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A generator of bivariate distributions: Properties, estimation and applications

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Abstract: In 2020, El-Morshedy et al. introduced a bivariate extension of the Burr type X generator (BBX-G) of distributions, and Muhammed presented a bivariate generalized inverted Kumaraswamy (BGIK) distribution. In this paper, we propose a more flexible generator of bivariate distributions based on the maximization process from an arbitrary three-dimensional baseline distribution vector, which is of interest for maintenance and stress models, and expands the BBX-G and BGIK distributions, among others. This proposed generator allows one to generate new bivariate distributions by combining non-identically distributed baseline components. The bivariate distributions belonging to the proposed family have a singular part due to the latent component which makes them suitable for modelling two-dimensional data sets with ties. Several distributional and stochastic properties are studied for such bivariate models, as well as for its marginals, conditional distributions and order statistics. Furthermore, we analyze its copula representation and some related association measures. The EM algorithm is proposed to compute the maximum likelihood estimations of the unknown parameters, which is illustrated by using two particular distributions of this bivariate family for modelling two real data sets.

Keywords: bivariate distribution generator; copula; reversed hazard gradient; maximum likelihood estimation; EM algorithm; multivariate distribution generator

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1. Introduction

Gumbel [1], Freund [2], and Marshall and Olkin [3] in their pioneering papers developed bivariate exponential distributions. Since then, an extensive amount of work has been done on these models and their different generalizations, which have played a crucial role in the construction of multivariate distributions and modelling in a wide variety of applications, such as physic, economy, biology, health, engineering, computer science, etc. Several continuous bivariate distributions can be found in Balakrishnan and Lai [4], and some generalizations and multivariate extensions have been studied by Franco and Vivo [5], Kundu and Gupta [6], Franco et al. [7], Gupta et al. [8], Kundu et al. [9], among others, and recently by Muhammed [10], Franco et al. [11], and El-Morshedy et al. [12], also see the references cited therein.

Kundu and Gupta [13] introduced a bivariate generalized exponential (BGE) distribution by using the trivariate reduction technique with generalized exponential (GE) random variables, which is based on the maximization process between components with a latent random variable, suitable for modelling of some stress and maintenance models. This procedure has also been applied in the literature to generate other bivariate distributions, for example, the bivariate generalized linear failure rate (BGLFR) given by Sarhan et al. [14], the bivariate log-exponentiated Kumaraswamy (BlogEK) introduced by Elsherpieny et al. [15], the bivariate exponentiated modified Weibull extension (BEMWE) given by El-Gohary et al. [16], the bivariate inverse Weibull (BIW) studied by Muhammed [17] and Kundu and Gupta [18], the bivariate Dagum (BD) provided by Muhammed [19], the bivariate generalized Rayleigh (BGR) depicted by Sarhan [20], the bivariate Gumbel-G (BGu-G) presented by Eliwa and El-Morshedy [21], the bivariate generalized inverted Kumaraswamy (BGIK) given by Muhammed [10], and the bivariate Burr typeX-G (BBX-G) proposed by El-Morshedy et al. [12]. Some associated inferential issues have been discussed in these articles, and all of them are based on considering the same kind of baseline components. In each of these bivariate models, the baseline components belong to the proportional reversed hazard rate (PRH) family with a certain underlying distribution (Gupta et al. [22] and Di Crescenzo [23]). It is worth mentioning that Kundu and Gupta [24] extended the BGE model by using components within the PRH family, called bivariate proportional reversed hazard rate (BPRH) family, and a multivariate extension of the BPRH model was studied by Kundu et al. [9].

The main aim of this paper is to provide a more flexible generator of bivariate distributions based on the maximization process from an arbitrary three-dimensional baseline continuous distribution vector, i.e., not necessarily identical continuous distributions. Hence, this proposed generator allows researchers and practitioners to generate new bivariate distributions even by combining non-identically distributed baseline components, which may be interpreted as a stress model or as a maintenance model. We refer to the bivariate models from this generator as the generalized bivariate distribution (GBD) family, which contains as special cases the aforementioned bivariate distributions. Note that a two-dimensional random variable (X_1, X_2), belonging to the GBD family, has dependent components due to a latent factor, and its joint cumulative distribution function (cdf) is not absolutely continuous, i.e., the joint cdf is a mixture of an absolutely continuous part and a singular part due to the positive probability of the event $X_1 = X_2$, whereas the line $x_1 = x_2$ has two-dimensional Lebesgue measure zero. In general, the maximum likelihood estimation (MLE) of the unknown parameters a GBD model cannot be obtained in closed form, and we propose to use EM algorithm to compute the MLEs of such parameters.

The rest of the paper is organized as follows. The construction of the GBD family is given in Section 2, we obtain its decomposition in absolutely continuous and singular parts and its joint probability density function (pdf). In Section 3, several special bivariate models are presented. The cdf and pdf of the marginals and conditional distributions are derived in Section 4, as well as for its order statistics. Some dependence and two-dimensional ageing properties for the GBD family, and stochastic properties of their marginals and order statistics are studied in Section 5, as well as its copula representation and some related association measures. The EM algorithm is proposed in Section 6, which is applied in Section 7, for illustrative purpose, to find the MLEs of particular models of the GBD family in the analysis of two real data sets. Finally, the multivariate extension is discussed in Section 8, as well as the concluding remarks. Some of the proofs are relegated to Appendix A for a fluent presentation of the results, and some technical details of the applications can be found in Appendix B.

2. The GBD family

In this section, we define the generalized bivariate distribution family as a generator system from any three-dimensional baseline continuous distribution, and then we provide its joint cdf, decomposition and joint pdf.

Let U_1, U_2 and U_3 be mutually independent random variables with any continuous distribution functions F_{U_1}, F_{U_2} and F_{U_3}, respectively. Let $X_1 = \max(U_1, U_3)$ and $X_2 = \max(U_2, U_3)$. Then, the random vector (X_1, X_2) is said to be a GBD model with baseline distribution vector $(F_{U_1}, F_{U_2}, F_{U_3})$.

Theorem 1. *Let (X_1, X_2) be a GBD model with baseline distribution vector $(F_{U_1}, F_{U_2}, F_{U_3})$, then its joint cdf is given by*

$$F(x_1, x_2) = F_{U_1}(x_1)F_{U_2}(x_2)F_{U_3}(z), \tag{1}$$

where $z = \min(x_1, x_2)$, for all $x_1, x_2 \in \mathbb{R}$.

Proof. It is immediate since

$$\begin{aligned} F(x_1, x_2) &= P(X_1 \leq x_1, X_2 \leq x_2) = P(\max(U_1, U_3) \leq x_1, \max(U_2, U_3) \leq x_2) \\ &= P(U_1 \leq x_1, U_2 \leq x_2, U_3 \leq \min(x_1, x_2)) = F_{U_1}(x_1)F_{U_2}(x_2)F_{U_3}(\min(x_1, x_2)). \end{aligned}$$

□

For instance, a stress model may lead to the GBD family, as in Kundu and Gupta [13]. Suppose a two-component system where each component is subject to an individual independent stress, say U_1 and U_2, respectively. The system has an overall stress U_3 which has been equally transmitted to both the components, independent of their individual stresses. Then, the observed stress for each component is the maximum of both, individual and overall stresses, i.e. $X_1 = \max(U_1, U_3)$ and $X_2 = \max(U_2, U_3)$, and (X_1, X_2) is a GBD model.

Analogously, a GBD model is also plausible for a maintenance model. Suppose a system has two components and it is assumed that each component has been maintained independently and also there is an overall maintenance. Due to component maintenance, the lifetime of the individual component is increased by a random time, say U_1 and U_2 respectively, and because of the overall maintenance, the lifetime of each component is increased by another random time U_3. Then, the increased lifetime of each component is the maximum of both, individual and overall maintenances, $X_1 = \max(U_1, U_3)$ and $X_2 = \max(U_2, U_3)$, respectively.

As mentioned before, a bivariate model belonging to the GBD family does not have an absolutely continuous cdf. Let us see now the decomposition of a GBD model as a mixture of bivariate absolutely continuous and singular cdfs, the proof is provided in Appendix A.

Theorem 2. *Let (X_1, X_2) be a GBD model with baseline distribution vector $(F_{U_1}, F_{U_2}, F_{U_3})$. Then,*

$$F(x_1, x_2) = \alpha F_s(x_1, x_2) + (1 - \alpha)F_{ac}(x_1, x_2) \tag{2}$$

where

$$F_s(x_1, x_2) = \frac{1}{\alpha} \int_{-\infty}^{z} F_{U_1}(u)F_{U_2}(u)dF_{U_3}(u) \tag{3}$$

and

$$F_{ac}(x_1, x_2) = \frac{1}{1 - \alpha} \left(F_{U_1}(x_1)F_{U_2}(x_2)F_{U_3}(z) - \int_{-\infty}^{z} F_{U_1}(u)F_{U_2}(u)dF_{U_3}(u) \right) \tag{4}$$

with $z = \min(x_1, x_2)$, are the singular and absolutely continuous parts, respectively, and

$$\alpha = \int_{-\infty}^{\infty} F_{U_1}(u)F_{U_2}(u)dF_{U_3}(u).$$

In addition, due to the singular part F_s in (2), the GBD family does not have a pdf with respect to the two-dimensional Lebesgue measure even when the distribution functions F_{U_1}, F_{U_2} and F_{U_3} are absolutely continuous. However, it is possible to construct a joint pdf for (X_1, X_2) through a mixture between a pdf with respect to the two-dimensional Lebesgue measure and a pdf with respect to the one-dimensional Lebesgue measure (the proof is provided in Appendix A).

Theorem 3. *If (X_1, X_2) is a GBD model with joint cdf given by (1), then the joint pdf with respect to μ, the measure associated with F, is*

$$f(x_1,x_2) = \begin{cases} f_1(x_1,x_2), & \text{if } x_1 < x_2 \\ f_2(x_1,x_2), & \text{if } x_1 > x_2 \\ f_0(x), & \text{if } x_1 = x_2 = x, \end{cases}$$

where

$$f_i(x_1,x_2) = f_{U_j}(x_j)\left(f_{U_i}(x_i)F_{U_3}(x_i) + F_{U_i}(x_i)f_{U_3}(x_i)\right), \text{ with } i \neq j \in \{1,2\},$$

and

$$f_0(x) = f_{U_3}(x)F_{U_1}(x)F_{U_2}(x),$$

when the pdf f_{U_i} of U_i exists, $i = 1,2,3$.

3. Special cases

In this section, we derive new bivariate models from Theorem 1, taking into account particular baseline distribution vectors $(F_{U_1}, F_{U_2}, F_{U_3})$.

Note that, if the baseline components U_is belong to the same distribution family, say F_U, then the proposed generator provides novel extended bivariate versions of that distribution F_U. Furthermore, under certain restrictions on the underlying parameters of each U_i, bivariate distributions given in the literature are obtained. From now on, it is assumed that all parameters of each F_{U_i} are positive unless otherwise mentioned.

Extended bivariate generalized exponential model. A random variable U follows a GE distribution, $U \sim GE(\theta, \lambda)$ (see, Gupta and Kundu [25]), if its cdf is given by

$$F_{GE}(u;\theta,\lambda) = \left(1 - e^{-\lambda u}\right)^{\theta}, \quad \text{for } u > 0.$$

If $U_i \sim GE(\theta_i, \lambda_i)$ $i = 1,2,3$, then the GBD model with GEs baseline distribution vector is an extended BGE model with $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ parameter vectors, denoted as $(X_1, X_2) \sim EBGE(\boldsymbol{\theta}, \boldsymbol{\lambda})$, being its joint cdf

$$F_{EBGE}(x_1,x_2) = F_{GE}(x_1;\theta_1,\lambda_1)F_{GE}(x_2;\theta_2,\lambda_2)F_{GE}(z;\theta_3,\lambda_3), \quad \text{for } x_1 > 0, x_2 > 0,$$

where $z = \min(x_1, x_2)$.

As a particular case, if $\lambda = \lambda_i$, $i = 1,2,3$, $(X_1, X_2) \sim BGE(\boldsymbol{\theta}, \lambda)$ given by Kundu and Gupta [13].

Extended bivariate proportional reversed hazard rate model. If $U_i \sim PRH(\theta_i)$ with base distribution F_{B_i} $i = 1,2,3$, i.e., its cdf can be expressed as $F_{U_i} = F_{B_i}^{\theta_i}$ (see, Gupta et al. [22] and Di Crescenzo [23]), then the GBD model with PRHs baseline distribution vector provides an extended BPRH model, $(X_1, X_2) \sim EBPRH(\boldsymbol{\theta}, \boldsymbol{\lambda})$, with $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ parameter vector of the PRH components and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ parameter vector of the underlying distributions F_{B_i}'s. From (1), its joint cdf is given by

$$F_{EBPRH}(x_1,x_2) = F_{B_1}^{\theta_1}(x_1;\lambda_1)F_{B_2}^{\theta_2}(x_2;\lambda_2)F_{B_3}^{\theta_3}(z;\lambda_3), \quad \text{for } x_1 > 0, x_2 > 0,$$

where $z = \min(x_1, x_2)$.

In particular, if the PRH components have the same base distribution, $F_B = F_{B_i}$ $i = 1,2,3$, then $(X_1, X_2) \sim BPRH(\boldsymbol{\theta}, \lambda)$ with baseline distribution $F_B(\cdot;\lambda)$ introduced by Kundu and Gupta [24].

Extended bivariate generalized linear failure rate model. It is said that a random variable U follows a GLFR distribution, $U \sim GLFR(\theta, \lambda, \gamma)$ (see, Sarhan and Kundu [26]), if its cdf is given by

$$F_{GLFR}(u;\theta,\lambda,\gamma) = \left(1 - \exp\left(-\lambda u - \frac{\gamma}{2}u^2\right)\right)^{\theta}, \quad \text{for } u > 0.$$

If $U_i \sim GLFR(\theta_i, \lambda_i, \gamma_i)$ $i = 1,2,3$, then the GBD model with GLFRs baseline distribution vector is an extended BGLFR model, $(X_1, X_2) \sim EBGLFR(\boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\gamma})$, with parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$, and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$, having joint cdf

$$F_{EBGLFR}(x_1, x_2) = F_{GLFR}(x_1; \theta_1, \lambda_1, \gamma_1) F_{GLFR}(x_2; \theta_2, \lambda_2, \gamma_2) F_{GLFR}(z; \theta_3, \lambda_3, \gamma_3), \quad \text{for } x_1 > 0, x_2 > 0,$$

where $z = \min(x_1, x_2)$.

When $\lambda_i = \lambda$ and $\gamma_i = \gamma$, $i = 1,2,3$, it is obtained that $(X_1, X_2) \sim BGLFR(\boldsymbol{\theta}, \lambda, \gamma)$ given by Sarhan et al. [14].

Extended bivariate log-exponentiated Kumaraswamy model. Let U be a random variable with logEK distribution, $U \sim logEK(\theta, \lambda, \gamma)$ (see, Lemonte et al. [27]), then its cdf

$$F_{logEK}(u; \theta, \lambda, \gamma) = \left(1 - \left(1 - \left(1 - e^{-u}\right)^{\lambda}\right)^{\gamma}\right)^{\theta}, \quad \text{for } u > 0.$$

If $U_i \sim logEK(\theta_i, \lambda_i, \gamma_i)$ $i = 1,2,3$, then the GBD model with logEKs baseline distribution vector is an extended BlogEK model, $(X_1, X_2) \sim EBlogEK(\boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\gamma})$ with parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$, and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$, and its joint cdf is given by

$$F_{EBlogEK}(x_1, x_2) = F_{logEK}(x_1; \theta_1, \lambda_1, \gamma_1) F_{logEK}(x_2; \theta_2, \lambda_2, \gamma_2) F_{logEK}(z; \theta_3, \lambda_3, \gamma_3), \quad \text{for } x_1 > 0, x_2 > 0,$$

where $z = \min(x_1, x_2)$.

Clearly, it can be seen that $(X_1, X_2) \sim BlogEK(\boldsymbol{\theta}, \lambda, \gamma)$ given by Elsherpieny et al. [15], when $\lambda_i = \lambda$ and $\gamma_i = \gamma$, $i = 1,2,3$.

Extended bivariate exponentiated modified Weibull extension model. A random variable U follows an EMWE distribution, $U \sim EMWE(\theta, \alpha, \beta, \lambda)$ (see, Sarhan and Apaloo [28]), if its cdf can be expressed as

$$F_{EMWE}(u; \theta, \alpha, \beta, \lambda) = \left(1 - \exp\left(\alpha\lambda\left(1 - e^{(u/\alpha)^{\beta}}\right)\right)\right)^{\theta}, \quad \text{for } u > 0.$$

If $U_i \sim EMWE(\theta_i, \alpha_i, \beta_i, \lambda_i)$ $i = 1,2,3$, then the GBD model with EMWEs baseline distribution vector is an extended BEMWE model, $(X_1, X_2) \sim EBEMWE(\boldsymbol{\theta}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\lambda})$ with $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$, and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ parameter vectors, and its joint cdf is given by

$$F_{EBEMWE}(x_1, x_2) = F_{EMWE}(x_1; \theta_1, \alpha_1, \beta_1, \lambda_1) F_{EMWE}(x_2; \theta_2, \alpha_2, \beta_2, \lambda_2) F_{EMWE}(z; \theta_3, \alpha_3, \beta_3, \lambda_3),$$

for $x_1 > 0$ and $x_2 > 0$, where $z = \min(x_1, x_2)$.

Note that if $\alpha_i = \alpha$, $\beta_i = \beta$ and $\lambda_i = \lambda$, $i = 1,2,3$, then $(X_1, X_2) \sim BEMWE(\boldsymbol{\theta}, \alpha, \beta, \lambda)$ given by El-Gohary et al. [16].

Extended bivariate inverse Weibull model. The cdf of the IW distribution (e.g., see Keller et al. [29]) is defined by

$$F_{IW}(u; \theta, \lambda) = e^{-\lambda u^{-\theta}}, \quad \text{for } u > 0.$$

If $U_i \sim IW(\theta_i, \lambda_i)$ $i = 1,2,3$, then the GBD model with IWs baseline distribution vector is an extended BIW model with $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ parameter vectors, denoted as $(X_1, X_2) \sim EBIW(\boldsymbol{\theta}, \boldsymbol{\lambda})$, and its joint cdf can be written as

$$F_{EBIW}(x_1, x_2) = e^{-\lambda_1 x_1^{-\theta_1} - \lambda_2 x_2^{-\theta_2} - \lambda_3 z^{-\theta_3}}, \quad \text{for } x_1 > 0, x_2 > 0,$$

where $z = \min(x_1, x_2)$.

In particular, $(X_1, X_2) \sim BIW(\theta, \boldsymbol{\lambda})$ studied by Muhammed [17] and Kundu and Gupta [18], when $\theta_i = \theta$ for $i = 1, 2, 3$.

Extended bivariate Dagum model. It is said that a random variable U follows a Dagum distribution [30], $U \sim D(\theta, \lambda, \gamma)$, if its cdf is given by

$$F_D(u; \theta, \lambda, \gamma) = (1 + \lambda u^{-\gamma})^{-\theta}, \text{ for } u > 0.$$

If $U_i \sim D(\theta_i, \lambda_i, \gamma_i)$ $i = 1, 2, 3$, then the GBD model with Dagum baseline distribution vector is an extended BD model with $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ parameter vectors, denoted as $(X_1, X_2) \sim EBD(\boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\gamma})$, having joint cdf

$$F_{EBD}(x_1, x_2) = F_D(x_1; \theta_1, \lambda_1, \gamma_1) F_D(x_2; \theta_2, \lambda_2, \gamma_2) F_D(z; \theta_3, \lambda_3, \gamma_3), \text{ for } x_1 > 0, x_2 > 0,$$

where $z = \min(x_1, x_2)$.

Note that, when $\lambda_i = \lambda$ and $\gamma_i = \gamma$ for $i = 1, 2, 3$, it is simplified to the model $(X_1, X_2) \sim BD(\boldsymbol{\theta}, \lambda, \gamma)$ defined by Muhammed [19].

Extended bivariate generalized Rayleigh model. The cdf of the GR distribution, also called Burr type X model [31], is

$$F_{GR}(u; \theta, \lambda) = \left(1 - e^{-(\lambda u)^2}\right)^{\theta}, \text{ for } u > 0.$$

If $U_i \sim GR(\theta_i, \lambda_i)$ $i = 1, 2, 3$, then the GBD model with GRs baseline distribution vector is an extended BGR model with $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ and $\boldsymbol{\theta}_B = (\lambda_1, \lambda_2, \lambda_3)$ parameter vectors, $(X_1, X_2) \sim EBGR(\boldsymbol{\theta}, \boldsymbol{\lambda})$, with joint cdf

$$F_{EBGR}(x_1, x_2) = F_{GR}(x_1; \theta_1, \lambda_1) F_{GR}(x_2; \theta_2, \lambda_2) F_{GR}(z; \theta_3, \lambda_3), \text{ for } x_1 > 0, x_2 > 0,$$

where $z = \min(x_1, x_2)$.

Hence, if $\lambda_i = \lambda$, $i = 1, 2, 3$, it is obtained that $(X_1, X_2) \sim BGR(\boldsymbol{\theta}, \lambda)$ given by Sarhan [20].

Extended bivariate Gumbel-G model. Alzaatrech et al. [32] proposed a transformed-transformer method for generating families of continuous distributions. From such method, it is said that a random variable U follows a Gumbel-G model, $U \sim Gu\text{-}G(\theta, \alpha, \boldsymbol{\theta}_G)$ if its cdf can be expressed as

$$F_{Gu-G}(u; G, \theta, \alpha, \boldsymbol{\lambda}) = \exp\left(-\theta \left(\frac{1 - G(u; \boldsymbol{\lambda})}{G(u; \boldsymbol{\lambda})}\right)^{\alpha}\right), \text{ for } u > 0$$

where G is the transformer distribution with parameter vector $\boldsymbol{\lambda}$. If $U_i \sim Gu\text{-}G(\theta_i, \alpha_i, \boldsymbol{\lambda}_i)$ $i = 1, 2, 3$, then the GBD model with Gu-Gs baseline distribution vector is an extended BGu-G model, $(X_1, X_2) \sim EBGu\text{-}G(\boldsymbol{\theta}, \boldsymbol{\alpha}, \boldsymbol{\lambda}_G)$, with parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, and $\boldsymbol{\lambda}_G = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)$, where $\boldsymbol{\lambda}_G$ encompasses all parameter vectors of G in each baseline component. Thus, its joint cdf is given by

$$F_{EBGu-G}(x_1, x_2) = F_{Gu-G}(x_1; G, \theta_1, \alpha_1, \boldsymbol{\lambda}_1) F_{Gu-G}(x_2; G, \theta_2, \alpha_2, \boldsymbol{\lambda}_2) F_{Gu-G}(z; G, \theta_3, \alpha_3, \boldsymbol{\lambda}_3),$$

for $x_1 > 0$, $x_2 > 0$, where $z = \min(x_1, x_2)$.

In particular, when $\alpha_i = \alpha$ and $\boldsymbol{\lambda}_i = \boldsymbol{\lambda}$ for $i = 1, 2, 3$, $(X_1, X_2) \sim BGu\text{-}G(\boldsymbol{\theta}, \alpha, \boldsymbol{\lambda})$ given by Eliwa and El-Morshedy [21].

Extended bivariate generalized inverted Kumaraswamy model. A random variable U is said to be a GIK distribution defined by Iqbal et al. [33], if its cdf is given by

$$F_{GIK}(u; \theta, \alpha, \gamma) = \left(1 - (1 + u^{\gamma})^{-\alpha}\right)^{\theta}, \text{ for } u > 0.$$

If $U_i \sim GIK(\theta_i, \alpha_i, \gamma_i)$ $i = 1, 2, 3$, then the GBD model with GIKs baseline distribution vector is an extended BGIK model, $(X_1, X_2) \sim EBGIK(\boldsymbol{\theta}, \boldsymbol{\alpha}, \boldsymbol{\gamma})$, with parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$, and its joint cdf can be written as

$$F_{EBGIK}(x_1, x_2) = F_{GIK}(x_1; \theta_1, \alpha_1, \gamma_1) F_{GIK}(x_2; \theta_2, \alpha_2, \gamma_2) F_{GIK}(z; \theta_3, \alpha_3, \gamma_3), \ \text{ for } x_1 > 0, x_2 > 0,$$

where $z = \min(x_1, x_2)$.

It is straightforward to see that $(X_1, X_2) \sim BGIK(\boldsymbol{\theta}, \alpha, \gamma)$ analyzed by Muhammed [10] when $\alpha = \alpha_i$ and $\gamma = \gamma_i$ for $i = 1, 2, 3$.

Extended bivariate Burr type X-G model. From the transformed-transformer method of Alzaatrech et al. [32], it is said that a random variable U follows a Burr X-G model, $U \sim BX\text{-}G(\theta, \boldsymbol{\lambda})$ if its cdf can be expressed as

$$F_{BX-G}(u; G, \theta, \boldsymbol{\lambda}) = \left(1 - \exp\left(-\left(\frac{G(u; \boldsymbol{\lambda})}{1 - G(u; \boldsymbol{\lambda})}\right)^2\right)\right)^{\theta}, \ \text{ for } u > 0$$

where $\boldsymbol{\lambda}$ is the parameter vector of the transformer distribution G.

If $U_i \sim BX\text{-}G(\theta_i, \boldsymbol{\lambda}_i)$ $i = 1, 2, 3$, then the GBD model with BX-Gs baseline distribution vector is an extended BBX-G model, $(X_1, X_2) \sim EBBX\text{-}G(\boldsymbol{\theta}, \boldsymbol{\lambda}_G)$, with parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$, and $\boldsymbol{\lambda}_G = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)$, where $\boldsymbol{\lambda}_G$ encompasses all parameter vectors of G in each baseline component. Then, its joint cdf can be expressed as

$$F_{EBBX-G}(x_1, x_2) = F_{BX-G}(x_1; \theta_1, \boldsymbol{\lambda}_1) F_{BX-G}(x_2; \theta_2, \boldsymbol{\lambda}_2) F_{BX-G}(z; \theta_3, \boldsymbol{\lambda}_3), \ \text{ for } x_1 > 0, x_2 > 0,$$

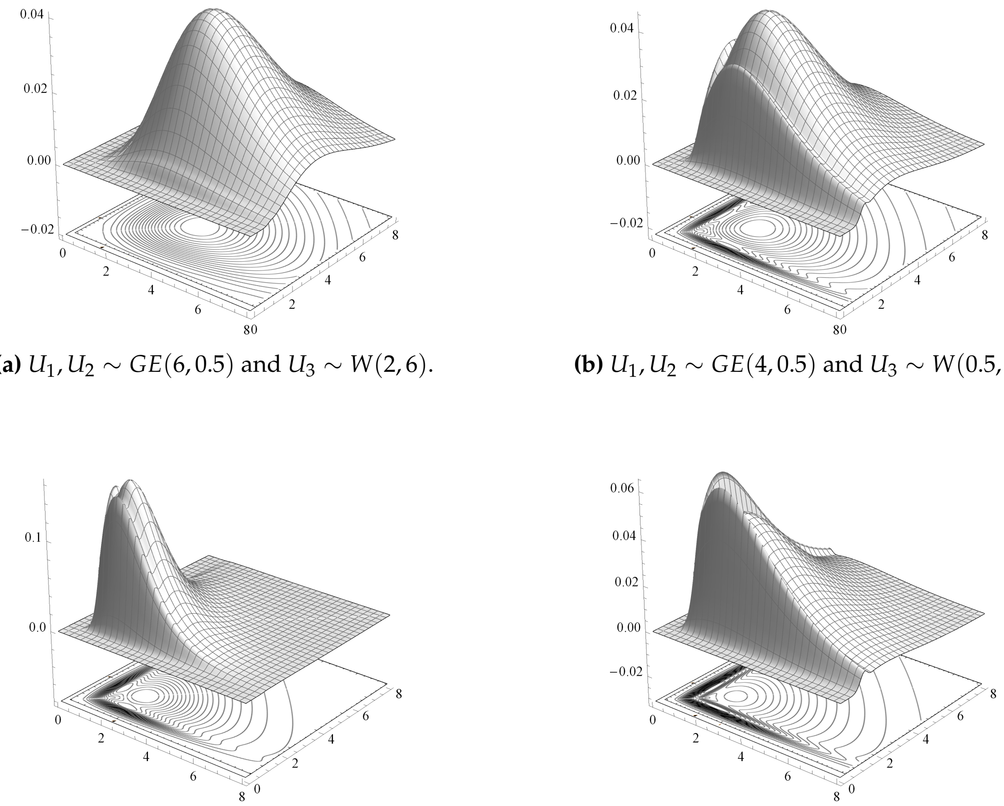
where $z = \min(x_1, x_2)$.

In particular, if $\boldsymbol{\lambda} = \boldsymbol{\lambda}_i$ for $i = 1, 2, 3$, then $(X_1, X_2) \sim BBX\text{-}G(\boldsymbol{\theta}, \boldsymbol{\lambda})$ introduced by El-Morshedy et al. [12].

GBD models from different baseline components. In addition, a GBD model can be derived from baseline components U_is belonging to different distribution families, which allows one to generate new bivariate distributions.

For illustrative purposes, Figures 1a-1d display 3D surfaces of different joint pdfs given by Theorem 3, along with their contour plots. Here, U_1 and U_2 are taken identically distributed $GE(\theta, \lambda)$ with different shape and scale parameter values, and U_3 having a Weibull distribution with scale parameter λ_3 and shape parameter $\alpha = 6$, $W(\lambda_3, 6)$.

Figure 1 shows that some of these GBD models are multi-modal bivariate models. It indicates a variety of shapes for the GBD family depending on the different baseline distribution components and for different parameter values.



(a) $U_1, U_2 \sim GE(6, 0.5)$ and $U_3 \sim W(2, 6)$.
(b) $U_1, U_2 \sim GE(4, 0.5)$ and $U_3 \sim W(0.5, 6)$.
(c) $U_1, U_2 \sim GE(6, 1)$ and $U_3 \sim W(1, 6)$.
(d) $U_1, U_2 \sim GE(3, 0.5)$ and $U_3 \sim W(0.5, 6)$.

Figure 1. Surface and contour plots of the joint pdf of GBD models (X_1, X_2) with different components (U_1, U_2, U_3).

4. Distributional properties

Here, we derive the marginal and conditional distributions of the GBD family, and the order statistics. Furthermore, some properties for particular baseline distribution vectors are provided.

4.1. Marginal and conditional distributions

From Theorem 1, it is easy to obtain the marginal cdfs of the components X_i's, which can be written as

$$F_{X_i}(x_i) = F_{U_i}(x_i) F_{U_3}(x_i), \text{ with } i = 1, 2, \tag{5}$$

and, when the pdf f_{U_i} of U_i exists, $i = 1, 2, 3$, the corresponding pdfs are given by

$$f_{X_i}(x_i) = f_{U_i}(x_i) F_{U_3}(x_i) + F_{U_i}(x_i) f_{U_3}(x_i), \text{ with } i = 1, 2. \tag{6}$$

For instance, we shall now suppose that U_is have PRH distributions, in order to provide some preservation results of the PRH property on the marginals, and its closure under exponentiation of the underlying distributions.

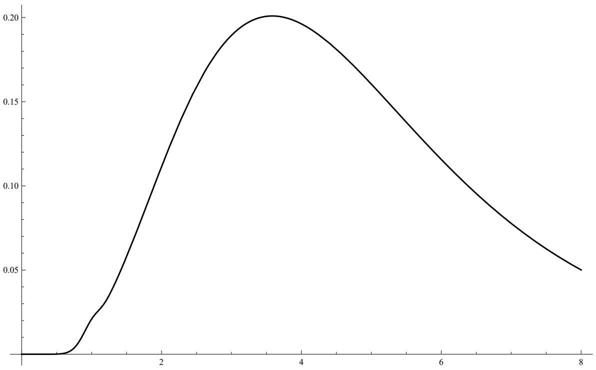
Proposition 1. *If (X_1, X_2) has a GBD model formed by $U_i \sim PRH(\theta_i)$ with baseline distribution F_{B_i} $(i = 1, 2, 3)$, then $X_i \sim PRH(\theta_i + \theta_3)$ with base distribution $F_{B_i^*} = F_{B_i}^{\theta_i/(\theta_i+\theta_3)} F_{B_3}^{\theta_3/(\theta_i+\theta_3)}$. Moreover, when the base distribution is common, $F_B = F_{B_i}$, then X_is also have the same baseline distribution F_B.*

Proof. It immediately follows from (5) and the EBPRH model, since $F_{U_i} = F_{B_i}^{\theta_i}$. $\square$

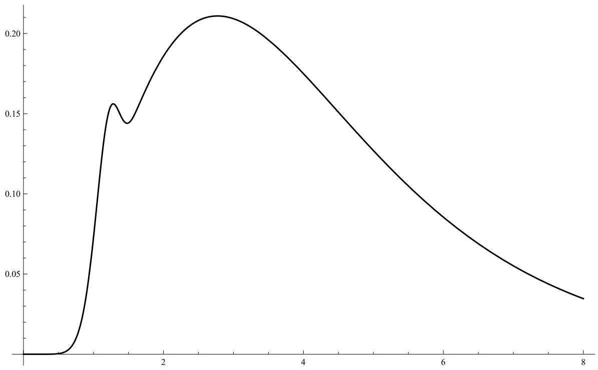
Corollary 1. *If $U_i \sim PRH(\theta_i)$ with base distribution F_{B_i}, having $F_{B_i} \sim PRH(\lambda_i)$ with base distribution $F_{\widetilde{B}_i}$ $(i = 1, 2, 3)$, then $X_i \sim PRH(\theta_i\lambda_i + \theta_3\lambda_3)$ with base distribution $F_{B_i^*} = F_{\widetilde{B}_i}^{\theta_i\lambda_i/(\theta_i\lambda_i+\theta_3\lambda_3)} F_{\widetilde{B}_3}^{\theta_3\lambda_3/(\theta_i\lambda_i+\theta_3\lambda_3)}$. Moreover, if $F_{\widetilde{B}} = F_{\widetilde{B}_i}$ $(i = 1, 2, 3)$, then X_is also have the same base distribution $F_{\widetilde{B}}$.*

In addition, Figure 2 displays the plots of the marginal pdfs of the GBD models depicted in Figures 1a-1d.

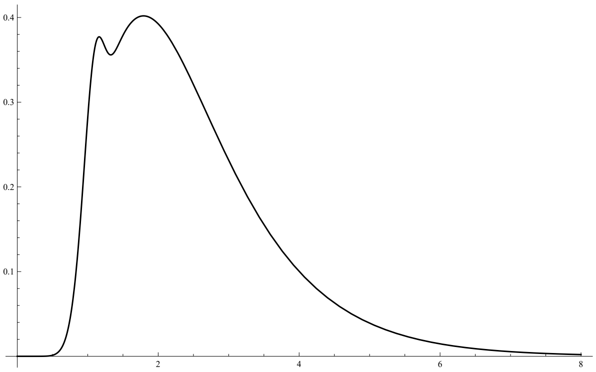
Note that Figures 2a-2d show some bimodal shapes for the marginal pdfs given by (6) of the GBD models represented in Figures 1a-1d, which also exhibit some multi-modal shapes of the joint pdfs. In this setting, Proposition 1 might be used to generate bimodal distributions from the marginals of the GBD family by mixing different baseline distribution components as in Figure 1.



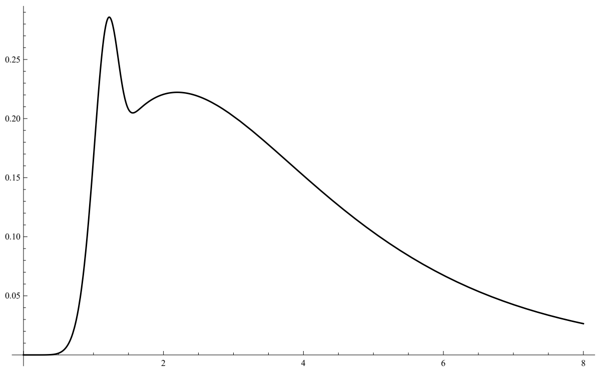
(a) $U_1, U_2 \sim GE(6, 0.5)$ and $U_3 \sim W(2, 6)$.



(b) $U_1, U_2 \sim GE(4, 0.5)$ and $U_3 \sim W(0.5, 6)$.



(c) $U_1, U_2 \sim GE(6, 1)$ and $U_3 \sim W(1, 6)$.



(d) $U_1, U_2 \sim GE(3, 0.5)$ and $U_3 \sim W(0.5, 6)$.

Figure 2. Plots of the marginal pdfs of the GBD models (X_1, X_2) with different components (U_1, U_2, U_3).

Furthermore, we provide some results about the conditional distributions of a GBD model whose proof can be found in Appendix A.

Theorem 4. *If (X_1, X_2) has a GBD model with baseline distribution vector $(F_{U_1}, F_{U_2}, F_{U_3})$, then*

1. *The conditional distribution of X_i given $X_j \le x_j$ $(i \ne j)$, say $F_{i|X_j \le x_j}$, is an absolutely continuous cdf given by*

$$F_{i|X_j \le x_j}(x_i) = \begin{cases} F_{U_i}(x_i)\frac{F_{U_3}(x_i)}{F_{U_3}(x_j)}, & \text{if } x_i < x_j \\ F_{U_i}(x_i), & \text{if } x_i \ge x_j \end{cases}.$$

2. *The conditional pdf of X_i given $X_j = x_j$ $(i \neq j)$, say $f_{i|X_j=x_j}$, is a convex combination of an absolutely continuous cdf and a degenerate cdf given by*

$$f_{i|X_j=x_j}(x_i) = \alpha_j I_{x_j}(x_i) + (1-\alpha_j) f_{i|x_j,ac}(x_i),$$

where I_{x_j} is the indicator function of the given point x_j, and $f_{i|x_j,ac}$ is the absolutely continuous part

$$f_{i|X_j=x_j,ac}(x_i) = \frac{1}{1-\alpha_j}\begin{cases} f_{X_i}(x_i)\frac{f_{U_j}(x_j)}{f_{X_j}(x_j)}, & \text{if } x_i < x_j \\ f_{U_i}(x_i), & \text{if } x_i > x_j \\ 0, & \text{if } x_i = x_j \end{cases}$$

and the mixing weight α_j is constant with respect to x_i

$$\alpha_j = F_{U_1}(x_j) F_{U_2}(x_j) \frac{f_{U_3}(x_j)}{f_{X_j}(x_j)}.$$

175 *4.2. Minimum and maximum order statistics*

176 Now we provide the cdfs of the maximum and minimum order statistics of a GBD model, which
177 may be interpreted as the lifetimes of parallel and series systems based on the components of (X_1, X_2).

Theorem 5. *If $T_1 = \min(X_1, X_2)$ and $T_2 = \max(X_1, X_2)$ of a GBD model (X_1, X_2) with baseline distribution vector $(F_{U_1}, F_{U_2}, F_{U_3})$, then their cdfs are given by*

$$F_{T_1}(x) = F_{U_3}(x) F_{U_{1:2}}(x) \text{ and } F_{T_2}(x) = F_{U_{3:3}}(x) \tag{7}$$

178 *where $U_{1:2} = \min(U_1, U_2)$ and $U_{3:3} = \max(U_1, U_2, U_3)$.*

179 **Proof.** It is trivial from (1) and (5) by taking into account that $F_{T_2}(x) = F(x,x)$ and $F_{T_1}(x) = F_{X_1}(x) +$
180 $F_{X_2}(x) - F_{T_2}(x)$. □

181 The pdfs f_{T_1} and f_{T_2} of the minimum and maximum statistics can be readily obtained by
182 differentiation of (7).
183 Furthermore, the PRH property is preserved by the maximum order statistic of a GBD model,
184 which is immediately derived from Theorem 5.

185 **Corollary 2.** *If $U_i \sim PRH(\theta_i)$ with baseline distribution F_{B_i} $(i = 1,2,3)$, then $T_2 \sim PRH(\theta)$ with base*
186 *$F_{B_{(2)}} = F_{B_1}^{\theta_1/\theta} F_{B_2}^{\theta_2/\theta} F_{B_3}^{\theta_3/\theta}$ and $\theta = \theta_1 + \theta_2 + \theta_3$. Moreover, when $F_B = F_{B_i}$ $(i = 1,2,3)$, then T_2 also has the*
187 *same base distribution F_B.*

188 5. Dependence and stochastic properties

189 In this section, we study various dependence and stochastic properties on the GBD family, its
190 marginals and order statistics, and its copula representation. Notions of dependence and ageing for
191 bivariate distributions can be found in Lai and Xie [34] and Balakrishnan and Lai [4], also see Shaked
192 and Shantikumar [35] for univariate and multivariate stochastic orders.

193 *5.1. GBD model*

194 **Proposition 2.** *If $(X_1, X_2) \sim$ GBD model, then (X_1, X_2) is positive quadrant dependent (PQD).*

195 **Proof.** From (1) and (5), it is readily to get that $F(x_1, x_2) \geq F_{X_1}(x_1) F_{X_2}(x_2)$, which is equivalent to say
196 that all random vector (X_1, X_2), having a GBD model, is PQD. □

An immediate consequence of the PQD property is that $Cov(X_1, X_2) > 0$. Other important bivariate dependence properties are the following, whose proofs are provided in Appendix A.

Proposition 3. *Let (X_1, X_2) be a random vector having a GBD model.*

1. *(X_1, X_2) is left tail decreasing (LTD).*
2. *(X_1, X_2) is left corner set decreasing (LCSD).*
3. *Its joint cdf F is totally positive of order 2 (TP_2).*

Proof. Note that F is TP_2 is equivalent to (X_1, X_2) is LCSD, which implies LTD (e.g., see Balakrishnan and Lai [4]). Thereby, we only have to prove (3). From the definition of TP_2 property, it is equivalent to check that the following inequality hold

$$\frac{F(\boldsymbol{x})F(\boldsymbol{x}')}{F(\boldsymbol{x} \vee \boldsymbol{x}')F(\boldsymbol{x} \wedge \boldsymbol{x}')} \leq 1, \tag{8}$$

for all $\boldsymbol{x}$ and $\boldsymbol{x}'$, where $\boldsymbol{x} \vee \boldsymbol{x}' = (\max(x_1, x_1'), \max(x_2, x_2'))$, and $\boldsymbol{x} \wedge \boldsymbol{x}' = (\min(x_1, x_1'), \min(x_2, x_2'))$. Hence, from (1), the inequality (8) can be expressed as

$$\frac{F_{U_3}(u)F_{U_3}(v)}{F_{U_3}(w)F_{U_3}(y)} \leq 1,$$

where $u = x_1 \wedge x_2$, $v = x_1' \wedge x_2'$, $w = (x_1 \vee x_1') \wedge (x_2 \vee x_2')$ and $y = u \wedge v$. Moreover, one can observe that $y \leq u \vee v = \max(u, v) \leq w$.

Therefore, when $u \leq v$, i.e., $y = u \leq v \leq w$, the inequality (8) can be simplified as follows

$$\frac{F_{U_3}(v)}{F_{U_3}(w)} \leq 1,$$

which is trivial, since $v \leq w$ and F_{U_3} is a cdf. An analogous development follows for $u > v$, which completes the proof. □

Let us see now some results related to the reversed hazard gradient of a random vector from the GBD family, which is defined as an extension of the univariate case, see Domma [36],

$$r(\boldsymbol{x}) = (r_1(\boldsymbol{x}), r_2(\boldsymbol{x})) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) \ln F(x_1, x_2)$$

where each $r_i(\boldsymbol{x})$ represents the reversed hazard function of $(X_i | X_j \leq x_j)$, $i \neq j = 1, 2$, and assuming that F is differentiable. In addition, it is said that (X_1, X_2) has a bivariate decreasing (increasing) reversed hazard gradient, BDRHG (BIRHG), if all components r_is are decreasing (increasing) functions in the corresponding variables.

Proposition 4. *If (X_1, X_2) has a GBD model with baseline distribution vector $(F_{U_1}, F_{U_2}, F_{U_3})$, then its reversed hazard gradient $r(\boldsymbol{x})$ is given by*

$$r_i(\boldsymbol{x}) = \begin{cases} r_{U_i}(x_i) + r_{U_3}(x_i), & \text{if } x_i < x_j \\ r_{U_i}(x_i), & \text{if } x_i \geq x_j \end{cases}$$

for $i \neq j = 1, 2$, when the reversed hazard function of U_i, $r_{U_i} = f_{U_i}/F_{U_i}$ exists, $i = 1, 2, 3$.

Proof. The proof is straightforward from the definition of reversed hazard rate function corresponding to the conditional cdf $F_{i|X_j \leq x_j}$ given by (1) of Theorem 4. □

Theorem 6. *Let* (X_1, X_2) *be a random vector having a GBD model. If* U_i*s have decreasing reversed hazard functions (DRH), then* $(X_1, X_2) \in BDRHG$.

Proof. It is straightforward from Proposition 4. $\square$

Note that Theorem 6 provides the closure of the DRH property under the formation of a GBD model. Thus, the bivariate extension of a DRH distribution F_U generated by the GBD family is BDRHG.

Nevertheless, it does not hold for the increasing reversed hazard (IRH) property, since both $r_i(\boldsymbol{x})$ given in Proposition 4 have a negative jump discontinuity at $x_i = x_j$ for $i \neq j = 1,2$. Therefore, if $U_i \in IRH$, then (X_1, X_2) cannot be BIRHG.

Finally, we present some interesting stochastic ordering results between bivariate random vectors of GBD type.

Theorem 7. *Let* $\boldsymbol{X} = (X_1, X_2)$ *and* $\boldsymbol{Y} = (Y_1, Y_2)$ *have GBD models with baseline distribution vectors* $(F_{U_1}, F_{U_2}, F_{U_3})$ *and* $(F_{V_1}, F_{V_2}, F_{V_3})$*, respectively. If* $U_i \leq_{st} V_i$ $(i = 1,2,3)$*, then* $\boldsymbol{X} \leq_{lo} \boldsymbol{Y}$.

Proof. The result immediately follows from the stochastic ordering between components and (1), since $U_i \leq_{st} V_i$ is equivalent to $F_{U_i}(x) \geq F_{V_i}(x)$, and the lower orthant ordering is defined by the inequality $F_{\boldsymbol{X}}(\boldsymbol{x}) \geq F_{\boldsymbol{Y}}(\boldsymbol{x})$ for all $\boldsymbol{x} = (x_1, x_2)$. $\square$

Corollary 3. *Let* $\boldsymbol{X} \sim EBPRH(\boldsymbol{\theta}, \boldsymbol{\lambda})$ *and* $\boldsymbol{Y} \sim EBPRH(\boldsymbol{\theta}^*, \boldsymbol{\lambda}^*)$ *with base distributions* F_{B_i} *and* $F_{B_i^*}$ $(i = 1,2,3)$*, respectively. If* $\theta_i \leq \theta_i^*$ *and* $F_{B_i} \leq_{st} F_{B_i^*}$ $(i = 1,2,3)$*, then* $\boldsymbol{X} \leq_{lo} \boldsymbol{Y}$.

Proof. It is obvious that $F_{B_i}^{\theta_i}(x_i) \geq F_{B_i}^{\theta_i^*}(x_i) \geq F_{B_i^*}^{\theta_i^*}(x_i)$, i.e. $U_i \leq_{st} V_i$, and then the proof readily follows from Theorem 7. $\square$

Remark 1. *From Corollary 3, if both EBPRH models are based on a common base distribution vector,* $F_{B_i} = F_{B_i^*}$ $(i = 1,2,3)$*, then it is only necessary that* $\theta_i \leq \theta_i^*$ *to be hold the lower orthant ordering.*

5.2. Marginals and order statistics

Now, we study some stochastic properties of the marginals and the minimum and maximum order statistics of the GBD model.

Firstly, from (5) and (6), the reversed hazard function of the marginal X_is can be expressed as

$$r_{X_i}(x) = \frac{f_{X_i}(x)}{F_{X_i}(x)} = r_{U_i}(x) + r_{U_3}(x),\ i = 1,2. \tag{9}$$

Therefore, the DRH (IRH) property is preserved to the marginals.

Theorem 8. *If* (X_1, X_2) *has a GBD model formed by* $U_i \in DRH$ $(i = 1,2,3)$*, then* $X_i \in DRH$ $(i = 1,2)$.

Remark 2. *Note that the IRH distributions have upper bounded support [37]. Thus, if any* U_i *is not upper bounded, its reversed hazard function is always decreasing at the end, and then the marginal cannot be IRH. Therefore, it is necessary that* $U_i \in IRH$ $(i = 1,2,3)$ *and they have the same upper bounds to be* $X_i \in IRH$ $(i = 1,2)$.

Example 1. *Suppose* U_i*s have extreme value distributions of type 3 with a common support,* $U_i \sim EV3(\beta, \lambda_i, k_i)$*, whose cdf is defined by*

$$F_{U_i}(u) = \exp\left(-\lambda_i(\beta - u)^{k_i}\right),\ \textit{for } u \leq \beta$$

and $F_{U_i}(u) = 1$ otherwise. Its reversed hazard function is given by

$$r_{U_i}(u) = \lambda_i k_i (\beta - u)^{k_i - 1}, \text{ for } u \in (-\infty, \beta],$$

which is increasing (decreasing) in its support for $k_i \leq (\geq) 1$. Thus, if $k_i \leq (\geq) 1$ ($i = 1,2,3$), then $U_i \in IRH(DRH)$, and consequently, $X_i \in IRH(DRH)$ ($i = 1,2$).

Example 2. *If (X_1, X_2) has an EBGE model, then its marginals are DRH, since r_{X_i} given by (9) is the sum of two decreasing functions because of each $U_i \sim GE(\theta_i, \lambda_i)$ is a $PRH(\theta_i)$ with exponential baseline distribution*

$$r_{U_i}(u) = \theta_i r_{Exp(\lambda_i)}(u) = \frac{\theta_i \lambda_i}{e^{\lambda_i u} - 1},$$

which is evidently a decreasing function. Here $Exp(\lambda)$ denotes an exponential random variable with mean $1/\lambda$.

Remark 3. *From (9), when the U_is have a common distribution F_U, then the marginals $X_i \sim PRH(2)$ with base distribution F_U. Therefore, $r_{X_i}(x) = 2r_U(x)$ have the same monotonicity. In particular, if $F_U \in DRH$ (IRH) then $X_i \in DRH$ (IRH).*

Remark 4. *From (9), if $U_i \sim PRH(\theta_i)$ with the same base distribution F_B, then $X_i \sim PRH(\theta_i + \theta_3)$ with base F_B, i.e., $r_{X_i}(x) = (\theta_i + \theta_3) r_B(x)$. Thus, Remark 3 also holds by using F_B instead of F_U.*

Secondly, the mean inactivity time (MIT), also called mean waiting time [37], of a random variable X is defined as

$$m_X(x) = E(x - X | X \leq x) = \int_{-\infty}^{x} \frac{F_X(y)}{F_X(x)} dy.$$

Thus, from (5), the MIT of the marginal X_is of a GBD model can be derived by

$$m_{X_i}(x) = \frac{1}{F_{U_i}(x) F_{U_3}(x)} \int_{-\infty}^{x} F_{U_i}(y) F_{U_3}(y) dy, \ i = 1,2. \tag{10}$$

Here, we shall focus on two particular cases of GBD models, having baseline components with monotonous MIT, which is preserved by the marginals.

Example 3. *Suppose $U_i \sim Exp(\lambda)$, then its MIT can be expressed as*

$$m_{U_i}(u) = \frac{u}{1 - e^{-\lambda u}} - \frac{1}{\lambda},$$

which is an increasing MIT function (IMIT), i.e. $U_i \in IMIT$. From (10), we obtain the MIT function of the marginals X_is for the bivariate exponential version of GBD type,

$$m_{X_i}(x) = \frac{2\lambda x - 3 + 4e^{-\lambda x}) - e^{-2\lambda x}}{2\lambda(1 - e^{-\lambda x})^2}.$$

Then, upon differentiation, $m'_{X_i}(x)$ has the same sign as the expression $1 - e^{-2\lambda x} - 2\lambda x e^{-\lambda x}$, which is positive, and therefore $X_i \in IMIT$ ($i = 1,2$).

Example 4. *Suppose $U_i \sim EV3(\beta, \lambda_i, k = 2)$, then its MIT can be expressed as*

$$m_{U_i}(u) = \frac{1}{e^{-\lambda_i(\beta - u)^2}} \int_{-\infty}^{u} e^{-\lambda_i(\beta - y)^2} dy = \frac{\Phi(u; \mu, \sigma_i)}{\phi(u; \mu, \sigma_i)}, \text{ for } u \leq \beta$$

where $\Phi(u;\mu,\sigma_i)$ and $\phi(u;\mu,\sigma_i)$ are the cdf and pdf of a normal model with $\mu=\beta$ and $\sigma_i=\frac{1}{\sqrt{2\lambda_i}}$, respectively. Moreover, taking into account that a random variable and its standardized version have PRH functions, and the standard normal distribution has the DRH property [38], we obtain that $U_i \in IMIT$.

Upon considering the cdf of U_is and (5), the marginal $X_i \sim EV3(\beta,\lambda_i+\lambda_3,2)$ ($i=1,2$). Thus, their MIT can be written as

$$m_{X_i}(x)=\frac{\Phi(x;\mu,\widetilde{\sigma}_i)}{\phi(x;\mu,\widetilde{\sigma}_i)},\ \text{for } x\le\beta$$

where $\widetilde{\sigma}_i=\frac{1}{\sqrt{2(\lambda_i+\lambda_3)}}$ for $i=1,2$, and consequently, $X_i \in IMIT$.

On the other hand, the following stochastic orderings among the three baseline components of two GBD models are preserved by their corresponding marginals. The proof immediately follows from the definitions of the stochastic orderings.

Theorem 9. *Let (X_1,X_2) and (Y_1,Y_2) have GBD models with base distribution vectors $(F_{U_1},F_{U_2},F_{U_3})$ and $(F_{V_1},F_{V_2},F_{V_3})$, respectively.*

1. *If $U_i\le_{st}V_i$ ($i=1,2,3$), then $X_i\le_{st}Y_i$ ($i=1,2$).*
2. *If $U_i\le_{rh}V_i$ ($i=1,2,3$), then $X_i\le_{rh}Y_i$ ($i=1,2$).*

Finally, we discuss some stochastic properties of the minimum and maximum order statistics of the GBD family. In this setting, from (7), the reversed hazard function of the maximum statistic T_2 of (X_1,X_2) of GBD type is determined by the sum of the reversed hazard rates of the baseline distribution vector:

$$r_{T_2}(x)=r_{U_1}(x)+r_{U_2}(x)+r_{U_3}(x) \tag{11}$$

when the pdf f_{U_i} of U_i exists, $i=1,2,3$. Hence, it is immediate the following result.

Theorem 10. *If $U_i\in DRH(IRH)$ ($i=1,2,3$), then $T_2\in DRH(IRH)$.*

Example 5. *Suppose $U_i\sim EV3(\beta,\lambda_i,k_i)$ ($i=1,2,3$). Then, the reversed hazard function of T_2 is given by*

$$r_{T_2}(x)=\sum_{i=1}^{3}\lambda_i k_i(\beta-x)^{k_i-1},$$

and therefore, if every $k_i\le(\ge)1$, $i=1,2,3$, then r_{T_2} is increasing (decreasing) in x, i.e. $T_2\in IRH(DRH)$.

Example 6. *If $U_i\sim GE(\theta_i,\lambda_i)$, then the maximum statistic of the EBGE model is DRH, $T_2\in DRH$, since (11) is the sum of three decreasing functions.*

Remark 5. *When U_is have a common distribution F_U, the GBD model has a maximum statistic whose cdf is F_U cube, and (11) can be written as $r_{T_2}(x)=3r_U(x)$. In particular, if $F_U\in DRH$ (IRH) then $T_2\in DRH$ (IRH).*

Remark 6. *From Corollary 2, if $U_i\sim PRH(\theta_i)$ with the same base distribution F_B, $T_2\sim PRH(\theta)$ with base F_B and $\theta=\theta_1+\theta_2+\theta_3$, i.e., $r_{T_2}(x)=\theta r_B(x)$. Thus, $T_2\in DRH$ (IRH) if and only if $F_B\in DRH$ (IRH).*

Furthermore, the MIT of the maximum statistic of a GBD model (X_1,X_2) can be derived by

$$m_{T_2}(x)=\frac{1}{F_{U_1}(x)F_{U_2}(x)F_{U_3}(x)}\int_{-\infty}^{x}F_{U_1}(y)F_{U_2}(y)F_{U_3}(y)dy,$$

for each specific baseline distribution vector $(F_{U_1},F_{U_2},F_{U_3})$, when the integral exists. For instance, we will consider a particular case, similar to one used in Example 4.

Example 7. *Suppose* (X_1, X_2) *has a GBD model with* $U_i \sim PRH(\theta_i)$ *and base distributions* $F_{B_i} \sim EV3(\beta, \lambda_i, k = 2)$ *for* $i = 1, 2, 3$*, then each component* $U_i \sim EV3(\beta, \theta_i\lambda_i, 2)$*, and consequently,* $U_i \in IMIT$ *for* $i = 1, 2, 3$*. Moreover, from Corollary 2, the maximum statistic* $T_2 \sim EV3(\beta, \theta^*, 2)$ *with* $\theta^* = \theta_1\lambda_1 + \theta_2\lambda_2 + \theta_3\lambda_3$*. Thus,* $T_2 \in IMIT$ *which is obtained along the same line as Example 4, since*

$$m_{T_2}(x) = \frac{\Phi(x; \beta, (2\theta^*)^{-1/2})}{\phi(x; \beta, (2\theta^*)^{-1/2})}, \text{ for } x \leq \beta.$$

Regarding the minimum statistic T_1 of (X_1, X_2) of GBD type, some preservation results are also obtained based on its reversed hazard rate r_{T_1}, the proofs are given in Appendix A, and from (7) r_{T_1} can be written as

$$r_{T_1}(x) = r_{U_{1:2}}(x) + r_{U_3}(x). \tag{12}$$

Theorem 11. *If* $U_i \in DRH$ $(i = 1, 2, 3)$ *and* $U_{1:2} \leq_{rh} U_i$ $(i = 1, 2)$*, then* $T_1 \in DRH$*.*

Corollary 4. *If* $U_i \in DRH$ $(i = 1, 2, 3)$ *and* $U_1 =_{st} U_2$*, then* $T_1 \in DRH$*.*

Example 8. *Suppose* $U_i \sim GE(\theta, \lambda)$ *for* $i = 1, 2$ *and* $U_3 \sim GE(\theta_3, \lambda_3)$*, then* $U_i \in DRH$*, and consequently,* $T_1 \in DRH$ *from Corollary 4.*

Remark 7. *Note that when* U_i*s have a common distribution* F_U*, (12) can be expressed as* $r_{T_1}(x) = r_U(x)(3 - 2/(2 - F_U(x)))$*, and from Corollary 4, it is immediate to have that if* $F_U \in DRH$ *then* $T_1 \in DRH$*.*

Theorem 12. *Let* (X_1, X_2) *be a GBD model. Then,* $T_1 \leq_{rh} T_2$*.*

Proof. From (11) and (12), the statement is equivalent to $r_{U_{1:2}}(x) \leq r_{U_{2:2}}(x)$, which readily follows from Theorem 1.B.56 of Shaked and Shanthikumar [35], since the baseline components U_is are independent. □

5.3. Copula and related association measures

Let us see now the copula representation of the GBD family and some related dependence measures of interest in the analysis of two-dimensional data.

It is well known that the dependence between the random variables X_1 and X_2 is completely described by the joint cdf $F(x_1, x_2)$, and it is often represented by a copula which describes the dependence structure in a separate form from the marginal behaviour. In this setting, from Sklar's theorem (e.g., see [39]), if its marginal cdfs F_{X_i}s are absolutely continuous, then the joint cdf has a unique copula representation for

$$F(x_1, x_2) = C\left(F_{X_1}(x_1), F_{X_2}(x_2)\right),$$

and reciprocally, if $F_{X_i}^{-1}$ is the inverse function of F_{X_i} $(i = 1, 2)$, then there exists a unique copula C in $[0, 1]^2$, such that

$$C(u_1, u_2) = F\left(F_{X_1}^{-1}(u_1), F_{X_2}^{-1}(u_2)\right).$$

Now we can derive the copula representation for the joint cdf of the GBD family as a function of its base distribution vector $(F_{U_1}, F_{U_2}, F_{U_3})$. In order to do this, by using (5), the joint cdf (1) can be expressed as

$$F(x_1, x_2) = F_{X_1}(x_1)F_{X_2}(x_2)\frac{F_{U_3}(\min(x_1, x_2))}{F_{U_3}(x_1)F_{U_3}(x_2)}.$$

and taking $u_i = F_{X_i}(x_i)$, the associated copula for an arbitrary base distribution vector $(F_{U_1}, F_{U_2}, F_{U_3})$ can be written as

$$C(u_1, u_2) = u_1 u_2 \frac{\min(A_1(u_1), A_2(u_2))}{A_1(u_1)A_2(u_2)}, \tag{13}$$

where

$$A_i(u_i) = F_{U_3}\left((F_{U_i} \times F_{U_3})^{-1}(u_i)\right),\ i = 1, 2,$$

which allows us to give an additional result.

Theorem 13. *Let $\boldsymbol{X} = (X_1, X_2)$ and $\boldsymbol{Y} = (Y_1, Y_2)$ be two GBD models with baseline distribution vectors $(F_{U_1}, F_{U_2}, F_{U_3})$ and $(F_{V_1}, F_{V_2}, F_{V_3})$, respectively. If $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the same associated copula and $U_i \leq_{st} V_i$, then $\boldsymbol{X} \leq_{st} \boldsymbol{Y}$.*

Proof. It is immediate by using Theorem 6.B.14 of Shaked and Shanthikumar [35] and (5), since $U_i \leq_{st} V_i$ implies $X_i \leq_{st} Y_i$. □

Corollary 5. *Let $\boldsymbol{X} = (X_1, X_2)$ and $\boldsymbol{Y} = (Y_1, Y_2)$ be two GBD models with common baseline distributions, F_U and F_V, respectively. If $U \leq_{st} V$, then $\boldsymbol{X} \leq_{st} \boldsymbol{Y}$.*

Note that (13) provides a general formula to establish the specific copula upon considering two particular continuous and increasing bijective functions A_1 and A_2 from $[0, 1]$ onto $[0, 1]$. Fang and Li [40] analyzed some stochastic orderings for an equivalent copula representation to (13) with interesting applications in network security and insurance. In the last section, we shall use the bivariate copula representation (13) to discuss the multivariate extension of the GBD family.

Furthermore, (13) may be considered a generalization of the Marshall-Olkin copula, as displayed in the following results whose proofs are omitted.

Corollary 6. *If (X_1, X_2) has a GBD model with a common base distribution F_U, then the copula representation of its joint cdf is*

$$C(u_1, u_2) = \min(u_1 u_2^{1/2}, u_1^{1/2} u_2).$$

Corollary 7. *If (X_1, X_2) has a GBD model with PRHs baseline distribution vector of the same base F_B, i.e., $(X_1, X_2) \sim BPRH(\theta_1, \theta_2, \theta_3)$, then the copula representation of its joint cdf is*

$$C(u_1, u_2) = \min\left(u_1 u_2^{\theta_2/(\theta_2+\theta_3)}, u_1^{\theta_1/(\theta_1+\theta_3)} u_2\right).$$

Some association measures for a bivariate random vector (X_1, X_2) of GBD type can be derived from the dependence structure described by the general expression (13) for each particular pair of continuous and increasing bijective functions A_1 and A_2 determined by the specific baseline distribution vector. For instance, for the special GBD models given in Corollaries 6 and 7, the measures of dependence namely Kendall's tau, Spearman's rho, Blomqvist's beta and tail dependence coefficients, see Nelsen [39] among others, can be calculated as follows.

Kendall's tau. The Kendall's τ is defined as the probability of concordance minus the probability of discordance between two pairs of independent and identically distributed random vectors, (X_1, X_2) and (Y_1, Y_2), as follows

$$\tau = P\left((X_1 - Y_1)(X_2 - Y_2) > 0\right) - P\left((X_1 - Y_1)(X_2 - Y_2) < 0\right),$$

and it can be calculated through its copula representation $C(u_1, u_2)$ by

$$\tau = 4E\left(C(\mathcal{U}_1, \mathcal{U}_2)\right) - 1 = 1 - 4\iint_{[0,1]^2} \frac{\partial C(u_1, u_2)}{\partial u_1}\frac{\partial C(u_1, u_2)}{\partial u_2} du_1 du_2 \tag{14}$$

with $\mathcal{U}_i$s uniform $[0,1]$ random variables whose joint cdf is C.

For example, if (X_1, X_2) has a GBD model with a common baseline F_U, upon substituting from the copula of Corollary 6 in (14), it is easy to check that Kendall's $\tau = 1/3$.

Analogously, from the copula given in Corollary 7 of the GBD model for $PRH(\theta_i)$ components with a common base F_B, the Kendall's τ coefficient (14) can be written as

$$\tau = \frac{\theta_3}{\theta_1 + \theta_2 + \theta_3}.$$

Spearman's rho. The Spearman's ρ coefficient measures the dependence by three pairs of independent and identically distributed random vectors, (X_1, X_2), (Y_1, Y_2) and (Z_1, Z_2). It is defined as

$$\rho = 3\left(P((X_1 - Y_1)(X_2 - Z_2) > 0) - P((X_1 - Y_1)(X_2 - Z_2) < 0)\right),$$

which can be computed by its copula representation $C(u_1, u_2)$ by

$$\rho = 12E\left(\mathcal{U}_1\mathcal{U}_2\right) - 3. \tag{15}$$

Thus, if there is a common base distribution as in Corollary 6, the Spearman's ρ coefficient between X_1 and X_2 is $\rho = 3/7$.

In the case of $U_i \sim PRH(\theta_i)$ with a common base distribution F_B, from (15) and Corollary 7, this association measure is

$$\rho = \frac{3\theta_3}{2\theta_1 + 2\theta_2 + 3\theta_3}$$

which coincides with one obtained by Kundu et al. [9] for this specific GBD model, $(X_1, X_2) \sim BPRH(\theta_1, \theta_2, \theta_3)$. As remarked by Kundu et al. [9] for the BPRH model, both coefficients, τ and ρ vary between 0 and 1 as θ_3 varies from 0 to ∞.

Blomqvist's beta. The Blomqvist's β coefficient, also called the medial correlation coefficient, is defined as the probability of concordance minus the probability of discordance between (X_1, X_2) and its median point, say (m_1, m_2), taking the following form:

$$\beta = P\left((X_1 - m_1)(X_2 - m_2) > 0\right) - P\left((X_1 - m_1)(X_2 - m_2) < 0\right) = 4F(m_1, m_2) - 1,$$

and from the copula of its joint cdf F, it can be expressed as

$$\beta = 4C(1/2, 1/2) - 1. \tag{16}$$

In the case of Corollary 6, it is immediate that the medial correlation coefficient between X_1 and X_2 is $\beta = \sqrt{2} - 1$ when it follows a GBD model with a common baseline distribution.

In the other case, from Corollary 7, it is also readily to obtain the Blomqvist's β coefficient (16) between the marginals of a BPRH model:

$$\beta = \begin{cases} 2^{\theta_3/(\theta_2+\theta_3)}, & \text{if } \theta_1 \le \theta_2 \\ 2^{\theta_3/(\theta_1+\theta_3)}, & \text{if } \theta_1 > \theta_2, \end{cases}$$

which takes values between 0 and 1 as θ_3 varies from 0 to ∞.

Tail dependence. The tail dependence measures the association of extreme events in both directions, the upper (lower) tail dependence λ_U (λ_L) provides an asymptotical association measurement in the upper (lower) quadrant tail of a bivariate random vector, given by (if it exists)

$$\lambda_U(\lambda_L) = \lim_{u \to 1^-(0^+)} P\left(X_2 > (\le) F_{X_2}^{-1}(u) | X_1 > (\le) F_{X_1}^{-1}(u)\right).$$

Similar to the above association coefficients, the tail dependence indexes can be calculated from the copula representation $C(u_1, u_2)$ of the joint cdf of (X_1, X_2), as follows

$$\lambda_U = 2 - \lim_{u \to 1^-} \frac{1 - C(u,u)}{1-u} \quad \text{and} \quad \lambda_L = \lim_{u \to 0^+} \frac{C(u,u)}{u}. \tag{17}$$

In particular, if (X_1, X_2) follows a GBD model with a common baseline distribution, upon substituting from the copula of Corollary 6 in (17), it is easy to check that $\lambda_L = 0$ and $\lambda_U = 1/2$.

In the case of $U_i \sim PRH(\theta_i)$ with the same base, from (17) and Corollary 7, it is clear that the tail dependence indexes of the BPRH model are $\lambda_L = 0$ and

$$\lambda_U = \begin{cases} \frac{\theta_3}{\theta_2+\theta_3}, & \text{if } \theta_1 \leq \theta_2 \\ \frac{\theta_3}{\theta_1+\theta_3}, & \text{if } \theta_1 > \theta_2, \end{cases}$$

which takes values between 0 and 1 as θ_3 varies from 0 to ∞.

6. Maximum likelihood estimation

In this section we address the problem of computing the maximum likelihood estimations (MLEs) of the unknown parameters based on a random sample. The problem can be formulated as follows. Suppose $\{(x_{1i}, x_{2i}); i = 1, \ldots, n\}$ is a random sample of size n from a GBD model, where it is assumed that for $j = 1, 2, 3$, U_j has the pdf $f_{U_j}(u; \boldsymbol{\theta}_j)$ and $\boldsymbol{\theta}_j$ is of dimension p_j. The objective is to estimate the unknown parameter vector $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3)$. We use the following partition of the sample:

$$I_1 = \{i : x_{1i} < x_{2i}\}, \;\; I_2 = \{i : x_{1i} > x_{2i}\}, \;\; I_0 = \{i : x_{1i} = x_{2i} = x_i\}.$$

Based on the above observations, the log-likelihood function becomes

$$\ell(\boldsymbol{\theta}) = \sum_{i \in I_0} \ln f_0(x_i; \boldsymbol{\theta}) + \sum_{i \in I_1} \ln f_1(x_{1i}, x_{2i}; \boldsymbol{\theta}) + \sum_{i \in I_2} \ln f_2(x_{1i}, x_{2i}; \boldsymbol{\theta}),$$

where $f_0(x_i; \boldsymbol{\theta})$, $f_1(x_{1i}, x_{2i}; \boldsymbol{\theta})$, $f_2(x_{1i}, x_{2i}; \boldsymbol{\theta})$ have been defined in Theorem 3.

Here, it is difficult to compute the MLEs of the unknown parameter vector $\boldsymbol{\theta}$ by solving a $p_1 + p_2 + p_3$ optimization problem. To avoid that, we suggest using the EM algorithm, and the basic idea is based on considering a random sample of size n from (U_1, U_2, U_3), instead of the random sample of size n from (X_1, X_2). From the observed sample $\{(x_{1i}, x_{2i}\}$, the sample $\{(u_{1i}, u_{2i}, u_{3i}); i = 1, \ldots, n\}$ has missing values as shown in Table 1. It is immediate that the MLEs of $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$ and $\boldsymbol{\theta}_3$ can be obtained by solving the following three optimization problems of dimensions p_1, p_2 and p_3, respectively,

$$\ell_j(\boldsymbol{\theta}_j) = \sum_{i=1}^{n} \ln f_{U_j}(u_{ji}; \boldsymbol{\theta}_j); \;\; j = 1, 2, 3,$$

which are computationally more tractable.

Table 1. Relation between (x_{1i}, x_{2i}) and (u_{1i}, u_{2i}, u_{3i}).

I_k	**Ordering of** U_j	X_1	X_2	**Missing**
I_0	$u_{1i} < u_{2i} < u_{3i}$	u_{3i}	u_{3i}	u_{1i}, u_{2i}
I_0	$u_{2i} < u_{1i} < u_{3i}$	u_{3i}	u_{3i}	u_{1i}, u_{2i}
I_1	$u_{1i} < u_{3i} < u_{2i}$	u_{3i}	u_{2i}	u_{1i}
I_1	$u_{3i} < u_{1i} < u_{2i}$	u_{1i}	u_{2i}	u_{3i}
I_2	$u_{2i} < u_{3i} < u_{1i}$	u_{1i}	u_{3i}	u_{2i}
I_2	$u_{3i} < u_{2i} < u_{1i}$	u_{2i}	u_{1i}	u_{3i}

From Table 1, if $i \in I_0$, then u_{3i} is known, and u_{1i} and u_{2i} are unknown. Similarly, if $i \in I_1$ ($i \in I_2$), then u_{2i} (u_{1i}) and $\max\{u_{1i}, u_{3i}\}$ ($\max\{u_{2i}, u_{3i}\}$) are known. Hence, in the E-step of the EM algorithm, the 'pseudo' log-likelihood function is formed by replacing the missing u_{ji} by its expected value, $u_{jim}(\boldsymbol{\theta})$, for $i = 1, \ldots, n$ and $j = 1, 2, 3$:

1. If $i \in I_0$, then
$$u_{jim}(\boldsymbol{\theta}) = E(U_j | U_j < x_i) = \frac{1}{F_{U_j}(x_i)} \int_{-\infty}^{x_i} u f_{U_j}(u) du, \ \ j = 1, 2.$$
2. If $i \in I_1$ and $j, k \in \{1, 3\}, j \neq k$, then
$$\begin{aligned} u_{jim}(\boldsymbol{\theta}) &= E(U_j | \max\{U_1, U_3\} = x_{1i}) \\ &= x_{1i} P(U_j > U_k) + P(U_j < U_k) \frac{1}{F_{U_j}(x_{1i})} \int_{-\infty}^{x_{1i}} u f_{U_j}(u) du \\ &= x_{1i} \int_{-\infty}^{\infty} f_{U_j}(u) F_{U_k}(u) du + \frac{1}{F_{U_j}(x_{1i})} \int_{-\infty}^{\infty} f_{U_k}(u) F_{U_j}(u) du \int_{-\infty}^{x_{1i}} u f_{U_j}(u) du. \end{aligned}$$
3. If $i \in I_2$ and $j, k \in \{2, 3\}, j \neq k$, then
$$\begin{aligned} u_{jim}(\boldsymbol{\theta}) &= E(U_j | \max\{U_2, U_3\} = x_{2i}) \\ &= x_{2i} P(U_j > U_k) + P(U_j < U_k) \frac{1}{F_{U_j}(x_{2i})} \int_{-\infty}^{x_{2i}} u f_{U_j}(u) du \\ &= x_{2i} \int_{-\infty}^{\infty} f_{U_j}(u) F_{U_k}(u) du + \frac{1}{F_{U_j}(x_{2i})} \int_{-\infty}^{\infty} f_{U_k}(u) F_{U_j}(u) du \int_{-\infty}^{x_{2i}} u f_{U_j}(u) du. \end{aligned}$$

Therefore, we propose the following EM algorithm to compute the MLEs of $\boldsymbol{\theta}$. Suppose at the k-th iteration of the EM algorithm, the value of $\boldsymbol{\theta}$ is $\boldsymbol{\theta}^{(k)} = (\boldsymbol{\theta}_1^{(k)}, \boldsymbol{\theta}_2^{(k)}, \boldsymbol{\theta}_3^{(k)})$, then the following steps can be used to compute $\boldsymbol{\theta}^{(k+1)}$:

E-step

- At the k-th step for $i \in I_0$, obtain the missing u_{1i} and u_{2i} as $u_{1im}(\boldsymbol{\theta}^{(k)})$ and $u_{2im}(\boldsymbol{\theta}^{(k)})$, respectively. For $i \in I_1$ obtain the missing u_{1i} and u_{3i} as $u_{1im}(\boldsymbol{\theta}^{(k)})$ and $u_{3im}(\boldsymbol{\theta}^{(k)})$, respectively. Similarly, for $i \in I_2$, obtain the missing u_{2i} and u_{3i} as $u_{2im}(\boldsymbol{\theta}^{(k)})$ and $u_{3im}(\boldsymbol{\theta}^{(k)})$, respectively.
- Form the 'pseudo' log-likelihood function as $\ell_s^{(k)}(\boldsymbol{\theta}) = \ell_{1s}^{(k)}(\boldsymbol{\theta}_1) + \ell_{2s}^{(k)}(\boldsymbol{\theta}_2) + \ell_{3s}^{(k)}(\boldsymbol{\theta}_3)$, where

$$\begin{aligned} \ell_{1s}^{(k)}(\boldsymbol{\theta}_1) &= \sum_{i \in I_0} \ln f_{U_1}(u_{1im}(\boldsymbol{\theta}^{(k)}); \boldsymbol{\theta}_1) + \sum_{i \in I_1} \ln f_{U_1}(u_{1im}(\boldsymbol{\theta}^{(k)}); \boldsymbol{\theta}_1) + \sum_{i \in I_2} \ln f_{U_1}(u_{1i}; \boldsymbol{\theta}_1) \\ \ell_{2s}^{(k)}(\boldsymbol{\theta}_2) &= \sum_{i \in I_0} \ln f_{U_2}(u_{2im}(\boldsymbol{\theta}^{(k)}); \boldsymbol{\theta}_2) + \sum_{i \in I_1} \ln f_{U_2}(u_{2i}; \boldsymbol{\theta}_2) + \sum_{i \in I_2} \ln f_{U_2}(u_{2im}(\boldsymbol{\theta}^{(k)}); \boldsymbol{\theta}_2) \\ \ell_{3s}^{(k)}(\boldsymbol{\theta}_3) &= \sum_{i \in I_0} \ln f_{U_3}(u_{3i}; \boldsymbol{\theta}_3) + \sum_{i \in I_1} \ln f_{U_3}(u_{3im}(\boldsymbol{\theta}^{(k)}); \boldsymbol{\theta}_3) + \sum_{i \in I_2} \ln f_{U_3}(u_{3im}(\boldsymbol{\theta}^{(k)}); \boldsymbol{\theta}_3). \end{aligned}$$

M-step

- $\boldsymbol{\theta}^{(k+1)} = (\boldsymbol{\theta}_1^{(k+1)}, \boldsymbol{\theta}_2^{(k+1)}, \boldsymbol{\theta}_3^{(k+1)})$ can be obtained by maximizing $\ell_{1s}^{(k)}(\boldsymbol{\theta}_1)$, $\ell_{2s}^{(k)}(\boldsymbol{\theta}_2)$ and $\ell_{3s}^{(k)}(\boldsymbol{\theta}_3)$ with respect to $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$ and $\boldsymbol{\theta}_3$, respectively.

Mainly for illustrative purposes, two particular GBD models will be applied in the next section to show the usefulness of the above EM algorithm. Firstly, we shall consider a GBD model with baseline components having the same distribution type and different underlying parameters. Secondly, we shall use a GBD model with baseline components from different distribution families. The technical details of both of them can be found in Appendix B.

7. Data analysis

In this section we present the analysis of two-dimensional data sets in order to show how the proposed EM algorithm can be applied to fit particular GBD models. For that, we shall suppose the following two models described in Appendix B: Model I is the GBD model with the exponential baseline distributions and different underlying parameters, $U_j \sim Exp(\lambda_j)$ $(j = 1, 2, 3)$. Model II is the GBD model with baseline components from Weibull and generalized exponential distributions, $U_1 \sim W(\lambda_1, \alpha_1)$, $U_2 \sim W(\lambda_2, \alpha_2)$ and $U_3 \sim GE(\alpha_3, \lambda_3)$.

7.1. Soccer data

We have analyzed a UEFA Champion's League data set [41], played during the seasons 2004-2005 and 2005-2006. This set represents the soccer data where at least one goal has been scored by a kick goal (penalty kick, foul kick or any other direct kick) by any team and one goal has been scored by the home team. Here in the bivariate data (X_1, X_2), X_1 represents the time in minutes of the first kick goal and X_2 represents the time in minutes scored by the home team. Clearly, all possibilities exist in the data set, namely $X_1 < X_2$, $X_1 > X_2$ and $X_1 = X_2$.

Meintanis [41] analyzed this data set using the Marshall-Olkin bivariate exponential model. The marginals of the Marshall-Olkin bivariate exponential distribution are exponential, and then they have constant hazard functions. A preliminary data analysis indicated that the empirical hazard function of both the marginals are increasing and their reversed hazard functions are decreasing. Hence, it may not be proper to use the Marshall-Olkin bivariate exponential model to analyze this data.

Example 9. *In order to use Model I, we have started the initial guess as* $\lambda_1^{(0)} = \lambda_2^{(0)} = \lambda_3^{(0)} = 1$. *The algorithm stops after eight iterations, the final estimates and the associated 95% confidence intervals are* $\widehat{\lambda}_1 = 0.03126$ (± 0.01121), $\widehat{\lambda}_2 = 0.04630$ (± 0.01563) *and* $\widehat{\lambda}_3 = 0.04269$ (± 0.01875), *with* -257.8871 *being the pseudo log-likelihood value. To check whether it has converged to the maximum or not, the performance of the EM algorithm may be compared with the experimental results obtained by using a quasi-Newton method for solving constrained non-linear optimization problem, which have been summarized in Appendix C as well as the corresponding ones to the subsequent examples.*

One natural question is whether Model I fits the bivariate data or not. We have computed the Kolmogorov-Smirnov (KS) distances with the corresponding p-values between the empirical and fitted cdfs for the marginals and the maximum order statistic. The results are reported in Table 2, and from them, we cannot reject the null hypothesis that this data is coming from the GBD model with exponential baseline distributions.

Example 10. *Let us consider now Model II. We have started the EM algorithm with the initial guesses as* $\alpha_1^{(0)} = \alpha_2^{(0)} = \alpha_3^{(0)} = 1$, $\lambda_1^{(0)} = 0.03$, $\lambda_2^{(0)} = 0.05$ *and* $\lambda_3^{(0)} = 0.04$. *The algorithm converges in nineteen iterations, the final estimates and the associated 95% confidence intervals are* $\widehat{\alpha}_1 = 1.2987$ (± 0.3124), $\widehat{\lambda}_1 = 0.0097$ (± 0.0005), $\widehat{\alpha}_2 = 0.8047$ (± 0.2823), $\widehat{\lambda}_2 = 0.0093$ (± 0.0021), $\widehat{\alpha}_3 = 1.0037$ (± 0.2879), $\widehat{\lambda}_3 = 0.0369$ (± 0.008), *with* -201.1141 *being the pseudo log-likelihood value.*

The KS distances with the corresponding p-values for the marginals and the maximum statistic are reported in Table 2. Thus, based on the p-values, we can say that the GBD model with two baseline Weibull distributions and the third GE one fits the data reasonably well.

Summarizing, it is clear both the GBD models provide good fit to the given data set and the EM algorithm also works quite effectively in both the cases. Now to compare Models I and II of Examples 9 and 10, which provide a better fit, we compute the Akaike's information criterion (AIC) and Bayesian information criterion (BIC) values and they are also presented in Table 2. Therefore, based on the AIC and BIC values, it is clear that Model I provides a better fit than Model II to the UEFA Champion's League data set.

Table 2. Goodness-of-fit results for UEFA Champion's League data.

GBD model	KS (p-value) X_1	X_2	$\max\{X_1, X_2\}$	AIC	BIC
Model I	0.1491 (0.3830)	0.1099 (0.7622)	0.1530 (0.3517)	604.8663	609.6990
Model II	0.0976 (0.8719)	0.0839 (0.9565)	0.1139 (0.7228)	708.5430	718.2085

7.2. Diabetic retinopathy data

Let us consider now the diabetic retinopathy data set [42], available in the R package "SurvCor" [43]. Such a data was investigated by the National Eye Institute to assess the effect of laser photocoagulation in delaying the onset of severe visual loss such as blindness in 197 patients with diabetic retinopathy. For each patient, one eye was randomly selected for laser photocoagulation and the other was given no treatment, used as the control. The times to blindness in both eyes were recorded in months and the censoring was caused by death, dropout, or the end of the study.

For illustrative purposes we have considered those patients for which complete data are available. Here X_1 denotes the time to the blindness of the untreated or control eye and X_2 denotes the time to blindness of the treated eye. Out of 197 patients, we have complete information of X_1 and X_2 for 38 patients.

Example 11. *As in Example 9, we have used Model I to analyze the data set. In this case we have also used the same initial guess as $\lambda_1^{(0)} = \lambda_2^{(0)} = \lambda_3^{(0)} = 1$. We have used the proposed EM algorithm, the iteration stops after 14 iterations, and the estimates of unknown parameters and the corresponding 95% confidence intervals are $\widehat{\lambda}_1 = 0.0653\ (\pm 0.0175)$, $\widehat{\lambda}_2 = 0.0737(\pm 0.0210)$ and $\widehat{\lambda}_3 = 0.1345(\pm 0.3879)$, with -172.2314 being the associated pseudo log-likelihood value.*

The KS distances with the corresponding p-values between the empirical and fitted cdfs for the marginals and the maximum statistic are presented in Table 3.

Example 12. *As in Example 10, we have analyzed the data set by using Model II. We have started the EM algorithm with the initial guesses $\alpha_1^{(0)} = \alpha_2^{(0)} = \alpha_3^{(0)} = 1$, $\lambda_1^{(0)} = 0.06$, $\lambda_2^{(0)} = 0.07$ and $\lambda_3^{(0)} = 0.13$. The algorithm stops after 27 iterations, the final estimates and the corresponding 95% confidence intervals are $\widehat{\alpha}_1 = 1.0937\ (\pm 0.2563)$, $\widehat{\lambda}_1 = 0.0447\ (\pm 0.0146)$, $\widehat{\alpha}_2 = 0.5851\ (\pm 0.1345)$, $\widehat{\lambda}_2 = 0.2369\ (\pm 0.0763)$, $\widehat{\alpha}_3 = 0.8995\ (\pm 0.2787)$, $\widehat{\lambda}_3 = 0.1898\ (\pm 0.0478)$, with -125.4519 being the associated pseudo log-likelihood value.*

The KS distances with the corresponding p-values for the marginals and the maximum order statistic are presented in Table 3.

From Table 3, we can also say that the estimated GBD models fit the diabetic retinopathy data reasonably well in both the cases. Moreover, we also present the AIC and BIC values of the two models in Table 3. Therefore, based on the AIC and BIC values, it is clear that Model I provides a better fit than Model II for the diabetic retinopathy data.

Table 3. Goodness-of-fit results for diabetic retinopathy data.

GBD model	KS (p-value) X_1	X_2	$\max\{X_1, X_2\}$	AIC	BIC
Model I	0.1033 (0.8244)	0.1848 (0.1598)	0.1229 (0.6310)	585.9757	590.8884
Model II	0.0920 (0.8960)	0.0952 (0.8706)	0.1152 (0.6778)	592.1515	601.9770

8. Discussion and conclusions

In this paper we have presented the generalized bivariate distribution family by a generator system based on the maximization process from any three-dimensional baseline continuous distribution vector with independent components, providing bivariate models with dependence structure.

For the proposed GBD family, several distributional and stochastic properties have been established. The preservation of the PRH property for the marginals and the maximum order statistic has been obtained. It has been shown the positive dependence between both marginals of the GBD models, some results about stochastic orders and on the preservation of the monotonicity of the reversed hazard function and of the mean inactivity time. Further, it has been discussed the copula representation of the GBD model, providing a general formula, and some related dependence measures have been also calculated for specific copulas of particular bivariate distributions of the GBD family. In addition, new bivariate distributions can be generated by combining independent baseline components from different distribution families, and several bivariate distributions given in the literature are derived as particular cases of the GBD family.

Note that, even in the simple case, the MLEs cannot be obtained in explicit forms, and it is required solving a multidimensional non-linear optimization problem. We have proposed to use an EM algorithm to compute the MLEs of the unknown parameters, and it is observed that the proposed EM algorithm perform quite satisfactorily in the two data analyses by using two different models of the GBD family. The experimental results summarized in Table A1 disclose such efficiency of the EM algorithm with the respect to a conventional numerical iterative procedure of Newton-type. In detail, Table A1 presents the experimental results obtained by the Broyden-Fletcher-Goldfarb-Shanno algorithm for maximizing the log-likelihood function, available in the R package "maxLik" [44].

It is worth mentioning that the bivariate copula representation (13) allows us to discuss its multivariate extension. Let U_is for $i = 1, ..., q+1$ be a set of $q+1$ mutually independent random variables with any continuous distribution functions, denoting by F_{U_i} the cdf of each U_i. Similarly to (1), the joint cdf of the q-dimensional random vector $(X_1, ..., X_q)$ with $X_i = \max(X_1, X_2)$ is given by

$$F(x_1, ...x_q) = F_{U_{q+1}}(\min(x_1, ..., x_q)) \prod_{i=1}^{q} F_{U_i}(x_i)$$

which can be considered as a generator of q-dimensional distribution models, called generalized multivariate distribution (GMD) family with baseline distribution vector $(F_{U_1}, ..., F_{U_{q+1}})$. Hence, the q-dimensional copula representation of this GMD family can be expressed as

$$C(u_1, ..., u_q) = \left(\prod_{i=1}^{q} u_i\right) \frac{\min_{i=1,...,q} A_i(u_i)}{\prod_{i=1}^{q} A_i(u_i)},$$

where

$$A_i(u_i) = F_{U_{q+1}}\left((F_{U_i} \times F_{U_{q+1}})^{-1}(u_i)\right), \text{ for } i = 1, ..., q.$$

From these q-dimensional joint cdf and copula, many distributional and stochastic properties established for the GBD family are extensible to the GMD family. Further, by using this generator of multivariate distributions, the special bivariate models given in Section 3 can be easily extended to the multivariate case, which contain multivariate versions of bivariate distributions given in the literature.

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Appendix A

Proof of Theorem 2. First, taking into account the event $A = (U_3 > \max(U_1, U_2))$, the joint cdf can be expressed as

$$F(x_1, x_2) = P(U_1 \leq x_1, U_2 \leq x_2, U_3 \leq \min(x_1, x_2)|A)P(A) \\ + P(U_1 \leq x_1, U_2 \leq x_2, U_3 \leq \min(x_1, x_2)|A')P(A')$$

where A' is the complementary event of A. For $z = min(x_1, x_2)$, note that

$$P(U_1 \leq x_1, U_2 \leq x_2, U_3 \leq z|A) = P(U_1 \leq x_1, U_2 \leq x_2, U_3 \leq z|U_1 < U_3, U_2 < U_3) \\ = P(U_1 \leq U_3, U_2 \leq U_3, U_3 \leq z) = \int_{-\infty}^{z} F_{U_1}(u)F_{U_2}(u)dF_{U_3}(u).$$

Hence, it is immediate that $F_s(x_1, x_2)$ given by (3) is a singular cdf as its mixed second partial derivatives are zero when $x_1 \neq x_2$.

Thus, $\alpha = P(A)$ may be established as follows

$$\alpha = P(U_3 > \max(U_1, U_2)) = \int_{-\infty}^{\infty} P(U_1 < u, U_2 < u)dF_{U_3}(u) \\ = \int_{-\infty}^{\infty} F_{U_1}(u)F_{U_2}(u)dF_{U_3}(u),$$

and consequently, the bivariate cdf $F(x_1, x_2)$ can be rewritten as (2), where the absolutely continuous part $F_{ac}(x_1, x_2)$ can be obtained by subtraction:

$$F_{ac}(x_1, x_2) = P(U_1 \leq x_1, U_2 \leq x_2, U_3 \leq \min(x_1, x_2)|A') \\ = \frac{1}{1-\alpha}\left(F(x_1, x_2) - \alpha F_s(x_1, x_2)\right) \\ = \frac{1}{1-\alpha}\left(F_{U_1}(x_1)F_{U_2}(x_2)F_{U_3}(z) - \int_{-\infty}^{z} F_{U_1}(u)F_{U_2}(u)dF_{U_3}(u)\right),$$

which completes the proof of the theorem. □

Proof of Theorem 3. Let μ, μ_s and μ_{ac} be the measures associated with F, F_s and F_{ac}, respectively. Obviously, μ_{ac} is an absolutely continuous measure with respect to the two-dimensional Lebesgue measure since

$$\mu_{ac}\left((-\infty, x_1] \times (-\infty, x_2]\right) = F_{ac}(x_1, x_2) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} f_{ac}(u, v)dudv$$

where the pdf associated to F_{ac} in (4), $f_{ac}(u, v) = \frac{\partial^2}{\partial u \partial v}F_{ac}(u, v)$, can be written as

$$f_{ac}(x_1, x_2) = \begin{cases} \frac{1}{1-\alpha}f_1(x_1, x_2), & \text{if } x_1 < x_2 \\ \frac{1}{1-\alpha}f_2(x_1, x_2), & \text{if } x_1 > x_2 \\ 0, & \text{if } x_1 = x_2 = x. \end{cases}$$

On the other hand, μ_s is given by

$$\mu_s\left((-\infty, x_1] \times (-\infty, x_2]\right) = F_s(x_1, x_2) = F_s(z, z) = \frac{1}{\alpha}\int_{-\infty}^{z} F_{U_1}(u)F_{U_2}(u)dF_{U_3}(u)$$

where $z = \min(x_1, x_2)$, and so it can be expressed as an absolutely continuous measure μ_s^* with respect to the one-dimensional Lebesgue measure on the projection onto the line $\mathbb{R}$ of the intersection between $(-\infty, x_1] \times (-\infty, x_2]$ and the line $x_1 = x_2$:

$$\mu_s\left((-\infty, x_1] \times (-\infty, x_2]\right) = \mu_s^*\left((-\infty, z]\right) = \int_{-\infty}^{z} f_s^*(u)du,$$

where $f_s^*(u) = \frac{1}{\alpha} F_{U_1}(u) F_{U_2}(u) f_{U_3}(u)$, which can be also written as $f_s^*(u) = \frac{1}{\alpha} f_0(u)$.

Further, it is trivial that the line $x_1 = x_2$ is a null set under the two-dimensional Lebesgue measure, and hence with respect to μ_{ac}. In addition, its complement $\{(x_1, x_2) \in \mathbb{R}^2 | x_1 \neq x_2\}$ is a null set with respect to μ_s, since its projection onto the line $\mathbb{R}$ is the empty set,

$$\mu_s\left(\{(x_1, x_2) \in \mathbb{R}^2 | x_1 \neq x_2\}\right) = \mu_s^*\left(\varnothing\right) = 0,$$

and consequently, the measures μ_s and μ_{ac} are mutually singular. Therefore, the measure associated with F

$$\begin{aligned}\mu\left((-\infty, x_1] \times (-\infty, x_2]\right) = F(x_1, x_2) &= \alpha \mu_s\left((-\infty, x_1] \times (-\infty, x_2]\right) \\ &+ (1-\alpha)\mu_{ac}\left((-\infty, x_1] \times (-\infty, x_2]\right)\end{aligned}$$

allows us to have the pdf of a GBD model with respect to μ, given by

$$f(x_1, x_2) = \alpha f_s^*(x_1) I_{(x_1 = x_2)}(x_1, x_2) + (1-\alpha) f_{ac}(x_1, x_2)$$

where $I_{(x_1=x_2)}$ is the indicator function of $x_1 = x_2$. Hence, it is easy to check that

$$\int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(u, v) d\mu = F(x_1, x_2)$$

for all $(x_1, x_2) \in \mathbb{R}^2$. $\square$

Proof of Theorem 4. From (1) and (5), the proof of (1) of Theorem 4 is straightforward.

In order to prove (2) of Theorem 4, from the joint pdf of a GBD model given in Theorem 3 and its marginal pdf (6), the conditional pdf $f_{i|X_j = x_j}$ can be expressed as

$$f_{i|X_j=x_j}(x_i) = \begin{cases} \frac{f_{X_i}(x_i) f_{U_j}(x_j)}{f_{X_j}(x_j)}, & \text{if } x_i < x_j \\ f_{U_i}(x_i), & \text{if } x_i > x_j \\ \frac{F_{U_1}(x_j) F_{U_2}(x_j) f_{U_3}(x_j)}{f_{X_j}(x_j)}, & \text{if } x_i = x_j, \end{cases}$$

by using the notation $\alpha_j = f_{i|X_j=x_j}(x_j)$, this conditional pdf can be readily rewritten as in the statement of Theorem 4. $\square$

Proof of Theorem 11. The reversed hazard function (12) of the minimum statistic can be rewritten as

$$r_{T_1}(x) = r_{U_1}(x) g_2(x) + r_{U_2}(x) g_1(x) + r_{U_3}(x),$$

where each g_i is a positive function ($i = 1, 2$) defined by

$$g_i(x) = 1 - \frac{F_{U_i}(x)}{F_{U_{1:2}}(x)}.$$

Here, observe that $U_{1:2} \leq_{rh} U_i$ implies the decreasing monotonicity of $g_i(x)$, and therefore r_{T_1} is a sum of three decreasing functions, which completes the proof. $\square$

Proof of Corollary 4. The proof readily follows along the same line as Theorem 11, taking into account that (12) can be simplified by using

$$r_{U_{1:2}}(x) = 2r_{U_i}(x)g_i(x)$$

where $g_i(x) = 1 - \frac{1}{2-F_{U_i}(x)}$ decreases in x. □

Appendix B

For practical implementation of the EM algorithm in the data analysis applications, we give the technical details of the EM algorithm for two particular GBD models, first with baseline component vector with the same distribution (Model I), and then, with different baseline distributions (Model II).

Model I.

Suppose $U_1 \sim Exp(\lambda_1)$, $U_2 \sim Exp(\lambda_2)$ and $U_3 \sim Exp(\lambda_3)$. To compute the MLEs of the unknown parameter vector $\boldsymbol{\theta} = (\lambda_1, \lambda_2, \lambda_3)$, one needs to solve a three dimensional optimization problem.

For implementation of the EM algorithm we need the following expected values:

1. If $i \in I_0$, then

$$u_{1im}(\boldsymbol{\theta}) = E(U_1|U_1 < x_i) = H(x_i; \lambda_1)$$
$$u_{2im}(\boldsymbol{\theta}) = E(U_2|U_2 < x_i) = H(x_i; \lambda_2),$$

where

$$H(x; \lambda) = \frac{1}{\lambda} - \frac{xe^{-\lambda x}}{1 - e^{-\lambda x}}.$$

2. If $i \in I_1$, then

$$u_{1im}(\boldsymbol{\theta}) = E(U_1|\max\{U_1, U_3\} = x_{1i}) = \frac{\lambda_3}{\lambda_1 + \lambda_3} x_{1i} + \frac{\lambda_1}{\lambda_1 + \lambda_3} H(x_{1i}; \lambda_1)$$
$$u_{3im}(\boldsymbol{\theta}) = E(U_3|\max\{U_1, U_3\} = x_{1i}) = \frac{\lambda_1}{\lambda_1 + \lambda_3} x_{1i} + \frac{\lambda_3}{\lambda_1 + \lambda_3} H(x_{1i}; \lambda_3).$$

3. If $i \in I_2$, then

$$u_{2im}(\boldsymbol{\theta}) = E(U_2|\max\{U_2, U_3\} = x_{2i}) = \frac{\lambda_3}{\lambda_2 + \lambda_3} x_{2i} + \frac{\lambda_2}{\lambda_2 + \lambda_3} H(x_{2i}; \lambda_2)$$
$$u_{3im}(\boldsymbol{\theta}) = E(U_3|\max\{U_2, U_3\} = x_{2i}) = \frac{\lambda_2}{\lambda_2 + \lambda_3} x_{2i} + \frac{\lambda_3}{\lambda_2 + \lambda_3} H(x_{2i}; \lambda_3).$$

Hence, the 'pseudo' log-likelihood function in this case becomes

$$\ell_s^{(k)}(\lambda_1, \lambda_2, \lambda_3) = \ell_{1s}^{(k)}(\lambda_1) + \ell_{2s}^{(k)}(\lambda_2) + \ell_{3s}^{(k)}(\lambda_3),$$

where

$$\ell_{1s}^{(k)}(\lambda_1) = n \ln \lambda_1 - \lambda_1 \left[\sum_{i \in I_0 \cup I_1} u_{1im}^{(k)} + \sum_{i \in I_2} x_{1i} \right]$$
$$\ell_{2s}^{(k)}(\lambda_2) = n \ln \lambda_2 - \lambda_2 \left[\sum_{i \in I_0 \cup I_2} u_{2im}^{(k)} + \sum_{i \in I_1} x_{2i} \right]$$
$$\ell_{3s}^{(k)}(\lambda_3) = n \ln \lambda_3 - \lambda_3 \left[\sum_{i \in I_1 \cup I_2} u_{3im}^{(k)} + \sum_{i \in I_0} x_i \right],$$

and the $u^{(k)}_{jim}$s are obtained from $u_{jim}(\boldsymbol{\theta})$, $j = 1,2,3$, by replacing $\boldsymbol{\theta} = (\lambda_1, \lambda_2, \lambda_3)$ with $\boldsymbol{\theta}^{(k)} = (\lambda^{(k)}_1, \lambda^{(k)}_2, \lambda^{(k)}_3)$. Therefore,

$$\begin{aligned}
\lambda_1^{(k+1)} &= \frac{n}{\left[\sum_{i \in I_0 \cup I_1} u^{(k)}_{1im} + \sum_{i \in I_2} x_{1i}\right]} \\
\lambda_2^{(k+1)} &= \frac{n}{\left[\sum_{i \in I_0 \cup I_2} u^{(k)}_{2im} + \sum_{i \in I_1} x_{2i}\right]} \\
\lambda_3^{(k+1)} &= \frac{n}{\left[\sum_{i \in I_1 \cup I_2} u^{(k)}_{3im} + \sum_{i \in I_0} x_{i}\right]}.
\end{aligned}$$

Note that, in this case, the maximization can be performed analytically at each M-Step.

Model II.

Suppose $U_1 \sim W(\lambda_1, \alpha_1)$, $U_2 \sim W(\lambda_2, \alpha_2)$ and $U_3 \sim GE(\alpha_3, \lambda_3)$. The pdf of a Weibull distribution $W(\lambda, \alpha)$ with scale parameter $\lambda > 0$ and the shape parameter $\alpha > 0$ can be written as

$$f_W(u; \lambda, \alpha) = \alpha \lambda u^{\alpha-1} e^{-\lambda u^{\alpha}}, \ \text{ for } u > 0,$$

and, zero otherwise. Similarly, the $GE(\alpha, \lambda)$ model defined in Section 3 has the pdf

$$f_{GE}(u; \alpha, \lambda) = \alpha \lambda e^{-\lambda u} (1 - e^{-\lambda u})^{\alpha - 1}; \ \text{ for } u > 0,$$

and, zero otherwise. Hence, one needs to solve a six dimensional optimization problem to compute the MLEs of the unknown parameter vector $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3)$ where each $\boldsymbol{\theta}_i$ represents the parameter vector of U_i.

We need the following expected values for implementation of the EM algorithm:

1. If $i \in I_0$, then
$$u_{jim}(\boldsymbol{\theta}) = E(U_j | U_j < x_i) = H_W(x_i; \alpha_j, \lambda_j), \ \ j = 1, 2,$$
where
$$H_W(x; \alpha, \lambda) = \frac{1}{1 - e^{-\lambda x^{\alpha}}} \int_0^{\lambda x^{\alpha}} \left(\frac{u}{\lambda}\right)^{1/\alpha} e^{-u} du.$$
2. If $i \in I_1$, then
$$\begin{aligned}
u_{1im}(\boldsymbol{\theta}) &= E(U_1 | \max\{U_1, U_3\} = x_{1i}) = p_{13} x_{1i} + (1 - p_{13}) H_W(x_{1i}; \alpha_1, \lambda_1) \\
u_{3im}(\boldsymbol{\theta}) &= E(U_3 | \max\{U_1, U_3\} = x_{1i}) = (1 - p_{13}) x_{1i} + p_{13} H_G(x_{1i}; \alpha_3, \lambda_3),
\end{aligned}$$
where $p_{13} = P(U_1 > U_3) = K(\alpha_1, \lambda_1)$ and
$$K(\alpha, \lambda) = \int_0^{\infty} \alpha \lambda x^{\alpha - 1} e^{-\lambda x^{\alpha}} (1 - e^{-\lambda_3 x})^{\alpha_3} dx, \ H_G(x; \alpha, \lambda) = x - \frac{1}{\lambda (1 - e^{-\lambda x})^{\alpha}} \int_0^{1 - e^{-\lambda x}} \frac{t^{\alpha}}{1 - t} dt$$
3. If $i \in I_2$, then
$$\begin{aligned}
u_{2im}(\boldsymbol{\theta}) &= E(U_2 | \max\{U_2, U_3\} = x_{2i}) = p_{23} x_{2i} + (1 - p_{23}) H_W(x_{2i}; \alpha_2, \lambda_2) \\
u_{3im}(\boldsymbol{\theta}) &= E(U_3 | \max\{U_2, U_3\} = x_{2i}) = (1 - p_{23}) x_{2i} + p_{23} H_G(x_{2i}; \alpha_3, \lambda_3),
\end{aligned}$$
where $p_{23} = P(U_2 > U_3) = K(\alpha_2, \lambda_2)$.

In this case, the terms of the 'pseudo' log-likelihood function $\ell_s^{(k)}(\boldsymbol{\theta})$ can be written as

$$
\begin{aligned}
\ell_{1s}^{(k)}(\alpha_1,\lambda_1) &= n\ln\alpha_1 + n\ln\lambda_1 + (\alpha_1-1)\left[\sum_{i\in I_0\cup I_1}\ln u_{1im}^{(k)} + \sum_{i\in I_2}\ln x_{1i}\right] \\
&\quad - \lambda_1\left[\sum_{i\in I_0\cup I_1}(u_{1im}^{(k)})^{\alpha_1} + \sum_{i\in I_2}x_{1i}^{\alpha_1}\right] \qquad \text{(A1)} \\
\ell_{2s}^{(k)}(\alpha_2,\lambda_2) &= n\ln\alpha_2 + n\ln\lambda_2 + (\alpha_2-1)\left[\sum_{i\in I_0\cup I_2}\ln u_{2im}^{(k)} + \sum_{i\in I_1}\ln x_{2i}\right] \\
&\quad - \lambda_2\left[\sum_{i\in I_0\cup I_2}(u_{2im}^{(k)})^{\alpha_2} + \sum_{i\in I_1}x_{2i}^{\alpha_2}\right] \qquad \text{(A2)} \\
\ell_{3s}^{(k)}(\alpha_3,\lambda_3) &= n\ln\alpha_3 + n\ln\lambda_3 + (\alpha_3-1)\left[\sum_{i\in I_1\cup I_2}\ln(1-e^{-\lambda_3 u_{3im}^{(k)}}) + \sum_{i\in I_0}\ln(1-e^{-\lambda_3 x_i})\right] \\
&\quad - \lambda_3\left[\sum_{i\in I_0}x_i + \sum_{i\in I_1\cup I_2}u_{3im}^{(k)}\right]. \qquad \text{(A3)}
\end{aligned}
$$

Therefore, $u_{1im}^{(k)}$, $u_{2im}^{(k)}$, $u_{3im}^{(k)}$ can be obtained from $u_{1im}(\boldsymbol{\theta})$, $u_{2im}(\boldsymbol{\theta})$ and $u_{3im}(\boldsymbol{\theta})$ by replacing $\boldsymbol{\theta} = (\alpha_1,\lambda_1,\alpha_2,\lambda_2,\alpha_3,\lambda_3)$ with $\boldsymbol{\theta}^{(k)} = (\alpha_1^{(k)},\lambda_1^{(k)},\alpha_2^{(k)},\lambda_2^{(k)},\alpha_3^{(k)},\lambda_3^{(k)})$. Thus, $\boldsymbol{\theta}_1^{(k+1)} = (\alpha_1^{(k+1)},\lambda_1^{(k+1)})$, $\boldsymbol{\theta}_2^{(k+1)} = (\alpha_2^{(k+1)},\lambda_2^{(k+1)})$ and $\boldsymbol{\theta}_3^{(k+1)} = (\alpha_3^{(k+1)},\lambda_3^{(k+1)})$ can be obtained by maximizing (A1), (A2) and (A3), respectively. Hence, we obtain them as follows:

$$
\begin{aligned}
\lambda_1^{(k+1)} &= \frac{n}{\sum_{i\in I_0\cup I_1}(u_{1im}^{(k)})^{\alpha_1^{(k+1)}} + \sum_{i\in I_2}x_{1i}^{\alpha_1^{(k+1)}}}, \\
\lambda_2^{(k+1)} &= \frac{n}{\sum_{i\in I_0\cup I_2}(u_{2im}^{(k)})^{\alpha_2^{(k+1)}} + \sum_{i\in I_1}x_{2i}^{\alpha_2^{(k+1)}}}, \\
\alpha_3^{(k+1)} &= -\frac{n}{\sum_{i\in I_1\cup I_2}\ln(1-e^{-\lambda_3^{(k+1)}u_{3im}^{(k)}}) + \sum_{i\in I_0}\ln(1-e^{-\lambda_3^{(k+1)}x_i})}, \\
\alpha_1^{(k+1)} &= \arg\max p_1(\alpha_1), \\
\alpha_2^{(k+1)} &= \arg\max p_2(\alpha_2), \\
\lambda_3^{(k+1)} &= \arg\max p_3(\lambda_3),
\end{aligned}
$$

where

$$
\begin{aligned}
p_1(\alpha_1) &= n\ln\alpha_1 - n\ln\left[\sum_{i\in I_0\cup I_1}(u_{1im}^{(k)})^{\alpha_1} + \sum_{i\in I_2} x_{1i}^{\alpha_1}\right] \\
&\quad + (\alpha_1 - 1)\left[\sum_{i\in I_0\cup I_1}\ln u_{1im}^{(k)} + \sum_{i\in I_2}\ln x_{1i}\right], \\
p_2(\alpha_2) &= n\ln\alpha_2 - n\ln\left[\sum_{i\in I_0\cup I_2}(u_{2im}^{(k)})^{\alpha_2} + \sum_{i\in I_1} x_{2i}^{\alpha_2}\right] \\
&\quad + (\alpha_2 - 1)\left[\sum_{i\in I_0\cup I_2}\ln u_{2im}^{(k)} + \sum_{i\in I_1}\ln x_{2i}\right], \\
p_3(\lambda_3) &= n\ln\lambda_3 - n\ln\left[-\sum_{i\in I_1\cup I_2}\ln(1-e^{-\lambda_3 u_{3im}^{(k)}}) - \sum_{i\in I_0}\ln(1-e^{-\lambda_3 x_i})\right] \\
&\quad - \lambda_3\left[\sum_{i\in I_0} x_i + \sum_{i\in I_1\cup I_2} u_{3im}^{(k)}\right] - \left[\sum_{i\in I_1\cup I_2}\ln(1-e^{-\lambda_3 u_{3im}^{(k)}}) + \sum_{i\in I_0}\ln(1-e^{-\lambda_3 x_i})\right].
\end{aligned}
$$

Note that, in this case, one needs to solve three one-dimensional optimization problems numerically at each M-Step.

Appendix C

Table A1. Summary of fitted GBD models for the two real data. EM rows are the parameters estimated with the EM algorithm for maximizing the pseudo log-likelihood function, along with the log-likelihood, AIC and BIC values, and BFGS rows corresponds to the results obtained by applying the Broyden-Fletcher-Goldfarb-Shanno algorithm for maximizing the log-likelihood function.

GBD model	θ α_1	λ_1	α_2	λ_2	α_3	λ_3	$\ell(\boldsymbol{\theta})$	**AIC**	**BIC**
				Soccer data					
Model I									
EM		0.03126		0.04630		0.04269	-299.4331	604.8663	609.6990
BFGS		0.03116		0.04636		0.04283	-299.4328	604.8656	609.6984
Model II									
EM	1.2987	0.0097	0.8047	0.0093	1.0037	0.0369	-348.2715	708.5430	718.2085
BFGS	1.3808	0.00698	0.5652	0.25469	1.53813	0.05219	-295.3057	602.6114	612.2770
				Diabetic retinopathy data					
Model I									
EM		0.0653		0.0737		0.1345	-289.9878	585.9757	590.8884
BFGS		0.06290		0.07181		0.14282	-289.9144	585.8288	590.7415
Model II									
EM	1.0937	0.0447	0.5851	0.2369	0.8995	0.1898	-290.0758	592.1515	601.9770
BFGS	1.1477	0.03920	0.7917	0.13923	0.41913	0.08272	-285.5795	583.1590	592.9846

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