PROBABILITY OF CORRECT SELECTION OF GAMMA VERSUS GE OR WEIBULL VERSUS GE BASED ON LIKELIHOOD RATIO STATISTIC

RAMESHWAR D. GUPTA

Department Computer Science and Applied Statistics
The University of New Brunswick, Saint John
Canada, E2L 4L5
E-mail: rdg@unbsj.ca

DEBASIS KUNDU AND ANUBHAV MANGLICK

Department of Mathematics
Indian Institute of Technology Kanpur
Pin 208016 India.
E-mail: kundu@iitk.ac.in

This paper proposes the use of likelihood ratio statistic in choosing between gamma and GE models or between Weibull and GE models. Probability of correct selections are obtained using Monte Carlo simulations for various sample sizes and for various model parameters. Simulation results indicate that it is easier to distinguish between GE and Weibull models than between GE and gamma models. Two real life data sets are analyzed. Interestingly, in both cases although in the literature gamma or Weibull model was used but based on the maximum likelihood values we select GE as the 'best' fitted model among these three distributions.

1 Introduction

Recently a new two-parameter distribution named as Generalized Exponential (GE) distribution or Exponentiated Exponential distribution has been introduced and studied quite extensively by two of the authors Gupta and Kundu [6,7,8,10]. The GE family has the following distribution function

$$F_{GE}(x;\alpha,\lambda) = (1-e^{-\lambda x})^{\alpha}; \qquad \alpha,\lambda > 0,$$

density function

$$f_{GE}(x;\alpha,\lambda) = \alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x}; \qquad \alpha,\lambda > 0, \tag{1}$$

survival function

$$S_{GE}(x;\alpha,\lambda) = 1-(1-e^{-\lambda x})^{\alpha}; \qquad \alpha,\lambda > 0,$$

and the hazard function

$$h_{GE}(x;\alpha,\lambda) = \frac{\alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x}}{1-(1-e^{-\lambda x})^{\alpha}}; \qquad \alpha,\lambda > 0.$$

Here α and λ are the shape and scale parameters respectively. It is observed in Gupta and Kundu [6,7] that the two-parameter GE distribution can be used quite effectively in analyzing many lifetime data particularly in place of two-parameter gamma or two-parameter Weibull distribution. The two-parameter GE distribution can have increasing, decreasing and constant failure rates depending on the shape parameter (Gupta and Kundu [6]). It is also observed by Gupta and Kundu [9] that the behavior of the two-parameter GE distribution is very much similar to a gamma distribution in many respects and in many cases the two-parameter GE model provides a *better fit* than the two-parameter gamma model or the two-parameter Weibull model in terms of maximum likelihood or minimum chi-square. Therefore, it is quite important to choose between a Weibull and GE models or between a gamma and GE models to analyze real life data sets. It is found to be very difficult to discriminate between these three models because all these three models are quite flexible and they overlap with each other in the sense that exponential distribution is a special case to all of them. Although, these three models may provide similar data fit for moderate sample sizes but it is still desirable to select the correct or more nearly correct model, since the inferences based on the model will often involve tail probabilities where the affect of the model assumption will be more critical. Therefore, even if large sample sizes are not available, it is still important to make best possible decision based on whatever data are available.

In this paper we propose to use the logarithm of the ratio of the maximum likelihood functions in choosing two overlapping distributions. The idea was originally proposed by Cox [3,4] in discriminating between two models and Bain and Englehardt [1] used it in choosing between gamma and Weibull models. It is observed that the probability of correct selection (PCS) depends only on the shape parameter of the distribution from which the data are coming. Since it is not possible to obtain the exact distributions of the likelihood ratio statistics, we obtain PCSs by using extensive Monte Carlo simulations for various sample sizes and for various model parameters. We use two real data sets to illustrate how the proposed methods can be used in practice.

Rest of the paper is organized as follows. In section 2, we provide the selection method based on likelihood ratio statistic. The PCSs are presented in section 3. We analyze two real data sets in section 4 and finally we conclude the paper in section 5.

2 Likelihood Ratio Statistic

Suppose $X_1, \ldots, X_n$ are independent and identically distributed random variables from any one of the three distributions. The density function of a GE random variable with the shape parameter α and the scale parameter λ will be denoted by (1). For both gamma and Weibull distributions we denote the shape parameter by β and the scale parameter by θ. The density function of a gamma random variable with the shape parameter β and the scale parameter θ will be denoted by $f_{GA}(x)$ as

$$f_{GA}(x) = \frac{\theta^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\theta x}; \qquad x, \beta, \theta > 0, \tag{2}$$

and similarly the density function of a Weibull random variable with the shape parameter parameter β and the scale parameter θ will be denoted by

$$f_{WE}(x) = \beta \theta^\beta x^{\beta-1} e^{-(x\theta)^\beta}.$$

Let us define

$$L_{GE}(\alpha, \lambda) = \prod_{i=1}^{n} f_{GE}(x_i), \qquad L_{GA}(\beta, \theta) = \prod_{i=1}^{n} f_{GA}(x_i)$$

and

$$L_{WE}(\beta, \theta) = \prod_{i=1}^{n} f_{WE}(x_i).$$

Now first consider choosing between gamma and GE models. The natural logarithm of the likelihood ratio statistic $T_1 = \ln(L_1)$, where

$$L_1 = \frac{L_{GE}(\hat{\alpha}, \hat{\lambda})}{L_{GA}(\hat{\beta}, \hat{\theta})}$$

and

$$T_1 = n \left[\ln(\hat{\alpha}\hat{\lambda}\tilde{X}) - \hat{\beta} \ln(\tilde{X}\hat{\theta}) - \frac{(\hat{\alpha} - 1)}{\hat{\alpha}} + \ln(\Gamma(\hat{\beta})) - \bar{X}(\hat{\lambda} - \hat{\theta}) \right]. \tag{3}$$

Here $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$, $\tilde{X} = (\prod_{i=1}^{n} X_i)^{\frac{1}{n}}$. Moreover, $\hat{\alpha}$ and $\hat{\lambda}$ are maximum likelihood estimators (MLEs) of α and λ if the data are assumed to come from a GE distribution and in this case (Gupta and Kundu [7]) they have the following relation

$$\hat{\alpha} = -\frac{n}{\sum_{i=1}^{n} \ln(1 - e^{-\hat{\lambda} X_i})}.$$

Similarly $\hat{\beta}$ and $\hat{\theta}$ are MLEs of β and θ if the data are assumed to come from a gamma distribution and in this case they satisfy the following equation

$$\hat{\theta} = \frac{\hat{\beta}}{\bar{X}}.$$

The proposed procedure is as follows. Choose a GE model if $T_1 > 0$, otherwise choose a gamma model. It is clear from (3) that if the data are coming from a GE population, the distribution of T_1 is free of λ and depends only on α and similarly if the data are coming from a gamma population, the distribution of T_1 is independent of θ and it depends only on β. The above assertions can be proved very easily. First let us assume that the data are coming from $GE(\alpha, \lambda)$. It implies that the distribution of λX_i is independent of λ. Moreover it follows from the Theorem 7.8.5 of Bain and Englehardt [2] that the distributions of $\frac{\hat{\lambda}}{\lambda}$ and $\frac{\hat{\theta}}{\lambda}$ are independent of λ. Since the distributions of $\hat{\alpha}$ and $\hat{\beta}$ are independent of λ, therefore the result follows immediately from the expression of T_1. Similar argument can be given when the data are coming from a gamma distribution then the distribution of T_1 is independent of θ and depends only on β. It implies that in the first case PCS is free from λ and depends only on α, whereas in the second case PCS is free from θ and depends only on β.

Now we consider the likelihood ratio statistic for discriminating between GE and Weibull models. Similarly as before, the natural logarithm of the likelihood ratio $T_2 = \ln(L_2)$, where

$$L_2 = \frac{L_{GE}(\hat{\alpha}, \hat{\lambda})}{L_{WE}(\hat{\beta}, \hat{\theta})}$$

and

$$T_2 = n\left[\ln\left(\frac{\hat{\alpha}\hat{\lambda}\tilde{X}}{\hat{\beta}}\right) - \frac{\hat{\alpha}-1}{\hat{\alpha}} - \hat{\lambda}\bar{X} - \hat{\beta}\ln(\hat{\theta}\tilde{X}) + 1\right]. \tag{4}$$

Here $\bar{X}$ and $\tilde{X}$ are same as defined before and $\hat{\alpha}$ and $\hat{\lambda}$ are MLEs of α and λ for GE distribution and similarly $\hat{\beta}$ and $\hat{\theta}$ are MLEs of β and θ for Weibull distribution. In the case of Weibull distribution, $\hat{\theta}$ and $\hat{\beta}$ satisfy the following relation

$$\hat{\theta} = \left(\frac{n}{\sum_{i=1}^{n} X_i^{\hat{\beta}}}\right)^{\frac{1}{\hat{\beta}}}.$$

We use the similar discrimination procedure as before, *i.e.* if $T_2 > 0$, choose GE distribution, otherwise choose Weibull distribution. In this case also it

Table 1. Probability of correct selection between GE and gamma distributions and when the data are generated from a GE distribution

$n \downarrow \alpha \rightarrow$	0.50	0.75	1.0	2.0	4.0	8.0	16.0
20	0.6610	0.5798	0.4942	0.4633	0.5081	0.5503	0.6012
40	0.6342	0.5870	0.5004	0.4812	0.5563	0.6155	0.6566
60	0.6282	0.5847	0.5039	0.5100	0.5841	0.6472	0.7031
80	0.6134	0.5822	0.5047	0.5194	0.6059	0.6740	0.7421
100	0.6134	0.5813	0.5059	0.5321	0.6242	0.7045	0.7641
200	0.6033	0.5760	0.4945	0.5699	0.6947	0.7860	0.8601

can be shown as before that if the data come from a GE population then the distribution of T_2 depends on α and independent of λ and if the data come from a Weibull population, then the distribution of T_2 depends only on β and independent of θ. It indicates that in this case also the PCS is independent of the corresponding scale parameter. It is difficult to compute the exact distributions of T_1 and T_2, therefore, we use Monte Carlo simulations to compute the probability of correct selections in the next section.

3 Probability of Correct Selections

In this section we use Monte Carlo simulations to compute the PCSs for different shape parameters and for different sample sizes. We consider different shape parameters namely 0.50, 0.75, 1.0, 2.0, 4.0, 8.0, 16.0 and also different sample sizes namely 20, 40, 60, 80, 100, 200. All the probabilities are calculated based on 10,000 replications. The exact details are provided below:

Case 1: Between GE and Gamma
First we generate a sample of size n from a $GE(\alpha, 1)$. From the given sample we obtain MLEs of α and λ (GE parameters), similarly we obtain MLEs of β and θ (gamma parameters). We compute T_1 and observe whether it is negative or positive. We replicate the process 10,000 times and compute the percentage of times T_1 is positive and that provides the probability of the correct selection. The same way we estimate the PCS between gamma and GE models when the data are coming from a Gamma(β, 1). We generate the sample from a Gamma(β, 1) and obtain the MLEs of α, λ, β and θ. In this case we compute the percentage of times T_1 is negative out of 10,000 replications. Results are reported in Tables 1 and 2.

Case 2: Between GE and Weibull

Table 2. Probability of correct selection between GE and gamma distributions when the data are generated from a gamma distribution

$n \downarrow \beta \rightarrow$	0.50	0.75	1.0	2.0	4.0	8.0	16.0
20	0.3655	0.4246	0.4977	0.5829	0.6104	0.6347	0.6653
40	0.4086	0.4317	0.5104	0.5948	0.6281	0.6900	0.7365
60	0.4370	0.4398	0.4999	0.5942	0.6607	0.7197	0.7849
80	0.4555	0.4498	0.4964	0.6051	0.6745	0.7544	0.8276
100	0.4519	0.4597	0.4998	0.6081	0.6943	0.7775	0.8542
200	0.5039	0.4855	0.4991	0.6319	0.7534	0.8598	0.9317

Table 3. Probability of correct selection between GE and Weibull distributions when the data are Generated from a GE distribution.

$n \downarrow \alpha \rightarrow$	0.50	0.75	1.0	2.0	4.0	8.0	16.0
20	0.6610	0.5643	0.5025	0.5731	0.6675	0.7283	0.7699
40	0.7258	0.5982	0.4976	0.6316	0.7520	0.8357	0.8811
60	0.7663	0.6157	0.4849	0.6829	0.8154	0.8929	0.9301
80	0.7856	0.6309	0.5042	0.7139	0.8582	0.9224	0.9582
100	0.8128	0.6430	0.5097	0.7382	0.8825	0.9496	0.9752
200	0.8849	0.7012	0.4933	0.8295	0.9614	0.9911	0.9981

In this case first we compute the PCS between GE and Weibull distributions and when the data are coming from a GE distribution. We generate a sample of size n from a $GE(\alpha, 1)$ and compute MLEs of α, λ (GE parameters) and β, θ (Weibull parameters). Replicate the process 10,000 times and observe the percentage of times T_2 is positive. Exactly the same way, we compute the PCSs between GE and Weibull distributions and when the data are coming from a Weibull distribution. The results are reported in Tables 3 and 4 [a].

Some of the points are quite clear from the Tables 1, 2, 3 and 4. First of all in all four cases when $\alpha = 1.0$, the PCS is close to .5 as it should be for all sample sizes. Because when $\alpha = 1.0$, all three distributions become exponential distribution therefore, between any two distributions the probability of choosing any particular one is 0.5. It is also observed that in all cases (except for $\alpha = 0.50$ or 0.75 in Table 1) as sample size n increases the PCS increases for all values of $\alpha \neq 1$. Moreover as $|\alpha - 1|$ increases the PCS increases from

[a] When $\beta = 4$, the Weibull distribution becomes almost symmetric. In this case it is too time consuming to compute MLEs of the GE parameters. In Table 4, for $\beta = 4$, the results are based on 1000 replications.

Table 4. Probability of correct selection between GE and Weibull distributions when the data are generated from a Weibull distribution.

$n \downarrow \beta \rightarrow$	0.50	0.75	1.0	2.0	4.0	8.0	16.0
20	0.6655	0.5155	0.5037	0.7061	0.8142	0.8738	1.0000
40	0.7159	0.5844	0.5045	0.7823	0.9185	0.9624	1.0000
60	0.7768	0.6169	0.5046	0.8395	0.9573	0.9868	1.0000
80	0.8233	0.6457	0.5069	0.8757	0.9768	0.9947	1.0000
100	0.8653	0.6700	0.5047	0.9006	0.9874	0.9979	1.0000
200	0.9492	0.7536	0.5085	0.9660	0.9992	1.0000	1.0000

0.5 in most of the cases. Surprisingly in Table 1, when the sample size 20 or 40 and the $\alpha = 2.0$, the PCSs are less than .5. It indicates that when the data come from GE and the shape parameter is greater than one but not very far from one then the likelihood ratio statistic can not distinguish properly whether the data are coming from a GE distribution or from a gamma distribution if the sample size is not very large. Similar pattern is also observed in Table 2, when the shape parameter is 0.50 or 0.75. It indicates that for certain ranges of the shape parameter it is really very difficult to distinguish between gamma and GE distributions. In Table 1, when $\alpha = 0.50$ or 0.75, the PCSs decrease as n increases. In Table 2, when $\alpha = 0.50$ and 0.75, the PCS are less than 0.5. Both these findings are quite counter intuitive and we can not justify them. We feel it might be due to the estimation procedures. It is known that when the shape parameters is less than one then the parameter estimations are quite difficult for both GE and gamma distributions. The likelihood functions are very flat and therefore the maximum likelihood estimators are quite unstable. The results might be the reflection of that. One of the referees has mentioned that it might be due to discretization of the cumulative distribution function.

Now comparing the results of Tables 1 and 2, it clear that the PCSs of Table 2 are more than the corresponding PCSs of Table 1 when the shape parameter is more than one. It clearly indicates that for the shape parameter greater than one, the likelihood ratio statistic can distinguish better between gamma and GE distributions if the data are coming from a gamma distribution than vice verse. It can be justified as follows. As the shape parameter increases the gamma density becomes symmetric but the GE density remains skewed as the shape parameter goes to infinity. Therefore, if the data are coming from a gamma distribution and the shape parameter is large then naturally PCS becomes higher.

Comparing the results between Tables 3 and 4, it is also observed that PCSs of Table 4 are more than the corresponding PCSs of Table 3. Therefore, between GE and Weibull distributions also the likelihood ratio statistic can distinguish better if the data are coming from a Weibull distribution than the other way. The same reason as above holds here also. Finally comparing the results between Tables 1, 2 and Tables 3, 4, it is clear that using the likelihood ratio statistic it is much easier to distinguish between Weibull and GE models than between gamma and GE models. It might be due to the fact that the Weibull density goes to symmetry much faster than the gamma density as the shape parameter increases. We believe that it should be true for any other statistics also and more work is needed in this direction.

4 Data Analysis

In this section we apply the above procedure in two data sets. Both these data sets have been used by several authors in the literature.

Data Set 1: Lieblein and Zelen [11] provided the following sample of size n = 23 to illustrate the use of Weibull model. These data indicate the endurance in millions of revolutions of deep-groove ball bearings. Thoman, Bain and Antle [12] mentioned that Weibull distribution should be used where as Bain and Englehardt [1] proposed to use gamma distribution. The data are as follows:
17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

For this data $\bar{X} = 72.12$ and $\tilde{X} = 63.46$. The MLEs of the GE parameters are $\hat{\alpha} = 5.2825$, $\hat{\lambda} = 0.0323$, gamma parameters are $\hat{\beta} = 4.0309$, $\hat{\theta} = 0.0558$ and Weibull parameters are $\hat{\beta} = 2.1033$, $\hat{\theta} = 0.0122$. Also $ln(L_{GE}(\hat{\alpha},\hat{\lambda}) =$ -112.9762, $ln(L_{GA}(\hat{\beta},\hat{\theta}) =$ -113.0272 and $ln(L_{WE}(\hat{\beta},\hat{\theta}) =$ -113.6886. Therefore, T_1 = 113.0272-112.9762 = .0510 and T_2 = 113.6886-112.9762 = 0.7124. Since both T_1 and T_2 are positive, therefore GE is preferable than gamma or Weibull distributions based on the likelihood ratio statistic. Moreover, from the tables it can be said that the PCS between gamma and GE is between 51% to 55% and the PCS between Weibull and GE is between 67%-73%.

Data Set 2: The following data represents the relief times of 20 patients receiving an analgesic. The data is obtained from Gross and Clark [5] (page; 105), where they fitted gamma distribution. The data are as follows:
1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0.

For this data set $\bar{X} = 1.90$ and $\tilde{X} = 1.80$. The MLEs of the GE parameters are as follows: $\hat{\alpha} = 36.6437$, $\hat{\lambda} = 2.2348$ and $\ln(L_{GE}(\hat{\alpha},\hat{\lambda}))$ = -16.2605. The MLEs of the gamma parameters are $\hat{\beta} = 9.6630$ and $\hat{\theta} = 5.0864$ and

$\ln(L_{GA}(\hat{\beta},\hat{\theta}))$ = -17.8186. Similarly we obtain the MLEs of the Weibull parameters as $\hat{\beta} = 2.7875$ and $\hat{\theta} = 0.4695$ and $\ln(L_{WE}(\hat{\beta},\hat{\theta}))$ = -20.5864. Therefore, T_1 = -16.2605 + 17.8186= 1.5581 and T_2 = -16.2605 + 20.5864 = 4.3259. Since T_1 and T_2 both are positive therefore in both cases we prefer GE models. The PCS between gamma and GE is more than 60% and the PCS between Weibull and GE is more than 77%.

5 Conclusions

In this paper we consider the likelihood ratio statistics to choose between GE and gamma model or between GE and Weibull models. It is very easy to use in practice. We could not obtain the exact distributions of the likelihood ratio statistics so we use Monte Carlo simulations to compute the probability of correct selections. The likelihood ratio statistics depend only on the shape parameters. It is observed that as the shape parameter deviates from one it becomes easier usually to distinguish between two distributions. It is also observed that it is much easier to distinguish between GE and Weibull distributions than between GE and gamma distributions. Two data sets are analyzed. Interestingly, in both cases although in the literature gamma or Weibull model was used but we select GE as the 'best' fitted model based on the maximum likelihood values. Another interesting question is to determine the minimum sample size n to discriminate between two distribution functions (either GE and gamma or GE and Weibull) for a given PCS. Work is in progress in that direction and it will be published elsewhere.

Acknowledgments

Part of the work of the first author was supported by a grant from the Natural Sciences and Engineering Research Council. The authors would like to thank two anonymous referees for some constructive suggestions.

References

1. L.J. Bain and M. Englehardt, Communications in Statistics Theory and Methods **9**, 375 (1980).
2. L.J. Bain and M. Englehardt, *Statistical Analysis of Reliability and Life-testing Models; Theory and Methods* (Marcel and Dekker Inc., New York, 1991)
3. D.R. Cox, Proceedings of the Fourth Berkeley Symposium in Mathematical Statistics and Probability **5**, 105 (1961).

4. D.R. Cox, Journal of the Royal Statistical Society Ser. B **24**, 406 (1962).
5. A.J. Gross and V.A. Clark *Survival Distributions; Reliability Applications in the Biomedical Sciences* (John Wiley and Sons, New York, 1975).
6. R.D. Gupta and D. Kundu, Australian and New Zealand Journal of Statistics **41**, 173 (1999).
7. R.D. Gupta and D. Kundu, Biometrical Journal **43**, 117 (2001).
8. R.D. Gupta and D. Kundu, Jour. Stat. Comp. Simul **69**, 315 (2001).
9. R.D. Gupta and D. Kundu, *Technical Report*(The University of New Brunswick, 2001).
10. R.D. Gupta and D. Kundu, Jour. Appl. Stat. Sc. , (2002).
11. J. Lieblein and M. Zelen, Jour. Res. Nat. Bur. Stand. **47**, 273 (1956).
12. D.R. Thoman, L.J. Bain and C.J. Antle, Technometrics **11**, 445 (1969).