Discriminating between Weibull and log-normal distributions in the presence of hybrid censoring

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Abstract

The Weibull and lognormal distributions are two of the most widely used models for analyzing lifetime data. Both share several properties, and for certain parameter ranges, their cumulative distribution functions can appear quite similar. However, choosing the more suitable distribution is essential for accurate inference. When data are subject to censoring, the problem becomes more complex. Here, we assume that the data come either from a Weibull or a lognormal distribution under hybrid censoring (considering both Type-I and Type-II hybrid schemes). To discriminate between the two models, we use the ratio (or equivalently, the difference) of their maximized log-likelihoods (RML). We further employ a recently developed method based on minimum density power divergence estimators (MDPDE) and compare its performance with the classical likelihood approach in both without outliers and with outliers settings, where it shows clear superiority. In addition, a Bayesian decision criterion is implemented and compared with these methods. The asymptotic distribution of the RML statistic is derived to compute the probability of correct selection and to determine the minimum required sample size for reliable discrimination. Simulation studies are carried out to evaluate how well the asymptotic results hold across different sample sizes, censoring levels, and censoring times. The asymptotic approximations perform well even for moderate sample sizes. Finally, we investigate the effect of model misspecification on key reliability measures, including the p^{th} quantile, mean residual life, reliability function, and prediction for future failures. A real data set is analyzed to illustrate the proposed methods.

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Keywords: Likelihood ratio test, asymptotic distribution, minimum density power divergence, probability of correct selection, Monte Carlo simulations.

1 Introduction

The log-normal and Weibull distributions are two of the most popular choices for analyzing skewed lifetime data. These distributions have several interesting properties, and their probability density functions (PDFs) can take on different shapes. Although both distributions may provide a similar fit to the middle portion of the data, it is crucial to select the more appropriate model. This is because inferences based on the model often involve tail probabilities, where the effects of model assumptions are significant. The challenge becomes more severe if the data are censored (see Section 2). Therefore, it is important to make the best possible choice based on the observed data.

The probability density function (PDF) and the cumulative distribution function (CDF) of a random variable X following a Weibull distribution, denoted as WE(α, β), are, respectively, given by

$$f_{\mathrm{WE}}(x;\alpha,\beta)=\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-(\frac{x}{\beta})^{\alpha}} \quad \text{and} \quad F_{\mathrm{WE}}(x;\alpha,\beta)=1-e^{-(\frac{x}{\beta})^{\alpha}},$$

where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter. Further, the PDF and the CDF of a random variable X following a lognormal distribution, denoted as LN(η, σ), are, respectively, given by

$$f_{\mathrm{LN}}(x;\eta,\sigma)=\frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(\ln x-\ln\eta)^2}{2\sigma^2}} \quad \text{and} \quad F_{\mathrm{LN}}(x;\eta,\sigma)=\Phi\left(\frac{\ln x-\ln\eta}{\sigma}\right),$$

where $\Phi(.)$ is the CDF of the standard normal variable, $\sigma > 0$ is the shape parameter, and $\eta > 0$ is the scale parameter. To demonstrate the similarity between the cumulative distribution functions (CDFs) of the two distributions, we kept the parameters of the Weibull distribution fixed at $(2, 1)$ and then found the parameters of the closest log-normal distribution by minimizing the Kolmogorov-Smirnov (KS) distance (An (1933)) between them. The resulting Kolmogorov-Smirnov (KS) distance was calculated to be 0.038. We then plotted the CDFs of both distributions, and the similarity between them can be seen in Fig. 1.

However, although the CDFs are similar, differences arise when we examine the probability density functions (PDFs) (Fig. 2a) and hazard functions (Fig. 2b) of the same distributions.The

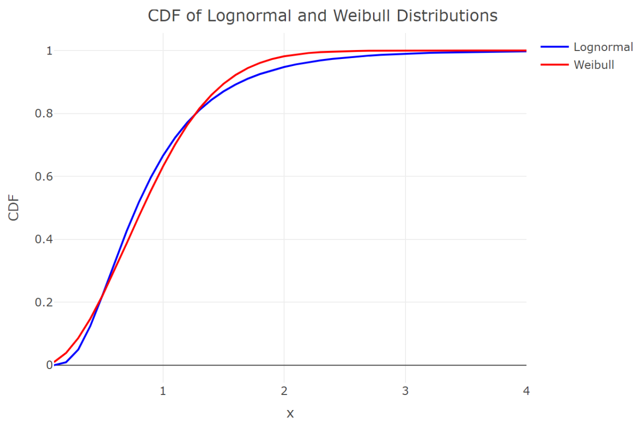
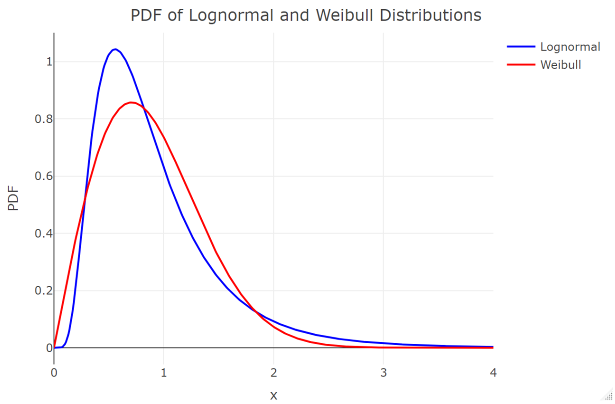


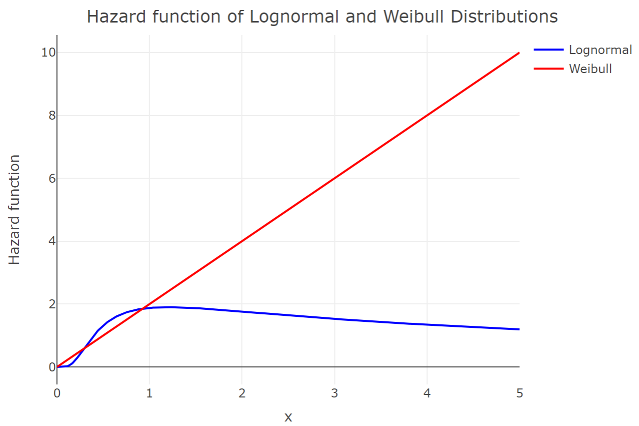
Figure 1: CDF of WE$(2, 1)$ and LN$(0.78, 0.58)$.

Hellinger distance (see Hellinger (1909)) between the two probability density functions (PDFs) was 0.020. Additionally, it is known that the log-normal distribution can exhibit an inverted bathtub-shaped hazard function, whereas the Weibull distribution's hazard function can be increasing, decreasing, or constant, depending on its shape parameter. In Section 2, we have briefly discussed the importance of selecting the appropriate model.

Thus, despite the similarities in the CDFs, it is crucial to use an appropriate model selection criterion. Discriminating between these two distributions becomes even more important when censoring is involved, as it increases the severity of the problem.



(a) PDF of WE$(2, 1)$ and LN$(0.78, 0.58)$.



(b) Hazard function of WE$(2, 1)$ and LN$(0.78, 0.58)$.

Figure 2: Difference in terms of shape of PDF and hazard function between WE(2,1) and LN$(0.78, 0.58)$.

Examining whether a set of given observations follows one of two (or more) probability distribution functions is a longstanding statistical challenge. In his seminal paper, Cox (1961) initially addressed this issue and also explored into the consequences of selecting an incorrect model. This problem was further developed by Atkinson (1969, 1970) , Dyer (1973), and Chen (1980). Since then extensive work has been done in discriminating between two or more distribution functions, see for example Dey and Kundu (2009) , Ahmad et al. (2017), and Raqab et al. (2018) (also see the references cited therein).

Several recent studies have made notable contributions in this area. For example, Abid and Kokonendji (2023) examined the problem of discrimination between Poisson-Tweedie and Poisson-exponential-Tweedie count models. Additionally, Basu and Kundu (2023) addressed the issue of discriminating between Birnbaum-Saunders and log-normal distributions, including the effects of model misspecification. Further work by Diyali et al. (2024) explored the problem of discrimination between inverse Weibull, lognormal, and inverse Gaussian distributions.Recently, Basu and Ng (2025) proposed a divergence-based method to discriminate between Weibull and generalized exponential distributions, emphasizing its robustness and superiority over the conventional ratio of maximized log-likelihoods, particularly in the presence of outliers. Several work has been done in studying effect of model mis-specification, for example: Wiens (1999), Khakifirooz et al. (2020) and the references cited therein.

While much of the classical work on model discrimination has been carried out under the assumption of complete samples, several contributions have addressed the problem in the presence of censoring. For example,Diyali et al. (2023) investigated discrimination between log–normal and log–logistic models under Type-II censoring using differences of maximized log-likelihoods and studied the corresponding asymptotic distribution and PCS. In a related direction, Dey and Kundu (2012) examined the log–normal versus Weibull case under Type-II censoring, while Kuş et al. (2019) extended the problem to progressive Type-II censoring. These studies, however, treat censoring schemes that are fixed in advance (Type-II or progressive) and primarily focus on the formulation of discrimination rules and their asymptotic performance.

In contrast, our work adapts and extends the existing asymptotic discrimination framework to the hybrid censoring context (both Type-I and Type-II hybrid designs). This allows for a unified treatment of censoring schemes where the number of observed failures is random, and explicit formulas are developed for the minimum required sample size to attain a prespecified protection level (probability of correct selection).

We further propose a Bayesian extension of this framework under hybrid censoring. While this approach builds upon the classical likelihood-based discrimination principle, such a Bayesian formulation under hybrid censoring has not yet been explored in the literature, and thus represents a methodological adaptation of existing ideas to a more general experimental setup.

Beyond this, we extend the divergence-based robust discrimination framework using the minimum density power divergence estimator (MDPDE), which, to the best of our knowledge, has previously been explored only for complete samples Basu and Ng (2025). Our findings show that the MDPDE-based approach attains a higher probability of correct selection than the classical likelihood method, particularly under data contamination.

In addition to the discrimination problem itself, we also investigate the impact of model misspecification on various reliability characteristics, such as the p-th quantile, mean residual life, predictions of future failures, and the reliability function, as detailed in later sections. In this way, our results extend and complement the existing literature on discrimination with censored data.

In reliability analysis, often the life testing experiments are terminated before the failures of all the units under consideration which leads to have observed data in the presence of some censoring scheme. One basic difference between the problem of discriminating under complete sample and that under hybrid case is that the probability of correct selection (PCS) under complete sample case is independent for the choice of scale parameter but under hybrid case that is not the case anymore.

This hybrid censoring scheme, which was originally introduced by Epstein (1954), has been used quite extensively in reliability acceptance test in Standard (1977). In this paper we have considered the problem of discriminating between Weibull and log-Normal distribution under hybrid censoring setup (considering both Type-I and Type-II hybrid setup). The importance of hybrid sampling is multifolded. If the data is coming under Type-I censoring, then there is a concern regarding no failure if T is taken quite small. Similarly, under Type-II censoring if number of failures is taken to be large enough then it is cost and time consuming. Thus, hybrid sampling solve these issues and serves importance in life-testing experiments. This problem is of particular significance as hybrid scheme inherently includes two special cases corresponding to discrimination under Type-I and Type-II censoring schemes. Consequently, the results obtained here can also be applied to these subcases independently. When the pre-specified censoring time T is sufficiently large, the hybrid censoring scheme effectively reduces to Type-II censoring, and conversely, for smaller values of T, it resembles the Type-I case. Moreover, we derive the asymptotic distribution of the ratio of maximized log-likelihoods (RML) under hybrid censoring, wherein the number of observed failures is a random variable. This necessitates a distinct treatment compared to existing results for fixed-sample cases.

One drawback of the Type-I hybrid censoring scheme lies in the fact that all inferential outcomes are derived under the assumption that there is a minimum of one observed failure.

Furthermore, there is the possibility of only a limited number of failures occurring within the predetermined time frame. To overcome this, Childs et al. (2003) proposed a hybrid censoring scheme known as the Type-II hybrid censoring scheme. In this work it has been discussed that this censoring scheme may be implemented when the experimenter specifies a requirement that a minimum number of failures must be observed, and they have already prepaid for utilizing the testing facility for a predetermined duration. In the event that failures are observed before this designated time, the experiment can extend up to the specified duration to maximize the utilization of the testing facility. However, if the r-th failure does not occur before the specified time, the experimenter would naturally opt to continue the experiment until the occurrence of the r-th failure. For a detailed comparison of the two hybrid censoring schemes, interested readers are directed to the work of Childs et al. (2003). For this reason we have considered the discrimination problem under both the setup i.e. Type-I hybrid censoring as well as Type-II hybrid censoring.

To illustrate, consider the Type-I hybrid setup. Let $x_1, x_2, \ldots, x_n$ be a random sample from either a log-normal or Weibull distribution. Let R $(\leq n)$ denote the prefixed number of failures and T the predetermined censoring time. Under the Type-I hybrid scheme, one of the following types of data is obtained:

- CASE I: $\{x_{(1)} < x_{(2)} < \ldots < x_{(R)}\}$ if $x_{(R)} < T$,
- CASE II: $\{x_{(1)} < x_{(2)} < \ldots < x_{(d)}\}$ if $d < R$ and $x_{(d)} < T < x_{(d+1)}$.

Thus, the experiment is terminated at $T_0 = \min(x_{(R)}, T)$, where $x_{(1)}, x_{(2)}, \ldots, x_{(R)}$ are order statistics. Based on the observed sample, the experimenter aims to discriminate whether the data arise from a Weibull or log-normal model. Similarly, for the Type-II hybrid setup, the experiment terminates at $T_0 = \max(x_{(R)}, T)$. We develop the corresponding discrimination criteria based on the observed hybrid sample in Section 3.

In summary, this study addresses the discrimination problem under hybrid censoring by providing a unified framework that encompasses both Type-I and Type-II schemes. The remainder of the paper is organized as follows. Section 2 outlines the motivation and significance of discriminating between the two distributions, particularly in the presence of censoring. Section 3 presents the different discrimination criteria considered in this study and develops the asymptotic distribution of the RML statistic. Section 4 focuses on determining the minimum required sample size for effective discrimination. Simulation results are discussed in Section 5. Section 6 examines the impact of model misspecification on the prediction of future failures. Section 7

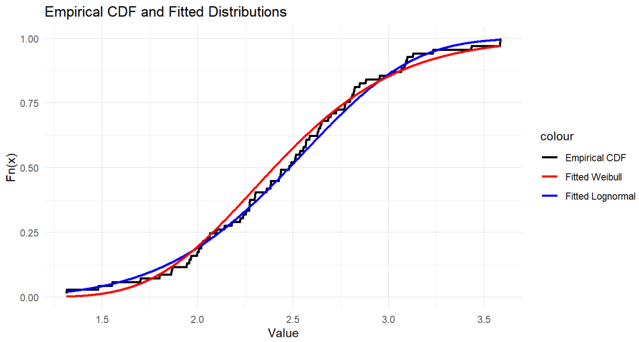


Figure 3: Empirical and fitted CDF for given real data.

illustrates the proposed methods through a real data analysis. Finally, Section 8 concludes the paper.

2 Motivation

In this section, we focus on why there is a need to discriminate between the two distributions by considering a real life data set obtained from Kundu and Gupta (2006). This data set, given in Table 1, represents the tensile strength (in GPA) of carbon fibers checked under tension at gauge of 20 millimeters in length.

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865
1.944	1.958	1.966	1.997	2.006	2.021	2.027	2.055
2.063	2.098	2.140	2.179	2.224	2.240	2.253	2.270
2.272	2.274	2.301	2.301	2.359	2.382	2.382	2.426
2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554
2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684
2.697	2.726	2.770	2.773	2.800	2.809	2.818	2.821
2.848	2.880	2.954	3.012	3.067	3.084	3.090	3.096
3.128	3.233	3.433	3.585	3.585			

Table 1: Tensile strength measures in GPA of 69 carbon fibres.

We checked if Weibull distribution could be used to fit the given data set or not. We obtained maximum likelihood estimators(MLE) as follows: $\hat{\alpha} = 5.50$ and $\hat{\beta} = 2.65$. We also computed the Kolmogorov-Smirnov (KS) distance between the empirical distribution function and the fitted distribution functions based on maximum likelihood estimates (MLEs). For the Weibull distribution, the KS distance was 0.05 with a p-value of 0.98. We performed a similar exercise with the log-normal distribution, obtaining MLEs of $\hat{\eta} = 2.39$ and $\hat{\sigma} = 0.21$. In this case, the KS distance was 0.07 with a p-value of 0.86. Both p-values suggest that both models fit the given data set well. Additionally, we plotted the empirical CDF for the given dataset along with the fitted CDFs for both models, and the similarity can be seen in Fig. 3. Based on this data, we considered two hybrid sampling scheme:

$$\text{Scheme 1} : R = 40 \text{ and } T = 2.5$$
$$\text{Scheme 2} : R = 50 \text{ and } T = 3.5$$

For scheme 1, it is observed that it is from Case II and for Scheme 2, it is from Case I. Now, even though both the model fits the given data, we discuss briefly the necessity of choosing the appropriate model. Along this direction, let us first look at the estimates of the p^{th} quantile under three cases i.e. when complete sample is observed and then under respective cases of censoring. Firstly, we computed p^{th} quantile for respective fitted distributions i.e. Weibull and log-normal on the basis of complete sample. Then performed the same calculations for both of the artificial hybrid schemes. The results are reported in Table 2.

Scheme	Model	0.50^{th} quantile	0.90^{th} quantile	0.99^{th} quantile
Complete sample	Weibull	2.47	3.08	3.49
	Log-Normal	2.39	3.15	3.94
	Relative Change	-0.0323	0.0227	0.1289
Scheme 1	Weibull	2.46	3.01	3.37
	Log-Normal	2.48	3.43	4.47
	Relative Change	0.0081	0.1395	0.3264
Scheme 2	Weibull	2.48	2.99	3.34
	Log-Normal	2.42	3.25	4.13
	Relative Change	-0.0241	0.0869	0.2365

Table 2: Estimates of p^{th} quantile under respective model assumption for real life data.

We can see that the effect of misspecification increases, as the value of estimates of p^{th} quantile under respective model assumption is different, as we move from complete sample to censoring and thus the problem of appropriate model selection becomes severe as censoring is introduced. We present some simulation results which shows effect of model misspecification on p^{th} quantile under hybrid setup.

Initially, data were generated from the $WE(2, 1)$ distribution, and the average maximum likelihood estimates (MLEs) of the pth quantile were obtained under the assumption that the data follow a Weibull model. Subsequently, the same data were fitted using a log-normal distribution, and the corresponding average MLEs of the pth quantile, along with their standard errors (given in parentheses), were computed. Each experiment was based on 1000 replications. The summarized results are reported in Table S1 of the supplementary material.

We can conclude from Table S1 that the average MLES of the p^{th} quantile are closer to the true values when assuming a Weibull distribution. The difference between the values obtained under the Weibull and log-normal models increases as we move towards the tail region (when the p^{th} quantile moves from 0.50 to 0.99). We performed a similar exercise by changing the choice of the parent distribution to log-normal, and the results are reported in Table S2 of the supplementary material. . In this case as well, we observe a considerable difference between the average estimates of the p^{th} quantile under the true model (log-normal) and the incorrect model (Weibull). This difference increases as we move towards the tail region.

We also examined the mean residual life (MRL) to assess the potential effects of model misspecification. The mean residual life (MRL) function for a continuous distribution is defined as:

$$\mathrm{MRL}(x) = \mathbb{E}[X - x \mid X > x]. \tag{1}$$

Using a similar approach as for the p^{th} quantile, we computed the MRL for varying values of x for the given dataset under the respective fitted distributions. This analysis was performed first for the complete data and then for the hybrid censoring scenarios. The findings are presented in Table 3.

Scheme	Model	$x = 2$	$x = 2.5$	$x = 3$
Complete Sample	Weibull	0.62	0.36	0.20
	Log-Normal	0.61	0.44	0.36
Scheme 1	Weibull	0.59	0.32	0.16
	Log-Normal	0.76	0.59	0.50
Scheme 2	Weibull	0.58	0.31	0.15
	Log-Normal	0.67	0.49	0.41

Table 3: Value of MRL(x) (equation (1)) for varying values of x under respective model assumption for real life data.

We observe that as the values of x increase, the difference between the estimated MRL values under the respective fitted distributions also increases for all three cases: complete sample, Scheme 1, and Scheme 2. The simulation results for the effect of model misspecification on the MRL values, under the respective choice of parent distribution, are presented (along with standard errors in parentheses) in Tables S3 and S4 of the supplementary material.

Based on the findings, we can conclude that as the value of x increases, the difference between the average estimates of MRL, computed under the true and incorrect models, also increases. Therefore, model misspecification has a significant effect on the value of MRL.

We now investigate the effect of model misspecification on the estimation of the reliability function, defined as

$$R(t) = P(T > t),$$

under a hybrid censoring scheme. For this purpose, we generated data from the $WE(2, 1)$ distribution and calculated the mean squared error (MSE) of the estimated reliability values at representative points of t. The results are reported in Table 4.

t	True $R(t)$	Model	$n = 30$	$n = 50$	$n = 100$
0.5	0.7788	Weibull	0.0045	0.0028	0.0012
		Lognormal	0.0053	0.0035	0.0017
1.0	0.3679	Weibull	0.0237	0.0138	0.0064
		Lognormal	0.0134	0.0120	0.0091
1.5	0.1054	Weibull	0.0090	0.0079	0.0045
		Lognormal	0.0266	0.0313	0.0295

Table 4: MSE of reliability estimates $R(t)$ when data is from $WE(2, 1)$ under 40% hybrid censoring ($T = 0.71$).

t	True $R(t)$	Model	$n = 30$	$n = 50$	$n = 100$
0.5	0.7559	Weibull	0.0052	0.0031	0.0014
		Lognormal	0.0054	0.0031	0.0014
1.0	0.5000	Weibull	0.0314	0.0159	0.0082
		Lognormal	0.0142	0.0072	0.0035
1.5	0.3426	Weibull	0.0512	0.0371	0.0253
		Lognormal	0.0173	0.0094	0.0046

Table 5: MSE of reliability estimates $R(t)$ when data is from $LN(0, 1)$ under 40% hybrid censoring ($T = 0.77$).

It can be seen that the MSE values are consistently smaller when the assumed model matches the true distribution (Weibull), whereas fitting the incorrect log-normal model leads to larger errors, particularly in the tail region. To further illustrate the effect of model misspecification, we repeated the experiment by generating data from the $LN(0,1)$ distribution and then estimating the reliability function under both log-normal and Weibull assumptions. The findings, summarized in Table 5, display a similar trend: the MSE values are much lower when the true log-normal model is fitted, while the misspecified Weibull model produces considerably higher errors. In both cases, we also observe that the MSE decreases as the sample size increases, which is expected since larger samples yield more accurate parameter estimates. However, even with larger sample sizes, the relative difference between the true and incorrect models remains substantial. These results highlight that model misspecification substantially increases the error in reliability estimation, and that this effect becomes more pronounced in the tail region. Hence, appropriate model selection is critical for reliability based inference, especially under censoring.

Now, let us move towards the prediction of future failures. Prediction of future observation holds importance when it comes to life testing experiments. Extensive work on prediction problem based on frequentist and Bayesian framework can be found in the literature. Kawsky and Rooln (1985)have applied maximum likelihood to the joint prediction of a future random variable and unknown parameter. Smith (1999) worked towards the properties of the different predictors based on Bayes and frequentist procedures for a class of parametric family under smooth loss functions. We have discussed point prediction along with effects of model misspecification on future prediction in depth in Section 6. Let $X = (x_{(1)}, x_{(2)} \dots x_{(r^*)})$ denotes a Type-I hybrid censored samples. We are interested in prediction of future failures $Y = X_{(s+r^*)}(s = 1, 2 \dots n - r^*)$ of all the $(n - r^*)$ censored units based on observed sample $X = (x_{(1)}, x_{(2)} \dots x_{(r^*)})$. For given data set, we estimated values of future failures using best unbiased predictor(BUP), under respective model assumptions (denoted as BUP_{WE} and BUP_{LN}) for varying values of $s = 5, 10$ and 15 under respective model assumption. Results are reported in Table 6.

Scheme	s	Exact Value	BUP_{WE}	BUP_{LN}
Scheme 1	5	2.56	2.57	2.60
	10	2.64	2.65	2.73
	15	2.77	2.73	2.88
Scheme 2	5	2.81	2.80	2.87
	10	3.01	2.92	3.07
	15	3.12	3.07	3.39

Table 6: Comparison of point prediction for real life data set.

We observe that the values of the BUP computed under each model assumption vary significantly as we project further into the future (as s ranges from 5 to 15). Additionally, we note that the BUP value computed under the Weibull model is closer to the actual value, suggesting that the Weibull model may be preferable.

Thus, we conclude that while both models fit the given dataset well, the p-th quantile, mean residual life, Reliability function and estimates of future predictions are sensitive to appropriate model selection. Therefore, discriminating between these two distributions under a hybrid setup is crucial.

3 Discrimination Criteria

We use the ratio of maximized log-likelihoods (RML) to discriminate between the two competing distributions. Suppose $(\hat{\alpha}, \hat{\beta})$ and $(\hat{\eta}, \hat{\sigma})$ are the MLEs of the Weibull parameters (α, β) and the log-normal parameters (η, σ), respectively, based on the available hybrid Type-I (or Type-II) censored sample. The ratio of maximized log-likelihoods, given in Equation (2), is employed for discrimination between the two distributions. This idea was originally proposed by Cox (1961, 1962) for distinguishing between separate models and later employed by Dumonceaux et al. (1973) to discriminate among different distributions. Bain and Engelhardt (1980) used this approach to discriminate between Weibull and Gamma distributions, and Gupta et al. (2002) applied it to overlapping families of distributions. Most existing works in this area have adopted this technique as their main decision criterion.

We choose Weibull or log-normal as the preferred model according to

$$T_N = \frac{L_{\mathrm{WE}}(\hat{\alpha}, \hat{\beta})}{L_{\mathrm{LN}}(\hat{\eta}, \hat{\sigma})}, \tag{2}$$

where $T_N > 1$ indicates a preference for the Weibull model, and $T_N < 1$ indicates a preference for the log-normal model.

Here,

$$\begin{aligned} L_{\mathrm{WE}}(\alpha, \beta) &= \sum_{i=1}^{r^*} \ln f_{\mathrm{WE}}(x_{(i)}; \alpha, \beta) + (n - r^*) \ln \left(1 - F_{\mathrm{WE}}(T_0; \alpha, \beta)\right), \\ L_{\mathrm{LN}}(\eta, \sigma) &= \sum_{i=1}^{r^*} \ln f_{\mathrm{LN}}(x_{(i)}; \eta, \sigma) + (n - r^*) \ln \left(1 - F_{\mathrm{LN}}(T_0; \eta, \sigma)\right), \end{aligned}$$

where r^* denotes the number of failures and $T_0 = \min\{x_{(R)}, T\}$. Thus,

$$T_0 = \begin{cases} x_{(R)}, & \text{if Case I,} \\ T, & \text{if Case II,} \end{cases} \qquad r^* = \begin{cases} R, & \text{if Case I,} \\ d, & \text{if Case II.} \end{cases}$$

From now on, we refer to T_N as the ratio of maximized log-likelihoods (RML) statistic. A similar discrimination criterion can be formulated for the hybrid Type-II censored sample. For the Type-II hybrid setup, the notations remain analogous, except that $T_0 = \max\{x_{(R)}, T\}$. Hence,

$$T_0 = \begin{cases} T, & \text{if Case I,} \\ x_{(R)}, & \text{if Case II,} \end{cases} \qquad r^* = \begin{cases} d, & \text{if Case I,} \\ R, & \text{if Case II.} \end{cases}$$

The exact distribution of T_N in Equation (2) is difficult to obtain; therefore, its asymptotic distribution is derived using existing results for Type-I and Type-II censoring, following the approach of Bhattacharyya (1985). It is shown in the next subsection that the asymptotic distribution of the RML statistic is approximately normal, and it is used to compute the probability of correct selection (PCS).

Next, we discuss the second method used for model discrimination, which is based on the Minimum Density Power Divergence Estimator (MDPDE), originally introduced by Basu et al. (1998). The MDPDE is obtained by minimizing the density power divergence between the true data-generating density $g(x)$ and the assumed model density $f_\theta(x)$, defined as

$$D_\varphi(g, f_\theta) = \int \left[f_\theta^{1+\varphi}(x) - \left(1 + \frac{1}{\varphi}\right) g(x) f_\theta^{\varphi}(x) + \frac{1}{\varphi} g^{1+\varphi}(x) \right] dx, \qquad \varphi \geq 0. \tag{3}$$

The tuning parameter φ controls the degree of robustness of the estimator. When $\varphi = 0$, the criterion reduces to the Kullback–Leibler divergence, and minimizing $D_\varphi(g, f_\theta)$ yields the maximum likelihood estimator (MLE). Since the true density $g(x)$ is unknown, it is replaced by the empirical distribution of the observed data, leading to the sample version

$$H_n(\theta) = \int f_\theta^{1+\varphi}(x)\, dx - \frac{1+\varphi}{\varphi} \cdot \frac{1}{n} \sum_{i=1}^{n} f_\theta^{\varphi}(x_i), \tag{4}$$

and the MDPDE $\hat{\theta}_\varphi$ is obtained by minimizing $H_n(\theta)$ with respect to θ.

Following Basu et al. (2006), for right-censored data, the empirical distribution in Eq. (4) is replaced by its consistent estimator $\tilde{G}_n(x) = 1 - \tilde{S}_n(x)$, where $\tilde{S}_n(x)$ denotes the Kaplan–Meier estimator of the survival function. Thus, the censored version of the objective function is

$$H_{n,\varphi}(\theta) = \int f_\theta^{1+\varphi}(x)\,dx - \left(1 + \frac{1}{\varphi}\right) \int f_\theta^{\varphi}(x)\,d\tilde{G}_n(x), \qquad \varphi \geq 0. \tag{5}$$

Here, $\tilde{G}_n(x)$ incorporates the censoring mechanism through the Kaplan–Meier jump probabilities, and the MDPDE $\hat{\theta}_\varphi$ is obtained by minimizing $H_{n,\varphi}(\theta)$ for each candidate model (Weibull and log-normal).

Using the minimized objective values from the two models, we define the MDPDE-based discrimination statistic as

$$d^*_\varphi\big(f_{\hat{\theta}_\varphi^{(\mathrm{WE})}}, f_{\hat{\theta}_\varphi^{(\mathrm{LN})}}\big) = H_{n,\varphi}\big(\hat{\theta}_\varphi^{(\mathrm{LN})}, \mathbf{x}\big) - H_{n,\varphi}\big(\hat{\theta}_\varphi^{(\mathrm{WE})}, \mathbf{x}\big), \tag{6}$$

where $\hat{\theta}_\varphi^{(\mathrm{WE})} = (\hat{\alpha}^{(d)}, \hat{\beta}^{(d)})$ and $\hat{\theta}_\varphi^{(\mathrm{LN})} = (\hat{\eta}^{(d)}, \hat{\sigma}^{(d)})$ are the MDPDEs under the Weibull and log-normal assumptions, respectively. The decision rule is to select the Weibull model if $d^*_\varphi\big(f_{\hat{\theta}_\varphi^{(\mathrm{WE})}}, f_{\hat{\theta}_\varphi^{(\mathrm{LN})}}\big) < 0$, and the log-normal model otherwise.

The third discrimination approach implemented in this study is the Bayesian method, as discussed in Cox (1961) (see also Araújo and Pereira (2007), Upadhyay and Peshwani (2003)). To the best of our knowledge, no existing research has considered model discrimination under any type of censoring using a Bayesian framework. It is of interest to compare the resulting probability of correct selection (PCS) between the Bayesian and the classical likelihood-based methods.

Let the vector of observations be Y, and let H_f and H_g denote the hypotheses that the pdf of Y is $f(y, \gamma)$ and $g(y, \psi)$, respectively, where γ and ψ are vectors of unknown parameters with $\gamma \in \Omega_\gamma$ and $\psi \in \Omega_\psi$. Let the prior probabilities of H_f and H_g be $\tilde{w}_f$ and $\tilde{w}_g$, respectively, with $\tilde{w}_f + \tilde{w}_g = 1$. Conditionally on H_f, let $p_f(\gamma)$ and $p_g(\psi)$ denote the prior pdfs of γ and ψ under H_f and H_g, respectively. If the observed value of Y is y, then the posterior odds for H_f versus H_g are

$$\frac{\tilde{w}_f \int_{\Omega_\gamma} f(y|\gamma) p_f(\gamma)\,d\gamma}{\tilde{w}_g \int_{\Omega_\psi} g(y|\psi) p_g(\psi)\,d\psi}.$$

The decision rule is: choose distribution f if

$$\widetilde{w}_f \int_{\Omega_\gamma} f(y|\gamma)p_f(\gamma)\,d\gamma > \widetilde{w}_g \int_{\Omega_\psi} g(y|\psi)p_g(\psi)\,d\psi,$$

and choose g otherwise. In our case, $\widetilde{w}_f = \widetilde{w}_g = \frac{1}{2}$.

For the log-normal distribution, we use Jeffreys' prior Zellner (1971); Tiao and Box (1973), while for the Weibull distribution we adopt a non-informative prior following Sun (1997), given by

$$\pi_{LN}(\mu,\sigma) = \frac{1}{\sigma}, \qquad \pi_{WE}(\alpha,\beta) = \frac{1}{\alpha\beta}.$$

Hence, the discrimination criterion is expressed as: if

$$\int_0^\infty \int_0^\infty L_{WE}(x;\alpha,\beta)\pi_{WE}(\alpha,\beta)\,d\alpha\,d\beta > \int_0^\infty \int_{-\infty}^\infty L_{LN}(x;\mu,\sigma)\pi_{LN}(\mu,\sigma)\,d\mu\,d\sigma, \tag{7}$$

then we choose the Weibull model; otherwise, we choose the log-normal model.

In terms of the Bayes factor (denoted B_n), the decision rule is defined as

$$B_n = \frac{\int_0^\infty \int_0^\infty L_{WE}(x;\alpha,\beta)\pi_{WE}(\alpha,\beta)\,d\alpha\,d\beta}{\int_0^\infty \int_{-\infty}^\infty L_{LN}(x;\mu,\sigma)\pi_{LN}(\mu,\sigma)\,d\mu\,d\sigma}. \tag{8}$$

If $B_n > 1$, we prefer Weibull; otherwise, log-normal.

It is important to emphasize that, in the simulation experiments, data were generated from a fixed true model—either Weibull or log-normal—with known parameter values. The prior distributions, $\pi_{WE}(\alpha,\beta)$ and $\pi_{LN}(\mu,\sigma)$, were employed solely for evaluating the marginal likelihoods in the Bayes factor computation, and not for data generation. Therefore, the Bayesian comparison is performed under a fixed data-generating framework to ensure consistency and comparability with the other two methods.

3.1 Asymptotic Distribution

The section deals with derivation of asymptotic distribution of RML statistic given by equation (2). The importance of establishing the asymptotic distribution is so that we are able to draw correspondence between probability of correct selection(PCS), obtained by Monte Carlo(MC) simulation, and that obtained by asymptotic distribution. Moreover, it would be easier for practical users to use the asymptotic mean and variances to compute PCS for required cases as well as to find minimum sample size needed to discriminate as discussed in Section 4. Before proceeding, we clarify that in both the simulation study and the development of asymptotic results, the censoring configuration is governed by the pair (p_1, p_2). Here, the $(1-p_1)$th quantile of the true distribution determines the degree of censoring (for example, a 40% degree of censoring corresponds to the 0.60th quantile), while the p_2th quantile specifies the censoring time T. This framework allows us to examine a wide range of Type-I and Type-II hybrid censoring scenarios. We will show that, when data is coming under Type-I hybrid censoring, the asymptotic distribution of T_N is approximately normally distributed for large n, some fixed censoring time T and number of failure R. Using the similar idea, asymptotic distribution of T_N, when data is coming under Type-II hybrid censoring, can be obtained. In this section, we use the following notations. Almost sure convergence will be denoted by a.s. For any functions, $f_1(.)$, $E_{LN}(f_1(U))$ and $V_{LN}(f_1(U))$ will denote the mean and variance of U under the assumption that U follows $LN(\eta, \sigma)$. Similarly, we define, $E_{WE}(f_1(U))$ and $V_{WE}(f_1(U))$ as mean and variance of U under the assumption that U follows $WE(\alpha, \beta)$. Moreover, if $f_2(.)$ and $f_1(.)$ are two functions, we define $Cov_{LN}(f_2(U), f_1(U))$ and $Cov_{WE}(f_2(U), f_1(U))$, as covariances where U follows $LN(\eta, \sigma)$ and $WE(\alpha, \beta)$, respectively.

3.2 Underlying distribution is Lognormal

As discussed earlier, under Type-I hybrid censoring, one of the following two cases may occur:
Case I: $\{x_{(1)} < x_{(2)} < \cdots < x_{(R)}\}$ if $x_{(R)} < T$,
Case II: $\{x_{(1)} < x_{(2)} < \cdots < x_{(d)}\}$ if $d < R$ and $x_{(d)} < T < x_{(d+1)}$.

Here, r^* denotes the number of observed failures, and the censoring time is given by $T_0 = \min\{x_{(R)}, T\}$. Accordingly,

$$T_0 = \begin{cases} x_{(R)}, & \text{if Case I,} \\ T, & \text{if Case II,} \end{cases} \qquad r^* = \begin{cases} R, & \text{if Case I,} \\ d, & \text{if Case II.} \end{cases}$$

Recalling Equation (2), the ratio of maximized log-likelihoods (RML) statistic under the hybrid Type-I setup can be written as

$$T_N = \sum_{i=1}^{r^*} \ln f_{\mathrm{WE}}(x_{(i)}; \hat{\alpha}, \hat{\beta}) + (n - r^*) \ln (1 - F_{\mathrm{WE}}(T_0; \hat{\alpha}, \hat{\beta}))$$
$$- \sum_{i=1}^{r^*} \ln f_{\mathrm{LN}}(x_{(i)}; \hat{\eta}, \hat{\sigma}) - (n - r^*) \ln (1 - F_{\mathrm{LN}}(T_0; \hat{\eta}, \hat{\sigma})).$$

Substituting the corresponding expressions for the Weibull and log-normal densities, we obtain

$$\begin{aligned} T_N = {} & r^*(\ln \alpha - \ln \beta) + (\alpha - 1) \sum_{i=1}^{r^*} \ln \left(\frac{x_{(i)}}{\beta} \right) - \sum_{i=1}^{r^*} \left(\frac{x_{(i)}}{\beta} \right)^{\alpha} \\ & - (n - r^*) \left(\frac{T_0}{\beta} \right)^{\alpha} + r^* \ln \sigma + r^* \ln \sqrt{2\pi} + \sum_{i=1}^{r^*} \ln x_{(i)} \\ & - \sum_{i=1}^{r^*} \frac{(\ln x_{(i)} - \eta)^2}{2\sigma^2} - (n - r^*) \left(1 - \Phi \left(\frac{\ln T_0 - \eta}{\sigma} \right) \right). \end{aligned} \tag{9}$$

Let $X_1, X_2, \ldots, X_n$ denote the lifetimes of n independent and identically distributed units drawn from $LN(\eta^0, \sigma^0)$. Under Type-I hybrid censoring, only the first r^* order statistics, denoted by $x_{(1)}, x_{(2)}, \ldots, x_{(r^*)}$, along with the censoring time $T_0 = \min\{x_{(R)}, T\}$, are observed as defined earlier. Here, X_i represents the lifetime of the ith unit before observation, and $x_{(i)}$ denotes its realized order statistic. Accordingly, the expectation and variance in the following expressions are taken with respect to the random variables X_i following the assumed log-normal model.

Therefore, if the data come from $LN(\eta^0, \sigma^0)$, we have

$$\begin{aligned} E(T_N | \eta^0, \sigma^0) = {} & E_{LN} \Bigg(r^*(\ln \alpha - \ln \beta) + (\alpha - 1) \sum_{i=1}^{r^*} \ln \left(\frac{X_i}{\beta} \right) - \sum_{i=1}^{r^*} \left(\frac{X_i}{\beta} \right)^{\alpha} \\ & - (n - r^*) \left(\frac{T_0}{\beta} \right)^{\alpha} + r^* \ln \sigma + r^* \ln \sqrt{2\pi} + \sum_{i=1}^{r^*} \ln X_i \\ & - \sum_{i=1}^{r^*} \frac{(\ln X_i - \eta)^2}{2\sigma^2} - (n - r^*) \left(1 - \Phi \left(\frac{\ln T_0 - \eta}{\sigma} \right) \right) \Bigg). \end{aligned} \tag{10}$$

Similarly, the variance can be expressed as

$$V(T_N|\eta^0,\sigma^0) = V_{LN}\Bigg(r^*(\ln\alpha - \ln\beta) + (\alpha-1)\sum_{i=1}^{r^*}\ln\left(\frac{X_i}{\beta}\right) - \sum_{i=1}^{r^*}\left(\frac{X_i}{\beta}\right)^{\alpha}$$
$$- (n-r^*)\left(\frac{T_0}{\beta}\right)^{\alpha} + r^*\ln\sigma + r^*\ln\sqrt{2\pi} + \sum_{i=1}^{r^*}\ln X_i$$
$$- \sum_{i=1}^{r^*}\frac{(\ln X_i - \eta)^2}{2\sigma^2} - (n-r^*)\left(1 - \Phi\left(\frac{\ln T_0 - \eta}{\sigma}\right)\right)\Bigg). \tag{11}$$

Our discrimination statistic T_N can therefore be expressed as:

$$T_N = \begin{cases} T_{N1}, & x_{(R)} > T, \\ T_{N2}, & x_{(R)} < T. \end{cases}$$

Here, T_{N1} corresponds to the RML statistic under the Type-I censoring setup and is given by

$$T_{N1} = \sum_{i=1}^{d}\ln f_{\mathrm{WE}}(x_{(i)};\hat\alpha,\hat\beta) + (n-d)\ln\left(1 - F_{\mathrm{WE}}(T;\hat\alpha,\hat\beta)\right)$$
$$- \sum_{i=1}^{d}\ln f_{\mathrm{LN}}(x_{(i)};\hat\eta,\hat\sigma) - (n-d)\ln\left(1 - F_{\mathrm{LN}}(T;\hat\eta,\hat\sigma)\right).$$

Similarly, T_{N2} corresponds to the RML statistic under the Type-II censoring setup and is given by

$$T_{N2} = \sum_{i=1}^{R}\ln f_{\mathrm{WE}}(x_{(i)};\hat\alpha,\hat\beta) + (n-R)\ln\left(1 - F_{\mathrm{WE}}(x_{(R)};\hat\alpha,\hat\beta)\right)$$
$$- \sum_{i=1}^{R}\ln f_{\mathrm{LN}}(x_{(i)};\hat\eta,\hat\sigma) - (n-R)\ln\left(1 - F_{\mathrm{LN}}(x_{(R)};\hat\eta,\hat\sigma)\right).$$

Note that, under Type-I censoring, the number of failures (d) is itself a random variable. To derive the asymptotic distribution of T_N, this randomness requires separate treatment, as detailed in the Appendix.

As discussed earlier, we can write

$$
\begin{aligned}
E(T_{N1}|\eta^0,\sigma^0) = & E_{LN}\Bigg(d(\ln\alpha - \ln\beta) + (\alpha-1)\sum_{i=1}^{d}\ln\left(\frac{X_i}{\beta}\right) - \sum_{i=1}^{d}\left(\frac{X_i}{\beta}\right)^{\alpha} \\
& - (n-d)\left(\frac{T}{\beta}\right)^{\alpha} + d\ln\sigma + d\ln\sqrt{2\pi} + \sum_{i=1}^{d}\ln X_i \\
& - \sum_{i=1}^{d}\left(\frac{(\ln X_{(i)}-\eta)^2}{2\sigma^2}\right) - (n-d)\Bigg(1-\Phi\left(\frac{\ln T-\eta}{\sigma}\right)\Bigg)\Bigg).
\end{aligned} \tag{12}
$$

Similarly,

$$
\begin{aligned}
V(T_{N1}|\eta^0,\sigma^0) = & V_{LN}\Bigg(d(\ln\alpha - \ln\beta) + (\alpha-1)\sum_{i=1}^{d}\ln\left(\frac{X_i}{\beta}\right) - \sum_{i=1}^{d}\left(\frac{X_i}{\beta}\right)^{\alpha} \\
& - (n-d)\left(\frac{T}{\beta}\right)^{\alpha} + d\ln\sigma + d\ln\sqrt{2\pi} + \sum_{i=1}^{d}\ln X_i \\
& - \sum_{i=1}^{d}\left(\frac{(\ln X_{(i)}-\eta)^2}{2\sigma^2}\right) - (n-d)\Bigg(1-\Phi\left(\frac{\ln T-\eta}{\sigma}\right)\Bigg)\Bigg).
\end{aligned} \tag{13}
$$

Let us define the asymptotic mean and variance of T_N under the log-normal model as

$$
AM_{LN} = \lim_{n\to\infty}\frac{E_{LN}(T_N|\eta^0,\sigma^0)}{n}, \qquad AV_{LN} = \lim_{n\to\infty}\frac{V_{LN}(T_N|\eta^0,\sigma^0)}{n}. \tag{14}
$$

Here, AM_{LN} and AV_{LN} denote the asymptotic mean and variance of T_N when data are generated from a log-normal distribution. Similarly, $E_{LN}(T_N|\eta^0,\sigma^0)$ and $V_{LN}(T_N|\eta^0,\sigma^0)$ represent the finite-sample mean and variance of T_N for large n, as stated in Theorem 3.1.

Using the proof of Theorem 3.1, we show that the asymptotic distribution of T_N under the Type-I hybrid setup coincides with that of T_{N1} (the RML statistic under Type-I censoring) when T is smaller than the p_1-th quantile of the parent distribution. Otherwise, it coincides with that of T_{N2} (the RML statistic under Type-II censoring). Here, $p_1 = \lim_{n\to\infty}\frac{R}{n}$, and the censoring time T corresponds to the p_2-th quantile of the parent distribution, i.e., $F(T) = p_2$.

Likewise, under the Type-II hybrid setup, the asymptotic distribution of T_N is identical to that of T_{N1} when T exceeds the p_1-th quantile, and to that of T_{N2} otherwise. Hence,

$$AM_{LN} = \begin{cases} AM_{LN}(p_2), & T < \zeta_{p1}, \\ AM_{LN}(p_1), & T \geq \zeta_{p1}, \end{cases}$$

where $AM_{LN}(p_2)$ denotes the asymptotic mean of T_{N1} (Type-I censoring), $AM_{LN}(p_1)$ denotes that of T_{N2} (Type-II censoring), and ζ_{p1} is the population p_1-th quantile.

Similarly,

$$AV_{LN} = \begin{cases} AV_{LN}(p_2), & T < \zeta_{p1}, \\ AV_{LN}(p_1), & T \geq \zeta_{p1}, \end{cases}$$

where $AV_{LN}(p_2)$ and $AV_{LN}(p_1)$ represent the asymptotic variances of T_{N1} and T_{N2}, respectively.

Finally,

$$AM_{LN}(p_2) = \lim_{n\to\infty} \frac{E_{LN}(T_{N1}|\eta^0, \sigma^0)}{n}, \qquad AV_{LN}(p_2) = \lim_{n\to\infty} \frac{V_{LN}(T_{N1}|\eta^0, \sigma^0)}{n}, \tag{15}$$

and

$$AM_{LN}(p_1) = \lim_{n\to\infty} \frac{E_{LN}(T_{N2}|\eta^0, \sigma^0)}{n}, \qquad AV_{LN}(p_1) = \lim_{n\to\infty} \frac{V_{LN}(T_{N2}|\eta^0, \sigma^0)}{n}. \tag{16}$$

Here, $E_{LN}(T_{Ni}|\eta^0, \sigma^0)$ and $V_{LN}(T_{Ni}|\eta^0, \sigma^0)$ denote the mean and variance of T_{Ni} $(i = 1, 2)$ for large n. The asymptotic distribution of T_{N2} (RML under Type-II censoring) has already been derived by Dey and Kundu (2012), and the corresponding expressions for $AM_{LN}(p_1)$ and $AV_{LN}(p_1)$ are provided in Section 2 of that paper. In this work, we extend these results by obtaining the expressions for $AM_{LN}(p_2)$ and $AV_{LN}(p_2)$, as presented in Equations (17) and (19).

Theorem 3.1 *Under the assumption that the Type-I hybrid censored data arise from $LN(\eta^0, \sigma^0)$, the statistic T_N is approximately normally distributed with mean $E_{LN}(T_N|\eta^0, \sigma^0)$ and variance $V_{LN}(T_N|\eta^0, \sigma^0)$ (see Equations (10) and (11)) for large n.*

To prove the above theorem, we first derive the asymptotic distributions of the RML statistics under Type-I and Type-II censoring, denoted by T_{N1} and T_{N2}, respectively. The asymptotic distribution of T_{N2} was established in Dey and Kundu (2012); we now derive that of T_{N1}.

Theorem 3.2 *If Type-I censored data are generated from $LN(\eta^0, \sigma^0)$, then T_{N1} is approxi-*

mately normally distributed with mean $E_{LN}(T_{N1}|\eta^0,\sigma^0)$ and variance $V_{LN}(T_{N1}|\eta^0,\sigma^0)$ (see Equations (12) and (13)) for large n.

To establish this result, we first present the following lemma.

Lemma 3.1 *If the Type-I censored data are drawn from $LN(\eta^0,\sigma^0)$, the following hold as $n\to\infty$:*
(a) $\hat\eta\to\eta^0$ a.s. and $\hat\sigma\to\sigma^0$ a.s., where

$$\begin{aligned}\Gamma_3(\eta^0,\sigma^0) &= \max_{u,v}\Gamma_3(u,v), \quad \textit{and}\\ \Gamma_3(u,v) &= \int_0^T \ln f_{LN}(x,u,v) f_{LN}(x,\eta^0,\sigma^0)\,dx + \ln\left(1-F_{LN}(T,u,v)\right)(1-F_{LN}(T,\eta^0,\sigma^0)).\end{aligned}$$

(b) $\hat\alpha\to\tilde\alpha$ a.s. and $\hat\beta\to\tilde\beta$ a.s., where

$$\begin{aligned}\Gamma_4(\tilde\alpha,\tilde\beta) &= \max_{u,v}\Gamma_4(u,v), \quad \textit{and}\\ \Gamma_4(u,v) &= \int_0^T \ln f_{WE}(x,u,v) f_{LN}(x,\eta^0,\sigma^0)\,dx + \ln\left(1-F_{WE}(T,u,v)\right)(1-F_{LN}(T,\eta^0,\sigma^0)).\end{aligned}$$

(c) Let

$$T^*_{N1} = \frac{L_{WE}(\tilde\alpha,\tilde\beta)}{L_{LN}(\eta^0,\sigma^0)}.$$

Then,

$$\frac{1}{\sqrt{n}}(T_{N1}-E_{LN}(T_{N1}|\eta^0,\sigma^0)) \textit{ is asymptotically equivalent to } \frac{1}{\sqrt{n}}(T^*_{N1}-E_{LN}(T^*_{N1})).$$

Proof of Lemma 3.1: See Appendix.
Proof of Theorem 3.2: By the Central Limit Theorem and part (b) of Lemma 3.1, $\frac{1}{\sqrt{n}}(T^*_{N1} - E_{LN}(T^*_{N1}))$ is asymptotically normal with mean zero and variance $V_{LN}(T^*_{N1})$. Using part (c) of Lemma 3.1, the desired result follows immediately.
Proof of Theorem 3.1: See Appendix.

It should be noted that $\tilde\alpha$ and $\tilde\beta$ depend on η^0, σ^0, and T. We refer to these as the misspecified Weibull parameters when the true data-generating process is log-normal.

We now provide the explicit expressions of $AM_{LN}(p_2)$ and $AV_{LN}(p_2)$ as defined in Equation (15). Using Theorem 1 of Bhattacharyya (1985), we have

$$\begin{aligned}
AM_{LN}(p_2) = &\int_0^T \ln\left(\frac{f_{WE}(x,\tilde{\alpha},\tilde{\beta})}{f_{LN}(x,1,1)}\right) f_{LN}(x,1,1)\,dx + (1-p_2)\ln\left(\frac{1-F_{WE}(x,\tilde{\alpha},\tilde{\beta})}{1-F_{LN}(x,1,1)}\right) \\
= &\left(\frac{1}{2}\ln 2\pi + \ln\tilde{\alpha} + \tilde{\alpha}\ln\frac{1}{\tilde{\beta}}\right)p_2 + \tilde{\alpha}E_{LN}((\ln Z)I_{Z\le T}) + \frac{1}{2}E_{LN}((\ln Z)^2 I_{Z\le T}) \\
- &\left(\frac{1}{\tilde{\beta}}\right)^{\tilde{\alpha}} E_{LN}(Z^{\frac{1}{\tilde{\alpha}}} I_{Z\le T}) - (1-p_2)h_1(T), \qquad (17)
\end{aligned}$$

where $Z \sim LN(1,1)$ and $I_{Z\le T}$ is an indicator function equal to 1 if $Z \le T$ and 0 otherwise. Furthermore,

$$h_1(T) = \left[\ln\left(1-\Phi(\ln T)\right) + \left(\frac{T}{\tilde{\beta}}\right)^{\tilde{\alpha}}\right]. \qquad (18)$$

The corresponding asymptotic variance is given by

$$AV_{LN}(p_2) = \tau + \frac{p_2(1-p_2)b^2}{f_{LN}^2(T,1,1)}, \qquad (19)$$

where

$$\tau = \int_0^T \left(\ln\frac{f_{WE}(x,\tilde{\alpha},\tilde{\beta})}{f_{LN}(x,1,1)}\right)^2 f_{LN}(x,1,1)\,dx - p_2^{-1}\left(\int_0^T \ln\left(\frac{f_{WE}(x,\tilde{\alpha},\tilde{\beta})}{f_{LN}(x,1,1)}\right) f_{LN}(x,1,1)\,dx\right)^2$$

and

$$\begin{aligned}
b = &-\ln\left(\frac{f_{WE}(T,\tilde{\alpha},\tilde{\beta})}{f_{LN}(T,1,1)}\right) f_{LN}(T,1,1) + p_2^{-1} f_{LN}(T,1,1)\int_0^T \ln\left(\frac{f_{WE}(x,\tilde{\alpha},\tilde{\beta})}{f_{LN}(x,1,1)}\right) f_{LN}(x,1,1)\,dx \\
&-(1-p_2)h_1^{'}(T),
\end{aligned}$$

where $h_1^{'}(T)$ denotes the derivative of $h_1(T)$ defined in Equation (18).

Similarly, the asymptotic behavior of T_N can be derived when the data are obtained under a Type-II hybrid censoring scheme.

Theorem 3.3 *Assume that the Type-II hybrid censored data arise from $LN(\eta^0,\sigma^0)$. Then, for large n, T_N is approximately normally distributed with mean $E_{LN}(T_N|\eta^0,\sigma^0)$ and variance $V_{LN}(T_N|\eta^0,\sigma^0)$.*

Proof of Theorem 3.3: The result follows directly from Theorem 3.2, using arguments similar to those employed in the proof of Theorem 3.1. The corresponding expressions for the mean $E_{LN}(T_N|\eta^0, \sigma^0)$ and variance $V_{LN}(T_N|\eta^0, \sigma^0)$ under the Type-II hybrid censoring scheme can be derived in an analogous manner to the Type-I case.

3.3 Underlying Distribution is Weibull

Since the development of this section follows similar ideas as in Subsection 3.2, we have derived the corresponding results and expressions by replacing the parent distribution from log-normal to Weibull. For brevity, this subsection has been moved to the supplementary material.

Finally, following Theorem 1 of Bhattacharyya (1985), we compute the asymptotic means (AM) and variances (AV) under both parent distributions, for various values of T (determined by p_2) and censoring proportions (determined by p_1). The computed values of AM_{LN}, AV_{LN}, AM_{WE}, and AV_{WE} under both Type-I and Type-II hybrid schemes are reported in Tables 7 and 8.

p_1	p_2	AM_{LN}	AV_{LN}	AM_{WE}	AV_{WE}
0.60	0.40	-0.0082	0.0124	0.0113	0.0369
	0.70	-0.0167	0.0255	0.0242	0.0814
	0.80	-0.0167	0.0255	0.0242	0.0814
0.50	0.40	-0.0082	0.0124	0.0113	0.0369
	0.70	-0.0121	0.0180	0.0174	0.0563
	0.80	-0.0121	0.0180	0.0174	0.0563
0.30	0.40	-0.0051	0.0078	0.0072	0.0221
	0.70	-0.0051	0.0078	0.0072	0.0221
	0.80	-0.0051	0.0078	0.0072	0.0221

Table 7: Value of asymptotic mean (AM) and asymptotic variance (AV) for respective parent distribution under Type-I hybrid setup.

p_1	p_2	AM_{LN}	AV_{LN}	AM_{WE}	AV_{WE}
0.60	0.40	-0.0167	0.0255	0.0242	0.0814
	0.70	-0.0233	0.0356	0.0336	0.1128
	0.80	-0.0319	0.0498	0.0454	0.1533
0.50	0.40	-0.0120	0.0181	0.0172	0.0563
	0.70	-0.0233	0.0356	0.0336	0.1128
	0.80	-0.0319	0.0498	0.0454	0.1533
0.30	0.40	-0.0081	0.0123	0.0115	0.0368
	0.70	-0.0233	0.0356	0.0336	0.1128
	0.80	-0.0319	0.0498	0.0454	0.1533

Table 8: Value of asymptotic mean (AM) and asymptotic variance (AV) for respective parent distribution under Type-II hybrid setup.

The asymptotic means and variances are presented in Tables 7 and 8. Identical values in the last two rows occur because, as established in Theorem 3.1, when $T \geq \zeta_p$, the asymptotic distribution of T_N coincides with that of T_{n2} (Type-II censoring). These values can be used for practical applications, such as calculating the probability of correct selection (PCS) using the asymptotic distribution. These values also allow for comparison with the PCS obtained from Monte Carlo (MC) simulations. Additionally, the asymptotic means and variances are useful for determining the minimum sample size required for discrimination, as discussed in the next section.

4 Minimum sample size determination

In this section, we propose a method to determine the minimum sample size needed to discriminate between the two distribution functions for a given user-specified probability of correct selection, considering that the censoring time T and the degree of censoring are known.

Suppose it is assumed that the data are coming from $LN(\eta^0, \sigma^0)$ and the censoring time T is chosen using p^{th} quantile (p_2) along with degree of censoring (depending on p_1). Since T_N is asymptotically normally distributed with mean $E_{LN}(T_N|\eta^0, \sigma^0)$ and variance $V_{LN}(T_N|\eta^0, \sigma^0)$ (obtained using equation (14),(17) and (19)) therefore, $PCS_{LN} = P(T_N \leq 0)$. Similarly, if it is assumed that the data are coming from $WE(\alpha^0, \beta^0)$, the PCS can be written as $PCS_{WE} = P(T_N \geq 0)$. Minimum sample size is determined for a given p and protection level α^*. Here protection level is referred to a desired probability of correct selection under respective cases. Therefore, for a given p, to determine the minimum sample size required to achieve at least α^*

protection level, we equate

$$\Phi\left(\frac{-n \times \mathrm{AM}_{\mathrm{LN}}(p)}{\sqrt{n \times \mathrm{AV}_{\mathrm{LN}}(p)}}\right) = \alpha^* \text{ and } \Phi\left(\frac{n \times \mathrm{AM}_{\mathrm{WE}}(p)}{\sqrt{n \times \mathrm{AV}_{\mathrm{WE}}(p)}}\right) = \alpha^*. \tag{20}$$

From equation (20), setting $z_{\alpha^*} = \Phi^{-1}(\alpha^*)$, we obtain

$$\frac{-n\,\mathrm{AM}_{LN}(p)}{\sqrt{n\,\mathrm{AV}_{LN}(p)}} = z_{\alpha^*}, \quad \frac{n\,\mathrm{AM}_{WE}(p)}{\sqrt{n\,\mathrm{AV}_{WE}(p)}} = z_{\alpha^*}.$$

Further simplifying the above equations, we obtain n_1 and n_2 as:

$$n_1 = \frac{z_{\alpha^*}^2 \mathrm{AV}_{\mathrm{LN}}(p)}{(\mathrm{AM}_{\mathrm{LN}}(p))^2} \text{ and } n_2 = \frac{z_{\alpha^*}^2 \mathrm{AV}_{\mathrm{WE}}(p)}{(\mathrm{AM}_{\mathrm{WE}}(p))^2},$$

here z_{α^*} is the α^*-th quantile point of a standard normal distribution. Therefore, we can take $n = max\{n_1, n_2\}$ the minimum sample size required to achieve at least α^* protection level for a given censoring time and degree of censoring.

Suppose, one wants a 0.95 protection level for a censoring time which is chosen as 0.40-th quantile and 0.40 degree of censoring, in that case minimum sample size$=max(498, 782)$ =782. Using Table 7 and 8, we provide minimum sample size to discriminate between two concerned distributions for a given protection level, degree of censoring (p_1 quantile) and censoring time (p_2 quantile). The results for minimum sample size under Type-I and Type-II hybdrid censoring are reported in Table 9 and 10, respectively.

α^*	$(0.60, 0.70)$	$(0.50, 0.70)$	$(0.60, 0.40)$	$(0.30, 0.40)$
0.90	229	306	475	701
0.95	378	504	782	1153
0.99	753	1007	1564	2308

Table 9: Minimum Sample Size to achieve a level of significance α^* for different values of (p_1 ,p_2) under Type-I hybrid censoring.

α^*	$(0.60, 0.70)$	$(0.50, 0.40)$	$(0.60, 0.40)$	$(0.30, 0.40)$
0.90	165	313	229	458
0.95	271	515	378	753
0.99	541	1030	753	1506

Table 10: Minimum Sample Size to achieve a level of significance α^* for different values of (p_1 ,p_2) under Type-II hybrid censoring.

We observe that as the protection level α^* increases, the minimum sample size needed to discriminate increases which is intuitive, as well. Moreover, the effective discrimination between log-normal and Weibull is possible if we have more observations toward the right tail. Thus, if choice of p_1 and p_2 is small we need larger n. Therefore, best choice of both p_1 and p_2 ,for discrimination problem to be easier, is 1 i.e. complete sample.

5 Numerical results

In this section, we present simulation experiments to study the behavior of the proposed model discrimination procedures under different hybrid censoring schemes. The primary objective is to examine how the Probability of Correct Selection (PCS), based on the asymptotic distributions derived in Section 3, performs across different sample sizes, censoring levels, and censoring times. We also compare the classical likelihood ratio statistic (T_N), the Bayesian decision criterion (see equation 8), and the Minimum Density Power Divergence Estimator (MDPDE)-based decision rule.

All simulations were carried out with a total of 1000 replications for each configuration of sample size ($n = 30, 50, 100, 120, 160$), censoring time ($T$), and degree of censoring (d.o.c). The censoring time T was chosen as the p_2-th quantile of the parent distribution, and the degree of censoring (d.o.c) was determined by the p_1-th quantile. PCS values were computed based on Monte Carlo simulations (denoted as MC), the asymptotic approximation (denoted as AS), the Bayesian decision rule, and the MDPDE-based criterion with varying values of tuning parameters φ.

5.1 Case I: Without Outliers

We first consider clean data without contamination. Samples were generated from the Weibull$(2, 1)$ and Lognormal$(1, 1)$ distributions under both Type-I and Type-II hybrid censoring setups. For each setup, the PCS was computed based on the decision rule using T_N, the asymptotic approximation, Bayesian inference, and the MDPDE approach. The results are presented in Tables 11 and 12. The results for the Type-I hybrid setup are reported in Table 11 and that for Type-II hybrid setup in Table S5. The first row (denoted as MC) represents the PCS values obtained through Monte Carlo simulations, the second row (AS) corresponds to the PCS computed using the asymptotic distribution derived in Section 3, the third row shows the PCS based on the Bayesian decision rule, and the last three rows report the PCS obtained from the MDPDE-based decision criterion. Across all cases, PCS increases steadily with the sample size (n) and decreases with higher degrees of censoring (d.o.c). Similarly, for fixed n and d.o.c, PCS declines as the censoring time T decreases, reflecting the expected loss of information due to early termination. The asymptotic PCS estimates correspond closely with the simulated results, especially for larger n, confirming the theoretical validity of the asymptotic approximation. The Bayesian decision criterion provides PCS values comparable to those obtained from T_N, showing its consistent performance. Moreover, the MDPDE-based decision rule with $\varphi = 0.001$ yields results nearly identical to the classical likelihood approach, as theoretically expected, while higher φ values gives slightly better PCS than other methods.

A similar analysis was carried out when the parent distribution was Lognormal$(1, 1)$. The corresponding PCS values for the Type-I and Type-II hybrid censoring schemes are presented in Tables 12 and S6, respectively. The observed patterns of PCS remain broadly consistent with those obtained under the Type-I scheme. The PCS improves with increasing sample size (n) and declines with higher degrees of censoring, reaffirming the expected behavior under information loss. Across different censoring times (T), lower T values generally correspond to smaller PCS, indicating reduced discriminative power due to early termination of the test. The asymptotic PCS values show close correspondence with the Monte Carlo results for moderate and large samples, supporting the reliability of the asymptotic formulation. Between the competing methods, the Bayesian and likelihood-based criteria perform similarly, while the divergence-based (MDPDE) approach exhibits stable and competitive behavior.

$d.o.c(\%)$	T	Method $\downarrow$ $n \rightarrow$	30	50	100	120	160
40	0.71	MC	0.583	0.623	0.715	0.736	0.791
		AS	(0.627)	(0.660)	(0.722)	(0.741)	(0.782)
		Bayes	(0.566)	(0.603)	(0.695)	(0.728)	(0.778)
		$\varphi = 0.001$	0.565	0.612	0.638	0.728	0.808
		$\varphi = 0.1$	0.588	0.665	0.668	0.751	0.825
		$\varphi = 0.2$	0.548	0.625	0.711	0.788	0.818
	1.09	MC	0.651	0.705	0.807	0.840	0.871
		AS	(0.679)	(0.726)	(0.802)	(0.824)	(0.858)
		Bayes	(0.614)	(0.686)	(0.790)	(0.838)	(0.877)
		$\varphi = 0.001$	0.641	0.648	0.773	0.812	0.829
		$\varphi = 0.1$	0.635	0.681	0.853	0.808	0.845
		$\varphi = 0.2$	0.725	0.744	0.876	0.872	0.889
	1.26	MC	0.653	0.705	0.808	0.838	0.877
		AS	(0.679)	(0.726)	(0.802)	(0.824)	(0.858)
		Bayes	(0.632)	(0.725)	(0.797)	(0.856)	(0.880)
		$\varphi = 0.001$	0.652	0.691	0.840	0.838	0.893
		$\varphi = 0.1$	0.736	0.684	0.810	0.835	0.903
		$\varphi = 0.2$	0.729	0.731	0.830	0.832	0.903
70	0.71	MC	0.545	0.581	0.684	0.698	0.706
		AS	(0.604)	(0.633)	(0.684)	(0.700)	(0.728)
		Bayes	(0.507)	(0.559)	(0.640)	(0.701)	(0.710)
		$\alpha = 0.001$	0.527	0.570	0.607	0.690	0.723
		$\alpha = 0.1$	0.550	0.623	0.637	0.713	0.740
		$\alpha = 0.2$	0.510	0.583	0.680	0.750	0.733
	1.09	MC	0.533	0.614	0.644	0.685	0.729
		AS	(0.604)	(0.633)	(0.684)	(0.700)	(0.728)
		Bayes	(0.523)	(0.551)	(0.638)	(0.671)	(0.693)
		$\varphi = 0.001$	0.523	0.557	0.610	0.657	0.687
		$\varphi = 0.1$	0.517	0.590	0.690	0.653	0.703
		$\varphi = 0.2$	0.607	0.653	0.713	0.717	0.747
	1.26	MC	0.524	0.591	0.665	0.693	0.714
		AS	(0.604)	(0.633)	(0.684)	(0.700)	(0.728)
		Bayes	(0.518)	(0.555)	(0.658)	(0.684)	(0.718)
		$\varphi = 0.001$	0.523	0.577	0.697	0.693	0.730
		$\varphi = 0.1$	0.547	0.570	0.667	0.690	0.740
		$\varphi = 0.2$	0.550	0.617	0.687	0.687	0.740

Table 11: Probability of correct selection (PCS) under Type-I hybrid censoring when data is generated from WE$(2, 1)$, comparing Monte Carlo (MC), approximate selection (AS), Bayesian, and divergence-based methods for different φ values, across censoring levels ($d.o.c = 40\%$ and 70%).

$d.o.c(\%)$	T	Method $\downarrow$ $n \rightarrow$	30	50	100	120	160
40	0.77	MC	0.6299	0.6780	0.7612	0.7758	0.8010
		AS	(0.6614)	(0.7032)	(0.7745)	(0.7955)	(0.8298)
		Bayes	(0.6672)	(0.6783)	(0.7655)	(0.7909)	(0.8018)
		$\varphi = 0.001$	0.610	0.658	0.741	0.756	0.781
		$\varphi = 0.1$	0.660	0.708	0.791	0.806	0.831
		$\varphi = 0.2$	0.680	0.728	0.811	0.826	0.851
	1.68	MC	0.6977	0.7558	0.8422	0.8579	0.8955
		AS	(0.7193)	(0.7715)	(0.8514)	(0.8730)	(0.9059)
		Bayes	(0.7016)	(0.7683)	(0.8255)	(0.8447)	(0.9144)
		$\varphi = 0.001$	0.678	0.736	0.822	0.838	0.876
		$\varphi = 0.1$	0.728	0.786	0.872	0.888	0.926
		$\varphi = 0.2$	0.748	0.806	0.892	0.908	0.946
	2.32	MC	0.7101	0.7614	0.8395	0.8617	0.9006
		AS	(0.7157)	(0.7689)	(0.8507)	(0.8728)	(0.9056)
		Bayes	(0.7300)	(0.7593)	(0.8502)	(0.8750)	(0.9009)
		$\varphi = 0.001$	0.690	0.741	0.820	0.842	0.881
		$\varphi = 0.1$	0.740	0.791	0.870	0.892	0.931
		$\varphi = 0.2$	0.760	0.811	0.890	0.912	0.951
70	0.77	MC	0.6160	0.6357	0.7085	0.7644	0.7717
		AS	(0.6225)	(0.6564)	(0.7155)	(0.7337)	(0.7644)
		Bayes	(0.6560)	(0.6887)	(0.7306)	(0.7822)	(0.7944)
		$\varphi = 0.001$	0.596	0.616	0.689	0.744	0.752
		$\varphi = 0.1$	0.646	0.666	0.739	0.794	0.802
		$\varphi = 0.2$	0.666	0.686	0.759	0.814	0.822
	1.68	MC	0.6497	0.6860	0.7462	0.7437	0.7562
		AS	(0.6225)	(0.6564)	(0.7155)	(0.7337)	(0.7644)
		Bayes	(0.6425)	(0.6811)	(0.7516)	(0.7608)	(0.7719)
		$\varphi = 0.001$	0.630	0.666	0.726	0.724	0.736
		$\varphi = 0.1$	0.680	0.716	0.776	0.774	0.786
		$\varphi = 0.2$	0.700	0.736	0.796	0.794	0.806
	2.32	MC	0.6371	0.6655	0.7206	0.7274	0.7714
		AS	(0.6225)	(0.6564)	(0.7155)	(0.7337)	(0.7644)
		Bayes	(0.6306)	(0.6694)	(0.7367)	(0.7604)	(0.7710)
		$\varphi = 0.001$	0.617	0.646	0.701	0.708	0.751
		$\varphi = 0.1$	0.667	0.696	0.751	0.758	0.801
		$\varphi = 0.2$	0.687	0.716	0.771	0.778	0.821

Table 12: Probability of correct selection (PCS) under Type-I hybrid censoring when data is generated from LN$(1, 1)$, comparing Monte Carlo (MC), approximate selection (AS), Bayesian, and divergence-based methods for different φ values, across censoring levels ($d.o.c = 40\%$ and 70%).

5.2 Cases with contamination

To assess robustness, we introduce data contamination into the simulation setup. In this framework, each observation originates from the true model with probability $(1 - \epsilon)$ and from a contamination model with probability ϵ, where ϵ denotes the expected contamination proportion. For each sample of size n, the number of contaminated observations follows a Binomial(n, ϵ) distribution.

We consider two contamination scenarios:

(a) True model: Weibull$(2, 1)$, contaminant: Lognormal$(1.0, 2.5)$

(b) True model: Lognormal$(0, 1)$, contaminant: Weibull$(1.0, 2.5)$

Simulations were conducted for contamination levels $\epsilon = 0.2$ and 0.4, using sample sizes $n = 30, 50$, and 100. The censoring time was fixed at $T = 6.0$, with the degree of censoring maintained at 40% throughout. PCS values were computed for the likelihood, Bayesian, and MDPDE-based decision rules. The results are shown in Tables 13 and 14.

In the present study, we consider the case where the true distribution follows a Weibull$(2, 1)$ model, while the contamination arises from a lognormal$(1.0, 2.5)$ model. Tables 13 present the simulated probability of correct selection (PCS) for the likelihood-based, Bayesian, and MDPDE-based decision rules across varying contamination levels ($\epsilon = 0.2, 0.4$) and sample sizes. These results highlight the robustness of the MDPDE method, particularly for moderate and large values of the tuning parameter φ, which helps stabilize the discrimination performance in the presence of contamination. Across the contamination experiments where the true model is Weibull$(2, 1)$ and the contaminant is lognormal, we find that MDPDE with moderate tuning parameters markedly outperforms the classical likelihood rule. In particular, $\varphi \in \{0.3, 0.4, 0.5\}$ yield the largest gains in PCS for contamination levels $\epsilon = 0.2$ and $\epsilon = 0.4$ over a range of sample sizes (30, 50, 100).

We also examined the reverse case where the true model is Lognormal$(0, 1)$ and the contamination comes from a Weibull$(1.0, 2.5)$ distribution (Table 14). The results show trends similar to those observed when the true model was Weibull$(2, 1)$. In both situations, the MDPDE-based rule remains more robust to contamination than the likelihood and Bayesian approaches. For moderate contamination levels ($\epsilon = 0.2, 0.4$) and larger sample sizes, moderate tuning values ($\varphi \approx 0.05$–0.07) yield higher PCS and are less affected by outliers. Overall, these findings confirm that the divergence-based method performs better than the other methods when model discrimination is carried out under contaminated conditions.

Contamination (ϵ)	n	Likelihood	Bayesian	MDPDE (for φ)		
				0.3	0.4	0.5
0.2	30	0.504	0.512	0.572	0.644	0.636
	50	0.564	0.570	0.648	0.604	0.626
	100	0.637	0.640	0.666	0.696	0.654
0.4	30	0.489	0.495	0.606	0.594	0.598
	50	0.471	0.476	0.620	0.624	0.634
	100	0.571	0.574	0.680	0.654	0.642

Table 13: Comparison of PCS values under lognormal contamination (True model: Weibull$(2, 1)$; Contamination model: Lognormal$(1.0, 2.5)$)

Contamination (ϵ)	n	Likelihood	Bayesian	MDPDE (for φ)		
				0.04	0.05	0.07
0.2	30	0.544	0.550	0.555	0.565	0.517
	50	0.532	0.540	0.590	0.583	0.565
	100	0.566	0.570	0.647	0.668	0.637
0.4	30	0.510	0.512	0.500	0.510	0.575
	50	0.548	0.550	0.532	0.578	0.595
	100	0.408	0.415	0.572	0.603	0.612

Table 14: Comparison of PCS values under Weibull$(1.0, 2.5)$ contamination (True model: Lognormal$(0, 1)$)

6 Effect of Model Misspecification on prediction for future failures

Let $X = (x_{(1)}, x_{(2)} \dots x_{(r^*)})$ denotes a Type-I hybrid censored samples. We are interested in prediction of future failures $Y = X_{(s+r^*)}(s = 1, 2 \dots n - r^*)$ of all the $(n - r^*)$ censored units based on observed sample $X = (x_{(1)}, x_{(2)} \dots x_{(r^*)})$ To study the effects of model misspecification on predcition for future failures, we have considered best unbiased predictor (BUP) of Y, which is given as :

$$\hat{Y}_{BUP} = E(Y|X) = \int_T^{\infty} y f(y|x, \theta)\, dy \quad ,$$

where $f(y|x, \theta)$ is conditional pdf of the model under which we are interested in estimation of future failure.

Now, we present the results of the effect of model misspecification on the prediction of future failures when data is generated from $WE(2,1)$. We compared the mean square error (MSE) values of the best unbiased predictor (BUP) of $X_{(s+d)}$ for varying values of s and under two different choices of censoring time (T). MSE_{WE} denotes the MSE computed for the estimation of BUP under the Weibull distribution, while MSE_{LN} denotes the MSE under the log-normal distribution. We considered two separate sub-cases: first with $n = 30$ and $R = 15$, followed by $n = 60$ and $R = 30$. The results are reported in Tables 15 and 16.

T	Model	$s = 3$	$s = 5$	$s = 10$	$s = 15$
0.70	MSE_{WE}	0.0089	0.0156	0.0541	0.1796
	MSE_{LN}	0.0099	0.0242	0.2318	2.4167
0.95	MSE_{WE}	0.0062	0.0126	0.0454	0.3309
	MSE_{LN}	0.0112	0.0365	0.3172	10.2742

Table 15: Effect of Model Misspecification on MSE of point predictions for $n = 30$ and $R = 15$.

T	Model	$s = 5$	$s = 10$	$s = 20$	$s = 30$
0.70	MSE_{WE}	0.0031	0.0076	0.0243	0.0841
	MSE_{LN}	0.0042	0.0163	0.1846	2.0559
0.95	MSE_{WE}	0.0026	0.0069	0.0287	0.2419
	MSE_{LN}	0.0058	0.0273	0.3253	12.4093

Table 16: Effect of Model Misspecification on MSE of point predictions for $n = 60$ and $R = 30$.

We did the similar exercise by changing the choice of parent distribution from $WE(2,1)$ to $LN(1,1)$ and the findings are reported in Table 17 and 18.

T	Model	$s = 3$	$s = 5$	$s = 10$	$s = 15$
0.77	MSE_{WE}	0.0363	0.1157	0.6724	4.6203
	MSE_{LN}	0.0274	0.0864	0.4234	1.3335
1.28	MSE_{WE}	0.0447	0.1152	1.0997	12.5341
	MSE_{LN}	0.0305	0.0627	0.3976	2.8627

Table 17: Effect of Model Misspecification on MSE of point predictions for $n = 30$ and $R = 15$.

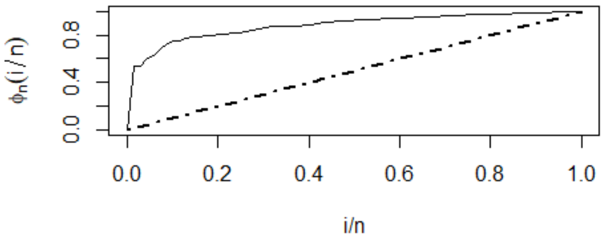


Figure 4: The empirical scaled TTT transform of the real life dataset

T	Model	$s = 5$	$s = 10$	$s = 20$	$s = 30$
0.77	MSE_{WE}	0.0174	0.0600	0.4845	5.1800
	MSE_{LN}	0.0086	0.0194	0.0929	1.0378
1.28	MSE_{WE}	0.0197	0.0771	0.8809	11.9795
	MSE_{LN}	0.0117	0.0318	0.2042	2.7971

Table 18: Effect of Model Misspecification on MSE of point predictions for $n = 60$ and $R = 30$.

From the above tables, we can conclude that when the parent distribution is Weibull, the MSE values under the incorrect model (log-normal in this case) are higher than those under the correct model. Moreover, as we move towards extreme future points (as s increases), the difference between the MSE values under both models increases. Similarly, when the parent distribution is log-normal, the effect of model misspecification on the MSE of BUP is evident. In both cases, as we move towards the endpoints, the MSE under the incorrect model increases significantly. Hence, we can conclude that model misspecification has a significant effect on the prediction of future failures.

7 Real data analysis

Considering the data set given in Section 2, we first computed maximized loglikelihood values for complete sample, as follows: $L_{WE} = -49.59$ and $L_{Ln} = -51.38$. Moreover, value of Bayes factor is 6.64. Both of the discrimination criterion suggests that appropriate model is Weibull. We further show the data analysis using Total Time on Test (TTT) transform(see Epstein and Sobel (1953), Barlow and Campo (1975)). Aarset (1987) showed that the scaled TTT transform is convex (concave) if the hazard rate is decreasing (increasing), and for bathtub (unimodal) hazard rates, the scaled TTT transform is first convex (concave) and then concave (convex). Using Fig. 4, we can see that the shape of the empirical scaled TTT transform is concave, which again indicates that the preferred model is Weibull. Now, introducing hybrid censoring for the given dataset, and for varying values of R and T, we make appropriate model selections based on both discrimination criteria: if $T_N > 0$, $d^*_\varphi < 0$ and $B_n > 1$, we prefer the Weibull model; otherwise, we prefer the log-normal distribution. The findings are reported in Table 19.

T	R			Divergence-based criterion (d^*_φ)			Preferred model
		T_N	B_n	$\varphi = 0.001$	$\varphi = 0.1$	$\varphi = 0.2$	
2.0	30	0.07	1.02	-0.0012	-0.0022	-0.0029	Weibull
2.5	20	0.15	1.07	-0.0023	-0.0032	-0.0037	Weibull
2.5	30	0.26	1.25	-0.0084	-0.0093	-0.0095	Weibull
3.0	40	0.72	1.97	-0.0098	-0.0105	-0.0104	Weibull
3.0	50	1.44	4.18	-0.0173	-0.0176	-0.0169	Weibull

Table 19: Comparison of value of RML, Bayesian and divergence-based discriminating statistics under hybrid censoring for real-life data

Therefore, we see that in the presence of hybrid censoring and for varying values of T and R, the preferred model remains Weibull. However, in the extreme case when $T = 2.0$ and $R = 30$, the values of the discrimination criteria are just off the cutoff ($T_N = 0.07$, $B_n = 1.02$ and $d^*_{0.001} = -0.0012$). The possible reason for this is that for this extreme censoring criterion, the sample size is 13, whereas the complete sample size was 69. To compute the probability of correct selection (PCS), we employed a parametric bootstrap approach (based on 1000 replications). Samples ($n = 69$) were generated from the fitted Weibull distribution for the given dataset, i.e., $WE(5.50, 2.65)$, across varying values of T and R. The resulting PCS values for both discrimination criteria are reported in Table 20.

T	R			Divergence-based PCS (for φ)		
		T_N	B_n	0.001	0.1	0.2
2.0	30	0.551	0.511	0.535	0.545	0.542
2.5	20	0.637	0.631	0.648	0.650	0.718
2.5	30	0.693	0.681	0.685	0.672	0.713
3.0	40	0.747	0.731	0.748	0.772	0.802
3.0	50	0.807	0.799	0.815	0.835	0.840

Table 20: Comparison of PCS based on T_N, B_n, and divergence-based criterion d^*_φ under hybrid censoring when data follow Weibull$(5.50, 2.65)$

We can conclude that PCS tends to decrease when the degree of censoring is highest (i.e., when R and T are at their extreme values). As R and T increase, PCS also increases.

8 Conclusion

This paper addresses the problem of discriminating between the log-normal and Weibull distributions under Type-I and Type-II hybrid censoring schemes. We employ the difference between maximized log-likelihoods as the decision criterion and derive the asymptotic distributions of the RML statistic for both hybrid setups. In addition, a divergence-based decision approach is implemented, which demonstrates superior performance compared to the classical method, particularly in the presence of outliers.

Through extensive Monte Carlo (MC) simulations, we compare the probability of correct selection (PCS) with asymptotic results across various sample sizes, censoring times, and degrees of censoring (d.o.c). The results indicate that asymptotic correspondence holds even for moderate sample sizes. By exploring both classical and Bayesian approaches, we find consistent outcomes between the two methods. Emphasizing the importance of correct model selection, we further investigate how model misspecification affects the p^{th} quantile, mean residual life (MRL), reliability function, and predictions of future failures.

Although our focus has been on discriminating between the Weibull and log-normal distributions, the proposed framework can be extended to any two members of the location-scale family or even to more than two competing models. Looking ahead, future research could focus on extending these discrimination methods to higher-dimensional settings. Moreover, developing asymptotic results for divergence-based decision criteria and determining the optimal choice of the tuning parameter φ remain promising directions for further investigation.

Acknowledgment

We are grateful to the Associate Editor and the reviewer for their valuable comments and suggestions, which were highly helpful in improving the quality of our work.

Appendix:

Before beginning the proof, we derive some asymptotic results regarding the number of failures (d), which are required to draw correspondence from the results of Bhattacharyya (1985). Suppose n units, with independent lifetimes and common CDF $F_X(x)$, are placed on a life test. Let d denote the number of units that failed on or before a pre-fixed time t_c, then d follows $Binomial(n, p)$, where $p = F_X(T)$. Hence,

$$\frac{d}{n} \xrightarrow{P} p \implies \frac{d}{n} = p + o_p(1), \tag{21}$$

where $\xrightarrow{P}$ denotes convergence in probability and $o_p(1)$ means convergence to zero in probability.

Proof of Lemma 3.1 (a) and (b):

The basic idea of the proof follows Theorems 1 and 2 of Bhattacharyya (1985). To use these results, define

$$g(x; \alpha, \beta, \eta^0, \sigma^0) = \ln\left(\frac{f_{WE}(x; \alpha, \beta)}{f_{LN}(x; \eta^0, \sigma^0)}\right),$$
$$h(x; \alpha, \beta, \eta^0, \sigma^0) = \ln\left(\frac{1 - F_{WE}(x; \alpha, \beta)}{1 - F_{LN}(x; \eta^0, \sigma^0)}\right).$$

Let $\gamma = (\alpha, \beta, \eta^0, \sigma^0)$ and $\mathcal{B}$ be a compact neighbourhood of γ. Then using Theorem 2 of Bhattacharyya (1985), we have

$$\frac{1}{n}\big(L_{WE}(\alpha, \beta) - L_{LN}(\eta^0, \sigma^0)\big) \longrightarrow \Gamma_1(\alpha, \beta) - \Gamma_2(\eta^0, \sigma^0), \tag{22}$$

with probability 1 and uniformly in $\gamma \in \mathcal{B}$, where

$$\Gamma_1(\alpha, \beta) - \Gamma_2(\eta^0, \sigma^0) = \int_0^T \ln\left(\frac{f_{WE}(x; \alpha, \beta)}{f_{LN}(x; \eta^0, \sigma^0)}\right) f_{LN}(x; \eta^0, \sigma^0)\, dx + q \ln\left(\frac{1 - F_{WE}(T; \alpha, \beta)}{q}\right).$$

We verified graphically that $\Gamma_1(\alpha, \beta)$ has a unique maximum at some $(\tilde{\alpha}, \tilde{\beta})$. For example, for $T = 0.77$, the unimodality of $\Gamma_1(\alpha, \beta)$ can be seen in Fig. 5.

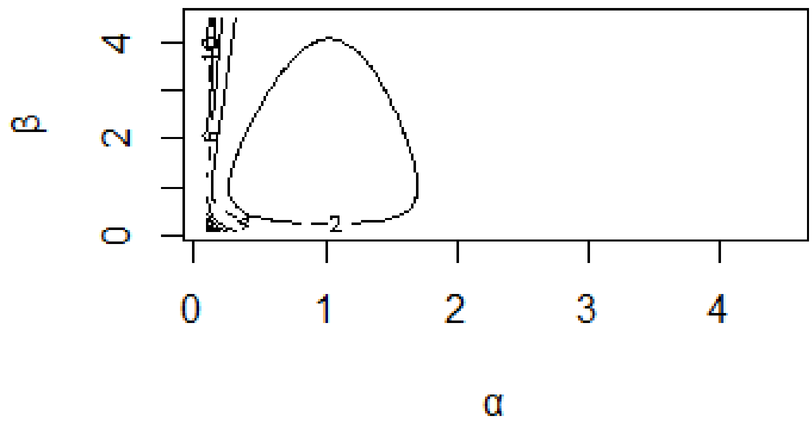


Figure 5: Contour plot of $\Gamma_1(\alpha,\beta)$ for $T = 0.77$.

Moreover,

$$L(\hat{\alpha},\hat{\beta}) = \sup_{(\alpha,\beta)\in\mathbb{R}^2} \frac{1}{n}\big(L_{WE}(\alpha,\beta) - L_{LN}(\eta^0,\sigma^0)\big).$$

Using Lemma 2.2 of White (1980), we have $\hat{\alpha} \to \tilde{\alpha}$ a.s. and $\hat{\beta} \to \tilde{\beta}$ a.s. This proves Lemma 3.1(b). A similar line of reasoning applies to Lemma 3.1(a).

Proof of Lemma 3.1(c):

We have,

$$\frac{T_{1n}}{n} = \frac{1}{n}\left[\sum_{i=1}^{d} g(x_{(i)},\hat{\gamma}) + (n-d)h(T,\hat{\gamma})\right]$$

and

$$\frac{T_{1*}}{n} = \frac{1}{n}\left[\sum_{i=1}^{d} g(x_{(i)},\tilde{\gamma}) + (n-d)h(T,\tilde{\gamma})\right]$$

here $\hat{\gamma} = (\hat{\alpha}, \hat{\beta}, \hat{\eta}, \hat{\sigma})^T$and $\tilde{\gamma} = (\tilde{\alpha}, \tilde{\beta}, \eta^0, \sigma^0)^T$ To prove Lemma 3.1(c), we need to show that :

$$\frac{1}{\sqrt{n}}[T_{1n} - E_{LN}(T_{1n}|\eta^0, \sigma^0) - T_{1*} + E_{LN}(T_{1*})] \xrightarrow{P} 0$$

In order to do so let us focus on the establishment of the statement:

$$\frac{1}{\sqrt{n}}[T_{1n} - T_{1*}] \xrightarrow{P} 0$$

followed by proving :

$$E_{LN}\bigg(\frac{1}{\sqrt{n}}[T_{1n} - T_{1*}]\bigg) \longrightarrow 0$$

$$\begin{aligned}
\frac{1}{\sqrt{n}}\left[T_{1n} - T_{1*}\right] &= \sqrt{n}\left\{\frac{1}{n}\sum_{i=1}^{d}\left[g(x_{(i)};\hat{\gamma}) - g(x_{(i)};\tilde{\gamma})\right] + \left(\frac{n-d}{n}\right)\left[h(T;\hat{\gamma}) - h(T;\tilde{\gamma})\right]\right\} \\
&= \sqrt{n}\left\{\frac{1}{n}\sum_{i=1}^{n}\left[g(x_i;\hat{\gamma}) - g(x_i;\tilde{\gamma})\right].\mathbf{1}_{x_i\le T} + \left(\frac{n-d}{n}\right)\left[h(T;\hat{\gamma}) - h(T;\tilde{\gamma})\right]\right\} \\
&= \sum_{j=1}^{4}\sqrt{n}(\hat{\gamma}_j - \tilde{\gamma}_j)\times\left[\frac{1}{n}\sum_{i=1}^{n}g_j^{'}(x_i;\tilde{\gamma}_j).\mathbf{1}_{x_i\le T} + \left(1 - \frac{d}{n}\right)h_j^{'}(T;\tilde{\gamma}_j) + o_p(1)\right] \\
&= \sum_{j=1}^{4}\sqrt{n}(\hat{\gamma}_j - \tilde{\gamma}_j)\times\left[\frac{1}{n}\sum_{i=1}^{n}g_j^{'}(x_i;\tilde{\gamma}_j).\mathbf{1}_{x_i\le T} + \Big(1 - p - o_p(1)\Big)h_j^{'}(T;\tilde{\gamma}_j) + o_p(1)\right]
\end{aligned}$$

Second last step appears using multivariate Taylor's expansion of $\left\{\frac{1}{n}\sum_{i=1}^{n}\left[g(x_i;\hat{\gamma})\right].\mathbf{1}_{x_i\le T} + (\frac{n-d}{n})\left[h(T;\hat{\gamma})\right]\right\}$ around $\tilde{\gamma}$. Here $o_p(1)$ means converges to zero in probability, where $g_j^{'} = \frac{\partial}{\partial\gamma_j}g(\cdot)$, $h_j^{'} = \frac{\partial}{\partial\gamma_j}h(\cdot)$, $\hat{\gamma}_j$ is the j-th component of $\hat{\gamma}$, and $\tilde{\gamma}_j$ is the j-th component of $\tilde{\gamma}$ for $j = 1, \ldots, 4$. Here $\sqrt{n}(\hat{\gamma}_j - \tilde{\gamma}_j)$ converges to a normal distribution with zero mean and some finite variance(using theorem 3.2 of White (1982)), therefore using Slutsky's theorem, we can write last step as:

$$\frac{1}{\sqrt{n}}\left[T_{1n} - T_{1*}\right] = \sum_{j=1}^{4}\left[\frac{1}{n}\sum_{i=1}^{n}g_j^{'}(x_i;\tilde{\gamma}_j).\mathbf{1}_{x_i\le T} + \Big(1 - p\Big)h_j^{'}(T;\tilde{\gamma}_j)\right]\sqrt{n}(\hat{\gamma}_j - \tilde{\gamma}_j) + o_p(1)$$

Now, using the weak law of large numbers and the fact that $h^{'}(\cdot)$ is a continuous function, we can write

$$\left[\frac{1}{n}\sum_{i=1}^{n} g_j^{'}(x_i;\tilde{\gamma}_j).\mathbf{1}_{x_i\leq T} + (1-p)h_j^{'}(T;\tilde{\gamma}_j)\right] \xrightarrow{P} E_{LN}\left(g_j^{'}(X);\gamma).\mathbf{1}_{X\leq T} + (1-p)h_j^{'}(T;\gamma)\right)\Big|_{\gamma=\tilde{\gamma}} \tag{23}$$

Using dominated convergence theorem (DCT) to interchange the operations of integration and differentiation, in right hand term of equation (9), we have,

$$\begin{aligned}
&E_{LN}\left(g_j^{'}(X;\gamma).\mathbf{1}_{X\leq T} + (1-p)h_j^{'}(T;\gamma)\right)\Big|_{\gamma=\tilde{\gamma}} \\
&= \frac{\partial}{\partial\gamma_j}\left[\int_0^T \ln\left(\frac{f_{WE}(x;\alpha,\beta)}{f_{LN}(x;\eta^0,\sigma^0)}\right) f_{LN}(x;\eta^0,\sigma^0)\,dx + q\ln\left(\frac{1-F_{WE}(T;\alpha,\beta)}{q}\right)\right]\Big|_{\gamma=\tilde{\gamma}} \\
&= 0
\end{aligned} \tag{24}$$

Using the fact that $\sqrt{n}(\hat{\gamma}_j - \tilde{\gamma}_j)$ converges to a normal distribution with zero mean and some finite variance(using theorem 3.2 of White (1982)), we have

$$\frac{1}{\sqrt{n}}[T_{1n} - T_{1*}] \xrightarrow{P} 0$$

Now let us consider the expectation term,

$$\begin{aligned}
&E_{LN}\left(\frac{1}{\sqrt{n}}\left[T_{1n} - T_{1*}\right]\right) \\
&= E_{LN}\left|\sqrt{n}\left\{\frac{1}{n}\sum_{i=1}^{d}\left[g(X_{(i)};\hat{\gamma}) - g(X_{(i)};\tilde{\gamma})\right] + \left(\frac{n-d}{n}\right)\left[h(T;\hat{\gamma}) - h(T;\tilde{\gamma})\right]\right\}\right| \\
&\leq \sum_{j=1}^{4} E_{LN}\left|\left[\frac{1}{n}\sum_{i=1}^{n} g_j^{'}(X_i;\tilde{\gamma}_j).\mathbf{1}_{X_i\leq T} + (1-p)h_j^{'}(T;\tilde{\gamma}_j)\right]\sqrt{n}(\hat{\gamma}_j - \tilde{\gamma}_j)\right| + o(1) \\
&\leq \sum_{j=1}^{4}\left[E_{LN}[\sqrt{n}(\hat{\gamma}_j - \tilde{\gamma}_j)]^2\; E_{LN}\left[\frac{1}{n}\sum_{i=1}^{n} g_j^{'}(X_i;\tilde{\gamma}_j).\mathbf{1}_{X_i\leq T} + (1-p)h_j^{'}(T;\tilde{\gamma}_j)\right]^2\right]^{\frac{1}{2}} + o(1)
\end{aligned}$$

Last step is written using Cauchy-Schwarz inequality. Since $E_{LN}[\sqrt{n}(\hat{\gamma}_j - \tilde{\gamma}_j)]^2 < \infty$ for all $j = 1 \dots 4$ (using theorem 3.2 of White (1982)), we have,

$$E_{LN}\left[\frac{1}{n}\sum_{i=1}^{n} g_j^{'}(X_i; \tilde{\gamma}_j).\mathbf{1}_{X_i \leq T} + (1-p)h_j^{'}(T; \tilde{\gamma}_j)\right]^2 \longrightarrow 0$$

Thus, $E_{LN}\left(\frac{1}{\sqrt{n}}[T_{1n} - T_{1*}]\right) \longrightarrow 0$
Hence result follows.

Proof of Theorem 3.1:

Using asymptotic results of T_{Nj} and following the ideas of Bhattacharyya (1985) (for asymptotic distribution of T_{N2}, see Dey and Kundu (2012)), we have

$$\frac{T_{Nj} - E_{\text{WE}}[T_{Nj}]}{\sqrt{V_{\text{WE}}[T_{Nj}]}} \xrightarrow{D} N(0,1), \quad j = 1, 2. \tag{25}$$

Let $W_N = \frac{T_N - E_{\text{WE}}[T_N]}{\sqrt{V_{\text{WE}}[T_N]}}$. Then,

$$\begin{aligned} F_{W_N}(w) &= P(W_N \leq w, X_{(R)} < T) + P(W_N \leq w, X_{(R)} > T) \\ &= P\left(\frac{T_{N2} - E_{\text{WE}}[T_{N2}]}{\sqrt{V_{\text{WE}}[T_{N2}]}} \leq w\right) P(X_{(R)} < T) + P\left(\frac{T_{N1} - E_{\text{WE}}[T_{N1}]}{\sqrt{V_{\text{WE}}[T_{N1}]}} \leq w\right) P(X_{(R)} > T). \end{aligned}$$

As $n \to \infty$,

$$\lim_{n\to\infty} F_{W_N}(w) = \Phi(w)p_1^* + \Phi(w)p_2^*,$$

where

$$p_1^* = \lim_{n\to\infty} P(X_{(R)} < T), \quad p_2^* = \lim_{n\to\infty} P(X_{(R)} > T).$$

Since $X_{(R)} = X_{\lfloor np \rfloor}$, the sample p-th quantile,

$$X_{\lfloor np \rfloor} \xrightarrow{P} \zeta_p,$$

where ζ_p is the population p-th quantile. Thus,

$$\lim_{n\to\infty} P(X_{(R)} < T) = \begin{cases} 0, & T < \zeta_p, \\ 1, & T \geq \zeta_p. \end{cases}$$

Hence, the asymptotic distribution of T_N is the same as that of T_{N1} (Type-I censoring) if $T < \zeta_p$, and that of T_{N2} (Type-II censoring) if $T \geq \zeta_p$.

Proof of Lemma S1.1:

The proof proceed along the same lines as that of Lemma 3.1 and is therefore omitted for brevity.

Proof of Theorem S1.1:

The proof proceed along the same lines as that of Theorem 3.1 and is therefore omitted for brevity.

Statements and Declarations:

- Funding:No funding was received for conducting this study
- Conflict of interest:The authors have no competing interests to declare that are relevant to the content of this article.

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