Meta-Analysis of Exponential Lifetime Data from Type-I Hybrid Censored samples

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Abstract

In this study, in order to do a life testing experiment, sampled units are divided into a prefixed number of groups with equal number of units. Units in all the groups are tested simultaneously and independently and, in each group the experiment is terminated as soon as a prefixed time elapses or a prefixed number of failures occurs. We provide the meta-analysis of an exponential lifetime data from Type-I hybrid censored samples. The main goal of this study is to obtain optimal schemes based on some optimality criteria by minimizing certain cost function that is based on a maximum likelihood estimator of mean lifetime. We provide the maximum likelihood estimator of mean lifetime and its probability density function under this set-up. Various optimal schemes have been provided by minimizing expected total cost incurred during the experiment as the raw moments can be obtained explicitly. Numerical results on bias and mean squared error of the maximum likelihood estimator have been reported. We also provide confidence intervals of the unknown parameter. For illustration, meta-analysis for a real data set of three groups is presented.

Key words: Exponential distribution; Type-I hybrid censoring; group sampling; confidence interval; cost function; optimal scheme.

1 Introduction

In reliability experiments, one of the most important issues is to shorten the duration of such reliability tests. Accelerating the life test by over stressing is one way of achieving it, but

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there are other ways as well, which allow us to shorten the duration of reliability test without changing the stress. Sudden death testing (SDT) approach is one such method among them. SDT was originally introduced by Johnson (1964) and then applied by Jun et al. (2006), Motyka (2007), Pascual and Meeker (1998), Rao (2009b), Rao (2009a) and many other researchers. In SDT approach, we divide the sample of n units into g number of groups containing m units in each group *i.e.*, $n = mg$ and each group is tested until the first failure occurs. As against this, a modified sudden death testing (MSDT) approach is proposed by Pascual and Meeker (1998), where each group is tested until a prefixed number of failures, say, r, occur for each group. SDT is special case of MSDT when $r = 1$.

In this study, we go along the lines of SDT and MSDT, but with a different notion. Here also, we have g number of groups containing m units in each and the experiments for life tests are implemented independently over all the groups. These experiments over the groups are terminated as soon as a prefixed number of failures r occurs or a prefixed time T elapses in each one of the groups. Hence we note that these experiments are conducted under the well-known Type-I hybrid censoring, in each group for the termination time of the experiment. It may be noted that it is MSDT if $T \to \infty$, whereas it is equivalent to SDT if we take $r = 1$ and $T \to \infty$. We obtain likelihood inference for the mean lifetime assuming the exponential distribution as the lifetime distribution. In other words, we consider, in detail, the life testing problem of g independent groups of samples and develop the inference for multi samples with Type-I hybrid censoring scheme under this set-up which is meta-analysis of Type-I hybrid censored data. In meta-analysis, we pool the results from two or more independent life testing samples (groups), to draw better inference about the lifetime distribution of the experimental units. Similar kind of work based on the multi-sample problem has been considered by Kateri et al. (2009, 2010) and Górny and Cramer (2020). These authors suggest to use the multi-sample problem and the related inferential results in meta-analysis. Moreover exact confidence interval (CI) for the mean lifetime is obtainable because the explicit form for the probability density function (PDF) of its maximum likelihood estimator (MLE) is known. But, it is not very convenient to find because of the complicated nature of the exact PDF of the MLE. Also, histogram of the MLE, based on the large sample size, shows that the MLE does not follow asymptotic normality results. Therefore, we provide CIs based on two other alternative ways. We have used a well-known Box-Cox transformation to approximate the probability distribution of MLE to the normal distribution in order to obtain the approximate CI. The Box-Cox transformation is helpful because it approximates the probability distribution of a random variable to the desirable normal distribution.

Moreover, since the raw moments are quick to find in their explicit forms using the conditional MGF, optimal sampling scheme is obtainable. In this particular set-up, a scheme is denoted by (g, m, r, T), so we provide optimal scheme (g_0, m_0, r_0, T_0), which is obtained by minimizing a certain cost function. Here, we focus on a cost function, which includes various components given by (1) number of groups; (2) size of the group; (3) number of failed units during the experiment; (4) time spent on the experiment; and (5) efficiency of the estimator used here to estimate the unknown parameter. In this study, this efficiency is measured through the mean squared error. An almost similar type of cost function has been considered by Bhattacharya et al. (2014). It is also observed that the expected total cost corresponding to the optimal scheme, when all the units have been put in a single group, is larger than the expected total cost corresponding to the optimal scheme, when we have more than one group. Therefore, it justifies to divide a sample into g groups instead of putting all the units in a single group.

The rest of the paper is organized as follows. Section 2 explains the model assumptions and provides an MLE of the mean lifetime along with its moment generating function (MGF), PDF and the expressions for raw moments. Section 3 provides CIs for the mean lifetime based on two methods. Different optimal schemes are reported in Section 4. For illustration purpose, data analysis for a real data set of three groups is provided in Section 5. Then, conclusion appears in Section 6. All the proofs have been provided in Appendices A and B.

2 Model Description and Parameter Estimation

In order to do a life testing experiment, the sampled units are divided into a pre-fixed number of groups with equal number of units. Suppose there are g groups having m units in each, so that the sample size $n = mg$. All the groups have been put to test independently for a lifetime experiment. Lifetimes of units are identically and independently distributed and follow exponential distribution with a scale parameter λ. The PDF of the lifetime distribution is given by

$$f_X(x;\lambda) = \lambda \; exp(-\lambda x), \; x > 0, \; \lambda > 0. \tag{1}$$

Let $\mathbf{X}_j = (X_{j1}, X_{j2}, \ldots, X_{jm})$ be a random sample of the lifetimes of m units in the j^{th} group, where $j = 1, 2, \ldots g$, and $X_{j(1)} \leq X_{j(2)} \leq \cdots \leq X_{j(m)}$ denote the order statistics of the above lifetimes. Define $\tau_j = \min(X_{j(r)}, T)$, where $1 \leq r \leq m$, is a prefixed positive integer and T is a prefixed time. We put all the groups on inspection and set the corresponding termination times for the g groups as $\tau_1, \tau_2, \ldots, \tau_g$, respectively. Note that the sampling plan within each of the g groups is subject to Type-I hybrid censoring scheme, so that we have a grouping of Type-I hybrid censored data.

Let D_j denote the number of failures in the j^{th} group, then $D_j \in \{0, 1, \ldots, r\}$. For the j^{th} group, joint PDF of $\mathbf{X}_j$ with $D_j = d_j$ is given by

$$f_{\mathbf{X}_j, D_j}(\mathbf{x}_j, d_j|\lambda) = \begin{cases} H_{j1}, & \text{if } d_j = 0 \\ H_{j2}, & \text{if } 1 \leq d_j \leq r-1 \\ H_{j3}, & \text{if } d_j = r, \end{cases}$$

where $\mathbf{x}_j$ denotes the realization of $\mathbf{X}_j$ with $j = 1, 2, \ldots, g$ and $H_{j1} = e^{-\lambda m T}$,

$$H_{j2} = \frac{m!}{(m-d_j)!} \lambda^{d_j} e^{-\lambda \left[\sum_{i=1}^{d_j} x_{j(i)} + (m-d_j)T \right]}, \text{ and } H_{j3} = \frac{m!}{(m-r)!} \lambda^r e^{-\lambda \left[\sum_{i=1}^{r} x_{j(i)} + (m-r) x_{j(r)} \right]}.$$

Hence, the overall likelihood function of the parameter λ based on the g independent groups, for $0 \leq d_j \leq r$, $j = 1, 2, \ldots, g$, can be written as:

$$L(\lambda) = \prod_{j=1}^{g} {H_{j1}}^{I_{j1}} {H_{j2}}^{I_{j2}} {H_{j3}}^{I_{j3}}, \tag{2}$$

where

$$I_{j1} = \begin{cases} 1, & \text{if } d_j = 0 \\ 0, & \text{if otherwise,} \end{cases} \quad I_{j2} = \begin{cases} 1, & \text{if } 1 \leq d_j \leq r-1 \\ 0, & \text{if otherwise,} \end{cases} \quad \text{and} \quad I_{j3} = \begin{cases} 1, & \text{if } d_j = r \\ 0, & \text{if otherwise.} \end{cases}$$

After simplifications it can be shown that

$$L(\lambda) \propto \lambda^{-\sum_{j=1}^{g}(d_j I_{j2}+rI_{j3})} exp\Big\{-\lambda\sum_{j=1}^{g}\Big(mTI_{j1}+\Big(\sum_{i=1}^{d_j}x_{j(i)}+(m-d_j)T\Big)I_{j2} + \Big(\sum_{i=1}^{r}x_{j(i)}+(m-r)x_{j(r)}\Big)I_{j3}\Big)\Big\},$$

We introduce some further notations as follows:

$$\begin{aligned}
Y_{j,T} &= \sum_{i=1}^{D_j} X_{j(i)} + (m-r)T,\\
Y_{j,r} &= \sum_{i=1}^{r} X_{j(i)} + (m-D_j)X_{j(r)},\\
D &= \sum_{j=1}^{g}(D_j I_{j2} + rI_{j3}).
\end{aligned}$$

Hence, for $0 \le d \le gr$,

$$L(\lambda) \propto \lambda^{-d} exp\Big\{-\lambda\sum_{j=1}^{g}\Big(mTI_{j1}+y_{j,T}I_{j2}+y_{j,r}I_{j3}\Big)\Big\},$$

where $d, y_{j,T}$ and $y_{j,r}$ are the realizations of $D, Y_{j,T}$ and $Y_{j,r}$, respectively. Note that D is total number of failures during the experiment. Denote mean lifetime of the units by θ, *i.e.*, $\theta = 1/\lambda$. When $D = 0$, MLE of θ, say, $\widehat{\theta}_D$ does not exist as the likelihood function is unbounded. But, when $D > 0$, the MLE exists and is given by

$$\widehat{\theta}_D = \frac{1}{D}\sum_{j=1}^{g}(mTI_{j1} + Y_{j,T}I_{j2} + Y_{j,r}I_{j3}).$$

Remark 2.1. *$\widehat{\theta}_D$ can be rewritten as follows:*

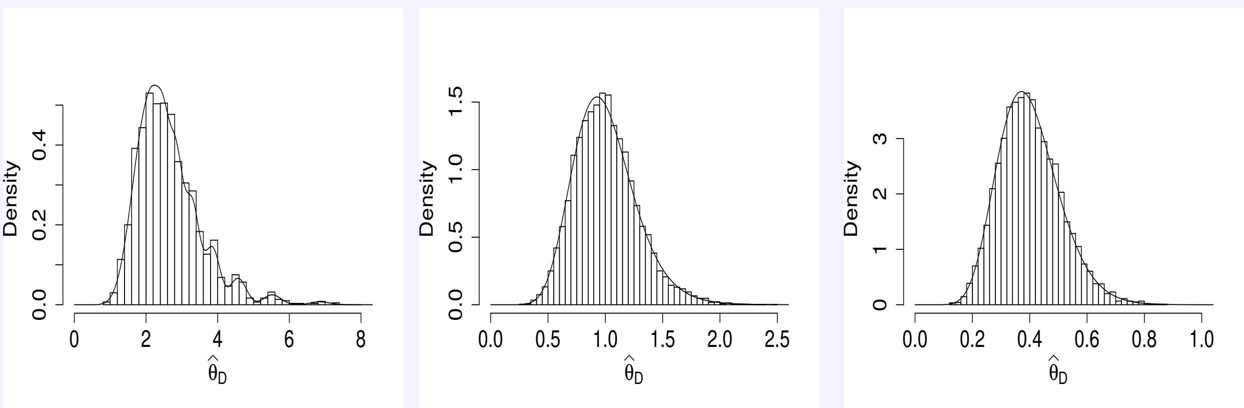
$$\widehat{\theta}_D = \frac{1}{D}\sum_{j=1}^{g}(mTI_{j1} + K_j\widehat{\theta}_{D_j}),$$

where $\widehat{\theta}_{D_j} = \big(Y_{j,T}I_{j2} + Y_{j,r}I_{j3}\big)/K_j$ and $K_j = D_jI_{j2} + rI_{j3}$. Note that $\widehat{\theta}_{D_j}$ is MLE of θ based on Type-I hybrid censored data in j^{th} group when $K_j > 0$. □

Górny and Cramer (2020) developed the inference for the multi samples under Type-I sequential HCS. The first part in Remark 3.4 of Górny and Cramer (2020) explains how the multi sample Type-I sequential HCS model specializes to our multi sample Type-I HCS model. Therefore, the explicit expression for the exact PDF of the MLE $\widehat{\theta}_D$ can be obtained from Theorem 4.3 of Górny and Cramer (2020) with a suitable choice of designing parameter values and it is given by

$$f_{\widehat{\theta}_D|D>0}(s) = (1-q^{mg})^{-1} \sum_{(d_1,d_2,\ldots d_g)\in S_1} \cdots \sum (-1)^d \Big(\prod_{i=1}^{g}\prod_{j=1}^{d_i}(\gamma^*_{i,j}T)\Big)$$
$$\times \big[\underline{\gamma}^*_{1,d_1+1}T\big]_{x_1}\big[\underline{\gamma}^*_{2,d_2+1}T\big]_{x_2}\cdots\big[\underline{\gamma}^*_{g,d_g+1}T\big]_{x_g} g\Big(s-\frac{1}{d}\sum_{i=1}^{g}x_i; d, \lambda d\Big)e^{-\lambda\sum_{i=1}^{g}x_i}.$$

Where $q=e^{-\lambda T}$, $S_1=\{(d_1,d_2,\ldots,d_g)\mid 0\le d_j\le r, \sum_{j=1}^{g}d_j=d, d>0\}$, $\gamma^*_{i,j}=m-j+1$ for $1\le j\le r$ and $\gamma^*_{i,j}=0$ for $j=r+1$, $\underline{\gamma}^*_{i,d_i+1}=(\gamma^*_{i,d_i+1}T,\gamma^*_{i,d_i}T,\ldots,\gamma^*_{i,d_i+1-j}T,\ldots,\gamma^*_{i,1}T)\in\mathbb{R}^{d_i+1}$ for any $1\le i\le g$ and $[t_0,t_1,\ldots,t_k]_x u(x)=\sum_{j=0}^{k}\frac{u(t_j)}{\sum_{i=0,i\ne j}^{k}(t_j-t_i)}$. Here notation $[t_0,t_1,\ldots,t_k]_x u(x)$ is used to denote the divided difference of an arbitrary function $u(x)$ with respect to the knots $t_0,t_1,\ldots,t_k$ for some $k\in\mathbb{N}$. Please check Section 2 and Remark 4.4 of Górny and Cramer (2020) for more information. Figure 1 shows the plots of the exact PDF along with the histogram of the MLE $\widehat{\theta}_D$ for various values of designing parameters.



(a) $\lambda=0.4$ **(b)** $\lambda=1$ **(c)** $\lambda=2.5$

Figure 1: Exact PDF plots along with the histogram of $\widehat{\theta}_D$ for different values of parameter λ with $g=2,\ m=10,\ r=7,\ T=2$.

In order to find $\text{MSE}(\widehat{\theta}_D)$ conditioned on $D>0$ that is used later in Section 4 for determining the optimal censoring scheme, the first two raw moments are to be obtained. Since the exact PDF of the MLE $\widehat{\theta}_D$ is complicated in nature, therefore obtaining the moments using the MGF seemed convenient. The alternative representation given in Remark 2.1 has been used in deriving conditional moment generating function of the MLE $\widehat{\theta}_D$. In the following theorem, the conditional MGF of $\widehat{\theta}_D$, defined as

$$M_{\widehat{\theta}_D|D>0}(s) = E(e^{s\widehat{\theta}_D}|D>0),\ s \in \mathbb{R}$$

has been provided below.

Theorem 2.1. *The conditional MGF of $\widehat{\theta}_D$ given $D>0$ can be expressed as follows:*

$$M_{\widehat{\theta}_D|D>0}(s) = \sum_{(d_1,d_2,\ldots d_g)\in S_1}\cdots\sum c(d_1,d_2,\ldots,d_g) M_{Z(d_1,d_2,\ldots,d_g)|(D_j=d_j,j=1,\ldots,g)}(s), \tag{3}$$

where

$$\begin{aligned} c(d_1,d_2,\ldots,d_g) &= \frac{\boldsymbol{P}(D_1=d_1,D_2=d_2,\ldots,D_g=d_g)}{\boldsymbol{P}(D>0)}, \\ Z(d_1,d_2,\ldots,d_g) &= \sum_{j=1}^{g} Z_{D_j}, \end{aligned}$$

and the random variable $Z_{D_j} = \frac{mTI_{j1}+K_j\widehat{\theta}_{D_j}}{d}$ for a given $D_j=d_j$ has MGF of the form

$$M_{Z_{D_j}|D_j=d_j}(s) = \begin{cases} e^{smT/d}, & \textit{if } d_j = 0 \\ \frac{1}{P(D_j=d_j)}\sum_{k=0}^{d_j}(-1)^k\binom{m}{d_j}\binom{d_j}{k}q^{m-d_j+k}\frac{e^{sT_{d_j,k}}}{(1-s/\lambda d)^{d_j}}, & \textit{if } 1\le d_j\le r-1 \\ \frac{1}{P(D_j=r)}\sum_{k=0}^{r-1}(-1)^k r\binom{m}{r}\binom{r-1}{k} & \\ \times\frac{1}{m-r+k+1}\left\{\frac{1}{(1-s/\lambda d)^r} - q^{m-r+k+1}\frac{e^{sT_k}}{(1-s/\lambda d)^r}\right\}, & \textit{if } d_j = r, \end{cases}$$

with $q=exp(-\lambda T)$, $T_{d_j,k}=(m-d_j+k)T/d$ and $T_k=(m-r+k+1)T/d$.

Proof. See Appendix A. □

Remark 2.2. *It can be observed that*

$$\sum_{(d_1,d_2,\ldots d_g)\in S_1}\cdots\sum c(d_1,d_2,\ldots,d_g) = 1 \qquad \square$$

The following theorem provides the moments of the MLE.

Theorem 2.2. *The k^{th} raw moment of the probability distribution of $\widehat{\theta}_D$ conditioned on $D > 0$, is given by*

$$E(\widehat{\theta}_D^k | D > 0) = \sum_{(d_1,d_2,\dots d_g)\in S_1}\cdots\sum c(d_1, d_2, \dots, d_g) E(Z^k(d_1, d_2, \dots, d_g) \mid D_1 = d_1, \dots, D_g = d_g).$$

Proof. Since the right hand side of equation (3) can be simplified as

$$M_{\widehat{\theta}_D|D>0}(s) = \sum_{k=0}^{\infty} \frac{s^k}{k!} E(\widehat{\theta}_D^k),$$

and similarly, left hand side of equation (3) can be expressed as

$$\begin{aligned}
&\sum_{(d_1,d_2,\dots d_g)\in S_1}\cdots\sum c(d_1, d_2, \dots, d_g) M_{Z(d_1,d_2,\dots,d_g)|(D_j=d_j, j=1,\dots,g)}(s) \\
&= \sum_{k=0}^{\infty} \frac{s^k}{k!} \Big\{ \sum_{(d_1,d_2,\dots d_g)\in S_1}\cdots\sum c(d_1, d_2, \dots, d_g) E(Z^k(d_1, d_2, \dots, d_g) \mid D_1 = d_1, \dots, D_g = d_g) \Big\},
\end{aligned}$$

therefore we must have the required result. $\square$

In order to find MSE($\widehat{\theta}_D$) conditioned on $D > 0$, the first two raw moments can be obtained using Theorem 2.2 and these are provided by

$$E(\widehat{\theta}_D | D > 0) = \sum_{(d_1,d_2,\dots d_g)\in S_1}\cdots\sum c(d_1, d_2, \dots, d_g) \Big\{ \sum_{j=1}^{g} E(Z_{D_j} | D_j = d_j) \Big\}$$

and

$$\begin{aligned}
E(\widehat{\theta}_D^2 | D > 0) = \sum_{(d_1,d_2,\dots d_g)\in S_1}\cdots\sum c(d_1, d_2, \dots, d_g) &\Bigg\{ \sum_{j=1}^{g} E(Z_{d_j}^2 | D_j = d_j) \\
&+ \sum_{\substack{j=1 \\ i\neq j}}^{g} \sum_{j=1}^{g} E(Z_{D_i} | D_i = d_i) E(Z_{D_j} | D_j = d_j) \Bigg\},
\end{aligned}$$

where $E(Z_{d_j} | D_j = d_j)$ and $E(Z_{d_j}^2 | D_j = d_j)$ have been obtained by using MGF of Z_{D_j} given

$D_j = d_j$ and the expressions are

$$E(Z_{d_j}|D_j = d_j) = \begin{cases} mT/d, & \text{if } d_j = 0 \\ \frac{1}{\mathrm{P}(D_j=d_j)} \sum_{k=0}^{d_j} (-1)^k \binom{m}{d_j}\binom{d_j}{k} q^{m-d_j+k}\left(\frac{d_j}{d}\theta + T_{d_j,k}\right), & \text{if } 1 \leq d_j \leq r-1 \\ \frac{1}{\mathrm{P}(D_j=r)} \sum_{k=0}^{r-1} (-1)^k r \binom{m}{r}\binom{r-1}{k} \frac{1}{m-r+k+1} & \\ \quad \times \left(\frac{r}{d}(1-q^{m-r+k+1})\theta - T_k q^{m-r+k+1}\right), & \text{if } d_j = r \end{cases}$$

and

$$\begin{aligned} & E(Z_{d_j}^2|D_j = d_j) \\ &= \begin{cases} (mT/d)^2, & \text{if } d_j = 0 \\ \frac{1}{\mathrm{P}(D_j=d_j)} \sum_{k=0}^{d_j} (-1)^k \binom{m}{d_j}\binom{d_j}{k} q^{m-d_j+k}\left(\frac{d_j(d_j+1)}{d^2}\theta^2 + \frac{2d_jT_{d_j,k}}{d}\theta + T_{d_j,k}^2\right), & \text{if } 1 \leq d_j \leq r-1 \\ \frac{1}{\mathrm{P}(D_j=r)} \sum_{k=0}^{r-1} (-1)^k r \binom{m}{r}\binom{r-1}{k} \frac{1}{m-r+k+1} & \\ \quad \times \left(\frac{r(r+1)}{d^2}(1-q^{m-r+k+1})\theta^2 - \frac{2rT_k}{d} q^{m-r+k+1}\theta - T_k^2 q^{m-r+k+1}\right), & \text{if } d_j = r. \end{cases} \end{aligned}$$

For the estimation purpose of the mean lifetime θ in meta-analysis, we give the CIs in Section 3.

3 Confidence Intervals for θ

Although, exact CI can be obtained using the exact PDF, but we present two other alternative convenient methods in this section. We provide the CI using the parametric bootstrap method in Section 3.1. Moreover, it may be seen graphically by plotting the histogram of the MLE for large sample size that the MLE does not satisfy the asymptotic normality results. Therefore, we use Box-Cox transformation to approximate the its probability distribution to the normal distribution. Hence, we also provide the approximate CI based on the Box-Cox transformation, in Section 3.2.

Table 1: Bootstrap CIs for θ when $T = 1$.

λ	g	m	r	Bias	MSE	95% AL	95% CP	99% AL	99% CP
0.5	3	4	3	0.4701	3.1314	6.7334	0.9475	8.9669	0.9734
	7			0.1625	0.6491	3.6332	0.9422	5.9882	0.9848
	7	4	3	0.1625	0.6491	3.6332	0.9422	5.9882	0.9848
		10		0.0302	0.2333	1.9618	0.9357	2.7052	0.9801
	7	4	3	0.1625	0.6491	3.6332	0.9422	5.9882	0.9848
			4	0.1667	0.6449	3.5753	0.9418	5.8807	0.9858
1	3	4	3	0.0842	0.2599	2.3931	0.9184	3.8856	0.9725
	7			0.0307	0.0739	1.1316	0.9418	1.6093	0.9858
	7	4	3	0.0307	0.0739	1.1316	0.9418	1.6093	0.9858
		10		0.0009	0.0484	0.8652	0.9368	1.1514	0.9812
	7	4	3	0.0307	0.0739	1.1316	0.9418	1.6093	0.9858
			4	0.0357	0.0706	1.1019	0.9461	1.5688	0.9867
2	3	4	3	0.0088	0.0336	0.7538	0.9180	1.0842	0.9658
	7			0.0034	0.0130	0.4565	0.9368	0.6142	0.9817
	7	4	3	0.0034	0.0130	0.4565	0.9368	0.6142	0.9817
		10		5.2707e-07	1.1905e-02	0.4262	0.9401	0.5615	0.9800
	7	4	3	0.0034	0.0130	0.4565	0.9368	0.6142	0.9817
			4	0.0067	0.0112	0.4244	0.9421	0.5765	0.9854

3.1 Bootstrap Confidence Interval

For this exercise, we fix $\lambda = 0.5, 1, 2$ and $T = 1, 2, 3$, and change the values of g, m and r to see the behavior of the average length (AL) of the $100(1-\alpha)\%$ CI and its corresponding coverage probability (CP). Here, we obtain 95% and 99% CIs. Results have been reported in Tables 1–3. The CP of the $100(1-\alpha)\%$ CI becomes closer to $(1-\alpha)$ and AL decreases, if any one among number of groups, number of units in each group, or the censoring number r, increases. We also provide the bias and MSE of the estimator $\widehat{\theta}_D$. As expected, the MSE decreases as any of the three numbers g, m or r increases (see Tables 1–3), which is due to the increase in number of observations in the experiment. It can also be seen that the results become better in terms of the bias, MSE and the AL of the CIs as T increases.

3.2 Confidence Interval based on Box-Cox Transformation

Box-Cox transformation is widely used to approximate the probability distribution of a random variable to the well known and desired normal distribution. It has been introduced by Box and Cox (1964), and is a power transformation of a random variable. Here, we apply the Box-Cox

Table 2: Bootstrap CIs for θ when $T = 2$.

λ	g	m	r	Bias	MSE	95% AL	95% CP	99% AL	99% CP
0.5	3	4	3	0.1684	1.0398	4.6340	0.9257	7.7283	0.9764
	7			0.0615	0.2959	2.2610	0.9392	3.2133	0.9852
	7	4	3	0.0615	0.2959	2.2610	0.9392	3.2133	0.9852
		10		0.0018	0.1936	1.7248	0.9412	2.2944	0.9811
	7	4	3	0.0615	0.2959	2.2610	0.9392	3.2133	0.9852
			4	0.0714	0.2824	2.2144	0.9450	3.1515	0.9871
1	3	4	3	0.0177	0.1346	1.5217	0.9178	2.1841	0.9700
	7			0.0069	0.0523	0.9144	0.9381	1.2202	0.9811
	7	4	3	0.0615	0.2959	0.9144	0.9381	1.2202	0.9811
		10		1.0541e-06	4.7620e-02	0.8585	0.9410	1.1323	0.9841
	7	4	3	0.0615	0.2959	0.9144	0.9381	1.2202	0.9811
			4	0.0135	0.0450	0.8520	0.9440	1.1598	0.9872
2	3	4	3	0.0003	0.0281	0.6540	0.9175	0.8882	0.9658
	7			0.0001	0.0119	0.4281	0.9344	0.5644	0.9794
	7	4	3	0.0001	0.0119	0.4281	0.9344	0.5644	0.9794
		10		1.4410e-13	1.1904e-02	0.4266	0.9362	0.5609	0.9804
	7	4	3	0.0001	0.0119	0.4281	0.9344	0.5644	0.9794
			4	0.0014	0.0093	0.3797	0.9427	0.5051	0.9838

Table 3: Bootstrap CIs for θ when $T = 3$.

λ	g	m	r	Bias	MSE	95% AL	95% CP	99% AL	99% CP
0.5	3	4	3	0.0755	0.6489	3.4431	0.9224	5.2546	0.9731
	7			0.0290	0.2316	1.9452	0.9352	2.6678	0.9815
	7	4	3	0.0290	0.2316	1.9452	0.9352	2.6678	0.9815
		10		7.2241e-05	1.9064e-01	1.7092	0.9370	2.2580	0.9790
	7	4	3	0.0290	0.2316	1.9452	0.9352	2.6678	0.9815
			4	0.0420	0.2093	1.8514	0.9471	2.5559	0.9875
1	3	4	3	0.0036	0.1169	1.3529	0.9138	1.8616	0.9660
	7			0.0014	0.0487	0.8707	0.9445	1.1587	0.9814
	7	4	3	0.0014	0.0487	0.8707	0.9445	1.1587	0.9814
		10		6.2248e-10	4.7619e-02	0.8536	0.9361	1.1229	0.9794
	7	4	3	0.0014	0.0487	0.8707	0.9445	1.1587	0.9814
			4	0.0061	0.0393	0.7858	0.9421	1.0484	0.9834
2	3	4	3	9.2002e-06	2.7791e-02	0.6506	0.9194	0.8625	0.9678
	7			3.6784e-06	1.1907e-02	0.4260	0.9361	0.5621	0.9792
	7	4	3	3.6784e-06	1.1907e-02	0.4260	0.9361	0.5621	0.9792
		10		9.9920e-16	1.1904e-02	0.4280	0.9384	0.5644	0.9792
	7	4	3	3.6784e-06	1.1907e-02	0.4260	0.9361	0.5621	0.9792
			4	0.0002	0.0090	0.3707	0.9415	0.4901	0.9822

transformation on the estimator $\widehat{\theta}_D$. Consider a family of transformations from $\widehat{\theta}_D$ to $\widehat{\theta}_D^{(\upsilon)}$, where

$$\widehat{\theta}_D^{(\upsilon)} = \begin{cases} \frac{\widehat{\theta}_D^{\upsilon}-1}{\upsilon}, & \text{if } \upsilon \neq 0 \\ \ln(\widehat{\theta}_D), & \text{if } \upsilon = 0, \end{cases} \tag{4}$$

such that a fixed value of the transformation parameter υ leads us to a particular transformation. Among this family of transformations, we try to find a transformation $\widehat{\theta}_D^{(\upsilon_0)}$, which approximately follows a normal distribution. By Taylor series expansion, ignoring higher order terms the mean and variance are as follows:

$$E(\widehat{\theta}_D^{(\upsilon_0)}) = \begin{cases} \frac{(E(\widehat{\theta}_D))^{\upsilon_0}-1}{\upsilon_0}, & \text{if } \upsilon_0 \neq 0 \\ \ln(E(\widehat{\theta}_D)), & \text{if } \upsilon_0 = 0, \end{cases}$$

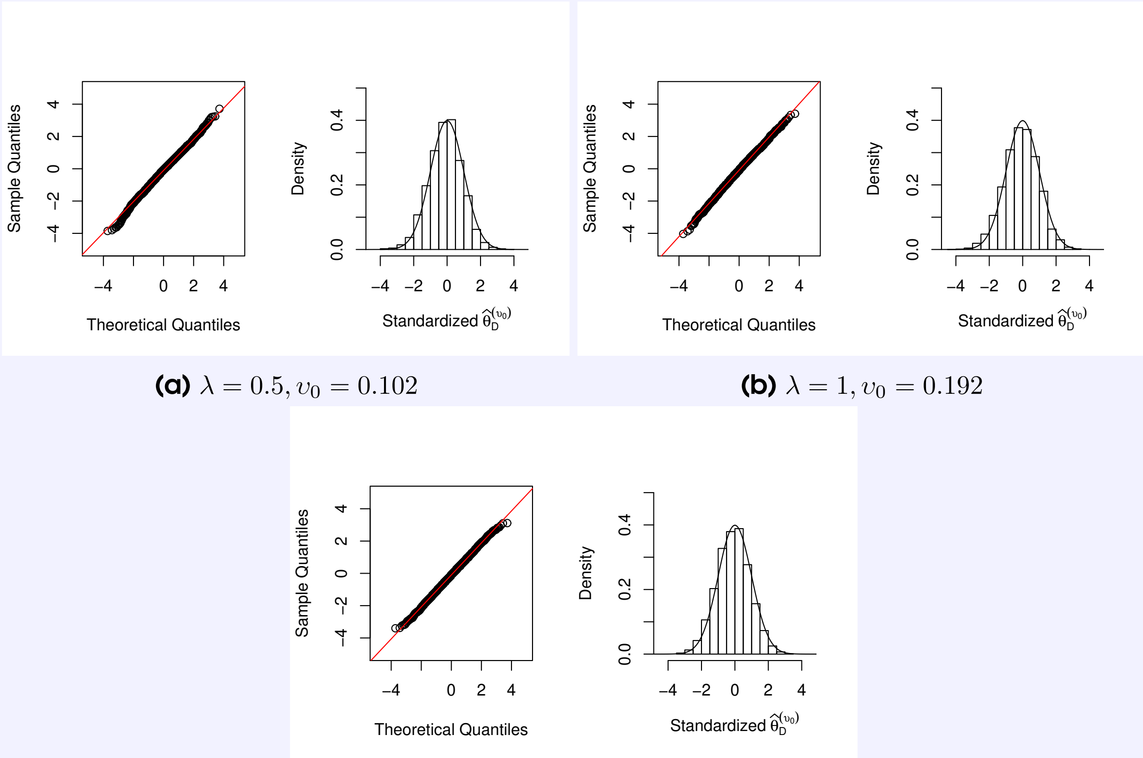
$$Var(\widehat{\theta}_D^{(\upsilon_0)}) = \begin{cases} (E(\widehat{\theta}_D))^{2(\upsilon_0-1)} Var(\widehat{\theta}_D), & \text{if } \upsilon_0 \neq 0 \\ \frac{Var(\widehat{\theta}_D)}{(E(\widehat{\theta}_D))^2}, & \text{if } \upsilon_0 = 0. \end{cases}$$

Let us denote $E(\widehat{\theta}_D^{(\upsilon_0)})$ and $Var(\widehat{\theta}_D^{(\upsilon_0)})$ by $h_1(\theta)$ and $h_2(\theta)$, respectively. Then, $(\widehat{\theta}_D^{(\upsilon_0)} - h_1(\theta))/\sqrt{h_2(\theta)}$ approximately follows standard normal distribution. $(\widehat{\theta}_D^{(\upsilon_0)} - h_1(\theta))/\sqrt{h_2(\widehat{\theta}_D)}$ also asymptotically follows the standard normal distribution as $h_2(\widehat{\theta}_D)/h_2(\theta) \to^P 1$ since $h_2(.)$ is a continuous function and $\widehat{\theta}_D$ is MLE of θ. Therefore, the $100(1-\alpha)\%$ CI for θ, denoted by $\left(\theta_L^{BC}, \theta_U^{BC}\right)$, is given by

$$(\theta_L^{BC}, \theta_U^{BC}) = \left(\left\{ 1 + \upsilon_0 \left(\widehat{\theta}_D^{(\upsilon_0)} - z_{1-\alpha/2}\sqrt{h_2(\widehat{\theta}_D)} \right) \right\}^{1/\upsilon_0}, \left\{ 1 + \upsilon_0 \left(\widehat{\theta}_D^{(\upsilon_0)} + z_{1-\alpha/2}\sqrt{h_2(\widehat{\theta}_D)} \right) \right\}^{1/\upsilon_0} \right) \tag{5}$$

It is possible to obtain equation (5) because we have $E(\widehat{\theta}_D) \approx \theta$, as the sample size becomes large.

We have used 12000 simulations in order to get AL and CP of the CIs obtained in Tables 5 – 6. Since υ_0 depends on λ, g, m, r and T, therefore the corresponding values of υ_0 also have been reported. We use the well known Kolmogorov Smirnov test (KS-test) for testing the normality



(a) $\lambda = 0.5, \upsilon_0 = 0.102$

(b) $\lambda = 1, \upsilon_0 = 0.192$

(c) $\lambda = 2, \upsilon_0 = 0.298$

Figure 2: Q-Q plots and Histograms along with the standard normal PDF plots varying λ when $g = 5, m = 8, r = 3, T = 2$

here. We provide some Q-Q plots and histograms of $(\widehat{\theta}_D^{(\upsilon_0)} - h_1(\theta))/\sqrt{h_2(\theta)}$ along with standard normal PDF for the verification of normality of $\widehat{\theta}_D^{(\upsilon_0)}$ in Figure 2. The corresponding KS-test statistic and P-value of Figure 2 for different values of λ have been reported in Table 4. The results will become more and more accurate as the sample size increases, as our approximation of CIs is for large sample sizes. The results reported in Tables 5 $-$ 6 become better in the sense of the AL, if g, m or r increases. AL becomes smaller if truncation time T is increased when λ is small and opposite behavior can been observed for larger values of λ.

In the next section, all inferential results about θ derived in Section 2 are used to obtain the

Table 4: KS-statistic and P-value corresponding to υ_0 for $\lambda = 0.5, 1, 2$.

λ	υ_0	KS-statistic	P-value
0.5	0.102	0.2780	0.9753
1	0.192	0.0216	0.9990
2	0.298	0.0299	0.9940

Table 5: CIs for θ based on Box-Cox transformation when $T = 2$.

λ	g	m	r	v_0	95% AL	95% CP	99% AL	99% CP
0.5	3	8	3	0.382	2.8307	0.9540	3.7516	0.9895
	5			0.102	2.1621	0.9525	2.9076	0.9902
	5	4	3	0.126	2.7623	0.9725	3.7432	0.9928
		8		0.102	2.1621	0.9525	2.9076	0.9902
	3	8	3	0.382	2.8307	0.9540	3.7516	0.9895
			7	-0.082	2.4155	0.9633	3.3203	0.9891
1	3	8	3	0.476	1.3064	0.9212	1.7192	0.9627
	5			0.192	1.0358	0.9413	1.3819	0.9806
	5	4	3	0.224	1.1001	0.9539	1.4668	0.9911
		8		0.192	1.0358	0.9413	1.3819	0.9806
	3	8	3	0.476	1.3064	0.9212	1.7192	0.9627
			7	0.252	0.9369	0.9526	1.2428	0.9894
2	3	8	3	0.526	0.6525	0.9244	0.8564	0.9651
	5			0.298	0.5134	0.9402	0.6806	0.9820
	5	4	3	0.132	0.5209	0.9434	0.6981	0.9833
		8		0.298	0.5134	0.9402	0.6806	0.9820
	3	8	3	0.526	0.6525	0.9244	0.8564	0.9651
			7	0.338	0.4340	0.9421	0.5731	0.9825

Table 6: CIs for θ based on Box-Cox transformation when $T = 3$.

λ	g	m	r	v_0	95% AL	95% CP	99% AL	99% CP
0.5	3	8	3	0.102	2.7860	0.9405	3.7955	0.9819
	5			0.508	1.3370	0.9323	1.7840	0.9730
	5	4	3	0.354	2.3225	0.9606	3.0744	0.9927
		8		0.508	1.3370	0.9323	1.7840	0.9730
	3	8	3	0.102	2.7860	0.9405	3.7955	0.9819
			7	-0.032	2.0645	0.9577	2.7973	0.9915
1	3	8	3	0.288	1.3370	0.9323	1.7840	0.9730
	5			0.248	1.0292	0.9393	1.3682	0.9798
	5	4	3	0.162	1.0492	0.9428	1.4034	0.9863
		8		0.248	1.0292	0.9393	1.3682	0.9798
	3	8	3	0.288	1.3370	0.9323	1.7840	0.9730
			7	0.418	0.8787	0.9430	1.1574	0.9830
2	3	8	3	0.360	0.6619	0.9210	0.8778	0.9690
	5			0.472	0.5052	0.9325	0.6645	0.9751
	5	4	3	0.400	0.5086	0.9354	0.6710	0.9752
		8		0.472	0.5052	0.9325	0.6645	0.9751
	3	8	3	0.360	0.6619	0.9210	0.8778	0.9690
			7	0.440	0.4295	0.9390	0.5654	0.9791

optimal schemes.

4 Optimal Scheme of (g, m, r, T)

In this section, we provide the optimal sampling scheme and therefore, we consider the following expected total cost incurred during the experiment denoted by TC:

$$TC = gC_g + mC_m + E(D)C_D + E(\tau)C_T + C_e \text{MSE}(\widehat{\theta}_D), \tag{6}$$

where

$$\begin{aligned}
C_g &= \text{cost per group}, \\
C_m &= \text{cost per unit in each group}, \\
C_D &= \text{cost on per unit of the expected number of total failed units}, \\
C_T &= \text{cost on per unit time of the total experimental time }, \\
C_e &= \text{ cost on per unit of efficiency of the estimator } \widehat{\theta}_D.
\end{aligned}$$

Here

$$\begin{aligned}
E(D) =& g\sum_{d=1}^{n}\sum_{d=0}^{d}(-1)^{(d-k)}\min(r,d)\binom{n}{d}\binom{d}{k}e^{-\lambda(n-k)T} \text{and} \\
E(\tau) =& \int_0^\infty \left[1 - \left\{I_{\{t>T\}} + I_{\{t<T\}}\left(\sum_{k=0}^{r-1}(-1)^{r-k-1}r\binom{m}{r}\binom{r-1}{k}\right.\right.\right. \\
&\left.\left.\left. \times\left(I_{\{m-k=0\}}\lambda t + I_{\{m-k\neq 0\}}\frac{1}{m-k}F_X(t;\lambda(m-k))\right)\right)\right\}^g\right] dt,
\end{aligned}$$

See Appendix B for derivations of $E(D)$ and $E(\tau)$.

Recall that a scheme is denoted by (g, m, r, T). So, an optimal scheme (g_0, m_0, r_0, T_0) can be obtained by minimizing the above expected total cost over all the schemes (g, m, r, T) such that

$$TC(g_0, m_0, r_0, T_0) = \min_{(g,m,r,T)} TC(g, m, r, T).$$

Here, expected total cost corresponding to the optimal scheme is minimum among all the schemes. Let us denote it by TC_0, *i.e.*, $TC_0 = TC(g_0, m_0, r_0, T_0)$. Note that the MLE $\widehat{\theta}_D$ is not an unbiased estimator of θ and therefore the MSE($\widehat{\theta}_D$) has been used in order to take care of efficiency in the expected total cost while giving the optimal schemes. It is natural to see the behavior of the optimal schemes and the corresponding expected total cost with respect to the costs C_g, C_m, C_D, C_T and C_e, and see how sensitive they are with respect to these costs. In order to do so, only one cost has been changed, among all costs, at a time, keeping others fixed. Some optimal schemes have been provided in Tables 7 − 8.

Table 7 provides the pattern of the optimal scheme with respect to the costs considered in equation (6), It can be observed that as the cost per group and cost per unit in each group increase, the optimal number of groups and the optimal number of units in each group decrease, respectively, which is obvious to happen. Also, the optimal value of time T_0 decreases as cost on per unit of the experimental time increases. In order to achieve high efficiency, it is necessary to have large number of groups and larger experimental time, which can be noted when the cost on efficiency C_e is large. Whenever any one of these costs increases, it is quite reasonable that the expected total cost corresponding to its optimal scheme will also increase. It is observed that g, m, r and T will decrease in order to have less number of failures in the experiment, if the cost on the expected number of failures C_D is increased.

In Table 8, we provide optimal schemes for various values of parameter λ. We see that the expected total cost corresponding the optimal scheme tends to increase as the mean lifetime θ increases or rate λ decreases. For some choices of the cost coefficients, it has also been observed that the optimal scheme, when all the units have been put in a single group, has larger expected total cost than the total expected cost corresponding to the optimal scheme when we have more than one group in this set up.

5 Data Analysis

In this section, we illustrate the proposed estimator through an example of a real data set previously analyzed in Birnbaum and Saunders (1969), representing the lifetimes in cycles 10-3 of aluminium 6061-T6 pieces cut in parallel angle with the rotation direction, oscillating at the rate of 18 cycles per second at maximum pressure 31,000 psi, with a total sample size of

Table 7: Optimal schemes (g_0, m_0, r_0, T_0) with their respective minimum total cost incurred for $\lambda = 0.5$.

C_g	C_m	C_D	C_T	C_e	g_0	m_0	r_0	T_0	TC_0
0.5					5	7	4	9.0010	48.4990
0.7					4	8	5	9.6760	49.4551
1					3	9	6	9.7510	50.4443
3	0.5	1	1	100	2	12	9	10.5010	55.2609
5					1	19	16	11.6260	58.9288
	0.2				1	25	18	7.2010	50.6677
	0.3				2	14	9	8.7850	52.7302
3	0.5	1	1	100	2	12	9	10.5100	55.2609
	1.0				2	10	8	13.0600	60.4602
	1.5				3	6	5	19.1500	63.7745
		0.5			2	14	11	10.8760	45.5315
		0.7			2	13	10	10.2760	49.7003
3	0.5	1	1	100	2	12	9	10.5010	55.2609
		1.5			2	10	7	10.5760	63.2411
		1.7			1	8	8	0.2772	65.0805
			0		2	9	9	34.3750	50.7222
			0.5		2	11	9	15.3760	53.5477
3	0.5	1	1	100	2	12	9	10.5010	55.2609
			2		2	13	8	7.5010	57.8122
			2.2		2	14	8	6.9760	58.2356
				10	1	4	4	0.5651	12.6569
				70	1	7	7	0.3191	47.4991
				75	2	10	7	9.9010	49.0982
3	0.5	1	1	100	2	12	9	10.5010	55.2609
				250	2	17	14	10.8760	81.9549
				370	3	15	12	11.4760	97.3898

101 units. The data is presented in Table 9. Suppose the lifetime characteristic in this example is denoted by Z. Then, for illustration, we made a transformation that is given by $(\frac{Z-65}{76.3222})^{3.2571}$. It is observed that it follows exponential distribution with the estimate of mean lifetime θ as 1.000002 with corresponding KS-statistic and P-value as 0.0524 and 0.9382, respectively. In order to proceed for the illustration of the proposed meta-analysis in this paper, three groups are randomly selected from the transformed exponential data that contain 25 observations each. In this case, $g = 3$ and $m = 25$ are already fixed and we obtain the CIs for mean lifetime θ varying r and T such that $r = 16,\ 20,\ 25$ and $T = 1,\ 2,\ 4$. CIs based on both

Table 8: Optimal schemes (g_0, m_0, r_0, T_0) with their respective minimum total cost incurred for various values of λ.

C_g	C_m	C_D	C_T	C_e	λ	g_0	m_0	r_0	T_0	TC_0
					0.3	3	15	10	12.8260	87.9715
					0.4	2	15	11	11.1760	67.5455
3	0.5	1	1	100	0.5	2	12	9	10.5010	55.2609
					0.6	2	9	7	10.8760	47.0488

Table 9: Data analysis: Data set.

70	90	96	97	99	100	103	104	104	105	107	108	108	108	109
109	112	112	113	114	114	114	116	119	120	120	120	121	121	123
124	124	124	124	124	128	128	129	129	130	130	130	131	131	131
131	131	132	132	132	133	134	134	134	134	134	136	136	137	138
138	138	139	139	141	141	142	142	142	142	142	142	144	144	145
146	148	148	149	151	151	152	155	156	157	157	157	157	158	159
162	163	163	164	166	166	168	170	174	196	212				

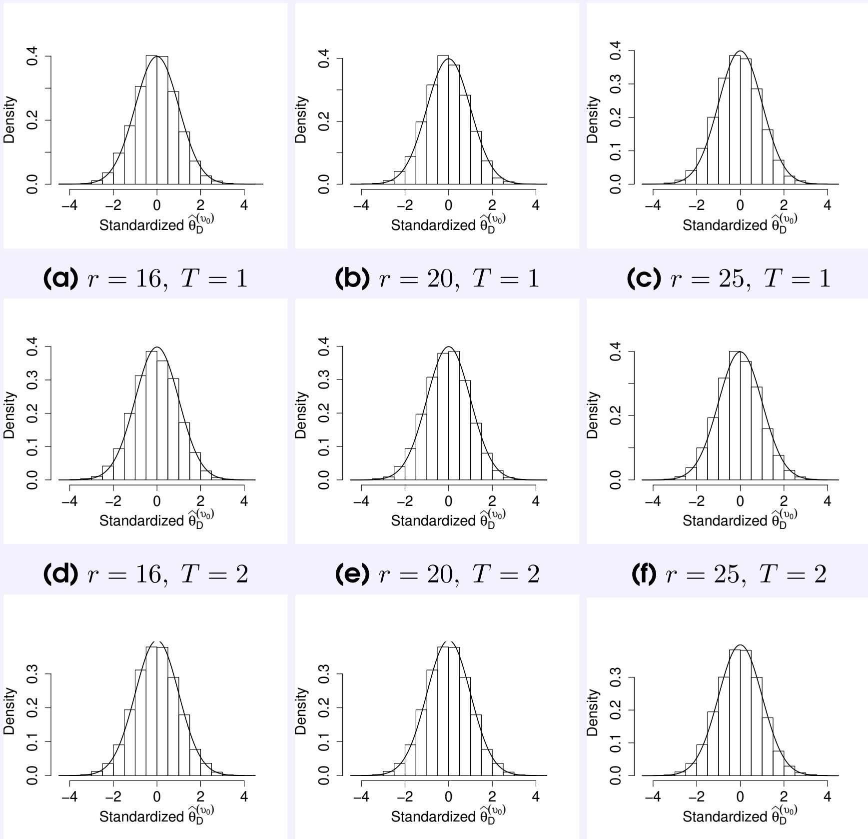
the methods, bootstrap and the other based on the Box-Cox transformation, are presented in Table 10. Since υ_0 depends on r and T, therefore its corresponding values are also reported in Table 10. Further, histograms of the standardized $\widehat{\theta}_D^{(\upsilon_0)}$ along with the standard normal PDF plots corresponding to reported υ_0 for various r and T are presented in Figure 3. It may be noted that the length of the CIs decrease whenever r and T are increased. In Table 11, we

Table 10: Data analysis: CIs for θ

.

Method	r	T	$\widehat{\theta}_D$	υ_0	CI 95%	CI 99%
Bootstrap	16	1	0.9653		(0.7097, 1.2930)	(0.6444, 1.4284)
	20		0.9278		(0.7052, 1.2395)	(0.6477, 1.3626)
	25		0.9278		(0.7068, 1.2395)	(0.6556, 1.3626)
	16	2	0.9930		(0.7307, 1.2910)	(0.6631, 1.3942)
	20		0.9164		(0.7024, 1.1630)	(0.6397, 1.2574)
	25		0.9396		(0.7353, 1.1893)	(0.6766, 1.2907)
	16	4	0.9930		(0.7307, 1.2905)	(0.6631, 1.3954)
	20		0.9164		(0.7024, 1.1580)	(0.6397, 1.2516)
	25		0.9233		(0.7289, 1.1473)	(0.6750, 1.2270)
Box-Cox	16	1	0.9653	-0.025	(0.7042, 1.2830)	(0.6413, 1.4110)
	20		0.9278	-0.415	(0.7062, 1.2521)	(0.6527, 1.3890)
	25		0.9278	-0.300	(0.7042, 1.2440)	(0.6493, 1.3734)
	16	2	0.9930	0.241	(0.7345, 1.2958)	(0.6668, 1.4073)
	20		0.9164	0.130	(0.7012, 1.1692)	(0.6450, 1.2632)
	25		0.9396	0.130	(0.7309, 1.1908)	(0.6750, 1.2822)
	16	4	0.9930	0.451	(0.7284, 1.2882)	(0.6561, 1.3921)
	20		0.9164	0.362	(0.6984, 1.1607)	(0.6389, 1.2473)
	25		0.9233	0.345	(0.7254, 1.1477)	(0.6702, 1.2259)

presented total cost incurred during the experiment for various values of T fixing r to be 20 with $C_g = 2,\ C_t = 0.5,\ C_D = 0.5$ varying the per unit cost on the number of units in each group, *i.e.*, C_m. The MSE is obtained using the estimate value of θ as 1.000002. It is observed that the grouping makes sense over the situation of having just a single group if we have sufficient cost on the number of units in each group as the total cost is smaller for $g = 3$ in such cases.



(a) $r = 16,\ T = 1$ **(b)** $r = 20,\ T = 1$ **(c)** $r = 25,\ T = 1$

(d) $r = 16,\ T = 2$ **(e)** $r = 20,\ T = 2$ **(f)** $r = 25,\ T = 2$

(g) $r = 16,\ T = 4$ **(h)** $r = 20,\ T = 4$ **(i)** $r = 25,\ T = 4$

Figure 3: Data analysis: Histograms of standardized $\widehat{\theta}_D^{(v_0)}$ along with the standard normal PDF plots corresponding to reported v_0 for various values of r and T.

Table 11: Data analysis: Total costs when $r = 20,\ T = 1$ with cost coefficients $C_g = 2,\ C_t = 0.5,\ C_D = 0.5$ varying C_m.

T	g	m	τ	D	$\widehat{\theta}_D$	MSE	C_m	TC
1	1	75	0.3440	21	1.0560	0.0500	1.5	137.6720
	3	25	1.0000	51	0.9278	0.0229		75.2226
							1.0	100.1720
								62.7226
							0.5	62.6720
								50.2226
							0.2	40.1720
								42.7226
2	1	75	0.3440	21	1.0560	0.0500	1.5	137.6720
	3	25	1.7107	60	0.9164	0.0172		78.6474
							1.0	100.1720
								66.1474
							0.5	62.6720
								53.6474
							0.2	40.1720
								46.1474

6 Conclusion

In this paper, we study the inference from a life testing experiment, where the lifetimes are assumed to follow the exponential distribution with a scale parameter. These inferences about the mean lifetime further can be used in the meta-analysis for the estimation purpose where we pool the studies of two or more independent life testing samples (groups) which are censored with Type-I hybrid censoring scheme. The MLE of the mean lifetime along with its MGF and raw moments, have been provided. Explicit expression for the exact PDF of the MLE of mean lifetime is obtained under this set-up. Further, we find the CIs for the mean lifetime based on parametric bootstrap method and Box-Cox transformation and the results are good in terms of bias, mean square error, AL and CP of the CIs. A real data set of three groups is illustrated through a data analysis.

Considering the total cost incurred during the experiment, optimal schemes have been reported varying the parameter and various costs, which have been included in the total cost function. It is also concluded, based on the reported results, that the optimal scheme when all the units have been put in a single group has larger expected total cost, than the total expected cost corresponding to the optimal scheme, when we have more than one group with the considered total cost.

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Appendices

A Proof of Theorem 2.1

$$
\begin{aligned}
&M_{\widehat{\theta}_D|D>0}(s) = E(e^{s\widehat{\theta}_D}|D>0)\\
&= \sum_{d=1}^{gr} E(e^{s\widehat{\theta}_D}|D=d)\mathbf{P}(D=d|D>0)\\
&= \sum_{\substack{d_1=0\\(d_1,d_2,\ldots,d_g)\in S_1}}^{r}\sum_{d_2=0}^{r}\cdots\sum_{d_g=0}^{r} E(e^{s\widehat{\theta}_D}|D_1=d_1,D_2=d_2,\ldots,D_g=d_g)\mathbf{P}(D_1=d_1,D_2=d_2,\ldots,D_g=d_g|D>0)\\
&= \frac{1}{\mathbf{P}(D>0)}\sum\cdots\sum_{(d_1,d_2,\ldots d_g)\in S_1} E(e^{s\widehat{\theta}_D}|D_1=d_1,D_2=d_2,\ldots,D_g=d_g)\\
&\qquad\qquad\qquad\qquad \times\mathbf{P}(D_1=d_1,D_2=d_2,\ldots,D_g=d_g)\\
&= \sum\cdots\sum_{(d_1,d_2,\ldots d_g)\in S_1}\frac{\mathbf{P}(D_1=d_1,D_2=d_2,\ldots,D_g=d_g)}{\mathbf{P}(D>0)}\\
&\qquad\qquad\qquad\qquad \times E(e^{s\left(\frac{1}{d}\sum\limits_{j=1}^{g}(mTI_{j1}+K_j\widehat{\theta}_{D_j})\right)}|D_1=d_1,D_2=d_2,\ldots,D_g=d_g)\\
&= \sum\cdots\sum_{(d_1,d_2,\ldots d_g)\in S_1} c(d_1,d_2,\ldots d_g)E(\prod_{j=1}^{g} e^{s\frac{\left(mTI_{j1}+K_j\widehat{\theta}_{D_j}\right)}{d}}|D_1=d_1,D_2=d_2,\ldots,D_g=d_g).
\end{aligned}
$$

As all the groups have been studied independently, the MGF can be rewritten as

$$
M_{\widehat{\theta}_D|D>0}(s) = \sum\cdots\sum_{(d_1,d_2,\ldots d_g)\in S_1} c(d_1,d_2,\ldots d_g)\Big\{\prod_{j=1}^{g} E(e^{s\frac{\left(mTI_{j1}+K_j\widehat{\theta}_{D_j}\right)}{d}}|D_j=d_j)\Big\}.
$$

Define a random variable $Z_{D_j}=\frac{mTI_{j1}+K_j\widehat{\theta}_{D_j}}{d}$ for given $D_j=d_j$ and denote its MGF $M_{Z_{D_j|D_j=d_j}}(s)$ by E_{d_j}. Hence

$$
\begin{aligned}
M_{\widehat{\theta}_D|D>0}(s) &= \sum\cdots\sum_{(d_1,d_2,\ldots d_g)\in S_1} c(d_1,d_2,\ldots d_g)\Big(\prod_{j=1}^{g} E_{d_j}\Big) \qquad (7)\\
&= \sum\cdots\sum_{(d_1,d_2,\ldots d_g)\in S_1} c(d_1,d_2,\ldots d_g)M_{\sum_{j=1}^{g} Z_{D_j}|(D_1=d_1,D_2=d_2,\ldots,D_g=d_g)}(s),
\end{aligned}
$$

where Z_{D_j}'s are independent and therefore we get (3). Now consider the quantity E_{d_j} for various fixed values of d_j's.

For $d_j = 0$,

$$E_{d_j} = E(e^{\frac{smTI_{j1}}{d}}|D_j = 0) = e^{\frac{smT}{d}}.$$

For $1 \leq d_j \leq r-1$,

$$\begin{aligned}
E_{d_j} &= E(e^{\frac{s\widehat{\theta}_{D_j}K_j}{d}}|D_j = d_j) \\
&= E(e^{\frac{s}{d}\widehat{\theta}_{D_j}K_j}|D_j = d_j) \\
&= \frac{e^{\frac{s}{d}(m-d_j)T}\left(1-e^{-\lambda T\left(1-\frac{s}{\lambda d}\right)}\right)^{d_j}}{\left(1-\frac{s}{\lambda d}\right)^{d_j}\left(1-e^{-\lambda T}\right)^{d_j}} \qquad \text{(See Lam (1988))} \\
&= \frac{\binom{m}{d_j}e^{-\lambda(m-d_j)T\left(1-\frac{s}{\lambda d}\right)}\left(1-e^{-\lambda T\left(1-\frac{s}{\lambda d}\right)}\right)^{d_j}}{\left(1-\frac{s}{\lambda d}\right)^{d_j}\mathrm{P}(D_j = d_j)}.
\end{aligned}$$

For $d_j = r$, it can be obtained from Lin et al. (2008)

$$\begin{aligned}
&E_{d_j} = E(e^{\frac{s\widehat{\theta}_{D_j}K_j}{d}}|D_j = r) \\
=&\frac{1}{\mathrm{P}(D_j = r)}r\binom{m}{r}\frac{1}{\left(1-\frac{sk_j}{\lambda dr}\right)^r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\frac{1}{m-r+k+1}\left(1-e^{-\lambda\left(1-\frac{sk_j}{\lambda dr}\right)(m-r+k+1)T}\right).
\end{aligned}$$

Hence

$$E_{Z_{D_j}} = \begin{cases}
e^{smT/d}, & \text{if } d_j = 0 \\
\frac{1}{\mathrm{P}(D_j=d_j)}\sum_{k=0}^{d_j}(-1)^k\binom{m}{d_j}\binom{d_j}{k}q^{m-d_j+k}\frac{e^{sT_{d_j,k}}}{(1-s/\lambda d)^{d_j}}, & \text{if } 1 \leq d_j \leq r-1 \\
\frac{1}{\mathrm{P}(D_j=r)}\sum_{k=0}^{r-1}(-1)^k r\binom{m}{r}\binom{r-1}{k} & \\
\frac{1}{m-r+k+1}\left\{\frac{1}{(1-s/\lambda d)^r} - q^{m-r+k+1}\frac{e^{sT_k}}{(1-s/\lambda d)^r}\right\}, & \text{if } d_j = r.
\end{cases} \tag{8}$$

B Computation of $E(\tau)$ and $E(D)$

Recall that the total experimental time τ is $\max(\tau_1, \tau_2, \ldots, \tau_g)$, therefore

$$E(\tau) = \int_0^\infty \mathrm{P}(\max(\tau_1, \tau_2, \ldots, \tau_g) > t)\, dt = \int_0^\infty \left\{1 - (\mathrm{P}(\tau_1 < t))^g\right\}\, dt. \tag{9}$$

Consider the quantity $\mathrm{P}(\tau_1 < t)$ separately,

$$\begin{aligned}\mathrm{P}(\tau_1 < t) =& \mathrm{P}(\min(X_{1(r)}, T) < t \mid X_{1(r)} < T) + \mathrm{P}(\min(X_{1(r)}, T) < t \mid X_{1(r)} > T) \\ =& I_{\{t>T\}} + I_{\{t<T\}}\mathrm{P}(X_{1(r)} < t).\end{aligned} \tag{10}$$

Here

$$\begin{aligned}&\mathrm{P}(X_{1(r)} < T) = \int_0^t \frac{m!}{(m-r)!(r-1)!}\lambda e^{-\lambda u} e^{-\lambda(m-r)u}(1-e^{-\lambda u})^{r-1}\, du \\ &= \sum_{k=0}^{r-1}(-1)^{r-k-1} r\binom{m}{r}\binom{r-1}{k}\int_0^\infty \lambda e^{-\lambda(m-k)u}\, du \\ &= \sum_{k=0}^{r-1}(-1)^{r-k-1} r\binom{m}{r}\binom{r-1}{k}\Big(I_{\{m-k=0\}}\lambda t + I_{\{m-k\neq 0\}}\frac{1}{m-k}F_X(t;\lambda(m-k))\Big)\end{aligned} \tag{11}$$

Combining equations (9), (10) and (11) we get the result. Now, since the total number of failures is given by $D = \sum_{j=1}^{g} K_j$, where K_j is number of the failures in j^{th} group and D_j's are identically distributed random variables, hence $E(D) = gE(K_1)$. See Lin et al. (2008) for the derivation of $E(K_1)$.