On Constrained Multi-Criteria based Planning for a Two-Parameter Exponential Distribution

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Abstract

In this article, we propose to plan an optimal life-testing experimental design under hybrid censoring scheme based on multi-criteria optimization, where two objectives are simultaneously optimized. In order to do this, we outline a strategy for simultaneously optimizing two objectives; the cost-based and variance-based objective models. We present a methodology and algorithm for implementing the multi-criteria optimization subject to certain two constraints. We consider first constraint on the overall budget to set-up a life-testing experiment and the second constraint is on the efficiency of lifetime model parameters' estimates based on the experimental design. In particular, a Type-II hybrid censoring scheme with a two-parameter exponential distribution as the lifetime model is illustrated. Afterwards, a sensitivity analysis regarding the constraints, model parameters and cost coefficients is performed. Furthermore, we examine the effects of model parameter mis-specification on the optimal designs, and demonstrate the applicability of the proposed approach using a real lifetime data set. Finally, it is shown that the proposed algorithm converges in a finite number of steps.

Keywords: multi-objective optimization; variance measure; cost model; life-testing planning; maximum likelihood estimation; hybrid censoring; design construction; design evaluation; Design optimality.

1 Introduction

Because of sophisticated advancements in engineering, we have highly reliable and expensive gadgets, therefore picking an ideal sample size is vital in reliability. Budget and time for

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reliability tests are usually limited. Data collection under such cost and time constraints is an addressable problem. It is crucial to choose designing parameters of a censoring scheme with cost and experiment duration in mind. Optimal censoring scheme provides optimal designing parameters based on some optimality criterion. Most optimal designs depend on model assumptions, hence a design that estimates model parameters effectively is often desired. Similarly, increased experimentation costs are not desirable. When an experiment has numerous aims, a design that satisfies them all is ideal. In reliability sampling plans, the Optimality criterion considers inspection expense, total cost on experimental time, cost due to failed units, and efficiency of the parameter estimation method for a probability model. The overall experimental time affects electricity, labour, and other expenditures, thus cost on the experimental time may include these costs. Sometimes an overall fixed cost is added to the main cost function, which usually includes installation and transportation costs for reliability experimentation facilities.

While constructing an optimal reliability sampling plan with complete or censored data, the design parameters are commonly chosen based on an optimality criterion. Typically, optimality in such settings is evaluated based on either "cost-based" objectives, which account for resource expenditure such as test duration or the number of items tested, or "variance-based" objectives, which reflect the statistical precision of the lifetime model-parameter estimates. In reliability, the usual practice is to consider the optimality criterion as variance-based or cost-based, or both. All three optimality criteria are used as single objective functions in a single-criterion optimization problem. Often, multi-objectives-based optimality criterion where both variance-based and cost-based components are optimized simultaneously is also of importance to get reliability plans under various censoring schemes. Multi-objective optimization is a major topic in operations research. The book by Ehrgott (2005) provides a brief overview of multi-objective optimization.

This study performs a multi-criteria optimization to determine the optimal designing parameters of Type-II hybrid censoring proposed by Childs et al. (2003). In Type-II HCS, an experiment is terminated by a time $\max\{X_{(r)}, T\}$ for a prefixed integer $1 \le r \le n$ and a prefixed time point $T > 0$. Traditionally, in the literature, single-objective optimization problems are common. When cost-based and variance-based models are combined, their cost coefficients are weighted to optimize both factors. In a real-world scenario, it is difficult to characterize the cost weight for variation component. Since, the problem is, how do you really decide how much a unit of variance is “worth” in cost terms? That is not straightforward in real-world applications. As a results, instead of forcing them into a single function, it makes more sense to treat both objectives as independent and optimize them separately/simultaneously to offer a more realistic and balanced framework.

This leads naturally to a multi-objective optimization framework. In multi-objective settings, optimizing each objective separately can lead to significantly different designs, where improving one aspect might severely compromise the other. Since, minimization of one objective (e.g., cost) in isolation may result in an unacceptably high value for the other (e.g., variance). Lexicographic approach (see Ehrgott (2005), page 7), where one criterion is given strict priority over the other, are also less appropriate in such cases, as they impose an artificial ordering that may not reflect the actual priorities of the experimenter. Instead, a simultaneous minimization approach is more suitable, where both cost and estimation precision are given equal importance, and trade-offs are managed through Pareto optimality. Where Pareto optimality refers to a situation in which no objective can be improved without worsening at least one other objective, leading to a set of non-dominated solutions known as the Pareto frontier.

Following the recommendations of Ehrgott (2005), and the implementation by Bhattacharya et al. (2020), this work adopts a minimax strategy to address the worst-case balance between competing objectives. Additionally, studies such as Lu et al. (2011) and have proposed alternative approaches for multi-criteria optimization in the context of experimental design. Lu et al. (2011) developed the Pareto Aggregating Point Exchange (PAPE) algorithm to identify non-dominated designs balancing various criteria. Similarly, Walsh et al. (2024) used particle swarm optimization to generate Pareto-optimal designs under two competing objectives. Although these methods were developed for different application areas, they demonstrate the effectiveness of Pareto-based approaches in achieving well-balanced solutions when dealing with multiple conflicting objectives.

Bhattacharya et al. (2014) designed optimal hybrid censoring to minimise experimental costs. The authors present a method for obtaining the optimal censoring schemes based on a scale-invariant cost function. They illustrate the proposed method for exponential and Weibull distributions. Bhattacharya et al. (2014) considered an optimality criterion that is a combination of variance-based and cost-based factors, which is a single-objective-based optimality criterion, whereas Bhattacharya et al. (2020) explored a multi-criteria-based optimality criterion to achieve the reliability plans under hybrid censoring scheme in which both variance-based and cost-based components are optimized simultaneously. The authors used the MLE of the lifetime model parameter to construct the variance-based optimality criteria. A-, T-, and D-optimality based measurements, which are considered by Ebrahimi (1988), Liski et al. (2002), Zhang and Meeker (2005), Kundu (2008), Dube et al. (2011) and Bhattacharya et al. (2014, 2020), and several other authors, are prominent variance measures for the optimality criterion function.

The primary objective of this paper is to provide the optimal hybrid censoring scheme based on multi-criteria optimization. To accomplish this, we outline a procedure for optimizing the cost-based and variance-based models simultaneously. The variance measure under consideration is similar to the variance measure considered by Kundu (2007) and Bhattacharya et al. (2020). The optimal criterion is constructed using the maximum likelihood estimator of lifetime model parameters. It is assumed that the lifetime distribution is a two-parameter exponential distribution. This study is motivated by real-world applications under clinical trials in biomedical research, electrical and industrial engineering, where lifetime data often follow the two parameter exponential model. For example, industrial products such as refrigerators, computers, and other electrical equipment are generally designed to operate reliably for a guaranteed period without failure. In such cases, their lifetimes are appropriately modeled as shifted random variables, accounting for a minimum failure-free operating time. Among various lifetime models, the two-parameter exponential distribution has been frequently employed in numerous research studies, including Yu and Peng (2014); Prajapat et al. (2023); Goel et al. (2025), as it provides a reasonable and practical choice for modeling lifetime data in many applied contexts. We provide a methodology and algorithm for addressing the optimization problem under specific constraints. First, we formulate the problem of interest and explore the optimization problem's methodology and implementation algorithm. The optimal hybrid censoring scheme is then obtained, along with a sensitivity analysis with respect to the model's parameters and cost coefficients. In addition, it is demonstrated that the proposed algorithm converges in a finite number of steps.

The rest of the content is organized as follows. Initially, we formulate the problem of interest in Section 2, where we will also explain the optimization problem's methodology. The optimal design of the HCS is then determined in Section 3. Section 4 implements the algorithm described in Subsection 4.2 in order to determine the optimal designs by conducting a sensitivity analysis with respect to the model's parameter and cost coefficients. A mis-specification analysis for the model parameters is also included later in Section 4.2. The conclusion of the study is provided in Section 5.

2 Problem Formulation

On the basis of the optimization of a specific criterion function, we will obtain the optimal HCS design in this study. In order to do this, it is rather typical in practice to take into account a criterion based on minimizing of any of the following:

- total expense incurred throughout the test,
- a measure of the variability of the estimators of the unknown parameters of the model for the lifetime random variable or
- both combined.

These optimization problems, though, only have one objective function. When a cost-based model and a variance-based model are combined, the two components are added using cost coefficients based on a set of weights, which undoubtedly optimizes both of the factors. However, Bhattacharya et al. (2014) highlighted that characterizing the cost weight for variation factor in a practical environment is difficult. Therefore, simultaneous optimization of each of these factors as separate objective functions is required and preferable. Here, we outline a methodology that simultaneously optimizes the cost-based and variance-based models while sticking to the experimenter's choice of cost and variance constraints. It is known as a multi-criteria optimization problem. The optimal criteria are built using the exact distribution of the MLE of the lifetime model parameter. The lifetime distribution has a two-parameter exponential distribution denoted by EXP(λ, μ), and the PDF is given by

$$f(x; \lambda, \mu) = \lambda e^{-\lambda(x-\mu)}, \; x \geq \mu, \; \lambda > 0, \; \mu > 0.$$

The location parameter $\mu(>0)$ and the scale parameter $\lambda(>0)$ are unknown in this case.

2.1 Formulation of Multi-Criteria Based Optimization

In designing a life-testing experiment, the overall cost of the experiment and the variance of the estimated lifetime model parameters are two crucial factors that the experimenter must prioritize. To determine the optimal design HCS, either a variance minimization criterion or a cost minimization criterion is applied. Here, we develop a procedure that simultaneously optimizes both criteria. Suppose n units are subjected to a life-testing experiment and their individual lifetime random variables are labeled by $X_1, X_2, \ldots, X_n$. As the units may be costly and there may be a limited time for inspection, it is essential to conduct a life-testing experiment that incorporates both time and failure limitations. The HCS addresses this need by imposing simultaneous restrictions on test duration and the number of failures. In this study, we focus specifically on the Type-II HCS, as introduced by Childs et al. (2003). For a prefixed number $1 \leq r \leq n$ and a given quantity $T > 0$, the life-testing experiment is continued by the time $\max\{X_{(r)}, T\}$ under Type-II HCS.

Let $\mathcal{D} = (n, r, T)$ represents the Type-II HCS's design parameters during the life-testing experiment. Consider the two important objective criteria functions. Suppose τ and M represent the duration and number of failures, respectively, of a life-testing experiment. Now, let us define the total anticipated cost of the experiment as a single objective criterion function:

$$\phi_1(\mathcal{D}) = C_0 + C_s n + C_T E(\tau) + C_D E(M), \tag{1}$$

where C_s is the inspection cost per unit of item, C_D is the cost per failed item, and C_T is the cost per unit of time of life-testing experiment, and C_0 is an overall constant cost that is independent of any designing parameters. The use of such cost-based criteria is well established in the literature on life-testing and has been previously adopted by Bhattacharya et al. (2020), among others.

Now, let us develop a second objective criterion function based on the uncertainty associated with estimating the unknown lifetime parameters. The second objective criterion function based on a variance measure is therefore provided by

$$\phi_2(\mathcal{D}) = \int_0^1 \mathrm{Var}(\widehat{X}_p)\, dp, \tag{2}$$

where X_p represents the pth percentile of the lifetime distribution and $\widehat{X}_p$ represents its MLE. In this case, when lifetime random variable follows two-parameter exponential distribution, if it is assumed that $\theta = 1/\lambda$, the MLE $\widehat{X}_p$ is $\widehat{\mu} + a_p\widehat{\theta}$ with $a_p = -\ln(1-p)$, where $\widehat{\mu}$ and $\widehat{\theta}$ are MLEs of parameters μ and θ. Mathematical expression for the MLEs of parameters μ and θ are provided in Section 3 later. Here, $\phi_2(\mathcal{D})$ provides a variance measure that is the average variance of the estimates for the p-th percentile across all percentile points. Interestingly, this quantity is independent of the percentile variable $p \in (0,1)$ and is depending only on n, r, and T. Kundu (2008) considers a similar form of variance measure, which is a modification of the variance measure proposed by Zhang and Meeker (2005). Zhang and Meeker (2005) considered variance of a certain percentile of the lifetime distribution. Whereas, Kundu (2008) introduced a weighted variance-based criterion that incorporates a non-negative weight function $W(p)$ defined on $(0,1)$, allowing certain percentiles to be emphasized more than others rather than treating all percentiles equally. $W(p)$ has to be choosen in advance depending on the problem in interest. For example, if someone is interested to give more stress at the center then more weight should be given at $p = .5$, on the other hand if tail probabilities are more important then proper weight can be given for large p. This extends earlier work by Zhang and Meeker (2005) and aligns with the broader class of variance-based optimality criteria, such as A- and D-optimality, previously explored by Ebrahimi (1988), Liski et al. (2002), and others in both single- and multi-criteria optimization frameworks. In light of these developments, we provide an additional comment in Remark 2.1 regarding the flexibility of choosing the percentile range of interest, which could motivate alternative criterion formulations in future studies. However, in the present work, we continue to use $\phi_2(\mathcal{D})$, a variance measure that assigns equal importance across all percentiles.

Remark 2.1. *Note that there may be circumstances where it is inappropriate to consider the entire range $p \in (0,1)$ in the variation of the quantile estimates for the variance measure. one may just be interested in a specific quantile range. For instance, the variance measure can be thought of in the following forms for $p_1, p_2 \in (0,1)$ with $p_1 < p_2$, where p_1 and p_2 are known: $\int_{p_1}^{p_2} Var(\widehat{X}_p)\, dp$ with in the range of interest (p_1, p_2); $\int_0^{p_1} Var(\widehat{X}_p)\, dp + \int_{p_2}^{1} Var(\widehat{X}_p)\, dp$ for the range $(0,p_1) \cup (p_2, 1)$, etc.*

Here, the criterion function $\phi_2(\mathcal{D})$ can be simplified as follows:

$$\begin{aligned}\phi_2(\mathcal{D}) &= \int_0^1 \text{Var}(\widehat{\mu} + a_p\widehat{\theta})\ dp \\ &= \text{Var}(\widehat{\mu}) \int_0^1 1\ dp + \text{Var}(\widehat{\theta})\Big(\int_0^1 a_p^2\ dp\Big) + 2\Big(\int_0^1 a_p\ dp\Big)\text{Cov}(\widehat{\mu}, \widehat{\theta}).\end{aligned}$$

Using $\int_0^1 1\ dp = 1$, $\int_0^1 a_p\ dp = 1$ and $\int_0^1 a_p^2\ dp = 2$, after simplification, it can be shown that

$$\begin{aligned}\phi_2(\mathcal{D}) &= \text{Var}(\widehat{\mu}) + 2\text{Var}(\widehat{\theta}) + 2\text{Cov}(\widehat{\mu}, \widehat{\theta}) \\ &= E(\widehat{\mu}^2) - (E(\widehat{\mu}))^2 + 2\big(E(\widehat{\theta}^2) - (E(\widehat{\theta}))^2\big) + 2\big(E(\widehat{\mu}\widehat{\theta}) - E(\widehat{\mu})E(\widehat{\theta})\big). \end{aligned} \tag{3}$$

Given the two objective functions, $\phi_1(\mathcal{D})$ and $\phi_2(\mathcal{D})$, it is desired to simultaneously minimize both criteria under two constraints. Consequently, our optimization problem can be viewed as a constrained multi-criteria-based optimization problem and stated as follows:

$$\begin{aligned}&\underset{\mathcal{D}}{\text{Minimize}}\ \text{Max}(\phi_1(\mathcal{D}), \phi_2(\mathcal{D})) \\ &\text{Subject to} \\ &\qquad \phi_1(\mathcal{D}) \leq C_b \\ &\qquad \phi_2(\mathcal{D}) \leq V_T,\end{aligned} \tag{4}$$

where, C_b and V_T represent a predetermined tolerable budget amount and targeted variance, respectively, permitted by the experimenter so that the reliability experiment is finished within a predetermined budget with a predetermined minimum experimental adequacy based on some estimation method. For simultaneous minimization, we cannot accept a high value for one criterion in exchange for a low value for the other criterion. As suggested by Bhattacharya et al. (2020), it is advisable to simultaneously achieve the minimum of the worst-case scenario for both objectives. For additional information, please view Ehrgott (2005). Now, in the next Section 2.2, we will demonstrate a procedure for solving this optimization problem.

2.2 Methodology of Optimization

Notice that the designing parameters $\mathcal{D}$ for the optimization problem defined by equation (4) consist of two discrete variables (i.e., n and r) and one continuous variable (i.e., T). In order to solve the optimization issue presented by equation (4), let us first analyze a simple example in which the values of variables n and r are fixed and known, and we are interested in determining the optimal value of variable T, i.e., $\mathcal{D} = T$. Hence, the optimization problem given by equation (4) simplifies to

$$
\begin{aligned}
&\underset{T}{\text{Minimize}}\ \text{Max}(\phi_1(T), \phi_2(T)) \\
&\text{Subject to} \\
&\qquad \phi_1(T) \le C_b \\
&\qquad \phi_2(T) \le V_T.
\end{aligned} \tag{5}
$$

It is appropriate and essential to scale the objective functions $\phi_1(T)$ and $\phi_2(T)$ to the range $[0,1]$ to ensure that the optimization makes practical sense. Given upper and lower limit values of T, say T_{min} and T_{max}, the new objective functions may be defined as follows:

$$\psi_1(T) = \frac{\phi_1(T) - \phi_1(T_{\min})}{\phi_1(T_{\max}) - \phi_1(T_{\min})}, \tag{6}$$

$$\psi_2(T) = \frac{\phi_2(T) - \{\phi_2(T)\}_{\min}}{\{\phi_2(T)\}_{\max} - \{\phi_2(T)\}_{\min}}. \tag{7}$$

As the criterion function $\phi_1(T)$ is a strictly increasing function of the variable T, $\phi_1(T_{\min})$ and $\phi_1(T_{\max})$ are employed in equation (6) for $\{\phi_1(T)\}_{\min}$ and $\{\phi_1(T)\}_{\max}$, respectively. In Subsection 4.1, we will cover the choices of $T_{\min}$ and $T_{\max}$ in extensive detail. Hence, the optimization problem in equation (5) is now

$$
\begin{aligned}
&\underset{T}{\text{Minimize}}\ \text{Max}(\psi_1(T), \psi_2(T)) \\
&\text{Subject to} \\
&\qquad \phi_1(T) \le C_b \\
&\qquad \phi_2(T) \le V_T.
\end{aligned} \tag{8}
$$

The region in the interval $T_{\min} \le T \le T_{\max}$ that satisfies the constraints $\phi_1(T) \le C_b$ and $\phi_2(T) \le V_T$ is referred to as the feasible region. Let us represent the feasible region with the symbol $\mathcal{D}_T^f$. Depending on the range of the feasible region, it is evident that there are three possible optimal solutions for the optimization problem. They are intersection points(s) of the two objective functions $\psi_1(T)$ and $\psi_2(T)$, local minima, and any feasible region boundary

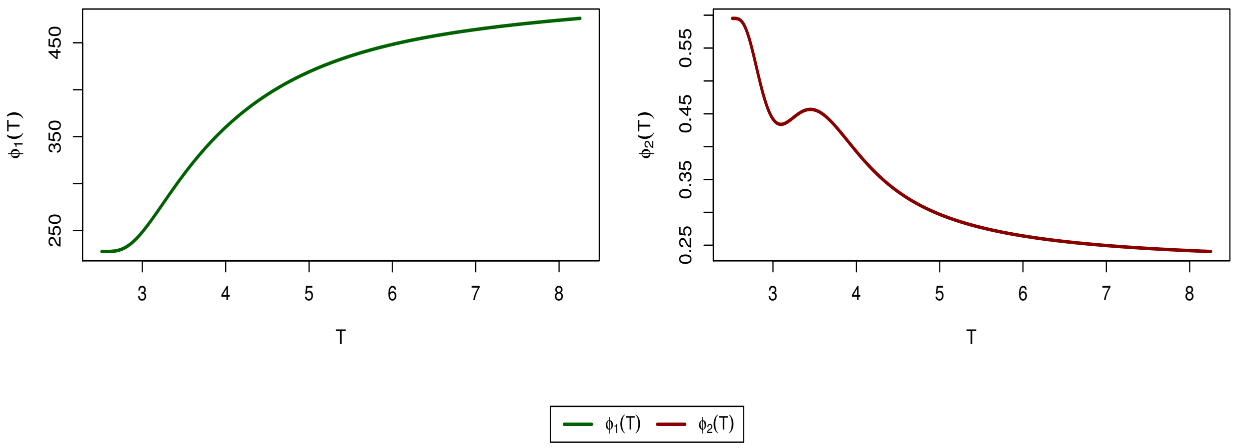


Figure 1: Plots for ϕ_1 and ϕ_2 functions with parameters $\lambda = 0.8$, $\mu = 2.5$ and coefficients $C_0 = 100$, $C_s = 1$, $C_T = 5$, $C_D = 25$.

points. See, for instance, Figure 2. Figure 2 depicts plots of the functions $\psi_1(T)$ and $\psi_2(T)$ along with feasible region and regions satisfied by both the constraints over a variety of C_b and V_t values. For all the plots depicted in Figures 1 and 2, we set $n = 13$ and $r = 4$. The values of $T_{\min}$ and $T_{\max}$ in this situation are 2.512563 and 8.256463, respectively. Figure 1 plots the original functions $\phi_1(T)$ and $\phi_2(T)$ to illustrate the regions satisfied by the constraints under the range $(T_{\min}, T_{\max})$ for various values of C_b and V_t. This figure can be used to determine the regions satisfied by the constraints for various values of C_b and V_t. There is also the possibility that the $\mathcal{D}_T^f$ is empty for all conceivable values of n and r; therefore, in this case, there is no solution to the optimization problem (see the last plot of Figure 2 for $C_b = 250$ and $V_t = 0.40$). If T^* is the optimal solution to the optimization problem represented by the equation (8), then we must have

$$\text{Max}(\psi_1(T^*), \psi_2(T^*)) \geq \text{Max}(\psi_1(T), \psi_2(T)) \ \forall \ T \in \mathcal{D}_T^f \quad . \tag{9}$$

We can get the optimal values of n and r by repeating the procedure outlined above for choosing the optimal value of T for every possible combination of n and r.

Section 3 comprises the implementation of the proposed method under Type-II HCS for obtaining the optimal censoring schemes described in Section 2.

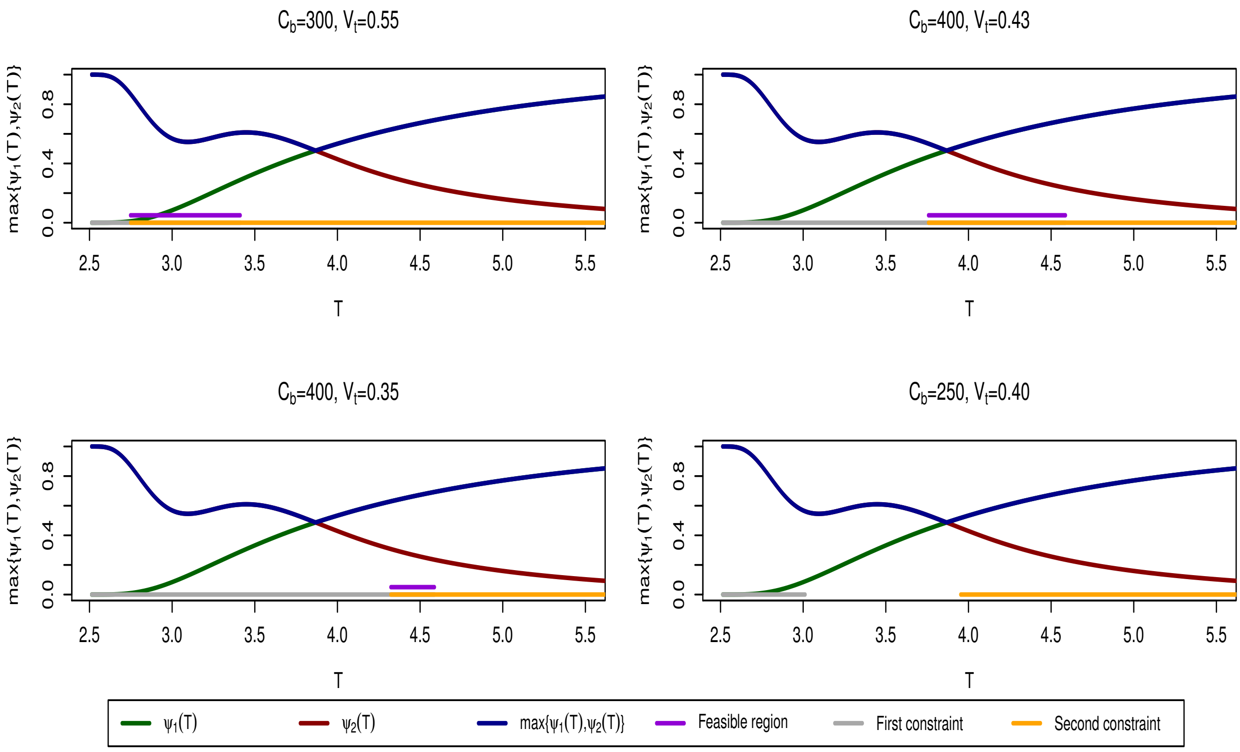


Figure 2: Plot for ψ_1 and ψ_2 functions with parameters $\lambda = 0.8, \mu = 2.5$ and coefficients $C_0 = 100, C_s = 1, C_T = 5, C_D = 25$ for various values of C_b and V_t.

3 Optimal Censoring Scheme under Type-II HCS

This section presents the implementation of the proposed methodology outlined in Section 2 for Type-II HCS. According to this censoring scheme, a life test experiment ends at time $\max\{X_{(r)}, T\}$. Let D represent the number of X_is that occur before the time T; then, $D = 0, 1, \ldots, r-1$ if $T \le X_{(r)}$ and $D = r, r+1 \ldots, n$ if $T > X_{(r)}$. The maximum number of failures throughout the experiment is therefore $\max\{r, D\}$. MLE of the model parameter(s) for $r = 1$ exists only when $D \ge 1$, however MLE of the model parameter(s) exists always for the case $r \ge 2$ (see Ganguly et al. (2012)). For $r = 1$, MLEs of μ and θ are $\widehat{\mu} = X_{(1)}$ and $\frac{1}{D}\Big(\sum_{i=1}^{D} X_{(i)} - nX_{(1)} + (n-D)T\Big)$, respectively. Whereas, when $r \ge 2$, they are expressed as follows

$$\widehat{\mu} = X_{(1)} \text{ and } \widehat{\theta} = \frac{1}{M}\Big(\sum_{i=1}^{M} X_{(i)} - nX_{(1)} + (n-M)\tau\Big),$$

with $M = \max\{r, D\}$ and $\tau = \max\{X_{(r)}, T\}$. As a result, the following are the two objective functions:

$$\phi_1(\mathcal{D}) = C_0 + C_s n + C_T E(\tau) + C_D E(M), \tag{10}$$

$$\phi_2(\mathcal{D}) = \begin{cases} \int_0^1 \text{Var}(\widehat{X}_p | D \geq 1)\, dp, & \text{for } r = 1 \\ \int_0^1 \text{Var}(\widehat{X}_p)\, dp, & \text{for } r \geq 2. \end{cases} \tag{11}$$

Where,

$$E(M) = \sum_{d=0}^{n} \sum_{l=0}^{d} (-1)^{d-k} \max\{r, d\} \binom{n}{d} \binom{d}{l} q^{n-l}, \tag{12}$$

$$E(\tau) = \sum_{l=0}^{r-1} (-1)^{r-l-1} r \binom{n}{r} \binom{r-1}{l} \left(T + \frac{1}{\lambda(n-l)} \right) \frac{1}{(n-l)} q^{n-l}$$
$$+ T \sum_{d=r}^{n} \sum_{l=0}^{d} (-1)^{d-k} \binom{n}{d} \binom{d}{l} q^{n-l}. \tag{13}$$

Recall from (3) that the final expression for $\phi_2(\mathcal{D})$ requires the first two raw moments of the MLEs $\widehat{\mu}$ and $\widehat{\theta}$. In both the situations of $r = 1$ and $r \geq 2$, we have derived the expression for $E(\widehat{\mu}\widehat{\theta})$. The additional moments $E(\widehat{\mu}), E(\widehat{\mu}^2), E(\widehat{\theta})$, and $E(\widehat{\theta}^2)$ may be obtained from Ganguly et al. (2012). In order to derive $E(\widehat{\mu}\widehat{\theta})$, we require the following theorem from Ganguly et al. (2012):

Theorem 3.1. *The joint MGF of MLE $(\widehat{\theta}, \widehat{\mu})$ are as follows:*

For $r = 1$

$$M_{\widehat{\theta},\widehat{\mu}|D\geq 1}(w_1, w_2) = (1 - q^n)^{-1} \Bigg[\frac{c_{10} e^{\mu_{10} w_1 + \mu w_2} - e_{10} e^{T w_2}}{\left(1 + \frac{w_1}{\lambda_{10}} - \frac{w_2}{\nu_0}\right)} + \frac{e^{\mu w_2} - q^n e^{T w_2}}{\left(1 - \frac{w_1}{\lambda_n}\right)^{\alpha_n} \left(1 - \frac{w_2}{\nu_{n-1}}\right)}$$
$$+ \sum_{d=2}^{n} \sum_{\substack{k=0 \\ k \neq n-1}}^{d-1} \frac{c_{dk} e^{\mu_{dk} w_1 + \mu w_2} - e_{dk} e^{T w_2}}{\left(1 - \frac{w_1}{\lambda_d}\right)^{\alpha_d} \left(1 + \frac{w_1}{\lambda_{dk}} - \frac{w_2}{\nu_k}\right)} \Bigg]$$

and for $r \geq 2$

$$M_{\widehat{\theta},\widehat{\mu}}(w_1, w_2) = \frac{q^n e^{T w_2}}{\left(1 - \frac{w_1}{\lambda_1}\right)^{\alpha_1} \left(1 - \frac{w_2}{\nu_{n-1}}\right)} + \frac{e^{\mu w_2} - q^n e^{T w_2}}{\left(1 - \frac{w_1}{\lambda_n}\right)^{\alpha_n} \left(1 - \frac{w_2}{\nu_{n-1}}\right)}$$
$$+ \sum_{d=1}^{n} \sum_{\substack{k=0 \\ k \neq n-1}}^{d-1} \frac{c_{dk} e^{\mu_{dk} w_1 + \mu w_2} - e_{dk} e^{T w_2}}{\left(1 - \frac{w_1}{\lambda_d}\right)^{\alpha_d} \left(1 + \frac{w_1}{\lambda_{dk}} - \frac{w_2}{\nu_k}\right)},$$

where

$$
\begin{aligned}
&m = \max\{r, d\} \qquad \alpha_d = m - 1, \qquad \lambda_d = \lambda m, d = 1, \dots, n, \\
&\nu_k = \lambda(k+1), k = 0, 1, \dots, n-1, \\
&\mu_{dk} = \frac{n-k-1}{m}(T - \mu), \qquad \lambda_{dk} = \frac{\lambda m(k+1)}{n-k-1}, \\
&c_{dk} = (-1)^{d-k-1}\binom{n}{d}\binom{d}{k+1}q^{n-k-1},\ k = 0, 1, \dots d-1,\ d = 1, \dots n, \\
&e_{dk} = (-1)^{d-k-1}\binom{n}{d}\binom{d}{k+1}q^{n},\ k = 0, 1, \dots d-1,\ d = 1, \dots n.
\end{aligned}
$$

Since

$$
E(\widehat{\mu}\widehat{\theta}) = \frac{\partial^2}{\partial w_1 \partial w_2} M_{\widehat{\theta},\widehat{\mu}}(w_1, w_2)\bigg|_{w_1=0, w_2=0},
$$

$E(\widehat{\mu}\widehat{\theta})$ can be obtained using the above MGF. The other required moments from Ganguly et al. (2012) are reported below after some minor typo corrections along with $E(\widehat{\mu}\widehat{\theta})$.

$$
\begin{aligned}
E(\widehat{\mu}) &= \mu + \frac{1}{n\lambda} \\
E(\widehat{\mu}^2) &= \mu^2 + \frac{2}{(n\lambda)^2} + 2\frac{\mu}{n\lambda} \\
E(\widehat{\theta}) &= \begin{cases} \theta + \theta A_1(\mu, \theta) + (1 - q^n)^{-1} B_1(\mu, \theta), & \text{for } r = 1 \\ \theta + \theta A_2(\mu, \theta) + B_1(\mu, \theta), & \text{for } r \geq 2 \end{cases} \\
E(\widehat{\theta}^2) &= \begin{cases} \theta^2 + \theta^2 C_1(\mu, \theta) + \theta D_1(\mu, \theta) + (1 - q^n)^{-1} E_1(\mu, \theta), & \text{for } r = 1 \\ \theta^2 + \theta^2 C_2(\mu, \theta) + \theta D_2(\mu, \theta) + E_1(\mu, \theta), & \text{for } r \geq 2 \end{cases} \\
E(\widehat{\mu}\widehat{\theta}) &= \begin{cases} \frac{n-1}{n^2}\theta^2 + (1 - q^n)^{-1}\Big[\theta^2 O_1(\mu, \theta) + \theta P_1(\mu, \theta) + \mu B_1(\mu, \theta)\Big], & \text{for } r = 1 \\ \frac{n-1}{n^2}\theta^2 + \theta^2 O_2(\mu, \theta) + \theta P_2(\mu, \theta) + \mu B_1(\mu, \theta), & \text{for } r \geq 2, \end{cases}
\end{aligned}
$$

where

$$A_1(\mu,\theta) = (1-q^n)^{-1}\Big[\sum_{\substack{d=2\\k\neq n-1}}^{n}\sum_{k=0}^{d-1}(c_{dk}-e_{dk})\Big\{\sum_{s=0}^{d-2}p_{ks}\frac{d-s-1}{d} - \big(1-\sum_{s=0}^{d-2}p_{ks}\big)\frac{n-k-1}{d(k+1)}\Big\}$$
$$-(c_{10}-e_{10})\frac{n-1}{r} - \frac{(1-q^n)}{n}\Big]$$

$$B_1(\mu,\theta) = \sum_{d=1}^{n}\sum_{k=0}^{d-1}c_{dk}\mu_{dk}$$

$$C_1(\mu,\theta) = (1-q^n)^{-1}\Big[\sum_{\substack{d=2\\k\neq n-1}}^{n}\sum_{k=0}^{d-1}(c_{dk}-e_{dk})\Big\{\sum_{s=0}^{d-2}p_{ks}\frac{(d-s)(d-s-1)}{d^2}$$
$$+2\Big(1-\sum_{s=0}^{d-2}p_{ks}\Big)\Big(\frac{n-k-1}{d(k+1)}\Big)^2\Big\} + 2(c_{10}-e_{10})(n-1)^2 - \frac{(1-q^n)}{n}\Big]$$

$$D_1(\mu,\theta) = 2(1-q^n)^{-1}\Big[\sum_{\substack{d=2\\k\neq n-1}}^{n}\sum_{k=0}^{d-1}c_{dk}\mu_{dk}\Big\{\sum_{s=0}^{d-2}p_{ks}\frac{d-s-1}{d} - \big(1-\sum_{s=0}^{d-2}p_{ks}\big)\frac{n-k-1}{d(k+1)}\Big\}$$
$$-(n-1)c_{10}\mu_{10}\Big]$$

$$E_1(\mu,\theta) = \sum_{d=1}^{n}\sum_{k=0}^{d-1}c_{dk}\mu_{dk}^2$$

$$O_1(\mu,\theta) = \sum_{\substack{d=2\\k\neq n-1}}^{n}\sum_{k=0}^{d-1}(c_{dk}-e_{dk})\frac{(d-1)(k+1)-2(n-k-1)}{d(k+1)^2}$$

$$P_1(\mu,\theta) = \frac{n-1}{n}(\mu-Tq^n) + \sum_{\substack{d=2\\k\neq n-1}}^{n}\sum_{k=0}^{d-1}(c_{dk}\mu-e_{dk}T)\frac{(d-1)(k+1)-(n-k-1)}{d(k+1)} + \frac{c_{dk}\mu_{dk}}{k+1}$$

$$A_2(\mu,\theta) = \sum_{\substack{d=1\\k\neq n-1}}^{n}\sum_{k=0}^{d-1}(c_{dk}-e_{dk})\Big\{\sum_{s=0}^{m-2}p_{ks}\frac{m-s-1}{m} - \big(1-\sum_{s=0}^{m-2}p_{ks}\big)\frac{n-k-1}{m(k+1)}\Big\} - \frac{q^n}{r} - \frac{(1-q^n)}{n}$$

$$C_2(\mu,\theta) = \sum_{\substack{d=1\\k\neq n-1}}^{n}\sum_{k=0}^{d-1}(c_{dk}-e_{dk})\Big\{\sum_{s=0}^{m-2}p_{ks}\frac{(m-s)(m-s-1)}{m^2} + 2\big(1-\sum_{s=0}^{m-2}p_{ks}\big)\frac{n-k-1}{m(k+1)}\Big\}$$
$$-\frac{q^n}{r} - \frac{(1-q^n)}{n}$$

$$D_2(\mu,\theta) = 2\sum_{d=1}^{n}\sum_{k=0}^{d-1}c_{dk}\mu_{dk}\Big\{\sum_{s=0}^{m-2}p_{ks}\frac{m-s-1}{m} - \big(1-\sum_{s=0}^{m-2}p_{ks}\big)\frac{n-k-1}{m(k+1)}\Big\}$$

$$O_2(\mu,\theta) = \frac{1}{n}\Big(\frac{1}{n}-\frac{1}{r}\Big)q^n + \sum_{\substack{d=1\\k\neq n-1}}^{n}\sum_{k=0}^{d-1}(c_{dk}-e_{dk})\frac{(m-1)(k+1)-2(n-k-1)}{m(k+1)^2}$$

$$P_2(\mu,\theta) = \frac{n-1}{n}\mu + \Big(\frac{1}{n}-\frac{1}{r}\Big)Tq^n + \sum_{\substack{d=1\\k\neq n-1}}^{n}\sum_{k=0}^{d-1}\Big\{(c_{dk}\mu-e_{dk}T)\frac{(m-1)(k+1)-(n-k-1)}{m(k+1)}$$
$$+\frac{c_{dk}\mu_{dk}}{k+1}\Big\}.$$

4 Numerical Results

The optimal censoring scheme is the censoring scheme that simultaneously minimizes the criteria $\phi_1(.)$ and $\phi_2(.)$ among all possible censoring schemes that fulfill the constraints. As stated previously, $\mathcal{D} = (n, r, T)$ represents the HCS's design parameters. Hence, a censoring scheme is represented as (n, r, T). Assume that the optimal censoring scheme to be denoted by (n_0, r_0, T_0).

To achieve an optimal censoring scheme, we first explore the bounds on design parameters based on the formulation of the optimization problem, and then provide a computational algorithm.

4.1 Bounds on Design Parameters of Censoring Schemes

It can be demonstrated that n_0 and r_0, the optimum values of n and r, are bounded. The proof is located in Appendix A. Further, $\phi_2(T)$ cannot be defined since $\psi_2(T)$ is a constant function of the variable T whenever $n = r$. Thus, these instances have been excluded from the region $\mathcal{D}$. Thereby, depending on Appendix A and this remark, we get $3 \le n_0 \le n_{\max}$, where $n_{\max} = [(\max\{C_b, V_t\} - C_0)/C_s]$ where $[y]$ is the largest integer less than or equal to y and where $1 \le r_0 < n$.

Please take note that the values of $T_{\min}$ and $T_{\max}$ may be considered as the lifetime distribution's lower and upper quantiles, respectively. In other words, set $T_{\min} = \mu - \frac{1}{\lambda}\ln(0.99)$ and $T_{\max} = \mu - \frac{1}{\lambda}\ln(0.01)$. Recall that under Type-II HCS, the MLE of the parameters of lifetime distribution always exists when $r \ge 2$, but when $r = 1$, the MLE exits if there is at least one failure, *i.e.*, $D \ge 1$. Consider the scenario when the event's probability, $D \ge 1$, is greater than 0.99, thus

$$P(D \ge 1) \ge 0.99. \tag{14}$$

From equation (14), we get the following inequality

$$T \ge \mu - \frac{1}{n\lambda}\ln(0.01). \tag{15}$$

As a result, for $r = 1$, we use $T_{\min} = \mu - \frac{1}{n\lambda}\ln(0.01)$ rather than setting it to the 0.99-th quantile of the lifetime distribution.

4.2 Algorithm

The following algorithm has been used to find the (n_0, r_0, T_0).

Algorithm Steps used for determining an optimal censoring scheme

Step 1. Fix n and r in a way that $3 \leq r \leq n_{\max}$ and $1 \leq r < n$.

Step 2. There are therefore $(n(n+1)/2) - 3 - (n-2)$ distinct pairs of n and r.

Step 3. Fix a pair (n, r) and determine the associated feasible region $\mathcal{D}_T^f$; then, if $\mathcal{D}_T^f$ is not empty, obtain a value of T in accordance with the equation (9), denoted $T_0^{n,r}$. Notice that it is possible for a pair (n, r) to have an empty $\mathcal{D}_T^f$, in which case we cannot have the corresponding value $T_0^{n,r}$.

Step 4. For every pair of Step 2 repeat Step 3. By repeating Step 3, we now have triplets (censoring schemes) $(n, r, T_0^{n,r})$ for distinct n and r values.

Step 5. Choose the scheme with the smallest value of the quantity $\text{Max}(\psi_1(T_0^{n,r}), \psi_2(T_0^{n,r}))$ among all $(n, r, T_0^{n,r})$ obtained in Step 4 and label it (n_0, r_0, T_0). (n_0, r_0, T_0) is the optimal censoring scheme.

Finally, using the aforementioned algorithm, we undertake an evaluation for the optimal censoring scheme design in reference to model parameters and cost coefficients in Subsection 4.3.

4.3 Sensitivity Analysis

With respect to cost coefficients and parameters, a sensitivity analysis is performed out by simply adjusting one coefficient at a time while maintaining the others constant. For the sake of illustration, we choose $(C_b,\ V_t) = (450,\ 0.14)$ and $(C_0,\ C_s,\ C_D,\ C_T) = (100,\ 1,\ 20,\ 5)$. All of our findings are based on the assumptions that $\lambda = 1$ and $\mu = 0$ as shown in Tables 1 and 2. Some qualitative observations have been drawn based on the values in these tables. The reader should note that all the optimal designs under Type-II HCS, as reported in Tables 1–3, exhibit both desirable total cost and efficiency while satisfying their respective constraints.

In particular, Table 1 shows that, at lower values of V_t, the sample sizes in the optimal designs tend to be larger due to the requirement of achieving higher efficiency. Consequently, the optimal values n_0 and r_0 tend to decrease as V_t increases. This pattern can be attributed to the fact that a larger number of failures improves the precision of estimation, therefore reducing the variance and enhancing the efficiency. Furthermore, when the constraint on efficiency is set at $V_t = 0.22$ and the budget limit is restricted to $C_b = 250$, no optimal design is found that simultaneously satisfies both stringent constraints. This is indicated as "Null" in the Table 1. On the other hand, when V_t is held fixed, smaller values of the budget limit C_b lead to a reduction in both the expected number of failures and the expected duration of the test. This is a direct consequence of tighter budgetary constraints. As C_b increases, both $E(D)$ and $E(\tau)$ rise accordingly. Interestingly, even when the budget limit C_b is decreased, the optimal sample size n_0 may still increase in order to satisfy the constraint on efficiency imposed by the fixed value of V_t. The proposed optimal designs based on the minimax approach depends highly on the choices of constraints, which is generally the case for any constrained optimization method. However, in practical applications, particularly in life-testing scenarios, the constraints (e.g., on cost or sample size, efficiency) are typically guided by real-world limitations, and are rarely so restrictive that no solution ("Null") is found. In our simulations, we were able to obtain viable solutions across a wide range of realistic settings.

Table 2 provides several qualitative insights into the behavior of optimal censoring schemes in relation to cost coefficients. For example, when C_s and C_D are increased, both r_0 and $\mathbb{E}(D)$ tend to decrease. This is because higher sampling inspection and failure-related costs lead to smaller optimal values of r, resulting in fewer observed failures in order to satisfy the budgetary constraints. However, the sample size n in the optimal design tends to increase in order to maintain the efficiency requirement imposed by the fixed value $V_t = 0.14$. On the other hand, when C_0, C_s, and C_D are held constant and C_T is increased, the expected number of failures, expected test duration, as well as r_0 and T_0, all tend to decrease. This inverse relationship is intuitive, as higher per-unit time costs discourage longer life-tests. The simultaneous decline in both r_0 and T_0 tends to occur from their combined contribution to the total experimental duration when the per unit cost component C_T is increased.

Table 1: Optimal sampling schemes while varying C_b and V_t marginally with fixed $C_0 = 100, C_s = 1, C_T = 5, C_D = 20$ and $(\lambda, \mu) = (1, 0)$.

V_t	C_b	n_0	r_0	T_0	$(\psi_1(\mathcal{D}_0), \psi_2(\mathcal{D}_0))$	$(\phi_1(\mathcal{D}_0), \phi_2(\mathcal{D}_0))$	$E(D)$	$E(\tau)$
0.30	204	10	4	0.4951	(0.0836, 0.5364)	(203.9879, 0.2991)	4.5538	0.5825
	208	9	4	0.6551	(0.1354, 0.4484)	(207.7709, 0.2960)	4.7577	0.7233
	215	9	4	0.7251	(0.1713, 0.4382)	(212.0283, 0.2943)	4.9576	0.7752
	220	7	5	1.1001	(0.1332, 0.2158)	(219.9423, 0.2965)	5.3231	1.2961
	230	7	5	1.1901	(0.1652, 0.1668)	(221.7437, 0.2938)	5.3995	1.3506
	250	7	6	1.4801	(0.1295, 0.1285)	(239.3290, 0.2788)	6.1640	1.8099
	300	7	6	1.4801	(0.1295, 0.1285)	(239.3290, 0.2788)	6.1640	1.8099
0.14	380	17	12	1.2551	(0.1511, 0.5258)	(379.9825, 0.1381)	12.8105	1.3543
	385	16	12	1.5701	(0.2405, 0.4694)	(384.9362, 0.1400)	13.0375	1.6373
	390	15	13	1.6901	(0.1151, 0.4403)	(389.9547, 0.1391)	13.2530	1.9788
	395	15	13	1.9701	(0.2139, 0.3337)	(394.9887, 0.1377)	13.4657	2.1349
	400	15	13	2.1451	(0.2851, 0.3192)	(398.6132, 0.1375)	13.6163	2.2575
	450	15	14	2.2751	(0.1848, 0.1841)	(411.8871, 0.1332)	14.1965	2.5914
	500	15	14	2.2751	(0.1848, 0.1841)	(411.8871, 0.1332)	14.1965	2.5914
0.22	250				— Null —			
0.23		12	6	0.7001	(0.1065, 0.5318)	(249.8883, 0.2294)	6.6976	0.7873
0.24		10	6	0.9801	(0.1622, 0.4253)	(249.9246, 0.2371)	6.7289	1.0693
0.25		9	6	1.2151	(0.2080, 0.3402)	(249.8361, 0.2450)	6.7157	1.3043
0.30		7	6	1.4801	(0.1295, 0.1285)	(239.3290, 0.2788)	6.1640	1.8099
0.40		5	3	0.7901	(0.1186, 0.1151)	(175.8109, 0.3991)	3.2991	0.9656
0.45		5	2	0.4801	(0.1008, 0.1014)	(155.2944, 0.4132)	2.3670	0.5907
0.14	450	15	14	2.2751	(0.1848, 0.1841)	(411.8871, 0.1332)	14.1965	2.5914
0.18		12	11	1.9851	(0.1503, 0.1516)	(347.0625, 0.1661)	11.1698	2.3334
0.20		10	9	1.7851	(0.1351, 0.1320)	(303.8977, 0.1986)	9.1593	2.1422
0.22		9	8	1.6901	(0.1323, 0.1323)	(282.3980, 0.2198)	8.1595	2.0416
0.24		9	8	1.6901	(0.1323, 0.1323)	(282.3980, 0.2198)	8.1595	2.0416
0.25		8	7	1.5901	(0.1306, 0.1305)	(260.8873, 0.2460)	7.1613	1.9322
0.30		7	6	1.4801	(0.1295, 0.1285)	(239.3290, 0.2788)	6.1640	1.8099

We now discuss sensitivity analysis with respect to the model-parameters presented in Table 3. The sensitivity of the optimal design parameters to changes in the scale parameter λ reveals several important trends. As λ increases and moves further away from the value 1, the optimal values of the design parameters n_0, r_0, and T_0 tend to decrease. This is because an increase in λ corresponds to a decrease in the mean lifetime of test items, resulting in a higher failure rate within a shorter time frame. Consequently, the experiment concludes more quickly, and fewer items are required, leading to a lower overall cost. In contrast, when the value of λ decreases from 1 to, say, 0.95 or 0.90, the mean lifetime of the items increases significantly. As a result, the experiment must run for a longer period to observe the required number of failures, which leads to an increase in both the expected number of failures and the expected test duration. Henceforth, the optimal design parameters n_0, r_0, and T_0, especially n_0 and r_0 along with its corresponding overall experimental cost and total cost $\phi_1(D_0)$ tend to increase.

Now, consider the sensitivity analysis with respect to the threshold parameter μ. As μ increases, both the sample size and the test duration tend to increase. This is because a larger threshold implies longer lifetimes for the items, thereby requiring the test to run longer to

Table 2: Optimal sampling schemes while varying C_s, C_T and C_D marginally with fixed $C_0 = 100, C_b = 450, V_t = 0.14$ and $(\lambda, \mu) = (1, 0)$.

		n_0	r_0	T_0	$(\psi_1(\mathcal{D}_0), \psi_2(\mathcal{D}_0))$	$(\phi_1(\mathcal{D}_0), \phi_2(\mathcal{D}_0))$	$E(D)$	$E(\tau)$
C_s	0.25	15	14	2.2751	(0.1848, 0.1841)	(400.6371, 0.1332)	14.1965	2.5914
	0.75	15	14	2.2751	(0.1848, 0.1841)	(400.6371, 0.1332)	14.1965	2.5914
	0.50	15	14	2.2751	(0.1848, 0.1841)	(404.3871, 0.1332)	14.1965	2.5914
	1.00	15	14	2.2751	(0.1848, 0.1841)	(411.8871, 0.1332)	14.1965	2.5914
	3.00	15	14	2.2751	(0.1848, 0.1841)	(411.8871, 0.1332)	14.1965	2.5914
	5.00	15	13	1.6901	(0.1151, 0.4403)	(449.9547, 0.1391)	13.2530	1.9788
	5.25	17	12	1.2101	(0.1311, 0.5408)	(449.9522, 0.1386)	12.7036	1.3261
C_D	5	15	14	2.3301	(0.1646, 0.1641)	(199.1847, 0.1331)	14.2154	2.6216
	10	15	14	2.3001	(0.1737, 0.1746)	(270.0746, 0.1331)	14.2050	2.6049
	20	15	14	2.2751	(0.1848, 0.1841)	(411.8871, 0.1332)	14.1965	2.5914
	22	15	14	2.2701	(0.1856, 0.1861)	(440.2297, 0.1332)	14.1948	2.5887
	24	15	13	2.0151	(0.2369, 0.3268)	(449.9116, 0.1376)	13.5037	2.1648
	26	17	12	1.1401	(0.1046, 0.5739)	(449.7218, 0.1399)	12.5496	1.2866
C_T	0	15	14	2.2201	(0.2073, 0.2079)	(398.5653, 0.1333)	14.1783	2.5628
	5	15	14	2.2751	(0.1848, 0.1841)	(411.8871, 0.1332)	14.1965	2.5914
	10	15	14	2.3001	(0.1737, 0.1734)	(425.1493, 0.1331)	14.2050	2.6049
	25	15	13	2.1451	(0.2185, 0.3192)	(443.7626, 0.1375)	13.6163	2.2575
	30	15	13	2.0101	(0.1682, 0.3275)	(449.8299, 0.1377)	13.4994	2.1614
	40	16	12	1.5751	(0.1660, 0.4693)	(442.6032, 0.1400)	13.0478	1.6412
	45	16	12	1.5601	(0.1564, 0.4695)	(449.6716, 0.1400)	13.0169	1.6296

accumulate sufficient failure information, as a result optimal design consists of larger values of T_0 and total cost $\phi_1(D_0)$. To maintain the budgetary and efficiency constraints (with fixed values of C_b and V_t) and per unit cost on experimental time (C_T), the design compensates by reducing the number of failures under the experiment, thereby decline in r_0.

Table 4 investigates how mis-specification of true model parameters (λ and μ) affects the optimal designs in terms of overall cost and estimation precision. In this table, $\mathcal{D}_0^{\text{True}}$ and $\mathcal{D}_0^{\text{MS}}$ denote the optimal designs obtained under the true and mis-specified parameter values, respectively. The criteria values corresponding to $\mathcal{D}_0^{\text{MS}}$ are evaluated under the true model parameter assumptions and reported as $(\phi_1(\mathcal{D}_0^{\text{MS}}), \phi_2(\mathcal{D}_0^{\text{MS}}))$. When true value of λ is 1 and it is overestimated (e.g., $\lambda = 1.5$), the resulting design reduces cost but leads to poor efficiency in the estimation due to fewer number of failures. On the contrary, underestimating λ (e.g., using 0.95) results in higher cost with only slight improvement in precision. Mis-specification of the threshold parameter μ also shows critical consequences. When μ is overestimated (e.g., using $\mu = 10$ instead of the true value $\mu = 5$), the optimal T_0 extends beyond the upper limit of the range of search, making the design impractical and more expensive. On the other hand, underestimating μ (e.g., setting it as $\mu = 0$) produces an invalid plan, as T_0 falls below the true threshold parameter value, violating the model assumptions.

Table 3: Optimal sampling schemes varying λ and μ marginally with $C_0 = 100, C_s = 1, C_T = 5, C_D = 20$ and $C_b = 450, V_t = 0.14$.

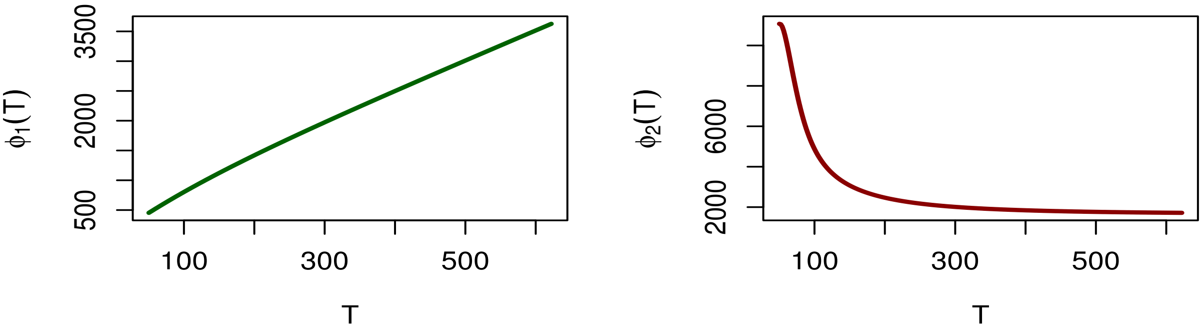
		n_0	r_0	T_0	$(\psi_1(\mathcal{D}_0), \psi_2(\mathcal{D}_0))$	$(\phi_1(\mathcal{D}_0), \phi_2(\mathcal{D}_0))$	$E(D)$	$E(\tau)$
λ	10.00	5	2	0.0460	(0.1114, 0.1221)	(152.0285, 0.0042)	2.3370	0.0578
	5.00	5	2	0.0970	(0.1209, 0.0922)	(153.0874, 0.0165)	2.3747	0.1188
	1.50	7	6	0.9817	(0.1373, 0.1339)	(236.2495, 0.1239)	6.1615	1.2041
	1.40	8	7	1.1322	(0.1373, 0.1339)	(258.0852, 0.1255)	7.1597	1.3784
	1.20	10	9	1.4834	(0.1379, 0.1353)	(302.0705, 0.1379)	9.1577	1.7831
	1.10	12	11	1.8041	(0.1524, 0.1512)	(345.9975, 0.1373)	11.1696	2.1210
	1.00	15	14	2.2751	(0.1848, 0.1841)	(411.8871, 0.1332)	14.1965	2.5914
	0.95	16	15	2.4806	(0.1934, 0.1936)	(434.0950, 0.1384)	15.2034	2.8054
	0.90	21	15	1.4812	(0.1651, 0.5545)	(449.8810, 0.1393)	16.0524	1.5666
μ	0	15	14	2.2751	(0.1848, 0.1841)	(411.8871, 0.1332)	14.1965	2.5914
	1	15	14	3.2751	(0.1848, 0.1841)	(416.8871, 0.1332)	14.1965	3.5914
	2	15	14	4.2751	(0.1848, 0.1841)	(421.8871, 0.1332)	14.1965	4.5914
	5	15	14	7.2751	(0.1848, 0.1841)	(436.8871, 0.1332)	14.1965	7.5914
	8	15	14	10.0551	(0.1188, 0.3016)	(449.9961, 0.1337)	14.1280	10.4873
	10	15	13	12.1451	(0.2851, 0.3192)	(448.6132, 0.1375)	13.6163	12.2575
	12	15	13	13.6901	(0.1151, 0.4403)	(449.9547, 0.1391)	13.2530	13.9788
	15	17	12	16.1501	(0.1062, 0.5684)	(449.8705, 0.1397)	12.5706	16.2919

Table 4: Impact on the criteria (overall cost and efficiency) due to mis-specification of the model parameters.

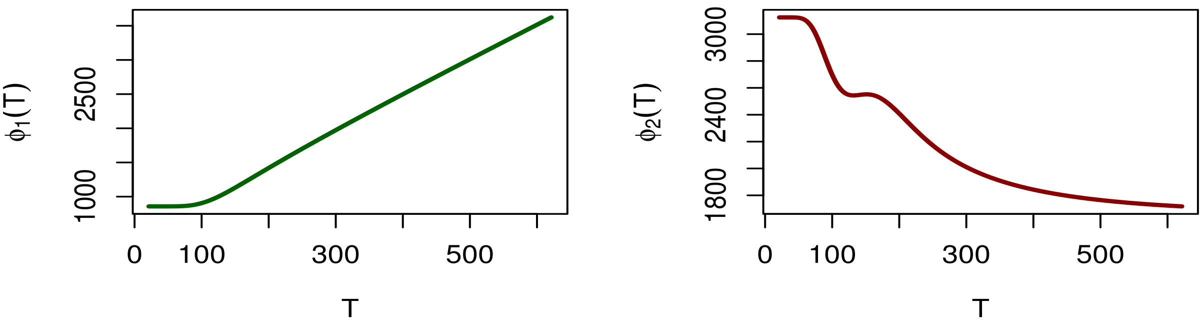
True vs Mis-specified		Designs and Criteria Values $\mathcal{D}_0^{\text{MS}}$, $(\phi_1(\mathcal{D}_0^{\text{MS}}), \phi_2(\mathcal{D}_0^{\text{MS}}))$ $\mathcal{D}_0^{\text{True}}$, $(\phi_1(\mathcal{D}_0^{\text{True}}), \phi_2(\mathcal{D}_0^{\text{True}}))$	Remarks under Mis-specification
$\lambda = 1$ vs $\lambda = 1.5$	Mis-specified:	$(7, 6, 0.9817), (235.93, 0.2901)$	Efficiency is significantly compromised due to less failures
	True:	$(16, 15, 2.2751), (411.89, 0.1332)$	
$\lambda = 1$ vs $\lambda = 0.95$	Mis-specified:	$(16, 15, 2.4806), (434.62, 0.1247)$	Higher cost with minimal gain in estimation precision.
	True:	$(16, 15, 2.2751), (411.89, 0.1332)$	
$\mu = 5$ vs $\mu = 10$	Mis-specified:	$(15, 13, 12.1451), (475.49, 0.1297)$	T_0 exceeds testing limits ($T_{\max} = 9.6052$), resulting in inefficient design beyond budget/time limits.
	True:	$(15, 14, 7.2751), (436.89, 0.1332)$	
$\mu = 5$ vs $\mu = 0$	Mis-specified:	$(15, 14, 2.2751), (-, -)$	$T_0 < \mu(= 2.2751)$ violates model assumptions; design becomes invalid.
	True:	$(15, 14, 7.2751), (436.89, 0.1332)$	

Overall, the proposed optimal designs exhibit moderate robustness to small deviations in model parameter values. However, substantial mis-specification can lead in either invalid or inefficient Type-II HCS experimental designs. This emphasizes the importance of accurately estimating the model parameters prior to design implementation. Therefore, accurate parameter estimation remains essential for ensuring practical, efficient and cost-effective life-testing plans.

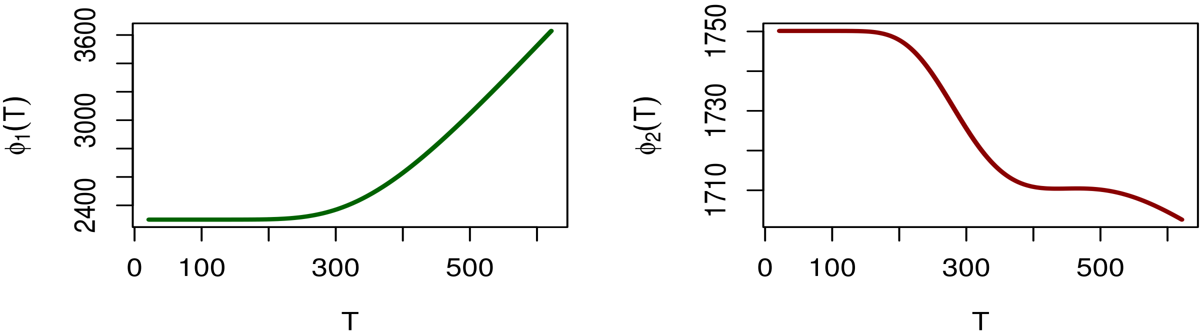
Now, we consider a real data set from Bain [1] to demonstrate the application of the proposed approach. Suppose a sample of 20 items is placed on test, and the test is terminated after 150 hours. Within this period, 13 failures are observed. The recorded failure times are: 3, 19, 23, 26, 37, 38, 41, 45, 58, 84, 90, 109, and 138. We treat this data set as preliminary information to obtain pre-estimates of the model parameters. Assuming that the failure time follows a two-parameter exponential distribution, the maximum likelihood estimates of the parameters are obtained as $\widehat{\mu} = 3$ and $\widehat{\lambda} = 0.0076$. Based on these pre-estimates, we determine the optimal designs under the Type-II HCS framework. The optimal designs for a life-testing experiment with $n = 20$ items are presented in Table 5, considering various constraint settings. For each design, the corresponding expected experimental duration, number of failures, total cost incurred, and the resulting precision are also reported. Although the cost coefficient values are typically chosen by the experimenter, for Table 5, we fix the cost coefficients as $C_0 = 100$, $C_s = 1$, $C_T = 5$, and $C_D = 20$. Table 5 presents the optimal design's behavior under different values of the budget constraint C_b and the variance threshold V_t, with a fixed sample size of $n = 20$. Before proceeding further, we examine the behavior of the criterion functions for various values of n and r. The plots of the criterion functions are presented in the Figure 3. In the top section of Table 5, V_t is fixed at 2500 while C_b is varied. The results show that increasing C_b leads to longer test runs and higher expected costs but generally improves the design's precision, as indicated by the decreasing values of the variance-based criterion $\phi_2(\mathcal{D}_0)$. The lower section of the table explores designs by increasing V_t while keeping $C_b = 1200$ fixed. As V_t increases from 2200 to 3000, the optimal censoring time T_0 fluctuates modestly and designing parameter r_0 decreases to result in fewer number of failures during the experiment, while the raise in the $\phi_2(\mathcal{D}_0)$ values, indicating reduced precision in estimating the the model parameters under the obtained designs. Meanwhile, the expected cost $\phi_1(\mathcal{D}_0)$ decreases with increasing V_t. Overall, this analysis illustrates how variations in constraints through C_b and V_t influence the trade-offs between cost, test duration, and efficiency in the optimal design under the proposed multi-criteria based planning.



(a) $r = 1$



(b) $r = 10$



(c) $r = 19$

Figure 3: Plots for criterion functions ϕ_1 and ϕ_2 based on the real data set, with cost coefficients $C_0 = 100$, $C_s = 1$, $C_T = 5$, and $C_D = 20$.

Table 5: Optimal designs under Type-II HCS for demonstration on real data.

V_t	C_b	n	r	T	$(\phi_1(\mathcal{D}_0)\ ,\ \phi_2(\mathcal{D}_0))$	$E(D)$	$E(\tau)$
2500	1000	20	11	120.1224	(999.7781, 2421.583)	12.3204	126.6740
	1200	20	12	148.5224	(1159.0126, 2278.368)	13.7066	152.9761
	1300	20	10	177.1224	(1299.8014, 2469.271)	14.6838	177.2249
	1325	20	9	181.5224	(1324.7297, 2487.132)	14.8517	181.5390
	1350	20	6	183.3224	(1335.0154, 2499.645)	14.9202	183.3225
	1400	20	7	191.1224	(1379.8640, 2439.268)	15.2126	191.1226
	2000	20	7	191.1224	(1379.8640, 2439.268)	15.2126	191.1226
	2350	20	7	191.1224	(1379.8640, 2439.268)	15.2126	191.1226
	2400	20	19	335.1224	(2399.8286, 1734.288)	19.1882	379.2130
	2500	20	19	365.2224	(2488.0565, 1731.087)	19.2679	396.5399
2200	1200	20	13	150.3224	(1199.849, 2172.493)	14.0869	159.6223
2400		20	12	148.5224	(1159.013, 2278.368)	13.7066	152.9761
2600		20	10	159.8224	(1199.705, 2540.140)	13.9532	160.1283
2700		20	8	160.1224	(1199.548, 2682.088)	13.9424	160.1400
2710		20	7	160.1224	(1199.448, 2708.098)	13.9410	160.1257
2720		20	6	160.1224	(1199.429, 2718.650)	13.9407	160.1229
2740		20	5	160.1224	(1199.426, 2722.223)	13.9407	160.1225
2800		20	1	153.7972	(1161.831, 2799.888)	13.6423	153.7972
2900		20	1	146.4972	(1118.077, 2898.849)	13.2796	146.4972
3000		20	1	139.9972	(1078.771, 2998.543)	12.9392	139.9972

5 Conclusion

We studied how to design a life-testing experiment under the optimal hybrid censoring scheme for a two-parameter exponential distribution under multi-criteria optimization, in which we simultaneously optimize multiple objectives under certain constraints. We examined why multi-criteria optimization and certain constraints must be taken into account while determining the optimal censoring scheme. To achieve the purpose of this research, a cost-based criterion and a variance-based criterion are optimized under the constraints of the overall cost incurred during the life-testing experiment and the efficiency of the estimators used to estimate the unknown parameters of the lifetime distribution. We present a method and an algorithm for such a constrained optimization problem. In addition, for demonstrative reasons, we examined a Type-II HCS life-testing experiment. On the basis of the proposed methodology and algorithm, optimal values for the designing parameters of the Type-II HCS are derived. In order to justify the suggested methodology, a sensitivity analysis with respect to cost coefficients and model parameters is performed by systematically modifying one coefficient/parameter while holding the others constant. Several qualitative observations based on the sensitivity analysis have been reported. In addition, further analysis is performed to assess the robustness of the optimal censoring schemes under potential mis-specification of true model parameters. The findings suggest that the proposed optimal designs demonstrate moderate robustness to minor deviations in parameter values. However, significant mis-specification may result in either invalid, less efficient or costly Type-II HCS experimental designs.

6 Declaration of Interest Statement

The authors did not receive any funding support from any organization for the submitted work. We hereby declare that the information provided here is accurate, and no apparent conflicts of interest that are relevant to the content of this article.

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A Boundedness of n_0 and r_0

Recall from equation (1) and equation (2) that

$$\phi_1(\mathcal{D}) = C_0 + C_s n + C_T E(\tau) + C_D E(D),$$

$$\phi_2(\mathcal{D}) = \int_0^1 \text{Var}(\widehat{X}_p)\, dp.$$

Here, we minimize $\phi_1(\mathcal{D})$ and $\phi_2(\mathcal{D})$ simultaneously, *i.e.*, our aim is to minimize $\max\{\phi_1(\mathcal{D}), \phi_2(\mathcal{D})\}$ subject to the constraints $\phi_1(\mathcal{D}) \le C_b$ and $\phi_2(\mathcal{D}) \le V_t$.

Let us denote the quantities $\max\{\phi_1(\mathcal{D}), \phi_2(\mathcal{D})\}$ and (n_0, r_0, T_0) by $\phi^*(\mathcal{D})$ and $\mathcal{D}_0$, respectively. At the optimal censoring scheme $\mathcal{D}_0$

$$\phi^*(\mathcal{D}_0) = \max\{\phi_1(\mathcal{D}_0), \phi_2(\mathcal{D}_0)\} \ge \phi_1(\mathcal{D}_0) \ge C_0 + C_s n_0 \tag{16}$$

since $C_T > 0$ and $C_D > 0$. Also, the constraints are satisfied at $\mathcal{D}_0$, therefore it implies that

$$\phi^*(\mathcal{D}_0) \le \max\{C_b, V_t\} \tag{17}$$

Equation equation (16) and equation (17) together imply

$$n_0 \le \frac{\max\{C_b, V_t\} - C_0}{C_s} \tag{18}$$

and obviously we must have $1 \le r \le n_0$.