1i}; \alpha)\big]. \end{aligned}$$

The Bayes estimators and credible intervals for the exponential, Weibull, and PCH baseline distributions are computed using the algorithms discussed in Section 3.

5 Testing of Hypothesis

In this section, we have considered the following testing of hypothesis problem:

$$H_0 : \lambda_1 = \lambda_2 \quad \text{and} \quad \lambda_4 = \lambda_5 \qquad \text{vs.} \qquad H_1 : \text{ at least one equality in } H_0 \text{ fails.} \tag{35}$$

Note that the above H_0 indicates that there is no difference between the two treatments. We have used the Bayes factor approach to address this problem. Under H_0 it is assumed that (λ_1, λ_3) has the Beta-Gamma prior

$$\pi_6(\lambda_1, \lambda_3|a_0, b_0, c_0, d_0) = \frac{\Gamma(c_0 + d_0)}{\Gamma(a_0)} [b_0(\lambda_1 + \lambda_3)]^{a_0 - c_0 - d_0} \frac{b_0^{c_0}}{\Gamma(c_0)} \lambda_1^{c_0 - 1} e^{-b_0 \lambda_1} \frac{b_0^{d_0}}{\Gamma(d_0)} \lambda_3^{d_0 - 1} e^{-b_0 \lambda_3} \tag{36}$$

and λ_4 has the prior $\pi_7(\lambda_4|c_1, d_1) \sim \text{Gamma}(c_1, d_1)$.

5.1 Complete observation

Note that for the complete observations, under H_0 the likelihood function is $l_0(\Theta_0, \alpha|D) = l_{10}(\Theta_0, \alpha|D) l_2(\alpha|D)$, where $\Theta_0 = (\lambda_1, \lambda_3, \lambda_4)^T$ and

$$\begin{aligned} l_{10}(\Theta_0, \alpha|D) = \quad & \lambda_1^{n_1+n_2} \lambda_3^{n_3} \lambda_4^{n_1+n_2} e^{-(2\lambda_1+\lambda_3)\left(\sum_{i \in I_1} H_B(y_{1i};\alpha) + \sum_{i \in I_2} H_B(y_{2i};\alpha) + \sum_{i \in I_3} H_B(y_i;\alpha) \right)} \\ & e^{-\lambda_4 \left(\sum_{i \in I_1} \left\{ H_B(y_{2i};\alpha) - H_B(y_{1i};\alpha) \right\} + \sum_{i \in I_2} \left\{ H_B(y_{1i};\alpha) - H_B(y_{2i};\alpha) \right\} \right)}. \end{aligned} \tag{37}$$

There will be two cases, known baseline distribution and unknown baseline distribution.

Known baseline distribution Here α is known. Therefore, under H_0, the marginal distribution of D is

$$
\begin{aligned}
l_{H_{01}}(D) = & \int\int\int l_{10}(\Theta_0|D)\pi_6(\lambda_1,\lambda_3|a_0,b_0,c_0,d_0)\pi_7(\lambda_4|c_1,d_1)d\lambda_1 d\lambda_3 d\lambda_4 \\
= & \frac{b_0^{a_0}\Gamma(c_0+d_0)d_1^{c_1}}{\Gamma(a_0)\Gamma(c_0)\Gamma(d_0)\Gamma(c_1)}\int\int\int\Big[\lambda_1^{n_1+n_2+c_0-1}\lambda_3^{n_3+d_0-1}\lambda_4^{n_1+n_2+c_1-1}(\lambda_1+\lambda_3)^{a_0-c_0-d_0} \\
& e^{-\lambda_1(b_0+2S_1)}e^{-\lambda_3(b_0+S_1)}e^{-\lambda_4(d_1+S_2+S_3)}\Big]d\lambda_1 d\lambda_3 d\lambda_4 \\
= & \frac{b_0^{a_0}\Gamma(c_0+d_0)d_1^{c_1}}{\Gamma(a_0)\Gamma(c_0)\Gamma(d_0)\Gamma(c_1)}\frac{n_1+n_2+c_1}{(d_1+S_2+S_3)^{n_1+n_2+c_1}}\int\int\Big[\lambda_1^{n_1+n_2+c_0-1}\lambda_3^{n_3+d_0-1} \\
& (\lambda_1+\lambda_3)^{a_0-c_0-d_0}e^{-\lambda_1(b_0+2S_1)}e^{-\lambda_3(b_0+S_1)}\Big]d\lambda_1 d\lambda_3
\end{aligned} \tag{38}
$$

where S_1, S_2 and S_3 have defined in Section 3.2. Under, H_1, we have considered the same priors as in Section 3.1. So the marginal of D is obtained as

$$
\begin{aligned}
l_{H_{11}}(D) = & \int\int l_1(\Theta|D)\pi_2(\lambda_1,\lambda_2,\lambda_3|a,b,a_1,a_2,a_3)\pi_3(\lambda_4|a_4,b_4)\pi_4(\lambda_5|a_5,b_5)d\Theta \\
= & \frac{b^a b_4^{a_4} b_5^{a_5}\Gamma(\tilde{a})}{\prod_{i=1}^5\Gamma(a_i)}\int\int\Big[\lambda_1^{n_1+a_1-1}\lambda_2^{n_2+a_2-1}\lambda_3^{n_3+a_3-1}\lambda_4^{n_2+a_4-1}\lambda_5^{n_1+a_5-1} \\
& e^{-(S_1+b)\lambda_1}e^{-(S_1+b)\lambda_2}e^{-(S_1+b)\lambda_3}e^{-(S_2+b_4)\lambda_4}e^{-(S_3+b_5)\lambda_5}\times\lambda^{a-\tilde{a}}\Big]d\Theta \\
= & \frac{b^a b_4^{a_4} b_5^{a_5}\Gamma(\tilde{a})}{\prod_{i=1}^5\Gamma(a_i)}\frac{\Gamma(n_2+a_4)\Gamma(n_1+a_5)}{(S_2+b_4)^{n_2+a_4}(S_2+b_5)^{n_1+a_5}}\int\int\int\Big[\lambda_1^{n_1+a_1-1}\lambda_2^{n_2+a_2-1}\lambda_3^{n_3+a_3-1} \\
& e^{-(S_1+b)\lambda}\lambda^{a-\tilde{a}}\Big]d\lambda_1\lambda_2\lambda_3
\end{aligned}
$$

Hence, the Bayes Factor (BF_1) is

$$
\text{BF}_1 = \frac{l_{H_{01}}(D)}{l_{H_{11}}(D)}
$$

Here, $l_{H_{01}}$ and $l_{H_{11}}$ do not have closed-form expressions, so they can be computed using the importance sampling technique for a given dataset and for the chosen hyperparameters.

Unknown baseline distribution Here α is unknown. Therefore, under H_0, the marginal distribution of D is

$$
l_{H_{02}}(D) = \int\int l_{10}(\Theta_0,\alpha|D)l_2(\alpha|D)\pi(\alpha)\pi_6(\lambda_1,\lambda_3|a_0,b_0,c_0,d_0)\pi_7(\lambda_4|c_1,d_1)d\lambda_1 d\lambda_3 d\lambda_4 d\alpha
$$

Under, H_1, we have considered the same priors as in Section 3.1. So the marginal of D is obtained as

$$
l_{H_{12}}(D) = \int\int l_1(\Theta_0\alpha|D)l_2(\alpha|D)\pi_2(\lambda_1,\lambda_2,\lambda_3|a,b,a_1,a_2,a_3)\pi_3(\lambda_4|a_4,b_4)\pi_4(\lambda_5|a_5,b_5)\pi(\alpha)d\Theta d\alpha
$$

Hence, the Bayes Factor (BF_2) is

$$\text{BF}_2 = \frac{l_{H_{02}}(D)}{l_{H_{12}}(D)}.$$

In this case, $l_{H_{02}}$ and $l_{H_{12}}$ does not have closed form, so it can be calculate using importance sampling technique for a given data set and a set of hyperparameters. We can compute the Bayes Factor for censored data in a similarly.

5.2 Censored observation

For the censored data, the likelihood function under null hypothesis H_0, is

$$l_{c0}(\Theta_0, \alpha|D) = l_{c10}(\Theta_0, \alpha|D) l_{c2}(\alpha|D)$$

where $\Theta_0 = (\lambda_1, \lambda_3, \lambda_4)$ and

$$\begin{aligned} l_{c10}(\Theta_0, \alpha|D) = \quad & \lambda_1^{m_1+m_2} \lambda_3^{m_3} \lambda_4^{m_4+m_5} e^{-\left(2\lambda_1+\lambda_3\right)\left[\sum_{i\in I_{11}\cup I_{21}} H_B(y_{1i};\alpha) + \sum_{i\in I_{12}\cup I_{32}} H_B(y_{2i};\alpha) + \sum_{i\in I_{13}\cup I_{43}} H_B(y_i;\alpha)\right]} \\ & e^{-\lambda_4\left[\sum_{i\in I_{11}\cup I_{21}} \left(H_B(y_{2i};\alpha) - H_B(y_{1i};\alpha)\right) + \sum_{i\in I_{12}\cup I_{32}} \left(H_B(y_{1i};\alpha) - H_B(y_{2i};\alpha)\right)\right]} \end{aligned} \tag{39}$$

For both known and unknown baseline distributions, the marginal distributions and Bayes factor under the null and alternative hypotheses coincide with those based on complete observations.

6 Simulation

In this section, a simulation study with complete observations is conducted to examine the performance of the proposed Bayes estimators and credible intervals under different baseline distributions and parameter values. Three baseline distributions are considered: (i) exponential, (ii) Weibull, and (iii) piecewise constant hazard. The Bayes estimates and credible intervals are computed based on 1000 replications and $N = 10000$ importance samples. To check the efficiency of the importance sampling procedure, we use simple weight-based diagnostics. Let $w_1, \ldots, w_N$ be the importance weights and $\tilde{w}_i = w_i / \sum_{j=1}^{N} w_j$ be the normalized weights. We compute the effective sample size (ESS) as $\text{ESS} = 1/\sum_{i=1}^{N} \tilde{w}_i^2$. In our analysis, ESS/N is close to 1, showing that the weights are well behaved and there is no serious weight degeneracy. We also compute the coefficient of variation (CV) of the weights, which is very small. Overall, these results indicate that the importance samples are efficient and the posterior estimates are stable.

6.1 Baseline distribution: exponential and Weibull

In case of exponential baseline distribution, we have considered the sample sizes 25, 50, 75 and 100. The following parameter values and the hyperparameters have been considered:

Set 1: $\lambda_1 = 0.03$, $\lambda_2 = 0.051$, $\lambda_3 = 0.015$, $\lambda_4 = 0.074$, $\lambda_5 = 0.057$. The corresponding hyperparameters: $a = 0.001$, $b = 0.001$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$, $b_4 = 0.001$, and $b_5 = 0.001$. (Tables 1, 2 and 3)

Set 2: $\lambda_1 = \lambda_2 = 0.038$, $\lambda_3 = 0.015$, $\lambda_4 = \lambda_5 = 0.066$. The corresponding hyperparameters: $a = 0.001$, $b = 0.001$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$, $b_4 = 0.001$, and $b_5 = 0.001$. (Tables 4, 5 and 6)

For the Weibull baseline distribution, we have considered the following parameter and hyperparameter values:

Set 3: $\lambda_1 = 0.19$, $\lambda_2 = 0.32$, $\lambda_3 = 0.097$, $\lambda_4 = 0.46$, $\lambda_5 = 0.34$, $\alpha = 1.1$. The corresponding Hyperparameters: $a = 0.001$, $b = 0.001$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$, $b_4 = 0.001$, $b_5 = 0.001$, $c = 12.1$, and $d = 11$. (Tables 7, 8 and 9).

Set 4: $\lambda_1 = \lambda_2 = 0.2$, $\lambda_3 = 0.25$, $\lambda_4 = \lambda_5 = 0.8$, $\alpha = 1.25$. The corresponding Hyperparameters: $a = 0.001$, $b = 0.001$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$, $b_4 = 0.001$, $b_5 = 0.001$, $c = 15.625$, and $d = 12.5$. (Tables 10, 11 and 12)

When the baseline is PCH, we have considered $M = 2$ and $M = 3$. For $M = 2$, we have taken $\tau_1 = 10$ and for $M = 3$, $\tau_1 = 15$, $\tau_2 = 30$. The parameter values have considered for $M = 2$ and $M = 3$ respectively

Set 5: $\lambda_1 = 0.09$, $\lambda_2 = 0.15$, $\lambda_3 = 0.045$, $\lambda_4 = 0.2$, $\lambda_5 = 0.15$, $c_1 = 0.31$, $c_2 = 0.69$. The corresponding Hyperparameters: $a = 0.001$, $b = 0.001$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$, $b_4 = 0.001$, $b_5 = 0.001$, and $\theta = (1, 1)$. (Tables 13, 15 and 15)

Set 6: $\lambda_1 = \lambda_2 = 0.2$, $\lambda_3 = 0.25$, $\lambda_4 = \lambda_5 = 0.8$, $c_1 = 0.0.09$, $c_2 = 0.33$, $c_3 = 0.58$. The corresponding Hyperparameters: $a = 0.001$, $b = 0.001$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$, $b_4 = 0.001$, $b_5 = 0.001$, and $\theta = (1, 1, 1)$. (Tables 16, 17 and 18)

In all cases considered , the mean squared error decreases as the sample size increases. Similarly, the average length of both the HPD and symmetric credible intervals decreases with increasing sample size. Moreover, the HPD intervals are consistently shorter than the corresponding symmetric intervals. Finally, the coverage probabilities are found to be close to the nominal levels in all cases. We have also conducted a prior sensitivity analysis by considering different choices of moderately informative and misspecified hyperparameters. The results indicate that the Bayes estimates of $\lambda_1, \ldots, \lambda_5$ remain largely stable across different prior specifications, demonstrating robustness of the proposed method. In contrast, the estimates of the Weibull shape parameter show greater sensitivity to the choice of hyperparameters.

Table 1: The average estimates (the associated mean squared error) for parameters in Set 1

n	λ_1	λ_2	λ_3	λ_4	λ_5
25	0.03524538	0.05655591	0.0165225	0.08127592	0.08043361
	(0.00017725)	(0.00027949)	(0.000076648)	(0.0004877922)	(0.001688493)
50	0.03548162	0.05397847	0.01646442	0.0784903	0.08165693
	(0.000095354)	(0.00011955)	(0.0000338068)	(0.0002695914)	(0.001074359)
75	0.03529128	0.05368103	0.01643105	0.07710862	0.08039329
	(0.000073376)	(0.0000788737)	(0.0000245986)	(0.0001894844)	(0.0008591837)
100	0.03493918	0.05375558	0.01628247	0.07703081	0.07917972
	(0.0000563187)	(0.0000629178)	(0.000017324)	(0.0001308397)	(0.0007090642)

Table 2: AL (CP) for 95% HPD intervals for parameters in Set 1

n	λ_1	λ_2	λ_3	λ_4	λ_5
25	0.0455	0.0591	0.0288	0.0851	0.1080
	(0.94)	(0.96)	(0.93)	(0.96)	(0.94)
50	0.0331	0.0412	0.02180	0.0605	0.0776
	(0.95)	(0.95)	(0.94)	(0.97)	(0.95)
75	0.0271	0.0336	0.0181	0.0486	0.0625
	(0.95)	(0.95)	(0.95)	(0.96)	(0.95)
100	0.0234	0.0292	0.0157	0.0419	0.0535
	(0.96)	(0.94)	(0.96)	(0.96)	(0.95)

Table 3: AL (CP) for 95% Symmetric intervals for parameters in Set 1

n	λ_1	λ_2	λ_3	λ_4	λ_5
25	0.0469	0.0603	0.0308	0.0868	0.1121
	(0.96)	(0.94)	(0.94)	(0.96)	(0.95)
50	0.0336	0.0417	0.0225	0.0613	0.0790
	(0.94)	(0.96)	(0.95)	(0.96)	(0.93)
75	0.0274	0.0339	0.0185	0.0490	0.0632
	(0.95)	(0.96)	(0.95)	(0.97)	(0.94)
100	0.0236	0.0294	0.0159	0.0423	0.0540
	(0.95)	(0.95)	(0.95)	(0.95)	(0.95)

7 Choice of cut points and data analysis

7.1 Choice of cut points

Earlier, it has been assumed that both the number of cut points and their positions were known. In practice, this is rarely the case, and the problem has received limited attention

Table 4: The average estimates (the associated mean squared error) for parameters in Set 2

n	λ_1	λ_2	λ_3	λ_4	λ_5
25	0.0413521	0.0421219	0.01861577	0.08551281	0.08406939
	(0.0001578195)	(0.0001689288)	(0.0000744159)	(0.001518075)	(0.001365964)
50	0.0417459	0.04182613	0.01729333	0.07574637	0.0746796
	(0.000085173)	(0.000085094)	(0.000032212)	(0.0004660099)	(0.0003761172)
75	0.04234654	0.04485402	0.01722576	0.07123323	0.0693818
	(0.000079978)	(0.000068298)	(0.000024729)	(0.0002257297)	(0.0001653067)
100	0.04344405	0.04340614	0.01836784	0.07244238	0.07185556
	(0.0000739835)	(0.000059591)	(0.000014418)	(0.0002126984)	(0.000155533)

Table 5: AL (CP) for 95% HPD intervals for parameters in Set 2

n	λ_1	λ_2	λ_3	λ_4	λ_5
25	0.0474	0.0479	0.0299	0.0978	0.0974
	(0.95)	(0.96)	(0.96)	(0.95)	(0.94)
50	0.0346	0.0346	0.0215	0.0629	0.0620
	(0.96)	(0.95)	(0.96)	(0.94)	(0.95)
75	0.0296	0.0305	0.0184	0.0487	0.0491
	(0.96)	(0.94)	(0.96)	(0.95)	(0.95)
100	0.0262	0.0262	0.01671	0.0439	0.0437
	(0.94)	(0.96)	(0.95)	(0.95)	(0.95)

Table 6: AL (CP) for 95% Symmetric intervals for parameters in Set 2

n	λ_1	λ_2	λ_3	λ_4	λ_5
25	0.0486	0.0492	0.0315	0.1004	0.1001
	(0.95)	(0.96)	(0.95)	(0.93)	(0.94)
50	0.0351	0.0352	0.0222	0.0639	0.0629
	(0.96)	(0.95)	(0.95)	(0.94)	(0.94)
75	0.0299	0.0308	0.0189	0.0492	0.0497
	(0.94)	(0.95)	(0.94)	(0.95)	(0.95)
100	0.0264	0.0264	0.0169	0.0443	0.0441
	(0.95)	(0.96)	(0.94)	(0.95)	(0.94)

in the literature. In this section, we present a Bayesian framework for estimating cut points when they are unknown.

Let τ be the maximum observed time point,

$$\tau = \max\{y_i, y_{1i}, y_{2i}; i = 1, \ldots, n\}$$

Apart from τ, estimation of the hazard function is not possible without further assump-

Table 7: The average estimates (the associated mean squared error) for parameters in Set 3

n	λ_1	λ_2	λ_3	λ_4	λ_5	α
25	0.1995897	0.3258452	0.09736655	0.4593465	0.3326005	1.129031
	(0.00519472)	(0.0106578)	(0.00275401)	(0.00319623)	(0.00242894)	(0.00103729)
50	0.1935924	0.3143088	0.09364148	0.4496803	0.3361917	1.128296
	(0.00263679)	(0.00435470)	(0.001087)	(0.00271758)	(0.00236553)	(0.000940602)
75	0.1950794	0.3148872	0.09216822	0.4483436	0.321636	1.127461
	(0.00002705)	(0.000039670)	(0.0000055517)	(0.00006891)	(0.00003215)	(0.0008814)
100	0.190708	0.312146	0.09785704	0.4461793	0.3329123	1.127801
	(0.0000098640)	(0.000018259)	(0.000001854)	(0.00002067)	(0.00002138)	(0.0002775)

Table 8: AL (CP) for 95% HPD intervals for parameters in Set 3

n	λ_1	λ_2	λ_3	λ_4	λ_5	α
25	0.2604	0.3410	0.1695	0.3076	0.2926	0.1262
	(0.95)	(0.96)	(0.95)	(0.98)	(0.98)	(0.98)
50	0.1844	0.2381	0.1236	0.2612	0.2533	0.1233
	(0.96)	(0.96)	(0.96)	(0.98)	(0.98)	(0.99)
75	0.1527	0.1968	0.1022	0.2345	0.2157	0.1208
	(0.97)	(0.96)	(0.98)	(0.99)	(0.98)	(0.99)
100	0.1317	0.1705	0.0925	0.2153	0.2052	0.1188
	(0.97)	(0.97)	(0.98)	(0.99)	(0.99)	(0.99)

Table 9: AL (CP) for 95% Symmetric intervals for parameters in Set 3

n	λ_1	λ_2	λ_3	λ_4	λ_5	α
25	0.2689	0.3483	0.1815	0.3108	0.2974	0.1266
	(0.95)	(0.95)	(0.95)	(0.98)	(0.98)	(0.99)
50	0.1878	0.2412	0.1279	0.2635	0.2565	0.1237
	(0.95)	(0.96)	(0.95)	(0.98)	(0.98)	(0.99)
75	0.1547	0.1987	0.1047	0.2363	0.2182	0.1212
	(0.96)	(0.96)	(0.96)	(0.98)	(0.99)	(0.99)
100	0.1331	0.1717	0.0942	0.2170	0.2072	0.1193
	(0.97)	(0.96)	(0.96)	(0.99)	(0.99)	(0.99)

tions. We model the cut points as the ordered arrival times of events in $[0, \tau]$ from a non-homogeneous Poisson process (NHPP) $\{N(t); t \geq 0\}$ with intensity function $u(t) = \delta t^{\delta-1}$, for $\delta > 0$. Since these arrival times are not observed, they are treated as latent (missing) variables. Conditional on δ and $M-1$, the joint density of the ordered arrival times $T_1, \ldots,$

Table 10: The average estimates (the associated mean squared error) for parameters in Set 4

n	λ_1	λ_2	λ_3	λ_4	λ_5	α
25	0.2024462	0.2045058	0.263843	0.7992495	0.8007952	1.283305
	(0.00564486)	(0.00478168)	(0.007600264)	(0.000891887)	(0.00076333)	(0.00131538)
50	0.2042641	0.2017424	0.2447375	0.7977674	0.7916068	1.279794
	(0.00321674)	(0.003008492)	(0.003004192)	(0.0001868905)	(0.0002025389)	(0.00104066)
75	0.1999402	0.2003999	0.2421109	0.7912395	0.7904181	1.278518
	(0.0000132)	(0.00001883)	(0.000019188)	(0.00002016512)	(0.000026459)	(0.00097518)
100	0.1996108	0.1974317	0.2486446	0.7937357	0.7953618	1.277454
	(0.0000118)	(0.00001423)	(0.00000966)	(0.00001408)	(0.00002479)	(0.000876168)

Table 11: AL (CP) for 95% HPD intervals for parameters in Set 4

n	λ_1	λ_2	λ_3	λ_4	λ_5	α
25	0.2705	0.2728	0.3149	0.3673	0.3684	0.1341
	(0.96)	(0.96)	(0.96)	(0.98)	(0.98)	(0.99)
50	0.1964	0.1949	0.2159	0.3498	0.3466	0.1316
	(0.96)	(0.97)	(0.97)	(0.99)	(0.99)	(0.99)
75	0.1596	0.1594	0.1762	0.3318	0.3321	0.1288
	(0.97)	(0.97)	(0.97)	(0.99)	(0.99)	(0.99)
100	0.1392	0.1382	0.1562	0.3212	0.3212	0.1268
	(0.97)	(0.97)	(0.97)	(0.99)	(0.99)	(0.99)

Table 12: AL (CP) for 95% Symmetric intervals for parameters in Set 4

n	λ_1	λ_2	λ_3	λ_4	λ_5	α
25	0.2801	0.2826	0.3236	0.3694	0.3709	0.1346
	(0.95)	(0.95)	(0.95)	(0.98)	(0.98)	(0.99)
50	0.1998	0.1987	0.2196	0.3519	0.3488	0.1320
	(0.95)	(0.95)	(0.95)	(0.98)	(0.98)	(0.99)
75	0.1617	0.1616	0.1783	0.3337	0.3340	0.1294
	(0.96)	(0.96)	(0.96)	(0.99)	(0.99)	(0.99)
100	0.1407	0.1397	0.1577	0.3232	0.3231	0.1272
	(0.96)	(0.96)	(0.96)	(0.99)	(0.99)	(0.99)

T_{M-1}, given $N(\tau) = M-1$, is

$$f_{T_1,\ldots,T_{M-1}|N(\tau)=M-1}(t_1,\ldots,t_{M-1}) = \frac{(M-1)!\delta^{M-1}}{\tau^{(M-1)\delta}} t_1^{\delta-1}\ldots t_{M-1}^{\delta-1};\ 0 < t_1 < \ldots < t_{M-1} < \tau \tag{40}$$

and zero otherwise. In the Bayesian approach, we assign prior distributions to all unknown

Table 13: The average estimates (the associated mean squared error) for parameters in Set 5

n	λ_1	λ_2	λ_3	λ_4	λ_5	c_1	c_2
25	0.09624685	0.1555908	0.05395494	0.2408837	0.2094545	0.3118677	0.6881323
	(0.001015)	(0.001862)	(0.000615)	(0.008137)	(0.016719)	(0.003659)	(0.003659)
50	0.09167736	0.1550339	0.05142827	0.2224331	0.178794	0.3140777	0.6859223
	(0.000809)	(0.001002)	(0.000309)	(0.004605)	(0.004275)	(0.002593)	(0.002593)
75	0.1000041	0.1510861	0.04775873	0.2145094	0.1741843	0.3058327	0.6941673
	(0.000511)	(0.000739)	(0.000197)	(0.002106)	(0.001871)	(0.001233)	(0.001233)
100	0.09341043	0.149516	0.04943435	0.2145031	0.1717686	0.3084021	0.6915979
	(0.000241)	(0.000344)	(0.0000935)	(0.001152)	(0.001005)	(0.001056)	(0.001056)

Table 14: AL (CP) for 95% HPD intervals for parameters in Set 5

n	λ_1	λ_2	λ_3	λ_4	λ_5	c_1	c_2
25	0.1257	0.1697	0.0884	0.2471	0.2772	0.2519	0.2519
	(0.94)	(0.94)	(0.98)	(0.92)	(0.94)	(0.95)	(0.95)
50	0.0852	0.1171	0.0628	0.1623	0.1669	0.1715	0.1715
	(0.95)	(0.94)	(0.97)	(0.93)	(0.95)	(0.96)	(0.96)
75	0.0735	0.0938	0.0486	0.1264	0.1257	0.1254	0.1254
	(0.95)	(0.94)	(0.94)	(0.95)	(0.93)	(0.95)	(0.95)
100	0.0598	0.0782	0.0427	0.1087	0.1103	0.1012	0.1012
	(0.94)	(0.97)	(0.94)	(0.94)	(0.97)	(0.95)	(0.95)

Table 15: AL (CP) for 95% Symmetric intervals for parameters in Set 5

n	λ_1	λ_2	λ_3	λ_4	λ_5	c_1	c_2
25	0.1329	0.1763	0.0952	0.2548	0.2907	0.2653	0.2623
	(0.96)	(0.96)	(0.97)	(0.92)	(0.92)	(0.95)	(0.95)
50	0.0878	0.1214	0.0656	0.1679	0.1733	0.1843	0.1816
	(0.94)	(0.94)	(0.98)	(0.91)	(0.92)	(0.94)	(0.94)
75	0.0762	0.0977	0.0510	0.1309	0.1302	0.1376	0.1353
	(0.96)	(0.95)	(0.96)	(0.95)	(0.94)	(0.94)	(0.94)
100	0.0623	0.0814	0.0445	0.1126	0.1157	0.1123	0.1106
	(0.95)	(0.95)	(0.96)	(0.94)	(0.93)	(0.95)	(0.95)

quantities, including δ and the cut points, and use the observed data to update these priors via the posterior distribution. We assign prior distributions as follows:

NHPP shape parameter:

$$\pi(\delta) = \delta \sim \text{Gamma}(a_\delta, b_\delta), \quad a_\delta > 0, b_\delta > 0.$$

Table 16: The average estimates (the associated mean squared error) for parameters in Set 6

n	λ_1	λ_2	λ_3	λ_4	λ_5	c_1	c_2	c_3
25	0.064872	0.096835	0.036789	0.135999	0.115956	0.062749	0.317226	0.620024
	(0.00052)	(0.00084)	(0.00047)	(0.00293)	(0.00549)	(0.00130)	(0.00487)	(0.00677)
50	0.061878	0.096956	0.033891	0.119404	0.109687	0.071862	0.322342	0.605796
	(0.00015)	(0.00039)	(0.00025)	(0.00083)	(0.00188)	(0.00059)	(0.00264)	(0.00371)
75	0.054883	0.090467	0.028445	0.115803	0.100636	0.073837	0.323039	0.603124
	(0.00008)	(0.00016)	(0.00003)	(0.00043)	(0.00077)	(0.00046)	(0.00101)	(0.00171)
100	0.059822	0.093253	0.028614	0.118129	0.092482	0.066807	0.324078	0.609114
	(0.00005)	(0.00012)	(0.00001)	(0.00016)	(0.00037)	(0.00012)	(0.00089)	(0.00077)

Table 17: AL (CP) for 95% HPD intervals for parameters in Set 6

n	λ_1	λ_2	λ_3	λ_4	λ_5	c_1	c_2	c_3
25	0.0832	0.1046	0.0582	0.1439	0.1489	0.0932	0.2663	0.2775
	(0.97)	(0.93)	(0.94)	(0.94)	(0.93)	(0.95)	(0.94)	(0.94)
50	0.0554	0.0678	0.0362	0.0786	0.0904	0.0650	0.1596	0.1767
	(0.96)	(0.93)	(0.95)	(0.94)	(0.92)	(0.96)	(0.95)	(0.94)
75	0.0379	0.0520	0.0277	0.0662	0.0748	0.0402	0.1375	0.1307
	(0.96)	(0.94)	(0.94)	(0.95)	(0.94)	(0.94)	(0.95)	(0.95)
100	0.0253	0.0358	0.0177	0.0429	0.0408	0.0242	0.0796	0.0749
	(0.95)	(0.95)	(0.94)	(0.95)	(0.95)	(0.94)	(0.94)	(0.96)

Table 18: AL (CP) for 95% Symmetric intervals for parameters in Set 6

n	λ_1	λ_2	λ_3	λ_4	λ_5	c_1	c_2	c_3
25	0.0880	0.1091	0.0623	0.1495	0.1563	0.1009	0.2744	0.2855
	(0.96)	(0.96)	(0.94)	(0.93)	(0.92)	(0.94)	(0.93)	(0.92)
50	0.0596	0.0709	0.0375	0.0819	0.0959	0.0733	0.1675	0.1852
	(0.97)	(0.95)	(0.94)	(0.94)	(0.93)	(0.95)	(0.94)	(0.94)
75	0.0379	0.0520	0.0277	0.0662	0.0748	0.0402	0.1375	0.1306
	(0.96)	(0.95)	(0.93)	(0.94)	(0.94)	(0.95)	(0.94)	(0.94)
100	0.0337	0.0437	0.0232	0.0574	0.0505	0.0332	0.1085	0.1013
	(0.96)	(0.96)	(0.94)	(0.93)	(0.94)	(0.94)	(0.95)	(0.93)

The joint PDF of the cut points conditional on δ:

$$\pi(\tau_1, \ldots, \tau_{M-1}|\delta) = \frac{(M-1)!\delta^{M-1}}{\tau^{(M-1)\delta}} \prod_{j=1}^{M-1} \tau_j^{\delta-1}.$$

So the posterior distribution will be

$$\Pi(\delta, \tau, \alpha) = \pi(\delta) \times \pi(\tau_1, \ldots, \tau_{M-1}|\delta) \times \pi(\alpha) \times l(\Theta, \alpha|D), \tag{41}$$

where α is the baseline parameter. As the resulting posterior does not have a closed-form expression, estimation is carried out using an importance sampling technique. In this approach, candidate values for δ and τ are generated from appropriate proposal distributions (for example, a Gamma distribution for δ and Dirichlet-based transformations for τ), and the normalized weights are then used to obtain posterior summaries, including the posterior means, standard deviations, and credible intervals for δ and the cut points.

7.2 Data Analysis

In this section, we analyze a bivariate diabetic retinopathy dataset to illustrate the application of the proposed model. The study was conducted to investigate the effectiveness of laser treatment in delaying the onset of blindness among patients with diabetic retinopathy. For each patient, one eye was randomly assigned to receive laser treatment, while the fellow eye received the traditional treatment. The time to vision loss was recorded for both eyes, where Y_1 denotes the time to vision loss in the laser-treated eye and Y_2 denotes the corresponding time for the other eye. In cases where vision loss was not observed during the study period due to death, dropout, or termination of the study, the observations were treated as censored and denoted by Y_1^* or Y_2^*. The dataset is publicly available in R as `data(retinopathy, package="survival")` and is also discussed in details in Samanta and Kundu [18].

The study sample consisted of 197 patients. Among these, 38 were complete cases with uncensored values for both eyes, while 80 were censored for both eyes. In addition, 54 patients had censoring in the laser-treated eye only, and 96 patients had censoring in the traditionally treated eye only. Based on these data, we aim to address two questions: (i) whether laser treatment significantly delays blindness, and (ii) whether blindness in one eye affects the time to blindness in the other eye.

We first consider the uncensored case, that is, when neither Y_1 nor Y_2 is censored. Here the sample size is $n = 38$, with 12 cases where $Y_1 < Y_2$, 20 cases where $Y_1 > Y_2$, and 6 cases where $Y_1 = Y_2$. Next, we consider the full dataset of 197 observations including censored cases.

7.3 Complete observations

Now we fit the bivariate load sharing model with different baseline distributions and compute the estimators of the parameters.

Baseline distribution: exponential We fitted the load-sharing model with the exponential baseline distribution. Here the hyperparameters we have taken as $a = 0.001$, $b = 0.001$,

$a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$, $b_4 = 0.001$, and $b_5 = 0.001$., which behave almost as non-informative priors (see [4]). For exponential, we follow the Algorithm 1 with $N = 10000$. The Bayes estimates and the associated standard errors (within brackets) are as follows: $\hat{\lambda}_1 = 0.0269(\pm 0.0077)$, $\hat{\lambda}_2 = 0.0453(\pm 0.0090)$, $\hat{\lambda}_3 = 0.0181(\pm 0.0064)$, $\hat{\lambda}_4 = 0.0739(\pm 0.0166)$, and $\hat{\lambda}_5 = 0.0469(\pm 0.0138)$. In Figure 2, we present the empirical Kaplan–Meier survival estimates for the two treatment groups together with the fitted marginal survival functions obtained under the exponential baseline distribution.

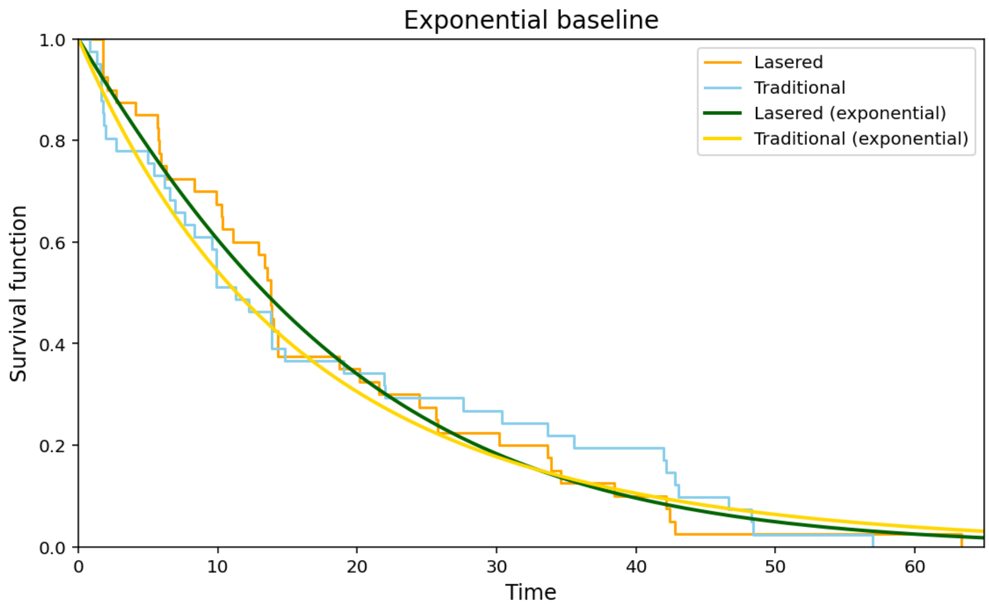

Figure 2: Survival functions of the time to blindness for laser-treated and traditional eye.

The 95% HPD and symmetric credible intervals for all the parameters are obtained as follows (Table 19): Now to test whether there is any significance difference between the two

Table 19: 95% HPD and Symmetric intervals with the exponential baseline for complete data

		λ_1	λ_2	λ_3	λ_4	λ_5
HPD	LL	0.0129	0.0262	0.0068	0.0429	0.0217
	UL	0.0426	0.0647	0.0306	0.1069	0.0730
SYM	LL	0.0139	0.0274	0.0078	0.0451	0.0243
	UL	0.0443	0.0662	0.0324	0.1102	0.0770

treatments we have performed the test (35). The corresponding Bayes factor for the data using almost non informative prior is 323202.6, hence we cannot reject the null hypothesis.

Baseline distribution: Weibull Next, we have also fitted the load sharing model with the baseline distribution as Weibull. In this case, similar as exponential, we have taken the hyperparameters as $a = 0.001$, $b = 0.001$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$, $b_4 = 0.001$, $b_5 = 0.001$ $c = 0.001$ and $d = 0.001$. The Bayes estimates and the associated standard errors are as follows: $\alpha = 1.1495(\pm 0.0097)$, $\hat{\lambda_1} = 0.0175(\pm 0.0051)$, $\hat{\lambda_2} = 0.0292(\pm 0.0066)$, $\hat{\lambda_3} = 0.0117(\pm 0.0042)$, $\hat{\lambda_4} = 0.0426(\pm 0.0096)$, and $\hat{\lambda_5} = 0.0261(\pm 0.0076)$. Figure 3 presents the empirical Kaplan–Meier survival estimates for the two treatment groups together with the fitted marginal survival functions obtained under the Weibull baseline distribution.

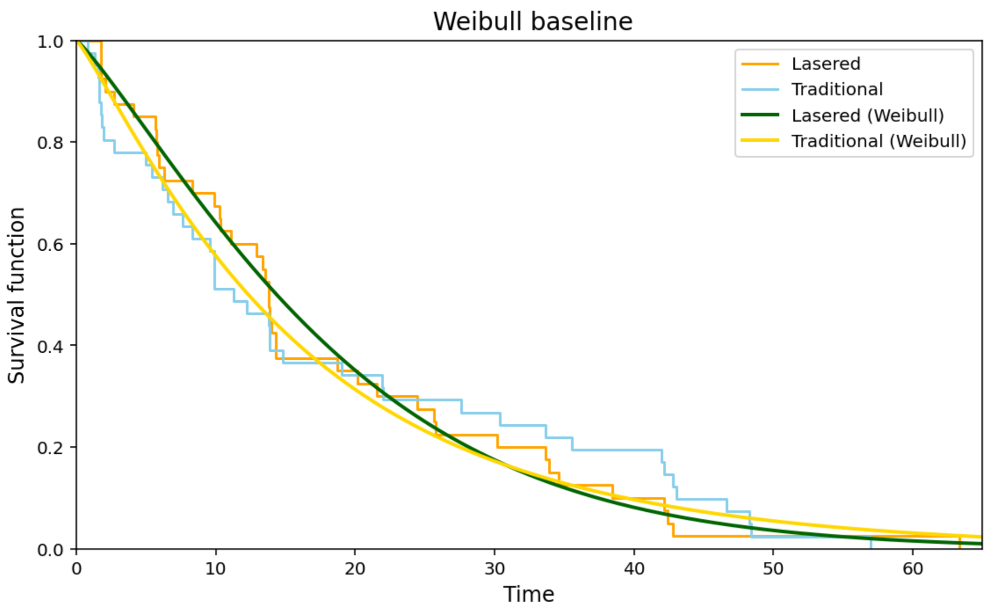

Figure 3: Survival functions of the time to blindness for laser-treated and traditional eye.

The credible intervals are obtained as follows (Table 20): In this case the testing the

Table 20: 95% HPD and Symmetric intervals with the Weibull baseline for complete data

		α	λ_1	λ_2	λ_3	λ_4	λ_5
HPD	LL	1.1308	0.0084	0.0168	0.0044	0.0249	0.0120
	UL	1.1688	0.0278	0.0423	0.0200	0.0620	0.0412
SYM	LL	1.1304	0.0089	0.0177	0.0050	0.0259	0.0133
	UL	1.1685	0.0289	0.0438	0.0212	0.0634	0.0430

hypothesis (35), the Bayes factor is 222769344 when we consider almost non informative priors, which implies H_0 cannot be rejected.

Baseline distribution: PCH When $M = 1$, the PCH model becomes exponential. Here we have fitted the model with $M = 2$ and 3. For both the M, we have taken

$\tau = \max\{X, Y\} = 63.33$. We have estimated the change points as given in Section 6.1 and we consider $a_\delta = 0.001$ and $b_\delta = 0.001$. When $M = 2$, the hyperparameters have been considered as $a = 0.001$, $b = 0.001$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$, $b_4 = 0.001$, $b_5 = 0.001$, and $\theta = (1, 1)$. The point estimates and the associated standard error are as follows: $\hat{\delta} = 1.9847(\pm 0.1288)$, $\hat{\tau_1} = 42.3771(\pm 0.0718)$, $\hat{\lambda_1} = 0.0369(\pm 0.0092)$, $\hat{\lambda_2} = 0.0745(\pm 0.0141)$, $\hat{\lambda_3} = 0.0268(\pm 0.0075)$, $\hat{\lambda_4} = 0.1187(\pm 0.0276)$, $\hat{\lambda_5} = 0.0695(\pm 0.0212)$, $\hat{c_1} = 0.7589(\pm 0.0287)$ and $\hat{c_2} = 0.2411\ (\pm 0.0287)$. Figure 4 shows the empirical Kaplan–Meier survival estimates for the two treatment groups along with the fitted marginal survival functions under PCH baseline distribution with $M = 2$. The credible intervals are as fol-

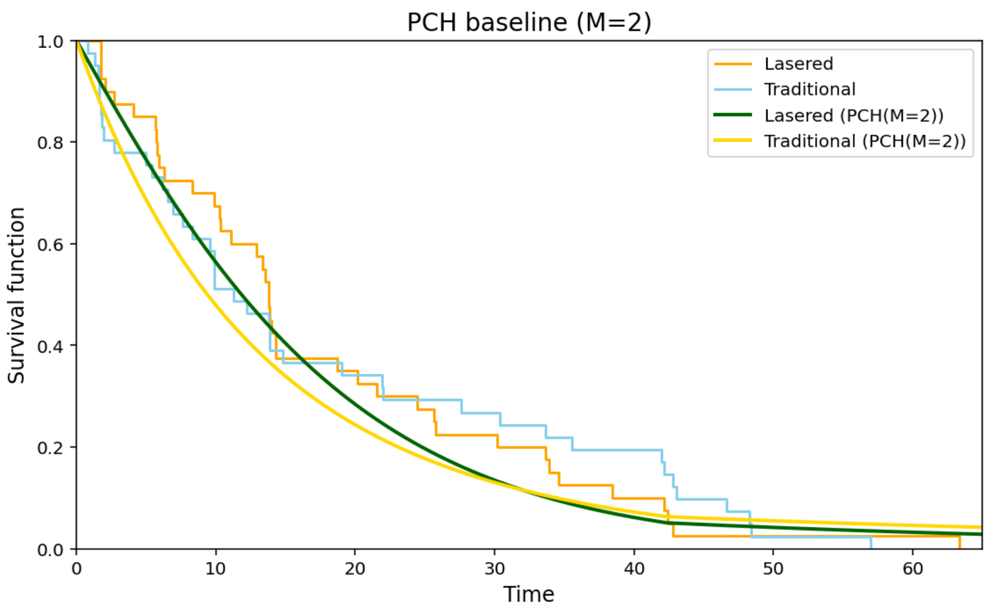

Figure 4: Survival functions of the time to blindness for laser-treated and traditional eye.

lows(Table 21):

Table 21: 95% HPD and Symmetric intervals with the PCH ($M = 2$) baseline for complete data

		δ	τ_1	λ_1	λ_2	λ_3	λ_4	λ_5	c_1	c_2
HPD	LL	1.7481	42.261	0.0169	0.0468	0.0096	0.0776	0.0344	0.7315	0.1841
	UL	2.2373	42.505	0.0591	0.0900	0.0369	0.1609	0.0897	0.8159	0.2684
SYM	LL	1.7481	42.260	0.0169	0.0449	0.0126	0.0718	0.0343	0.7316	0.1679
	UL	2.2372	42.501	0.0591	0.0899	0.0447	0.1608	0.0925	0.8320	0.2684

We have performed the test (35) to verify whether there is any significance difference between the two treatments or not. The Bayes factor is 514757.4, so we can not reject H_0.

When $M = 3$, the hyperparameters have been considered as $a = 0.001$, $b = 0.001$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$, $b_4 = 0.001$, $b_5 = 0.001$, and $\theta = (1, 1, 1)$. The point estimates (standard error) are obtained as follows $\hat{\delta} = 3.0716(\pm 0.2184)$, $\hat{\tau}_1 = 42.1641(\pm 0.1309)$, $\tau_2 = 55.2362(\pm 0.2259)$, $\hat{\lambda}_1 = 0.0623(\pm 0.0122)$, $\hat{\lambda}_2 = 0.0969\ (\pm 0.0214)$, $\hat{\lambda}_3 = 0.0348(\pm 0.0203)$, $\hat{\lambda}_4 = 0.1673(\pm 0.0320)$, $\hat{\lambda}_5 = 0.0891(\pm 0.0192)$, $\hat{c}_1 = 0.4619(\pm 0.0113)$, $\hat{c}_2 = 0.3585(\pm 0.0353)$, and $\hat{c}_3 = 0.1797(\pm 0.0367)$. In Figure 5, the empirical Kaplan–Meier survival estimates for the two treatment groups are compared with the fitted marginal survival functions obtained under the PCH baseline distribution with $M = 3$. The credible intervals are given in Table

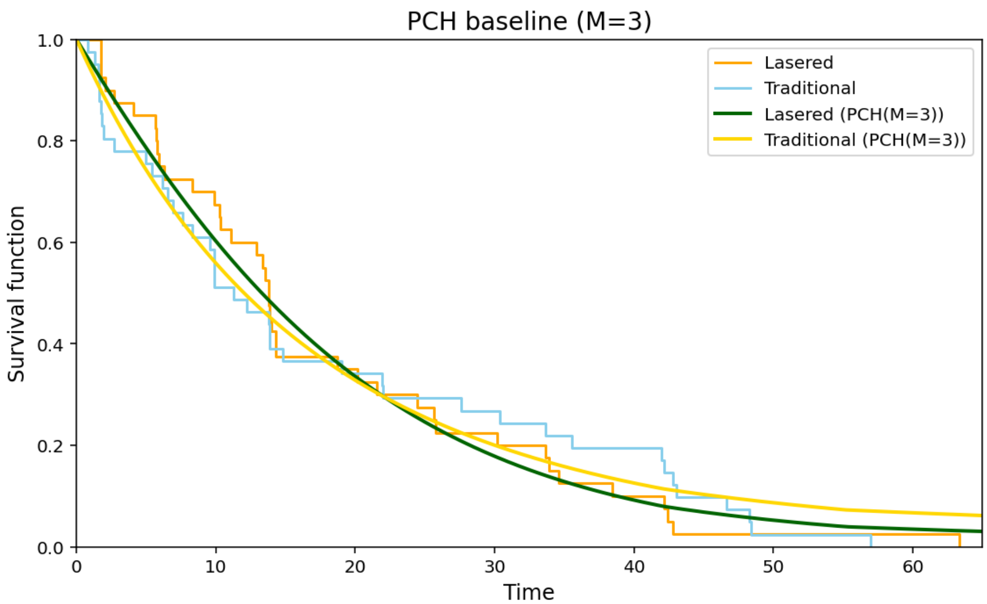

Figure 5: Survival functions of the time to blindness for laser-treated and traditional eye.

22: In this case, for testing the hypothesis (35), the Bayes factor is 1774.913 so we cannot

Table 22: HPD and Symmetric intervals with the PCH ($M = 3$) baseline for complete data

		δ	τ_1	τ_2	λ_1	λ_2	λ_3	λ_4	λ_5	c_1	c_2	c_3
HPD	LL	2.6793	41.875	54.943	0.0366	0.0711	0.0176	0.0906	0.0517	0.4525	0.3141	0.1176
	UL	3.5361	42.417	55.644	0.0791	0.1309	0.0827	0.1886	0.1272	0.4839	0.4367	0.2342
SYM	LL	2.6793	41.876	54.943	0.0366	0.0719	0.0170	0.0906	0.0517	0.4525	0.2707	0.1176
	UL	3.5362	42.418	55.644	0.0889	0.1543	0.0827	0.2417	0.1312	0.4941	0.4231	0.2549

reject the null hypothesis H_0. Now to see which model is the best fitted one, we have presented the AIC, BIC and 5-fold log posterior density (LPD) values in Table 23. Based on the AIC, BIC and LPD values PCH($M = 3$) is the best fitted model to the complete data set.

Table 23: AIC, BIC and LPD values for the different models for complete data set

Criterion	Exponential	Weibull	PCH ($M = 2$)	PCH($M = 3$)
AIC	606.1147	666.3937	556.7361	474.2156
BIC	714.1147	810.3937	736.7361	690.2156
LPD	-302.9373	-296.3846	-280.6311	-237.3069

7.4 Complete and Censored Observations

We fit the bivariate load-sharing model with different baseline distributions to the complete and censored data and compute the corresponding estimators of the model parameters. All hyperparameters are taken to be the same as those used for the complete observations.

Baseline distribution: exponential The Bayes estimates and the associated standard errors are as follows: $\hat{\lambda}_1 = 0.0051(\pm 0.0009)$, $\hat{\lambda}_2 = 0.0137(\pm 0.0016)$, $\hat{\lambda}_3 = 0.0158(\pm 0.0017)$, $\hat{\lambda}_4 = 0.0105(\pm 0.0023)$, and $\hat{\lambda}_5 = 0.0184(\pm 0.0051)$. The 95% HPD and symmetric credible intervals for all the parameters are obtained as follows (Table 24): Here, the value of the

Table 24: 95% HPD and Symmetric intervals with the exponential baseline for censored data

		λ_1	λ_2	λ_3	λ_4	λ_5
HPD	LL	0.0032	0.0106	0.0125	0.0059	0.0090
	UL	0.0069	0.0168	0.0191	0.0148	0.0285
SYM	LL	0.0034	0.0108	0.0126	0.0065	0.0098
	UL	0.0071	0.0170	0.0192	0.0154	0.0296

Bayes factor is 0.0722, which implies that H_0 should be rejected.

Baseline distribution: Weibull The Bayes estimators and the associated errors are as follows: $\hat{\alpha} = 1.0067(\pm 0.0069)$, $\hat{\lambda}_1 = 0.0038(\pm 0.0007)$, $\hat{\lambda}_2 = 0.0103(\pm 0.0012)$, $\hat{\lambda}_3 = 0.0117(\pm 0.0013)$, $\hat{\lambda}_4 = 0.0101(\pm 0.0022)$, $\hat{\lambda}_5 = 0.0177(\pm 0.00492)$. The credible intervals are given in Table 25. The Bayes factor for the Weibull baseline is 0.0002, which implies H_0

Table 25: HPD and Symmetric intervals with the Weibull baseline for censored data

		α	λ_1	λ_2	λ_3	λ_4	λ_5
HPD	LL	0.9944	0.0024	0.0080	0.0092	0.0059	0.0087
	UL	1.0205	0.0052	0.0127	0.0142	0.0147	0.0276
SYM	LL	0.9959	0.0025	0.0080	0.0093	0.0061	0.0093
	UL	1.0226	0.0053	0.0127	0.0144	0.0149	0.0285

should be rejected.

Baseline distribution: PCH($M = 2$) The point estimates and the associated standard error are as follows: $\hat{\delta} = 1.9782(\pm 0.1391)$, $\hat{\tau_1} = 38.38013(\pm 0.0944)$, $\hat{\lambda_1} = 0.0136(\pm 0.0008)$, $\hat{\lambda_2} = 0.0372(\pm 0.0021)$, $\hat{\lambda_3} = 0.0333(\pm 0.0009)$, $\hat{\lambda_4} = 0.0188(\pm 0.0008)$, $\hat{\lambda_5} = 0.0258(\pm 0.0056)$, $\hat{c_1} = 0.3967(\pm 0.0041)$ and $\hat{c_2} = 0.6033(\pm 0.0041)$. The credible intervals are as follows(Table 26):

Table 26: HPD and Symmetric intervals with the PCH ($M = 2$) baseline for complete and censored data

		δ	τ_1	λ_1	λ_2	λ_3	λ_4	λ_5	c_1	c_2
HPD	LL	1.7202	38.175	0.0108	0.0309	0.0331	0.0176	0.0245	0.3954	0.5946
	UL	2.2332	38.560	0.0138	0.0378	0.0352	0.0188	0.0294	0.4054	0.6045
SYM	LL	1.7202	38.176	0.0108	0.0309	0.0331	0.0176	0.0244	0.3954	0.5878
	UL	2.2333	38.560	0.0138	0.0378	0.0352	0.0189	0.0575	0.4088	0.6045

For this baseline, the value of the Bayes factor is 1.194931e-4, so we reject the H_0.

Baseline distribution: PCH($M = 3$) The point estimates and the associated standard error are as follows: $\hat{\delta} = 2.9255(\pm 0.2084)$, $\hat{\tau_1} = 26.7931(\pm 0.5276)$, $\hat{\tau_2} = 54.0394(\pm 0.1098)$, $\hat{\lambda_1} = 0.0156(\pm 0.0003)$, $\hat{\lambda_2} = 0.0431(\pm 0.0007)$, $\hat{\lambda_3} = 0.0495(\pm 0.0005)$, $\hat{\lambda_4} = 0.0302(\pm 0.0007)$, $\hat{\lambda_5} = 0.0719(\pm 0.0026)$, $\hat{c_1} = 0.3443(\pm 0.0022)$, $\hat{c_2} = 0.3084(\pm 0.0081)$ and $\hat{c_3} = 0.3473(\pm 0.0061)$. The credible intervals are given in Table 27: The Bayes factor is 6.484857e-5, the H_0 should

Table 27: HPD and Symmetric intervals with the PCH ($M = 3$) baseline for complete and censored data

		δ	τ_1	τ_2	λ_1	λ_2	λ_3	λ_4	λ_5	c_1	c_2	c_3
HPD	LL	2.5351	25.825	53.836	0.0144	0.0367	0.0467	0.0259	0.0512	0.3248	0.2786	0.3259
	UL	3.3264	27.631	54.256	0.0174	0.0508	0.0515	0.0339	0.1134	0.3575	0.3493	0.3638
SYM	LL	2.5352	25.825	53.837	0.0145	0.0367	0.0467	0.0254	0.0512	0.3248	0.2785	0.3258
	UL	3.3264	27.631	54.256	0.0174	0.0508	0.0515	0.0339	0.1134	0.3575	0.3493	0.3638

be rejected. The AIC, BIC and LPD (5-fold) values for the complete and censored observations are presented in Table 28. In this case also based on the AIC, BIC and LPD values, PCH($M = 3$) is the best fitted model to the complete and censored data set. they are as follows;

Compared with the MLEs in [13], the standard errors for $\lambda_1, \ldots, \lambda_5$ are nearly the same. In contrast, the Bayesian approach produces considerably lower MSEs for the Weibull baseline parameter α. Similar improvements are observed for δ, the cut points τ_i, and the hazard constants c_i for the PCH baseline distribution, where the Bayesian estimates achieve smaller MSEs than the MLEs.

In addition to this, we also compared the proposed load-sharing model with the BPH model. Note that when the load factors are set to $\delta = 1$ and $\gamma = 1$, our model reduces to

Table 28: AIC, BIC and LPD values for the different models for complete and censored data set

Criterion	Exponential	Weibull	PCH ($M = 2$)	PCH($M = 3$)
AIC	2374.603	2531.423	2005.249	1868.285
BIC	3349.603	3506.423	3373.249	2156.285
LPD	-1187.327	-1201.836	-1030.902	-607.4358

the BPH (Marshall–Olkin–type) model, and the likelihood also reduces to the corresponding BPH likelihood. Using this fact, we computed AIC, BIC, and the 5-fold LPD for both complete and censored data (presented in the tables 29 and 30). The results show that for complete observations, the BPH model gives smaller AIC, BIC, and LPD values than the load-sharing model. However, when censoring is present, the load-sharing model produces smaller AIC, BIC, and LPD values, suggesting a better fit in the censored case.

Table 29: AIC, BIC and LPD Values for different BPH models for complete data

Criteria	Exponential	Weibull	PCH ($M = 2$)	PCH ($M = 3$)
AIC	604.3086	657.8692	555.4385	472.2384
BIC	676.3086	801.8692	699.4385	652.2384
LPD	-302.528	-293.1733	-279.6841	-236.7148

Table 30: AIC, BIC anf LPD Values for different BPH models for complete and Censored data

Criteria	Exponential	Weibull	PCH ($M = 2$)	PCH ($M = 3$)
AIC	2393.136	2569.670	2027.826	1884.359
BIC	3368.136	3544.670	3392.826	2173.359
LPD	-1194.005	-1205.187	-1039.803	-651.315

Conclusion

In this paper, we have proposed a Bayesian framework for the bivariate load-sharing model. One important advantage of the proposed approach is that it allows a data-driven baseline distribution. There is no need to assume any particular form of the baseline hazard function; for example, whether it is increasing, decreasing, or of any other specific shape, the data itself can capture the structure. Hence, the model behaves like a semi-parametric model, and its implementation is straightforward in practice. Another advantage of the proposed methodology is that it can handle censored observations naturally, while also accommodating complete data, which makes it more widely applicable. In the case of complete observations, the analysis suggests that laser treatment does not have any significant effect in delaying blindness, as it was observed by Kundu [13]. However, when censored observations are included, the results indicate that laser treatment has a significant effect compared to the traditional treatment. In this paper, we have mainly developed the Bayesian inference of the unknown parameters and obtained both point and interval estimates using suitable priors. It will be interesting to explore extensions of this framework to more general baseline distributions, alternative censoring schemes, or hierarchical structures. More work is needed in that direction.

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Declaration

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Appendix

The explicit expressions of D_{ij}, E_{ij} and F_j are obtained by Kundu (2025) as follows:

$$
\begin{aligned}
D_{11} &= \sum_{w=1}^{n_{111}} y_{111w} + \tau_1 \sum_{m=2}^{M} n_{11m} \\
D_{12} &= \sum_{w=1}^{n_{112}} (y_{112w} - \tau_1) + (\tau_2 - \tau_1) \sum_{m=3}^{M} n_{11m} \\
\vdots\ & \ \vdots
\end{aligned}
$$

$$
\begin{aligned}
D_{1(M-1)} &= \sum_{w=1}^{n_{11(M-1)}} (y_{11(M-1)w} - \tau_{M-2}) + (\tau_{M-1} - \tau_{M-2}) n_{11M} \\
D_{1M} &= \sum_{w=1}^{n_{11M}} (y_{11Mw} - \tau_{M-1}) \\
D_{21} &= \sum_{w=1}^{n_{121}} y_{121w} + \tau_1 \sum_{m=2}^{M} n_{12m} \\
D_{22} &= \sum_{w=1}^{n_{122}} (y_{122w} - \tau_1) + (\tau_2 - \tau_1) \sum_{m=3}^{M} n_{12m} \\
\vdots &\ \ \vdots \\
D_{2(M-1)} &= \sum_{w=1}^{n_{12(M-1)}} (y_{12(M-1)w} - \tau_{M-2}) + (\tau_{M-1} - \tau_{M-2}) n_{12M} \\
D_{2M} &= \sum_{w=1}^{n_{12M}} (y_{12Mw} - \tau_{M-1})
\end{aligned}
$$

$$
\begin{aligned}
E_{11} &= \sum_{w=1}^{n_{211}} y_{211w} + \tau_1 \sum_{m=2}^{M} n_{21m} \\
E_{12} &= \sum_{w=1}^{n_{212}} (y_{212w} - \tau_1) + (\tau_2 - \tau_1) \sum_{m=3}^{M} n_{21m} \\
\vdots &\ \ \vdots \\
E_{1(M-1)} &= \sum_{w=1}^{n_{21(M-1)}} (y_{21(M-1)w} - \tau_{M-2}) + (\tau_{M-1} - \tau_{M-2}) n_{21M} \\
E_{1M} &= \sum_{w=1}^{n_{21M}} (y_{21Mw} - \tau_{M-1}) \\
E_{21} &= \sum_{w=1}^{n_{221}} y_{221w} + \tau_1 \sum_{m=2}^{M} n_{22m} \\
E_{22} &= \sum_{w=1}^{n_{222}} (y_{222w} - \tau_1) + (\tau_2 - \tau_1) \sum_{m=3}^{M} n_{22m} \\
\vdots &\ \ \vdots
\end{aligned}
$$

$$
\begin{aligned}
E_{2(M-1)} &= \sum_{w=1}^{n_{22(M-1)}} (y_{22(M-1)w} - \tau_{M-2}) + (\tau_{M-1} - \tau_{M-2}) n_{22M} \\
E_{2M} &= \sum_{w=1}^{n_{22M}} (y_{22Mw} - \tau_{M-1}) \\
F_1 &= \sum_{w=1}^{n_{11}} y_{1w} + \tau_1 \sum_{m=2}^{M} n_{1m} \\
F_2 &= \sum_{w=1}^{n_{12}} (y_{2w} - \tau_1) + (\tau_2 - \tau_1) \sum_{m=3}^{M} n_{1m} \\
\vdots &\quad \vdots \\
F_{(M-1)} &= \sum_{w=1}^{n_{1(M-1)}} (y_{(M-1)w} - \tau_{M-2}) + (\tau_{M-1} - \tau_{M-2}) n_{1M} \\
F_M &= \sum_{w=1}^{n_{1M}} (y_{Mw} - \tau_{M-1})
\end{aligned}
$$