Robust Parameter Estimation of Newton's Rings

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Abstract

Estimation of physical parameters from the noisy image of a Newton's ring is an important problem in optics. Different methods are available in the literature to estimate unknown parameters from a given image. Among different estimators, it is well known that the least squares estimators are the most efficient estimators in the sense that the asymptotic variances of the least squares estimators reach the Cramer-Rao lower bound when the noise is Gaussian. But, it is well known that the least squares estimators are not robust. They are very sensitive to the outliers. Even in the presence of few outliers the performances of the least squares estimators affect significantly. In this case some of the robust estimators like least absolute deviation estimators or Huber-M estimators may be used. But they are quite difficult to implement in practice. Moreover, they may not work very well when there is outliers in the data set. In this paper we have proposed weighted least squares estimators which can be implemented quite easily and they perform very similar to the least absolute deviation estimators or Huber-M estimators when there are outliers in the data set. Moreover, they perform very similar to the least squares estimators when there is no outliers in the data set. We have derived the asymptotic properties of the weighted least squares estimators. Extensive simulations have been performed to compare the performances of the different estimators. Implementation of the proposed weighted least squares estimators have been suggested. Finally we have provided one illustrative example to show how the proposed method can be used in practice.

Keywords: Sinusoidal model; chirp model; chirp like model; least squares estimators, asymptotic distribution; Cramer-Rao bound; consistent estimators; sequential procedures.

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1 Introduction

Estimating the physical parameters of a Newton's ring is an important problem in optics. It is also well known to be a difficult problem, see for example Nascov [19], Gorthi and Rastogi [5, 6, 21] and the references cited therein. The intensity variation of Newton's ring pattern can be written mathematically as follows;

$$I(x, y) = I_0 + A \cos \left\{ \frac{2\pi}{\lambda R}[(x - x_0)^2 + (y - y_0)^2] + \pi \right\}. \tag{1}$$

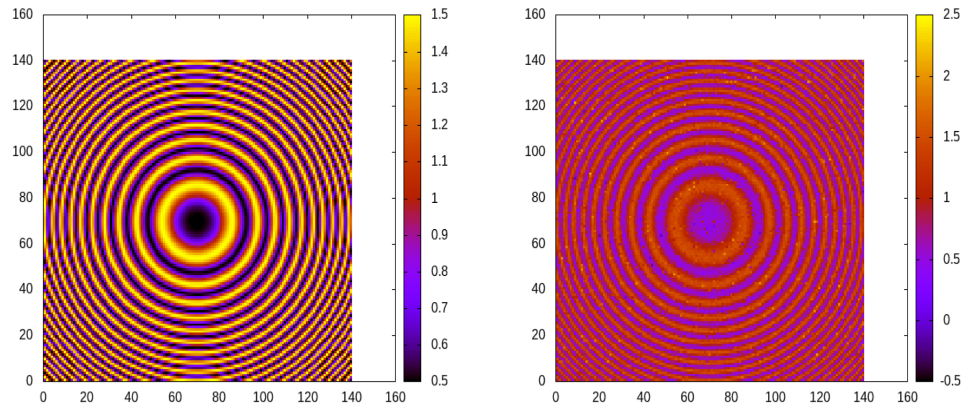
Here I_0 is the mean intensity, A is the amplitude of the cosine variation of the fringes, λ is the intensity of light wavelength, R is the radius of curvature of the spherical surface, (x_0, y_0) is the center of Newton's rings. The center of the Newton's rings may be known or unknown, but λ is always known. Hence, the problem is to estimate I_0, A and R if the center is known, other we need to estimate the center also.

Often in practice, we do not observe intensity variation $I(x, y)$; they are always corrupted with additive noise. Hence, the problem becomes quite complicated. Mathematically, the problem can be formulated as follows. Based on the following observations $\{y(m, n)\}$, where

$$\begin{aligned} y(m, n) &= A^0 + B^0 \cos(\alpha^0(m^2 + n^2) + \beta^0 m + \gamma^0 n) + e(m, n); \\ & \qquad m = 1, \ldots, M; \quad n = 1, \ldots, N, \end{aligned} \tag{2}$$

estimate the unknown parameters A^0, B^0, α^0, β^0 and γ^0 under the suitable error assumptions. The assumptions of the error components will be made explicitly later. The model (2) is a special case of the two-dimensional chirp model, see for example Lu et al. [16], Simeunovic and Djurovic [24], Grover et al. [7], Shukla et al. [23] and the references cited therein. In this case the chirp parameters are same in both direction. Typical Newton's rings with and without noise are presented in Figure 1.

An extensive amount of work has been done in providing different estimation procedure of the unknown parameters of the model (2). The least squares estimators (LSEs) method



(a) (b)

Figure 1: (a) Newton's ring with out any noise, (b) Newton's ring with additive noise.

was proposed by Fan and Liu [3], some of the other methods were proposed by Nascov et al. [18], Kemao et al. [12], Lu et al. [16], Guo and Li [8], Zhang et al. [27], see also Aslam et al. [1], Cao et al. [2] and Radhika et al. [20] for some relevant work related to this. The model (2) is a typical 2-D non-linear regression model, see for example Seber and Wild [22]. The problem is known to be a numerically challenging problem. Among the different estimators, the most efficient estimators are the LSEs. It may be mentioned that the model (2) does not satisfy the standard sufficient conditions of Jennrich [11] or Wu [26] which are needed to establish the LSEs to be consistent and asymptotically normally distributed. For this model it can be shown that the LSEs are consistent and the asymptotic variances reach the Cramer-Rao lower bound under the assumption of Gaussian errors but the theoretical properties of the other estimators are not known. Although the LSEs are the most efficient estimators, it is well known that the they are not robust. Even if there are few outliers in the data set, the performances of the LSEs affect significantly. Some of the other robust estimators are; for example least absolute deviation estimators (LADEs) or Huber's M-estimators (HMEs), see for example Huber [10]. It is well known that both the estimators are difficult to implement in practice even in one dimension. Further, establishing the theoretical properties of the LADEs and HMEs are quite challenging. They usually need stronger error assumptions than what are needed for the LSEs to be consistent and asymptotically normal, see for example Kim et al. [13] or Lahiri et al. [15] and the references cited there in. Moreover, it is also observed that if there is no outliers in the data set, then the performances of LADEs or HMEs are worse than the LSEs in terms of the mean squared errors (MSEs).

The main aim of this present manuscript is to provide an efficient estimation procedure of the parameters of a Newton's ring when the noise is corrupted with some outliers. It is not uncommon in practice to observe signals with outliers, see for example Fu et al. [4] and the references cited therein. In this paper we have proposed a novel 2-D weighted least squares (WLS) method to estimate the unknown parameters in presence of outliers. The implementation of the proposed method is quite simple compared to the other robust estimators like LADEs or HMEs. Moreover, under the same set of assumptions as the LSEs, we will prove the consistency and the asymptotic normality properties of the WLS estimators (WLSEs). Extensive simulations have been performed to compare the performances of the different estimators, and it is observed that the performances of the WLSEs are better than LSEs and they are at par with the LADEs and HMEs when the outliers are present. It is observed that the performances of the WLSEs depend on the choice of the weight functions, and we have proposed how to choose the proper weight function in practice. One illustrative example has been presented to show how the proposed method can be used in practice.

The main contributions of this paper are the following. First of all the theoretical properties of the LSEs are not available in the literature. Based on our general results the theoretical properties of the LSEs can be obtained. We have addressed the robust estimation procedure of the unknown parameters of a Newton's rings which has not been considered in the literature. We have provided a very novel estimation procedure which produces robust estimators in presence of impulsive noise. The implementation of the proposed procedure is quite simple compared to the other robust estimation procedures. We have established the theoretical properties of the proposed robust estimators. It is well known that the existing robust estimators like LADEs or HMEs perform quite poorly compared to the LSEs, if there is no impulsive noise. The proposed robust estimators behave very similarly to the LSEs if there is no impulsive noise and perform better than the LSEs in presence of impulsive noise.

The rest of the paper is organized as follows. In Section 2 we have presented the model assumptions and the WLSEs. The asymptotic properties of the WLSEs are provided in Section 3. In Section 4 the simulation results and in Section 5 the choice of weights functions and an illustrative example have been presented. Finally in Section 6 we conclude the paper.

2 Weighted Least Squares Estimators

We consider two cases separately; (i) when the center of the Newton's ring is known and (ii) when the center of the Newton's ring is unknown. In both the cases it is assumed that the error random variables $e(m,n)$'s are independent identically distributed (i.i.d.) Gaussian random variables with mean zero and variance σ_0^2. Suppose the weight function $w(s,t)$ is a polynomial of degree k, defined for $0 \le s,t \le 1$, and it can be written as follows:

$$w(s,t) = c_{00} + c_{10}s + c_{01}t + c_{20}s^2 + c_{11}st + c_{02}t^2 + \ldots + c_{k0}s^k + \ldots + c_{0k}t^k. \qquad (3)$$

It is assumed that the coefficients of the above $w(s,t)$ are such that

$$\inf_{0 \le s,t \le 1} w(s,t) > \psi > 0. \qquad (4)$$

2.1 Model 1: Center of the Newton's Ring is Known

In this case the center of the Newton's ring is assumed to be known, and with out loss of generality it can be assumed to be $(0,0)$. Hence, the model (2) takes the form

$$y(m,n) = A^0 + B^0 \cos(\alpha^0(m^2+n^2)) + e(m,n); \quad m = 1,\ldots,M; \quad n = 1,\ldots,N. \qquad (5)$$

Here it is assumed that $A^0 \neq 0$, $B^0 \neq 0$ and there exists a $K < \infty$, such that $|A^0| \leq K$, $|B^0| \leq K$ and $0 < \alpha^0 < \pi$. The weighted residual sum of squares can be defined as

$$Q_1(A, B, \alpha) = \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} w\left(\frac{m}{M}, \frac{n}{N}\right) \left(y(m,n) - A - B\cos(\alpha(m^2+n^2))\right)^2. \qquad (6)$$

Hence, the WLSEs can be obtained as the argument minimum of $Q_1(A, B, \alpha)$. Let us use the following notations for further development;

$$a_{11}^1 = \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} w\left(\frac{m}{M}, \frac{n}{N}\right), \quad a_{12}^1 = a_{21}^1 = \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} w\left(\frac{m}{M}, \frac{n}{N}\right) \cos(\alpha(m^2+n^2)),$$

$$a_{22}^1 = \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} w\left(\frac{m}{M}, \frac{n}{N}\right) \cos^2(\alpha(m^2+n^2)),$$

$$b_1^1 = \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} w\left(\frac{m}{M}, \frac{n}{N}\right) y(m,n), \quad b_2^1 = \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} w\left(\frac{m}{M}, \frac{n}{N}\right) y(m,n)\cos(\alpha(m^2+n^2)).$$

Then it can be easily seen that for fixed α, the WLSEs of A^0 and B^0, say $\widehat{A}(\alpha)$ and $\widehat{B}(\alpha)$, respectively, can be obtained as

$$\widehat{A}(\alpha) = \frac{b_1^1 a_{22}^1 - b_2^1 a_{12}^1}{a_{11}^1 a_{22}^1 - a_{12}^1 a_{21}^1} \quad \text{and} \quad \widehat{B}(\alpha) = \frac{b_2^1 a_{11}^1 - b_1^1 a_{21}^1}{a_{11}^1 a_{22}^1 - a_{12}^1 a_{21}^1},$$

respectively. The WLSE of α^0, say $\widehat{\alpha}$ can be obtained as the argument minimum of

$$\begin{aligned} R_1(\alpha) &= Q(\widehat{A}(\alpha), \widehat{B}(\alpha), \alpha) \\ &= \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} w\left(\frac{m}{M}, \frac{n}{N}\right) \left(y(m,n) - \widehat{A}(\alpha) - \widehat{B}(\alpha)\cos(\alpha(m^2+n^2))\right)^2. \qquad (7) \end{aligned}$$

Once we get $\widehat{\alpha}$, the WLSEs of A^0 and B^0 can be obtained as $\widehat{A}(\widehat{\alpha})$ and $\widehat{B}(\widehat{\alpha})$, respectively, Hence, the WLSEs of the unknown parameters of the model (5) can be obtained by solving only one 1-D optimization problem. Using the number theory results (see Lahiri [14], it can be shown that

$$\begin{aligned} \lim_{M,N\to\infty} a_{11}^1 &= \int_0^1 \int_0^1 w(s,t)dsdt = a_0(\text{ say}), \\ \lim_{M,N\to\infty} a_{12}^1 &= \lim_{M,N\to\infty} a_{12} = 0, \\ \lim_{M,N\to\infty} a_{22}^1 &= \frac{1}{2}\int_0^1 \int_0^1 w(s,t)dsdt = \frac{1}{2}a_0(\text{ say}). \end{aligned}$$

Hence, for large M and N, $\widehat{A}(\alpha)$ and $\widehat{B}(\alpha)$ can be approximated by

$$\begin{aligned}
\widetilde{A} &= \frac{1}{a_0 MN} \sum_{m=1}^{M} \sum_{n=1}^{N} w\left(\frac{m}{M}, \frac{n}{N}\right) y(m,n), \\
\widetilde{B}(\alpha) &= \frac{2}{a_0 MN} \sum_{m=1}^{M} \sum_{n=1}^{N} w\left(\frac{m}{M}, \frac{n}{N}\right) y(m,n) \cos(\alpha(m^2+n^2)),
\end{aligned}$$

respectively. Now we discuss about the WLSEs of the unknown parameters of a Newton's ring when the center is not known.

2.2 Model 2: Center of the Newton's Ring is Not Known

In this case it is assumed that the center of the Newton's ring is not known. Hence, it is assumed that the observed $y(m,n)$ has the form (2). The problem is to estimate the unknown parameters namely; A^0, B^0, α^0, β^0 and γ^0 based on the sample $\{(y(m,n); m = 1, \ldots, M; n = 1, \ldots, N.\}$, under the same set of model assumptions as Model 1. Here, it is further assumed that $0 < \beta^0, \gamma^0 < \pi$. We can define the weighted residual sum of squares in this case also as

$$Q_2(A,B,\alpha,\beta,\gamma) = \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} w\left(\frac{m}{M}, \frac{n}{N}\right) \left(y(m,n) - A - B\cos(\alpha(m^2+n^2) + \beta m + \gamma n)\right)^2 . \tag{8}$$

The WLSEs of A^0, B^0, α^0, β^0 and γ^0 can be obtained as the argument minimum of $Q_2(A,B,\alpha,\beta,\gamma)$. Hence, similarly as before if we use the following notations;

$$\begin{aligned}
a_{11}^2 &= \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} w\left(\frac{m}{M}, \frac{n}{N}\right), \\
a_{12}^2 &= a_{21}^2 = \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} w\left(\frac{m}{M}, \frac{n}{N}\right) \cos(\alpha(m^2+n^2) + \beta m + \gamma n), \\
a_{22}^2 &= \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} w\left(\frac{m}{M}, \frac{n}{N}\right) \cos^2(\alpha(m^2+n^2) + \beta m + \gamma n),
\end{aligned}$$

$$
\begin{aligned}
b_1^2 &= \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right) y(m,n),\\
b_2^2 &= \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right) y(m,n)\cos(\alpha(m^2+n^2)+\beta m+\gamma n),
\end{aligned}
$$

then for fixed α, the WLSEs of A^0 and B^0, $\widehat{A}(\alpha,\beta,\gamma)$ and $\widehat{B}(\alpha,\beta,\gamma)$, respectively, can be obtained as

$$
\widehat{A}(\alpha) = \frac{b_1^2 a_{22}^2 - b_2^2 a_{12}^2}{a_{11}^2 a_{22}^2 - a_{12}^2 a_{21}^2} \quad \text{and} \quad \widehat{B}(\alpha) = \frac{b_2^2 a_{11}^2 - b_1^2 a_{21}^2}{a_{11}^2 a_{22}^2 - a_{12}^2 a_{21}^2}.
$$

The WLSE of α^0, β^0 and γ^0, say $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\gamma}$, respectively, can be obtained as the argument minimum of

$$
\begin{aligned}
R_2(\alpha,\beta,\gamma) &= Q_2(\widehat{A}(\alpha),\widehat{B}(\alpha),\alpha)\\
&= \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right)\left(y(m,n)-\widehat{A}(\alpha)-\widehat{B}(\alpha)\cos(\alpha(m^2+n^2)+\beta m+\gamma n)\right)^2.
\end{aligned}
\tag{9}
$$

Hence, in this case the WLSEs of α^0, β^0 and γ^0 can be obtained by solving a three-dimensional optimization problem. Similarly as before using the results of Lahiri [14], we obtain

$$
\begin{aligned}
\lim_{M,N\to\infty} a_{11}^2 &= a_{11}^1 = \int_0^1\int_0^1 w(s,t)dsdt = a_0(\text{ say}),\\
\lim_{M,N\to\infty} a_{12}^2 &= \lim_{M,N\to\infty} a_{12}^2 = 0,\\
\lim_{M,N\to\infty} a_{22}^2 &= \frac{1}{2}\int_0^1\int_0^1 w(s,t)dsdt = \frac{1}{2}a_0(\text{ say}).
\end{aligned}
$$

Hence, in this case, for large M and N, $\widehat{A}(\alpha,\beta,\gamma)$ and $\widehat{B}(\alpha,\beta,\gamma)$ can be approximated by

$$
\begin{aligned}
\widetilde{A} &= \frac{1}{a_0MN}\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right) y(m,n),\\
\widetilde{B}(\alpha,\beta,\gamma) &= \frac{2}{a_0MN}\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right) y(m,n)\cos(\alpha(m^2+n^2)+\beta m+\gamma n).
\end{aligned}
$$

3 Asymptotic Properties

In this section we provide the asymptotic properties of the WLSEs of the unknown parameters for both Model 1 and Model 2.

3.1 Model 1

We need the following assumptions for further development.

Assumption 1: There exists a K, such that $|A^0| < K$, $|B^0| < K$ and $0 < \alpha^0 < \pi$.

Assumption 2: The random variables $e(m,n)$'s are independent and identically distributed (i.i.d.) normal random variables with mean 0 and variance σ^2.

Assumption 3: The weight function $w(s,t)$ has the form (3) for some fixed k, and it satisfies the restriction (4).

Assumption 4: $M \to \infty$, $N \to \infty$ and $M/N \to 1$ as $\min\{M,N\} \to \infty$.

Theorem 1: Under Assumptions 1,2,3 and 4, if $\widehat{A}$, $\widehat{B}$ and $\widehat{\alpha}$ are the WLSEs of A^0, B^0 and α^0, respectively for model (5), then the WLSEs are consistent of their respective parameters.

Proof: See in the Appendix A. ∎

Now we would like to provide the asymptotic distribution of $\widehat{A}$, $\widehat{B}$ and $\widehat{\alpha}$. We need to introduce the following notations for that purpose. For $p,q = 0,1,2,\ldots$, let us define

$$u_{pq} = \int_0^1 \int_0^1 s^p t^q w(s,t) ds dt, \quad \text{and} \quad v_{pq} = \int_0^1 \int_0^1 s^p t^q w^2(s,t) ds dt. \tag{10}$$

The 3×3 diagonal matrix $\boldsymbol{D}_3$ is defined as follows:

$$\boldsymbol{D}_3 = \operatorname{diag}\left\{M^{1/2}N^{1/2}, M^{1/2}N^{1/2}, (M^{5/2}N^{1/2} + M^{1/2}N^{5/2})\right\}. \tag{11}$$

The matrix $\mathbf{\Sigma}_3$ is

$$\mathbf{\Sigma}_3 = \begin{bmatrix} 2v_{00} & 0 & 0 \\ 0 & v_{00} & 0 \\ 0 & 0 & \frac{1}{2}(v_{40} + v_{04} + 2v_{22}) B^{0^2} \end{bmatrix}$$

and $\mathbf{\Gamma}_3$ is defined as follows;

$$\mathbf{\Gamma}_3 = \begin{bmatrix} 2u_{00} & 0 & 0 \\ 0 & u_{00} & 0 \\ 0 & 0 & \frac{1}{2}(u_{40} + u_{04} + 2u_{22}) B^{0^2} \end{bmatrix}$$

Theorem 2: Under Assumptions 1,2,3 and 4,

$$\left(\widehat{A} - A^0, \widehat{B} - B^0, \widehat{\alpha} - \alpha^0\right) \boldsymbol{D}_3 \xrightarrow{d} N_3\left(\mathbf{0}, 2\sigma^2 \mathbf{\Gamma}_3^{-1} \mathbf{\Sigma}_3 \mathbf{\Gamma}_3^{-1}\right).$$

Here, $\xrightarrow{d}$ means convergence in distribution and $N_3(\boldsymbol{a}, \boldsymbol{A})$ denotes a 3-variate normal distribution with mean vector $\boldsymbol{a}$ and dispersion matrix $\boldsymbol{A}$.

Proof: See in the Appendix A. ∎

Comment: Note that when $w(s,t) = 1$, then the WLSEs become the LSEs. In case of LSEs, $\mathbf{\Gamma}_3 = \mathbf{\Sigma}_3$. From Theorems 1 and 2, we immediately obtain that if $\widehat{A}$, $\widehat{B}$ and $\widehat{\alpha}$ are the LSEs of A^0, B^0 and α^0, respectively, then they are consistent and

$$\left(\widehat{A} - A^0, \widehat{B} - B^0, \widehat{\alpha} - \alpha^0\right) \boldsymbol{D}_3 \xrightarrow{d} N_3\left(\mathbf{0}, 2\sigma^2 \mathbf{\Sigma}_3^{-1}\right),$$

where

$$\mathbf{\Sigma}_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{14}{45} B^{0^2} \end{bmatrix}.$$

3.2 Model 2

We need the following assumptions for further development.

Assumption 5: There exists a K, such that $|A^0| < K$, $|B^0| < K$ and $0 < \alpha^0, \beta^0, \gamma^0 < \pi$.

Theorem 3: Under Assumptions 2,3,4 and 5, if $\widehat{A}$, $\widehat{B}$, $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\gamma}$ are the WLSEs of A^0, B^0, α^0, β^0 and γ^0, respectively for model (2), then the WLSEs are consistent estimators of the respective parameters.

Proof: Proof follows along the same line as the proof of Theorem 1, hence it is avoided. ∎

The 5×5 diagonal matrix $\boldsymbol{D}_5$ is defined as follows:

$$\boldsymbol{D}_5 = \operatorname{diag}\left\{M^{1/2}N^{1/2}, M^{1/2}N^{1/2}, (M^{5/2}N^{1/2} + M^{1/2}N^{5/2}), M^{3/2}N^{1/2}, M^{1/2}N^{3/2}\right\}. \quad (12)$$

The matrix $\boldsymbol{\Sigma}_5$ is

$$\boldsymbol{\Sigma}_5 = \begin{bmatrix} 2v_{00} & 0 & 0 & 0 & 0 \\ 0 & v_{00} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(v_{40}+v_{04}+2v_{22})B^{0^2} & \frac{1}{2}(v_{30}+v_{12})B^{0^2} & \frac{1}{2}(v_{21}+v_{03})B^{0^2} \\ 0 & 0 & \frac{1}{2}(v_{30}+v_{12})B^{0^2} & B^{0^2}v_{30} & \frac{1}{2}B^{0^2}v_{11} \\ 0 & 0 & \frac{1}{2}(v_{21}+v_{03})B^{0^2} & \frac{1}{2}B^{0^2}v_{11} & B^{0^2}v_{03} \end{bmatrix}$$

and $\boldsymbol{\Gamma}_5$ is defined as follows;

$$\boldsymbol{\Gamma}_5 = \begin{bmatrix} 2u_{00} & 0 & 0 & 0 & 0 \\ 0 & u_{00} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(u_{40}+u_{04}+2u_{22})B^{0^2} & \frac{1}{2}(u_{30}+u_{12})B^{0^2} & \frac{1}{2}(u_{21}+u_{03})B^{0^2} \\ 0 & 0 & \frac{1}{2}(u_{30}+u_{12})B^{0^2} & B^{0^2}u_{30} & \frac{1}{2}B^{0^2}u_{11} \\ 0 & 0 & \frac{1}{2}(u_{21}+u_{03})B^{0^2} & \frac{1}{2}B^{0^2}u_{11} & B^{0^2}u_{03} \end{bmatrix}$$

Theorem 4: Under Assumptions 2,3, 4 and 5,

$$\left(\widehat{A} - A^0, \widehat{B} - B^0, \widehat{\alpha} - \alpha^0, \widehat{\beta} - \beta^0, \widehat{\gamma} - \gamma^0\right) \boldsymbol{D}_5 \xrightarrow{d} N_5\left(\mathbf{0}, 2\sigma^2 \boldsymbol{\Gamma}_5^{-1}\boldsymbol{\Sigma}_5\boldsymbol{\Gamma}_5^{-1}\right).$$

Proof: See Appendix B. ∎

Comment: For Model 2 from Theorems 3 and 4, we obtain that if $\widehat{A}$, $\widehat{B}$, $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\gamma}$ are the LSEs of A^0, B^0, α^0, β^0 and γ^0, respectively, then they are consistent and

$$\left(\widehat{A} - A^0, \widehat{B} - B^0, \widehat{\alpha} - \alpha^0, \widehat{\beta} - \beta^0, \widehat{\gamma} - \gamma^0\right) \boldsymbol{D}_5 \xrightarrow{d} N_5\left(\mathbf{0}, 2\sigma^2 \boldsymbol{\Sigma}_5^{-1}\right),$$

where

$$\boldsymbol{\Sigma}_5 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{14}{45}B^{0^2} & \frac{5}{24}B^{0^2} & \frac{5}{24}B^{0^2} \\ 0 & 0 & \frac{5}{24}B^{0^2} & \frac{1}{4}B^{0^2} & \frac{1}{8}B^{0^2} \\ 0 & 0 & \frac{5}{24}B^{0^2} & \frac{1}{8}B^{0^2} & \frac{1}{4}B^{0^2} \end{bmatrix}.$$

4 Simulation Results

The simulations have been performed to examine how the proposed methods work in practice. We have performed the simulations for both Model 1 and Model 2. In all the cases the error random variables $e(m,n)$'s are i.i.d. $N(0,\sigma_0^2)$. In each case the outliers are present in the middle portion of the data and they are $N(0,\sigma_1^2)$. The following parameter sets have been used for Model 1 and Model 2, respectively.

Set 1: $A = B = 1, \alpha = 1.25\pi = 3.926991$,

Set 2: $A = B = 1, \alpha = 1.25\pi = 3.926991, \beta = 0.75\pi = 2.356194, \gamma = 0.25\pi = 0.785398$.

In all the cases, we have used $\sigma_0^2 = 1.0$ and $\sigma_1^2 = 6.25$. We have presented the number of outliers (NOL) within brackets below the sample size in the first column of each table. We have used two different weight functions as follows:

Weight Function 1 (WLSE-1): $w(t,s) = 1.0 + (t-0.5)^2 + (s-0.5)^2$

Weight Function 2 (WLSE-2): $w(t,s) = 1.0 + (t-0.75)^2 + (s-0.75)^2$.

Note that for WLSE-1, $w(t,s)$ takes the minimum value at (0.5,0.5), and the function increases as (t,s) deviates from $(0.5,0.5)$, i.e. it puts less weight in the middle of the data set. On the other hand for WLSE-2, $w(t,s)$ takes the minimum value at $(0.75,0.75)$, and the function increases as (t,s) deviates from $(0.75,0.75)$ For comparison purposes we have used the least squares estimators (LSEs), the least absolute deviation estimators (LADEs) and Huber-M estimators (HMEs) also. We have also presented the case when there are no outliers in the data set and error variance is $\sigma_0^2 = 1$. In each case we have presented the average estimates and the mean squared errors based on 1000 replications. The results for Model 1 are presented in Tables 1 and 2 and for Model 2, the results are presented in Tables 3 and 4. In Tables 1 and 3 we have presented the results when there are outliers in the data set and in Tables 2 and 4, we have presented the results when there are no outliers in the data set.

Now let us discuss about the computational issues of the different methods. Note that for Model 1, the LSEs and the WLSEs can be obtained by solving a one-dimensional optimization problem, where as for Model 2, one needs to solve a three-dimensional optimization problem. In case of LADEs or HMEs for Model 1, one needs to solve a three-dimensional optimization problem, and for Model 2, it is a five-dimensional optimization problem. Hence, computationally LSEs and WLSEs can be obtained in a more convenient manner compared to LADEs or HMEs.

Some of the points are observed from the numerical experiments. It is observed in all the cases that as the sample size increases for all the methods the biases and the mean squared errors decrease. It verifies the consistency properties of all the estimates considered here. It is also observed that all the estimates provide unbiased estimates of the corresponding parameters. In most of the cases the biases are quite negligible. It is observed that for both the models the WLSE-1s, LADEs and HMEs provide estimators which have lower mean squared errors than the LSEs, in most of the cases when there are outliers in the data set. It may be mentioned that the outliers are present in the middle portion of the data set. When there are no outliers are present the LSEs provide the lowest mean squared errors, but the MSEs corresponding to WLSE-1 and WLSE-2 are quite close to the LSEs. On the other hand the mean squared errors for the LADEs and HMEs are more than the WLSE-1s or WLSE-2s. Comparing WLSE-1 and WLSE-2, it is observed that the mean squared errors of the WLSE-2s are always more than the WLSE-1s when there are outliers in the data set. Hence, comparing everything if it is known that the outliers are from the middle portion of the data set, it is recommended that WLSE-1 can be used.

Form all these extensive simulation experiments we have observed that the WLSEs perform very well compared to the LSEs if there are outliers in the data set. In fact the performances of the proposed WLSEs are comparable with the other robust estimators like LADEs and HMEs. Computationally, the proposed WLSEs are easier to implement than the LADEs and HMEs. On the other hand when there is no outliers in the data set, LADEs and HMEs perform quite poorly compared to the LSEs, where as the the WLSEs perform very similarly as the LSEs.

$M = N$ (NOL)	Parameter	LSE	WLSE-1	WLSE-2	LAD	HME
25	A	1.00305	1.00383	1.00360	1.00153	1.00229
		(1.179E-02)	(1.055E-02)	(1.246E-02)	1.111E-02)	(1.112E-02)
(49)	B	0.99387	0.99978	0.99979	0.99835	0.99443
		(3.069E-02)	(2.824E-02)	(3.264E-02)	(2.900E-02)	(2.911E-02)
	α	3.92699	3.92698	3.92698	3.92699	3.92698
		(9.529E-05)	(9.446E-05)	(9.809E-05)	(9.493E-05)	(9.457E-05)
50	A	1.00013	1.00006	1.00021	0.99999	1.00020
		(1.272E-03)	(1.035E-03)	(1.638E-03)	1.111E-03)	(1.114E-03)
(125)	B	0.99888	0.99765	0.99826	0.99918	0.99898
		(1.045E-02)	(9.469E-03)	(1.078E-02)	(9.498E-03)	(9.495E-03)
	α	3.92699	3.92699	3.92699	3.92699	3.92698
		(1.571E-05)	(1.224E-05)	(1.342E-05)	(1.299E-05)	(1.239E-05)
75	A	1.00005	1.00011	0.99997	0.99998	0.99999
		(4.482E-04)	(4.327E-04)	(3.924E-04)	(4.374E-04)	(4.349E-04)
(529)	B	0.99885	0.99888	0.99859	0.99857	0.99891
		(3.696E-03)	(3.134E-03)	(4.143E-03)	(3.139E-03)	(3.136E-03)
	α	3.92699	3.92699	3.92699	3.92699	3.92699
		(4.694E-06)	(3.619E-06)	(3.792E-06)	(3.680E-06)	(3.768E-06)
100	A	1.00002	1.00001	1.00002	1.00000	1.00000
		(4.241E-05)	(3.101E-05)	(5.146E-05)	(3.111E-05)	(3.071E-05)
(961)	B	0.99992	0.99980	0.99981	0.99987	0.99984
		(2.473E-04)	(1.916E-04)	(2.765E-04)	(2.035E-04)	(2.078E-04)
	α	3.92699	3.92699	3.92699	3.92699	3.92699
		(1.963E-06)	(1.508E-06)	(1.563E-06)	(1.579E-06)	(1.576E-06)

Table 1: The average estimates (AE) and the associated mean squared errors (MSE) for parameter set 1, of the different methods for different sample sizes ($M = N$) and for different number of outliers (NOL) and when the outliers are present in the middle portion of the data set.

$M = N$	Parameter	LSE	WLSE-1	WLSE-2	LAD	Huber-M
25	A	1.00218	1.00290	1.00360	1.00313	1.00210
		(8.025E-03)	(8.175E-03)	(8.198E-03)	9.976E-03)	(9.971E-03)
	B	0.99697	1.00050	1.00059	0.99867	0.99498
		(2.305E-02)	(2.314E-02)	(2.334E-02)	(2.643E-02)	(2.634E-02)
	α	3.92699	3.92698	3.92698	3.92699	3.92698
		(8.682E-05)	(8.746E-05)	(8.736E-05)	(1.113E-04)	(9.674E-05)
50	A	1.00028	1.00021	1.00030	1.00026	1.00041
		(8.279E-04)	(8.300E-04)	(8.315E-04)	1.057E-03)	(1.059E-03)
	B	0.99884	0.99855	0.99794	0.99921	0.99891
		(8.161E-03)	(8.205E-03)	(8.211E-03)	(9.319E-03)	(1.001E-02)
	α	3.92699	3.92699	3.92699	3.92699	3.92699
		(1.128E-05)	(1.199E-05)	(1.216E-05)	(1.422E-05)	(1.398E-05)
75	A	0.99992	1.00017	1.00004	1.00001	1.00001
		(2.543E-04)	(2.617E-04)	(2.621E-04)	(3.561E-04)	(3.014E-04)
	B	0.99905	0.99882	0.99879	0.99905	0.99933
		(3.259E-03)	(3.325E-03)	(3.351E-03)	(3.785E-03)	(3.881E-03)
	α	3.92699	3.92699	3.92699	3.92699	3.92699
		(3.356E-06)	(3.367E-06)	(3.365E-06)	(4.225E-06)	(3.578E-06)
100	A	0.99998	1.00002	1.00000	0.99999	0.99997
		(3.334E-05)	(3.415E-05)	(3.422E-05)	(5.041E-05)	(5.056E-05)
	B	0.99994	0.99990	0.99992	0.99992	0.99994
		(1.219E-04)	(1.227E-04)	(1.237E-04)	(2.271E-04)	(2.261E-04)
	α	3.92699	3.92699	3.92699	3.92699	3.92699
		(1.306E-06)	(1.402E-06)	(1.398E-06)	(1.831E-06)	(1.713E-06)

Table 2: The average estimates (AE) and the associated mean squared errors (MSE) for Set 1, of the different methods for different sample sizes ($M = N$) when there is no outliers.

5 Choice of Weight Function and an Example

5.1 Choice of Weight Function

It is an important problem in practice. In our extensive simulations it has been observed that if the outliers are from the middle portion of the data set, then we can choose the weight function properly, and the method performs very well. In fact it has been observed that if it is known apriori the position of the outliers, then the weight function can be chosen accordingly. But if it is not known in advance where the outliers are present, the problem becomes quite difficult. In this case we choose a class of weight functions which put less weights at different places. In practice we suggest to use 6-8 different weight functions, and compute the residual sum of squares for each weight function. Finally choose that particular weight function for which the residual sum of squares is minimum. We suggest to keep the weight function $w(s,t) = 1$ also in this class. We illustrate this method using the following illustrative example.

5.2 Illustrative Example

In this section we provide one illustrative example to show how the above method can be used in practice when we do not know where the outliers are present. The Newton ring without any noise has been generated in Figure 1 (a). In practice we observe the noise corrupted Newton ring as provided in Figure 1 (b). In this noise corrupted Newton ring, there are 15% outliers, where the noise standard deviation is 500 times of the original noise standard deviation. The outliers have been generated randomly over the entire data set.

We have used the WLSEs method to estimate the original Newton ring parameters. The following weight functions have been used.

$$
\begin{aligned}
&\text{W-1}: && w(t,s) = 1 \\
&\text{W-2}: && w(t,s) = 1 + (s-0.5)^2 + (t-0.5)^2 \\
&\text{W-3}: && w(t,s) = 1 - (s-0.5)^2 - (t-0.5)^2 \\
&\text{W-4}: && w(t,s) = 1 + (s-0.5)^2 + (t-0.5)^2 + (s-0.5)*(t-0.5) \\
&\text{W-5}: && w(t,s) = 1 + (s-0.75)^2 + (t-0.75)^2 \\
&\text{W-6}: && w(t,s) = 1 + (s-0.75)^2 + (t-0.75)^2 + (s-0.75)*(t-0.75) \\
&\text{W-7}: && w(t,s) = 1 + (s-0.25)^2 + (t-0.25)^2 \\
&\text{W-8}: && w(t,s) = 1 + (s-0.25)^2 + (t-0.75)^2.
\end{aligned}
$$

Different weight functions have been chosen in such a manner so that they have different shapes. We have calculated the WLSEs for each weight functions, and the residual sum of squares based on W-8 provides the smallest residual sum of squares 219.8806. The LSEs provide the residual sum of squares as 223.3178. Based on the LSEs and the W-8 weight function, we have estimated the Newton ring. The estimated Newton rings are presented in Figure 2(a) and 2(b), respectively. Clearly, Figure 2(b) provides better prediction of Figure 1 (a) compared to Figure 2(a).

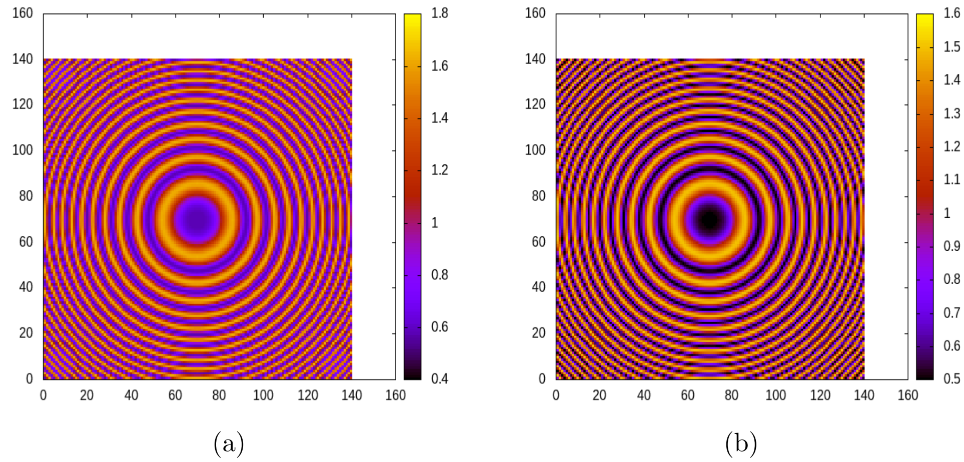


Figure 2: (a) Predicted Newton's ring with LSEs, (b) Predicted Newton's ring with WLSEs.

6 Conclusions

In this paper we have considered the estimation of Newton rings parameters when they are corrupted with impulsive noise. It is assumed that the outliers variance can be significantly larger than the noise variance. We have proposed a very novel weighted least squares method to estimate the unknown parameters of a noise corrupted Newton ring. The asymptotic properties of the WLSEs have been developed under the same set of assumptions as it is needed for the LSEs to be consistent and asymptotically normal. Extensive simulations have been performed to show the performances of the proposed estimators and the performances are quite satisfactory. It is observed that the performances of the WLSEs depend on the choice of the weight functions. We have proposed a novel method to choose a proper weight function from a class of weight functions. We have illustrated the method with an example. It is observed that the WLSEs provide a better fit than the LSEs. It is observed that the implementation of the proposed WLSEs is quite simple in practice. Computationally, they are less expensive than the other existing robust estimators. The performances of the WLSEs are comparable with the other robust estimators when there are outliers in the data set. If there is no outliers in the data set the performances of the WLSEs are comparable with the LSEs. Hence, we recommend to use the WLSEs in all cases.

Data Availability and Conflict of Interest Statement

No real data has been used in this paper. The author does not have any conflict of interest in the preparation of this manuscript.

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Appendix A: Proof of Asymptotic Properties

To prove theorem 1, we need the following lemmas.

Lemma 1: Under Assumption 1, as $\min\{M, N\} \to \infty$, for any non-negative integers p and

q,

$$\sup_{0<\alpha,\beta<\pi} \left| \frac{1}{M^{p+1}N^{q+1}} \sum_{m=1}^{M} \sum_{n=1}^{N} m^p n^q e(m,n) \cos(m^2\alpha + n^2\beta) \right| \longrightarrow 0, \quad a.s.$$

Proof: It can be obtained from the Lemma 5.2.3 of Lahiri [14]. ∎

Lemma 2: Under Assumption 1, as $\min\{M,N\} \to \infty$, for any integers $p \geq 0$ and $q \geq 0$,

$$\sup_{0<\alpha<\pi} \left| \frac{1}{M^{p+1}N^{q+1}} \sum_{m=1}^{M} \sum_{n=1}^{N} m^p n^q e(m,n) \cos(\alpha(m^2 + n^2)) \right| \longrightarrow 0, \quad a.s.$$

Proof: It follows from Lemma 1. ∎

Lemma 3: Under Assumptions 1 and 2, as $\min\{M,N\} \to \infty$,

$$\sup_{0<\alpha<\pi} \left| \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} w\left(\frac{m}{M}, \frac{n}{N}\right) e(m,n) \cos(\alpha(m^2 + n^2)) \right| \longrightarrow 0, \quad a.s.$$

Proof: It can be obtained using Lemma 2. ∎

Further, we use the following notations: $\boldsymbol{\theta} = (A, B, \alpha)$, $\boldsymbol{\theta}^0 = (A^0, B^0, \alpha^0)$ and $\boldsymbol{\Theta} = [-K, K] \times [-K, K] \times [0, \pi]$.

Lemma 4: Let For any given $\delta > 0$,

$$\liminf_{M,N \to \infty} \inf_{|\boldsymbol{\theta} - \boldsymbol{\theta}^0| > \delta} \left\{ Q_1(\boldsymbol{\theta}) - Q_1(\boldsymbol{\theta}^0 \right\} > 0.$$

Proof: Let us denote $f(m,n,\boldsymbol{\theta}) = A + B\cos(\alpha(m^2 + n^2))$. Now consider

$$\begin{aligned}
Q_1(\boldsymbol{\theta}) - Q_1(\boldsymbol{\theta}^0) &= \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right)\left(y(m,n) - f(m,n,\boldsymbol{\theta})\right)^2 - \\
&\quad \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right) e^2(m,n) \\
&= \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right)\left(f(m,n,\boldsymbol{\theta}^0) - f(m,n,\boldsymbol{\theta}) + e(m,n)\right)^2 - \\
&\quad \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right) e^2(m,n) \\
&= \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right)\left(f(m,n,\boldsymbol{\theta}^0) - f(m,n,\boldsymbol{\theta})\right)^2 + \\
&\quad \frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right)\left(f(m,n,\boldsymbol{\theta}^0) - f(m,n,\boldsymbol{\theta})\right) e(m,n).
\end{aligned}$$

For any given $\delta >$, using Lemma 5, it follows that

$$\begin{aligned}
&\liminf_{M,N\to\infty} \inf_{|\boldsymbol{\theta}-\boldsymbol{\theta}^0|>\delta} \left\{Q_1(\boldsymbol{\theta}) - Q_1(\boldsymbol{\theta}^0)\right\} = \\
&\liminf_{M,N\to\infty} \inf_{|\boldsymbol{\theta}-\boldsymbol{\theta}^0|>\delta} \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right)\left(f(m,n,\boldsymbol{\theta}^0) - f(m,n,\boldsymbol{\theta})\right)^2 \geq \\
&\liminf_{M,N\to\infty} \inf_{|\boldsymbol{\theta}-\boldsymbol{\theta}^0|>\delta} \frac{\psi}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} \left(f(m,n,\boldsymbol{\theta}^0) - f(m,n,\boldsymbol{\theta})\right)^2.
\end{aligned}$$

Note that the last inequality is due to (4). Consider the following sets:

$$\Gamma_1 = \{\boldsymbol{\theta} : \boldsymbol{\theta} \in \boldsymbol{\Theta}, |A - A^0| > \delta/\sqrt{3}\}, \quad \Gamma_2 = \{\boldsymbol{\theta} : \boldsymbol{\theta} \in \boldsymbol{\Theta}, |B - B^0| > \delta/\sqrt{3}\},$$

$$\Gamma_3 = \{\boldsymbol{\theta} : \boldsymbol{\theta} \in \boldsymbol{\Theta}, |\alpha - \alpha^0| > \delta/\sqrt{3}\}.$$

Note that

$$\{\boldsymbol{\theta} : \boldsymbol{\theta} \in \boldsymbol{\Theta}, |\boldsymbol{\theta} - \boldsymbol{\theta}^0| > \delta\} \subset \Gamma_1 \cup \Gamma_2 \cup \Gamma_3.$$

Hence

$$\liminf_{M,N\to\infty} \inf_{\boldsymbol{\theta}\in\Gamma_1} \frac{\psi}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} \left(f(m,n,\boldsymbol{\theta}^0) - f(m,n,\boldsymbol{\theta})\right)^2 = \frac{\psi|A - A^0|^2}{3} > 0.$$

Along the same line, it is observed for $k=2,3$,

$$\liminf_{M,N\to\infty}\inf_{\boldsymbol{\theta}\in\Gamma_k}\frac{\psi}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\left(f(m,n,\boldsymbol{\theta}^0)-f(m,n,\boldsymbol{\theta})\right)^2>0.$$

Hence, the result follows. ∎

Proof of Theorem 1: Let us denote $\widehat{\boldsymbol{\theta}}=(\widehat{A},\widehat{B},\widehat{\alpha})$ as the WLSEs of $\boldsymbol{\theta}$. In this proof only we denote it by $\widehat{\theta}_{MN}$ to emphasize that it is a function of M and N. Note that for any M and N,

$$Q(\widehat{\boldsymbol{\theta}}_{MN})-Q(\boldsymbol{\theta}^0)<0. \tag{13}$$

Suppose, $\widehat{\boldsymbol{\theta}}_{MN}$ is not a consistent estimate of $\boldsymbol{\theta}^0$. Since $\widehat{\boldsymbol{\theta}}_{MN}\in\boldsymbol{\Theta}$, there exists a subsequence $\{M_kN_k\}$, such that $\widehat{\boldsymbol{\theta}}_{M_kN_k}\to\boldsymbol{\theta}'\neq\boldsymbol{\theta}^0$. Now due to (13),

$$Q(\boldsymbol{\theta}')-Q(\boldsymbol{\theta}^0)\le 0.$$

this contradicts Lemma 6. Hence, Theorem 1 is proved. ∎

Proof of Theorem 2: To prove Theorem 2, we need the Taylor series expansion of

$$S_1(\boldsymbol{\theta})=MNQ_1((\boldsymbol{\theta})=\sum_{m=1}^{M}\sum_{n=1}^{N}w\left(\frac{m}{M},\frac{n}{N}\right)\left(y(m,n)-A-B\cos(\alpha(m^2+n^2))\right)^2,$$

as follows

$$S_1'(\widehat{\boldsymbol{\theta}})-S_1'(\boldsymbol{\theta}^0)=(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^0)S_1''(\overline{\boldsymbol{\theta}}). \tag{14}$$

Here

$$S_1'(\boldsymbol{\theta})=\left(\frac{\partial S_1(\boldsymbol{\theta})}{\partial A},\frac{\partial S(\boldsymbol{\theta})}{\partial B},\frac{\partial S_1(\boldsymbol{\theta})}{\partial \alpha}\right),\quad S_1''(\boldsymbol{\theta})=\begin{bmatrix}\frac{\partial^2 S_1(\boldsymbol{\theta})}{\partial A^2} & \frac{\partial^2 S_1(\boldsymbol{\theta})}{\partial A\partial B} & \frac{\partial^2 S_1(\boldsymbol{\theta})}{\partial A\partial \alpha}\\ \frac{\partial^2 S_1(\boldsymbol{\theta})}{\partial B\partial A} & \frac{\partial^2 S_1(\boldsymbol{\theta})}{\partial B^2} & \frac{\partial^2 S_1(\boldsymbol{\theta})}{\partial B\partial \alpha}\\ \frac{\partial^2 S_1(\boldsymbol{\theta})}{\partial \alpha\partial A} & \frac{\partial^2 S_1(\boldsymbol{\theta})}{\partial \alpha\partial B} & \frac{\partial^2 S_1(\boldsymbol{\theta})}{\partial \alpha^2}\end{bmatrix}$$

and $(\overline{\boldsymbol{\theta}})$ is a point on the line joining $\widehat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}^0$. Since, $S'(\widehat{\boldsymbol{\theta}})=0$, from (14) we obtain

$$-S_1'(\boldsymbol{\theta}^0)\boldsymbol{D}_3^{-1}=(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^0)\boldsymbol{D}_3\boldsymbol{D}_3^{-1}S_1''(\overline{\boldsymbol{\theta}})\boldsymbol{D}_3^{-1}. \tag{15}$$

Here the diagonal matrix $\boldsymbol{D}_3$ is same as defined before. Observe that

$$\begin{aligned}
\frac{\partial S_1(\boldsymbol{\theta}^0)}{\partial A} &= -2\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right) e(m,n) \\
\frac{\partial S_1(\boldsymbol{\theta}^0)}{\partial B} &= -2\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right) e(m,n)\cos(\alpha^0(m^2+n^2)) \\
\frac{\partial S_1(\boldsymbol{\theta}^0)}{\partial \alpha} &= 2B\sum_{m=1}^{M}\sum_{n=1}^{N} (m^2+n^2) w\left(\frac{m}{M},\frac{n}{N}\right) e(m,n)\sin(\alpha^0(m^2+n^2))
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 S_1(\boldsymbol{\theta}^0)}{\partial A^2} &= 2\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right) \\
\frac{\partial^2 S_1(\boldsymbol{\theta}^0)}{\partial B^2} &= 2\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right)\cos^2(\alpha^0(m^2+n^2)) \\
\frac{\partial^2 S_1(\boldsymbol{\theta}^0)}{\partial \alpha^2} &= 2B^2\sum_{m=1}^{M}\sum_{n=1}^{N} (m^2+n^2)^2 w\left(\frac{m}{M},\frac{n}{N}\right)\sin^2(\alpha^0(m^2+n^2)) \\
\frac{\partial^2 S_1(\boldsymbol{\theta}^0)}{\partial A\partial B} &= 2\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right)\cos(\alpha^0(m^2+n^2)) \\
\frac{\partial^2 S_1(\boldsymbol{\theta}^0)}{\partial A\partial \alpha} &= -2B^0\sum_{m=1}^{M}\sum_{n=1}^{N} (m^2+n^2) w\left(\frac{m}{M},\frac{n}{N}\right)\sin(\alpha^0(m^2+n^2)) \\
\frac{\partial^2 S_1(\boldsymbol{\theta}^0)}{\partial B\partial \alpha} &= -2B^0\sum_{m=1}^{M}\sum_{n=1}^{N} (m^2+n^2) w\left(\frac{m}{M},\frac{n}{N}\right)\sin(\alpha^0(m^2+n^2))\cos(\alpha^0(m^2+n^2)).
\end{aligned}$$

Observe that

$$S_1'(\boldsymbol{\theta}^0)\boldsymbol{D}_3^{-1} \xrightarrow{d} N_3\left(\mathbf{0}, 2\sigma^2\boldsymbol{\Sigma}_3\right),$$

and

$$\begin{aligned}
\lim_{M,N\to\infty}\frac{1}{MN}\frac{\partial^2 S_1(\boldsymbol{\theta}^0)}{\partial A^2} &= 2u_{00}, \quad \lim_{M,N\to\infty}\frac{1}{MN}\frac{\partial^2 S_1(\boldsymbol{\theta}^0)}{\partial B^2} = u_{00}, \\
\lim_{M,N\to\infty}\frac{1}{MN(M^2+N^2)^2}\frac{\partial^2 S_1(\boldsymbol{\theta}^0)}{\partial \alpha^2} &= \frac{1}{2}B^{0^2}(u_{40}+u_{04}+2u_{22}), \quad \lim_{M,N\to\infty}\frac{1}{MN}\frac{\partial^2 S_1(\boldsymbol{\theta}^0)}{\partial A\partial B} = 0, \\
\lim_{M,N\to\infty}\frac{1}{MN}\frac{\partial^2 S_1(\boldsymbol{\theta}^0)}{\partial A\partial \alpha} &= 0, \quad \lim_{M,N\to\infty}\frac{1}{MN}\frac{\partial^2 S_1(\boldsymbol{\theta}^0)}{\partial B\partial \alpha} = 0.
\end{aligned}$$

Hence, the result follows. ∎

Appendix B

Proof of Theorem 4: The proof follows along the same way as the Proof of Theorem 2. We need to expand $S_2(\boldsymbol{\theta})$ by Taylor series.

$$S_2(\boldsymbol{\theta}) = MNQ((\boldsymbol{\theta}) = \sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M}, \frac{n}{N}\right)\left(y(m,n) - A - B\cos(\alpha(m^2+n^2) + \beta m + \gamma n)\right)^2.$$

We further need the following quantities similarly as in Theorem 2. The results can be obtained in a very similar manner by using the Central Limit Theorem properly.

$$\begin{aligned}
\frac{\partial S_2(\boldsymbol{\theta}^0)}{\partial A} &= -2\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M}, \frac{n}{N}\right) e(m,n) \\
\frac{\partial S_2(\boldsymbol{\theta}^0)}{\partial B} &= -2\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M}, \frac{n}{N}\right) e(m,n)\cos(\alpha^0(m^2+n^2) + \beta^0 m + \gamma^0 n) \\
\frac{\partial S_2(\boldsymbol{\theta}^0)}{\partial \alpha} &= 2B\sum_{m=1}^{M}\sum_{n=1}^{N} (m^2+n^2) w\left(\frac{m}{M}, \frac{n}{N}\right) e(m,n)\sin(\alpha^0(m^2+n^2) + \beta^0 m + \gamma^0 n) \\
\frac{\partial S_2(\boldsymbol{\theta}^0)}{\partial \beta} &= 2B\sum_{m=1}^{M}\sum_{n=1}^{N} m w\left(\frac{m}{M}, \frac{n}{N}\right) e(m,n)\sin(\alpha^0(m^2+n^2) + \beta^0 m + \gamma^0 n) \\
\frac{\partial S_2(\boldsymbol{\theta}^0)}{\partial \gamma} &= 2B\sum_{m=1}^{M}\sum_{n=1}^{N} n w\left(\frac{m}{M}, \frac{n}{N}\right) e(m,n)\sin(\alpha^0(m^2+n^2) + \beta^0 m + \gamma^0 n)
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial A^2} &= 2\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right) \\
\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial B^2} &= 2\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right)\cos^2(\alpha^0(m^2+n^2)+\beta^0 m+\gamma^0 n) \\
\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial \alpha^2} &= 2B^2\sum_{m=1}^{M}\sum_{n=1}^{N}(m^2+n^2)^2 w\left(\frac{m}{M},\frac{n}{N}\right)\sin^2(\alpha^0(m^2+n^2)+\beta^0 m+\gamma^0 n) \\
\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial \beta^2} &= 2B^2\sum_{m=1}^{M}\sum_{n=1}^{N} m^2 w\left(\frac{m}{M},\frac{n}{N}\right)\sin^2(\alpha^0(m^2+n^2)+\beta^0 m+\gamma^0 n) \\
\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial \gamma^2} &= 2B^2\sum_{m=1}^{M}\sum_{n=1}^{N} n^2 w\left(\frac{m}{M},\frac{n}{N}\right)\sin^2(\alpha^0(m^2+n^2)+\beta^0 m+\gamma^0 n) \\
\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial A\partial B} &= 2\sum_{m=1}^{M}\sum_{n=1}^{N} w\left(\frac{m}{M},\frac{n}{N}\right)\cos(\alpha^0(m^2+n^2)+\beta^0 m+\gamma^0 n) \\
\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial A\partial \alpha} &= -2B^0\sum_{m=1}^{M}\sum_{n=1}^{N}(m^2+n^2) w\left(\frac{m}{M},\frac{n}{N}\right)\sin(\alpha^0(m^2+n^2)+\beta^0 m+\gamma^0 n) \\
\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial B\partial \alpha} &= -2B^0\sum_{m=1}^{M}\sum_{n=1}^{N}(m^2+n^2) w\left(\frac{m}{M},\frac{n}{N}\right)\times \\
&\qquad \sin(\alpha^0(m^2+n^2)+\beta^0 m+\gamma^0 n)\cos(\alpha^0(m^2+n^2)+\beta^0 m+\gamma^0 n). \\
\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial \alpha\partial \beta} &= 2B^2\sum_{m=1}^{M}\sum_{n=1}^{N} m(m^2+n^2) w\left(\frac{m}{M},\frac{n}{N}\right)\sin^2(\alpha^0(m^2+n^2)+\beta^0 m+\gamma^0 n) \\
\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial \alpha\partial \gamma} &= 2B^2\sum_{m=1}^{M}\sum_{n=1}^{N} n(m^2+n^2) w\left(\frac{m}{M},\frac{n}{N}\right)\sin^2(\alpha^0(m^2+n^2)+\beta^0 m+\gamma^0 n) \\
\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial \beta\partial \gamma} &= 2B^2\sum_{m=1}^{M}\sum_{n=1}^{N} mn w\left(\frac{m}{M},\frac{n}{N}\right)\sin^2(\alpha^0(m^2+n^2)+\beta^0 m+\gamma^0 n)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{M,N\to\infty}\frac{1}{MN}\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial A^2} &= 2u_{00}, \quad \lim_{M,N\to\infty}\frac{1}{MN}\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial B^2} = u_{00},\\
\lim_{M,N\to\infty}\frac{1}{MN(M^2+N^2)^2}\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial \alpha^2} &= \frac{1}{2}B^{0^2}(u_{40}+u_{04}+2u_{22}), \quad \lim_{M,N\to\infty}\frac{1}{MN}\frac{\partial^2 S_1(\boldsymbol{\theta}^0)}{\partial A\partial B} = 0,\\
\lim_{M,N\to\infty}\frac{1}{MN}\frac{\partial^2 S_1(\boldsymbol{\theta}^0)}{\partial A\partial \alpha} &= 0, \quad \lim_{M,N\to\infty}\frac{1}{MN}\frac{\partial^2 S_1(\boldsymbol{\theta}^0)}{\partial B\partial \alpha} = 0\\
\lim_{M,N\to\infty}\frac{1}{M^3N}\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial \beta^2} &= B^{0^2}u_{30}, \quad \lim_{M,N\to\infty}\frac{1}{MN^3}\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial \gamma^2} = B^{0^2}u_{03},\\
\lim_{M,N\to\infty}\frac{1}{M^2N(M^2+N^2)}\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial \alpha\partial\beta} &= \frac{1}{2}B^{0^2}(u_{30}+u_{12}),\\
\lim_{M,N\to\infty}\frac{1}{MN^2(M^2+N^2)}\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial \alpha\partial\gamma} &= \frac{1}{2}B^{0^2}(u_{21}+u_{03}),\\
\lim_{M,N\to\infty}\frac{1}{M^2N^2}\frac{\partial^2 S_2(\boldsymbol{\theta}^0)}{\partial \beta\partial\gamma} &= \frac{1}{2}B^{0^2}u_{11},
\end{aligned}
$$

References

[1] Aslam, M.S., Radhika, T., Chandrasekar, A., Zhu, Q. (2024), "Improved Event-Triggered-Based Output Tracking for a Class of Delayed Networked T–S Fuzzy Systems" *International Journal of Fuzzy Systems*, vol. 26, 1–14.

[2] Cao, Y., Chandrasekar, A., Radhika, T., Vijayakumar, V. (2024), "Input-to-state stability of stochastic Markovian jump genetic regulatory networks", *Mathematics and Computers in Simulation*, vol. 222, 174–187.

[3] Fan, C. and Liu, C. (2015), "A Novel Algorithm for Circle Curve Fitting Based on the Least Square Method by the points of the Newton's rings", *IEEE International Conference on Computers, Communications, and Systems*, 256 – 260.

[4] Fu, L., Zhang, M., Liu, Z. and Li, H. (2019), "Robust Frequency Estimation of Multi-sinusoidal Signals Using Orthogonal Matching Pursuit with Weak Derivatives Criterion", *Circuits, Systems, and Signal Processing*, vol. 38, 1194–1205.

[5] Gorthi, S.S. and Rastogi, P. (2009), "Estimation of phase derivatives using discrete chirp-Fourier-transform-based method", *Optics Letters*, vol. 34, 2396–2398.

[6] Gorthi, S.S. and Rastogi, P. (2009), "Simultaneous measurement of displacement, strain and curvature in digital holographic interferometry using high-order instantaneous moments", *Optical Express*, vol. 17, 17784–17791.

[7] Grover, R., Kundu, D. and Mitra, A. (2018), "Asymptotic of approximate least squares estimators of parameters of two-dimensional chirp signal", *Journal of Multivariate Analysis*, vol. 168, 211 –220.

[8] Guo, Y. and Li, B.Z. (2018), "Novel method for parameter estimation of Newton's rings based on CFRFT and ER-WCA", *Signal Processing*, vol. 144, 118–126.

[9] Guo, Z., Lu, M-F., Wu, J-M., Jin, X-W., Zhang, F. and Tao, R. (2019), "Fast FRFT-based method for estimating physical parameters from Newton's rings", *Applied Optics*, vol. 58, No. 14, 3926 – 3931.

[10] Huber, P. (1981), *Robust Statistics*, John Wiley, New York.

[11] Jennrich, R.I. (1969), "Asymptotic properties of the nonlinear least squares estimators", *Annals of Mathematical Statistics*, vol. 40, 633 – 643.

[12] Kemao, Q., Wang, H. and Gao, W. (2008), "Windowed Fourier transform for fringe pattern analysis: theoretical analyses", *Applied Optics*, vol. 47, 5408–5419.

[13] Kim, T.S., Kim, H.K. and Choi, H.C. (2000), "Asymptotic properties of LAD estimators of a nonlinear time series regression model", *Journal of Korean Statistical Society*, vol. 29, 2, 187–199.

[14] Lahiri, A. (2011) *Estimators of Parameters of Chirp Signals and Their Properties*, Ph.D. Thesis, Indian Institute of Technology Kanpur, India, https://etd.iitk.ac.in:8443/jspui/handle/123456789/12724.

[15] Lahiri, A., Kundu, D. and Mitra, A. (2014), "On least absolute deviation estimator of one dimensional chirp model", *Statistics*, vol. 48, no. 2, 405 – 420.

[16] Lu, M.F., Ni, G.Q., Bai, T.Z., Tao, R. and Zhang, F. (2013), "Method for suppressing the quantization error of Newton's rings fringe pattern", *Optical Engineering*, vol. 52, 103105.

[17] Mangulis, V. (1965), *Handbook of series for scientists and engineers*, New York, Academic Press.

[18] Nascov, V., Apostol, D. and Garoi, F. (2007), "Statistical processing of Newton's rings using discrete Fourier analysis", *Optical Engineering*, vol. 46, 028201.

[19] Nascov, V. Dobroiu, A. and Apostol, D. (2004), "Statistical errors on Newton fringe pattern digital processing", —*Proc. SPIE*, 5581, 788–796.

[20] Radhika, T., Chandrasekar, A., Vijayakumar, V. and Zhu, Q. (2023), "Analysis of Markovian Jump Stochastic Cohen–Grossberg BAM Neural Networks with Time Delays for Exponential Input-to-State Stability", *Neural Processing Letters*, vol. 55, 11055–11072.

[21] Rajshekhar, G. and Rastogi, P. (2012), "Fringe analysis: premise and perspectives", *Optics Laser Engineering*, vol. 50, iii-X.

[22] Seber, G.A.F. and Wild, C.J. (1989), *Nonlinear Regression*, Wiley Interscience, New York, USA.

[23] Shukla, A., Grover, R., Kundu, D. and Mitra, A. (2022), "Approximate least squares estimators of a two-dimensional chirp model", *Journal of Multivariate Analysis*, Vol. 192, Artilce No. 105045.

[24] Simeunovic, M. and Djurovic, I. (2016), "Parameter estimation of multicomponent 2D polynomial-phase signals using the 2D PHAF-based approach, *IEEE Transactions on Signal Processing*, vol. 64, pp.771–782.

[25] Vinogradov, I.M. (1954), "The method of trigonometrical sums in the theory of num- bers", *Interscience*, Translated from Russian. Revised and annotated by K.F. Roth and Anne Davenport. Reprint of the 1954 translation. Dover Publications,Inc., Mineola, NY, 2004

[26] Wu, C.F.J.(1981), "Asymptotic theory of the nonlinear least squares estimation", *Annals of Statistics*, vol. 9, 501–513.

[27] Zhang, T., Lu, M.F., Wu, J.M., He, W., Zhang, F. and Tao, R. (2024), "Towards a more practical analysis of Newton's rings using deep learning", *Applied Optics*, vol. 63, No. 7, 1854 – 1866.

$M = N$ (NOL)	Parameter	LSE	WLSE-1	WLSE-2	LAD	Huber-M
25	A	1.00024	1.00061	1.00027	1.00041	1.00074
		(4.839E-03)	(3.663E-03)	(7.709E-03)	3.679E-03)	(3.678E-03)
(49)	B	1.00038	1.00029	0.99846	0.99925	0.99899
		(2.373E-02)	(2.074E-02)	(3.245E-02)	(2.167E-02)	(2.111E-02)
	α	3.92700	3.92694	3.92694	3.92700	3.92695
		(4.174E-04)	(3.881E-04)	(4.949E-04)	(3.997E-04)	(4.010E-04)
	β	2.35597	2.35713	2.35725	2.35601	2.35711
		(9.656E-03)	(9.087E-03)	(1.010E-02)	(9.111E-03)	(9.211E-03)
	γ	0.78546	0.78614	0.78622	0.78528	0.78586
		(7.913E-03)	(7.300E-03)	(8.771E-03)	(7.330E-03)	(7.381E-03)
50	A	1.00002	1.00001	1.00009	0.99998	0.99999
		(2.601E-04)	(2.314E-04)	(3.391E-04)	2.304E-04)	(2.416E-03)
(125)	B	1.00007	1.00009	1.00032	0.99953	0.99982
		(2.898E-03)	(2.412E-03)	(3.456E-03)	(2.462E-03)	(2.425E-03)
	α	3.92699	3.92699	3.92699	3.92699	3.92699
		(2.939E-05)	(2.404E-05)	(3.899E-05)	(2.344E-05)	(2.432E-05)
	β	2.35608	2.35611	2.35630	2.35619	2.35617
		(1.472E-03)	(1.268E-03)	(2.178E-03)	(1.247E-03)	(1.391E-03)
	γ	0.78536	0.78533	0.78544	0.78544	0.78538
		(8.290E-04)	(8.081E-04)	(9.213E-04)	(8.121E-03)	(7.987E-04)
75	A	0.99998	0.99999	0.99999	0.99999	0.99999
		(5.873E-05)	(4.784E-05)	(5.118E-05)	(5.075E-05)	(4.801E-04)
(529)	B	0.99979	0.99977	0.99977	0.99982	0.99982
		(6.107E-04)	(4.933E-04)	(5.267E-04)	(5.062E-04)	(5.109E-04)
	α	3.92699	3.92699	3.92699	3.92699	3.92699
		(5.940E-06)	(5.338E-06)	(5.897E-06)	(5.379E-06)	(5.411E-06)
	β	2.35619	2.35620	2.35617	2.35619	2.35621
		(2.745E-04)	(2.451E-04)	(3.101E-04)	(2.431E-04)	(2.414E-04)
	γ	0.78542	0.78544	0.78541	0.78542	0.78542
		(1.873E-04)	(1.457E-04)	(2.136E-04)	(1.428E-04)	(1.462E-04)
100	A	0.99999	0.99999	0.99999	0.99999	0.99999
		(1.233E-05)	(1.113E-05)	(1.312E-05)	(1.132E-05)	(1.117E-05)
(961)	B	0.99993	0.99923	0.99991	0.99991	0.99991
		(1.604E-04)	(1.242E-04)	(1.378E-04)	(1.266E-04)	(1.284E-04)
	α	3.92699	3.92699	3.92699	3.92699	3.92699
		(1.972E-06)	(1.698E-06)	(1.998E-06)	(1.698E-06)	(1.677E-06)
	β	2.35619	2.35620	2.35621	2.35620	2.35620
		(9.466E-05)	(8.154E-05)	(9.018E-05)	(8.123E-04)	(8.197E-04)
	γ	0.78541	0.78542	0.78541	0.78541	0.78541
		(4.984E-05)	(3.777E-05)	(3.887E-05)	(3.797E-05)	(3.781E-05)

Table 3: The average estimates (AE) and the associated mean squared errors (MSE) for parameter set 2, of the different methods for different sample sizes ($M = N$) and for different number of outliers (NOL) and when the outliers are present in the middle portion of the data set.

$M = N$	Parameter	LSE	WLSE-1	WLSE-2	LAD	Huber-M
25	A	1.00036	1.00024	1.00062	1.00057	1.00030
		(3.189E-03)	(3.391E-03)	(3.399E-03)	4.127E-03)	(4.101E-03)
	B	0.99958	0.99855	1.00044	0.99958	0.99967
		(1.852E-02)	(1.851E-02)	(1.934E-02)	(2.237E-02)	(2.111E-02)
	α	3.92700	3.92697	3.92698	3.92699	3.92695
		(3.297E-04)	(3.304E-04)	(3.312E-04)	(4.122E-04)	(4.100E-04)
	β	2.35596	2.35678	2.35748	2.35656	2.35711
		(7.969E-03)	(7.964E-03)	(8.028E-02)	(9.611E-03)	(9.411E-03)
	γ	0.78531	0.78566	0.78605	0.78533	0.78586
		(5.846E-03)	(5.925E-03)	(5.971E-03)	(7.094E-03)	(6.381E-03)
50	A	1.00000	0.99999	0.99997	0.99996	0.99998
		(2.289E-04)	(2.294E-04)	(2.334E-04)	2.493E-03)	(2.515E-03)
	B	0.99991	0.99969	0.99956	0.99965	0.99991
		(2.102E-03)	(2.122E-03)	(2.177E-03)	(2.592E-03)	(2.497E-03)
	α	3.92699	3.92699	3.92694	3.92699	3.92699
		(2.220E-05)	(2.441E-05)	(2.507E-05)	(3.084E-05)	(2.898E-05)
	β	2.35616	2.35619	2.35619	2.35613	2.35614
		(1.046E-03)	(1.096E-03)	(1.192E-03)	(1.356E-03)	(1.301E-03)
	γ	0.78538	0.78546	0.78546	0.78544	0.78544
		(6.842E-04)	(6.926E-04)	(6.942E-04)	(7.356E-04)	(7.345E-04)
75	A	1.00000	1.00000	0.99998	0.99999	0.99999
		(4.344E-05)	(4.470E-05)	(4.518E-05)	(4.753E-04)	(4.801E-04)
	B	0.99992	0.99977	0.99981	0.99982	0.99979
		(4.500E-04)	(4.733E-04)	(4.767E-04)	(5.407E-04)	(5.389E-04)
	α	3.92699	3.92699	3.92699	3.92699	3.92699
		(4.718E-06)	(4.831E-06)	(4.859E-06)	(6.604E-06)	(6.611E-06)
	β	2.35618	2.35620	2.35621	2.35623	2.35621
		(2.358E-04)	(2.381E-04)	(2.378E-04)	(2.717E-04)	(2.788E-04)
	γ	0.78540	0.78543	0.78543	0.78542	0.78542
		(1.234E-04)	(1.249E-04)	(1.246E-04)	(1.511E-04)	(1.562E-04)
100	A	0.99999	0.99999	0.99999	0.99999	0.99999
		(1.376E-05)	(1.326E-05)	(1.346E-05)	(1.423E-05)	(1.445E-05)
	B	0.99993	0.99993	0.99991	0.99991	0.99993
		(1.330E-04)	(1.366E-04)	(1.359E-04)	(1.466E-04)	(1.467E-04)
	α	3.92699	3.92699	3.92699	3.92699	3.92699
		(1.667E-06)	(1.731E-06)	(1.724E-06)	(2.089E-06)	(2.101E-06)
	β	2.35619	2.35620	2.35620	2.35620	2.35620
		(9.658E-05)	(9.683E-05)	(9.687E-05)	(1.001E-04)	(1.111E-04)
	γ	0.78541	0.78541	0.78541	0.78541	0.78541
		(4.393E-05)	(4.412E-05)	(4.421E-05)	(4.628E-05)	(4.612E-05)

Table 4: The average estimates (AE) and the associated mean squared errors (MSE) for Set 2, of the different methods for different sample sizes ($M = N$) when there is no outliers.