On Three-Parameter Generalized Exponential Distribution

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Abstract

In this article, we consider the estimation of the three-parameter generalized exponential distribution. In presence of the unknown location parameter the usual maximum likelihood estimators do not exist for $0 < \alpha < 1$. The maximum product of spacings method serves as good alternative since it always exists and yields consistent estimators in the entire parametric space. We develop the asymptotic distribution of the proposed estimators along-with a detailed discussion about the computational intricacies involved in implementing the product of spacing method. Extensive simulations have been performed to demonstrate the effectiveness of the proposed method for α in the range $(0, 1)$, in addition to a comparative study with some standard techniques known to provide consistent estimators, even for $0 < \alpha < 1$. Furthermore, two real data sets have been analysed to demonstrate the applicability of the proposed method.

Keywords: Maximum product of spacings, non-regular estimable condition, percentile bootstrap intervals.

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1 Introduction

The cumulative distribution function (CDF) and the probability density function (PDF) of a generalized exponential distribution (GED) are

$$F(x,\alpha,\lambda,\mu)=\begin{cases}\left(1-e^{-\lambda(x-\mu)}\right)^{\alpha}; & x>\mu,\alpha>0,\lambda>0\\ 0\ ; & otherwise\end{cases} \tag{1.1}$$

$$f(x,\alpha,\lambda,\mu)=\begin{cases}\alpha\lambda\left(1-e^{-\lambda(x-\mu)}\right)^{\alpha-1}e^{-\lambda(x-\mu)}; & x>\mu,\alpha>0,\lambda>0\\ 0\ ; & otherwise\end{cases} \tag{1.2}$$

respectively. Here, α, λ, μ are the shape, scale and location parameters, respectively. Gupta and Kundu [1999] proposed this model as an extension of the standard two parameter exponential distribution and demonstrated its similarities with the Gamma and Weibull distributions. The exponential distribution has been widely used in the reliability and life testing studies since long due to its structural simplicity, computational ease and explicit solution to several real life situations pertaining to censoring, stress-strength reliability, etc. However, a major drawback of the exponential distribution is that it has a constant hazard function. The GED overcomes this drawback with the introduction of a shape parameter that allows the existence of monotone hazard function. Regardless of the values of μ and λ, the GED exhibits decreasing, increasing and constant hazard function for $\alpha<1$, $\alpha>1$ and $\alpha=1$, respectively. Clearly, when $\alpha=1$, the GED reduces to the two-parameter exponential distribution. In the introductory article of the GED, various measures were derived like the moments, moment generating function, distribution of sums, and many more, with the assumption $\mu=0$. Also, the maximum likelihood (ML) estimators for α,λ and μ were derived for complete and Type-II censored cases. The asymptotic distribution of these estimators were obtained for $\alpha>2$. It should be noted that for $0<\alpha<1$, the ML estimators do not exist. Hence, if the values of α is unknown and the true distribution lies in the above mentioned region, then, the MLE's cannot be obtained. The main problem is that there are paths in the parameter space with location parameter $\mu\to x_{(1)}$ which results in an infinite likelihood and hence non-existent MLE's, similar to the three-parameter Weibull and gamma distributions. However, if the location parameter μ is known, then it becomes a regular family and the ML estimators of α and λ can be obtained for all $\alpha > 0, \lambda > 0$.

The two-parameter GED has been extensively explored in connection to gamma, Weibull and Log-normal distributions, see for example Kundu et al. [2005]. Stress-strength reliability, entropy, distributions of sums, products and ratios of GED variates along with many other properties has been elicited by AL-Hussaini and Ahsanullah [2015]. Best linear unbiased estimators of location and scale of location shifted GED, under the assumption of known shape parameter, based on the order statistics was discussed by Raqab and Ahsanullah [2001]. Bayesian estimation of the parameters and reliability function in presence of informative prior based on importance sampling, coupled with prediction of future observations through M-H algorithm within Gibbs sampler was derived by Raqab and Madi [2005]. Inferential work on stress-strength reliability of three parameter GED, in addition to asymptotic distribution and related bootstrap and highest posterior density intervals has been appraised in Raqab et al. [2008]. A detailed discussion on the existing work and further statistical developments on the two parameter generalized exponential distribution may be found in the review article by Nadarajah [2011]. It is evident from the available literature is that, even though substantial work has been directed for the two-parameter GED, yet, very few work dealt the estimation of the location shifted GED, all parameters being unknown, for shape parameter lying in the range $(0, 2]$. Since MLE is known to exist for $\alpha \geq 1$, pivot of this article is to obtain consistent and reliable estimators of the unknown parameters for $\alpha < 1$.

Cheng and Amin [1983] and Ranneby [1984] independently suggested the use of the maximum product of spacings (MPS) method, especially suited for distributions with an unknown threshold. An important characteristic of the MPS estimators is that they are consistent and efficient in many cases where the ML estimators may not exist. Moreover, when MLE's exists, convergence of the MPS estimates to the ML estimates in probability can be easily derived. This intrigued the authors to study the behaviour of the MPS estimators in details for the GE distribution. Thus, the primary interest of this article lies in developing an efficient estimation method capable of furnishing consistent estimates when ML technique does not exist.

The implementation of the MPS method requires obtaining an ordered sample $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ of size n. For brevity, we denote $x_{(i)}$ as x_i from now on. The differences of the cumulative distribution function (CDF), evaluated for a pair of consecutive order statistic gives rise to spacings. The MPS estimator is that value of the parameter which maximises the geometric mean of the spacings derived from the observed sample. Since, the CDF is bounded, the product of spacings is also bounded and it attains the maximum *iff* all the spacings are equal in magnitude. Thus, for a given distribution, the most probable sample generates spacings of more or less equal in magnitude whereas any other sample from some other distribution will not have this property.

Although the MPS technique guarantees consistent estimators for the location, shape and scale, yet, the algorithm needed to compute the MPS estimators is computationally quite challenging. A careful probe of the joint density function and CDF is needed due to sensitivity of the algorithms for the choice of the initial values. Most of the standard algorithm may converge to a local maximum rather than the global maximum. Another important goal of this manuscript is to explore and discuss the challenges met to implement the optimization algorithm in case of the three-parameter GED, in addition to derivation of the asymptotic properties of the MPS estimators.

The performance of the proposed estimators are analysed through a simulation study in terms of root mean square errors (RMSE) and lengths of boot-strap intervals along-with a comparative analysis with some existing standard methods, known to provide reasonable estimators for such cases.

The rest of the manuscript is organized as follows. In Section 2 we provide the MPS estimators for the three-parameter GED. The implementation of the algorithm and simulation results are presented in Section 3 and Section 4, respectively. In Section 5 we provide the analyses of two real data sets, and finally we conclude the paper in Section 6. Also, the asymptotic distribution of the proposed MPS estimators are derived in the appendix.

2 Estimation using Maximum Product of Spacings method

In this section, we derive the MPS estimators of α, λ and μ based on a ordered sample $\{x_1, \ldots, x_n\}$ obtained from the three-parameter GED having the PDF (1.1). Based on the ordered observations, the spacings can be defined as

$$D_i = F(x_i; \alpha, \lambda, \mu) - F(x_{i-1}; \alpha, \lambda, \mu); \quad i = 1, 2, \cdots, n+1,$$

here $x_0 = \mu$ and $x_{n+1} = \infty$. Hence, the MPS estimators are obtained by maximizing the product of spacings, i.e.

$$S(\alpha, \lambda, \mu) = \left\{ \prod_{i=1}^{n+1} D_i \right\}^{1/(n+1)}. \tag{2.1}$$

Maximizing the product of spacings (PS) function is equivalent to maximizing its natural logarithm or minimizing the negative of the same. Thus, the log-PS function $H(\alpha, \lambda, \mu)$ is obtained as

$$\begin{aligned} H(\alpha, \lambda, \mu) &= \ln\left(S(\alpha, \lambda, \mu)\right) \\ &= \frac{1}{n+1} \sum_{i=1}^{n+1} \ln(F(x_i; \alpha, \lambda, \mu) - F(x_{i-1}; \alpha, \lambda, \mu)) \\ &= \frac{1}{n+1} \sum_{i=1}^{n+1} \ln\left\{ \left(1 - e^{-\lambda(x_i - \mu)}\right)^{\alpha} - \left(1 - e^{-\lambda(x_{i-1} - \mu)}\right)^{\alpha} \right\} \end{aligned} \tag{2.2}$$

Evaluating the partial derivatives of $H(\alpha, \lambda, \mu)$ with respect to α, λ and μ and equating them to 0, we obtain the normal equations as:

$$\frac{\partial H}{\partial \alpha} = \frac{1}{n+1} \left\{ \begin{array}{l} \sum\limits_{i=1}^{n+1} \dfrac{\left(1 - e^{-\lambda(x_i - \mu)}\right)^{\alpha} \ln\left(1 - e^{-\lambda(x_i - \mu)}\right)}{\left(1 - e^{-\lambda(x_i - \mu)}\right)^{\alpha} - \left(1 - e^{-\lambda(x_{i-1} - \mu)}\right)^{\alpha}} \\ -\dfrac{\left(1 - e^{-\lambda(x_{i-1} - \mu)}\right)^{\alpha} \ln\left(1 - e^{-\lambda(x_{i-1} - \mu)}\right)}{\left(1 - e^{-\lambda(x_i - \mu)}\right)^{\alpha} - \left(1 - e^{-\lambda(x_{i-1} - \mu)}\right)^{\alpha}} \end{array} \right\} = 0 \tag{2.3}$$

$$\frac{\partial H}{\partial \lambda} = \frac{\alpha}{n+1} \sum_{i=1}^{n+1} \left\{ \begin{array}{l} \dfrac{(x_i - \mu)(1 - e^{-\lambda(x_i - \mu)})^{\alpha - 1} e^{-\lambda(x_i - \mu)}}{\left(1 - e^{-\lambda(x_i - \mu)}\right)^{\alpha} - \left(1 - e^{-\lambda(x_{i-1} - \mu)}\right)^{\alpha}} \\ -\dfrac{(x_{i-1} - \mu)(1 - e^{-\lambda(x_{i-1} - \mu)})^{\alpha - 1} e^{-\lambda(x_{i-1} - \mu)}}{\left(1 - e^{-\lambda(x_i - \mu)}\right)^{\alpha} - \left(1 - e^{-\lambda(x_{i-1} - \mu)}\right)^{\alpha}} \end{array} \right\} = 0 \tag{2.4}$$

$$\frac{\partial H}{\partial \mu} = \frac{\alpha \lambda}{n+1} \sum_{i=1}^{n+1} \frac{\left\{(1 - e^{-\lambda(x_{i-1} - \mu)})^{\alpha - 1} e^{-\lambda(x_{i-1} - \mu)} - (1 - e^{-\lambda(x_i - \mu)})^{\alpha - 1} e^{-\lambda(x_i - \mu)}\right\}}{\left(1 - e^{-\lambda(x_i - \mu)}\right)^{\alpha} - \left(1 - e^{-\lambda(x_{i-1} - \mu)}\right)^{\alpha}} = 0 \tag{2.5}$$

Clearly, the above equations cannot be solved explicitly, and one needs to use some numerical techniques to solve them. One needs to be careful about the choice of the initial guesses and the convergence to a local maximum rather than the global maximum. Hence, to solve these equations, one needs to study the nature of the log-PS function extensively over the whole parameter space.

Note that because of the complicated nature of the log-PS function, it is difficult to check the uni-modality of the function analytically. Also, because of three dimensions, it is not possible to check the uni-modality of the log-PS function graphically in this case. A simple technique to proceed with the maximization is by considering the location parameter as a known constant. This reduces the log-PS function as a function of two unknown and hence, for a given value of μ, the PS estimates $\widetilde{\lambda}(\mu), \widetilde{\alpha}(\mu)$ may be computed numerically as solution of the equations $\partial H(\alpha, \lambda|\mu, x)/\partial\alpha = 0$ and $\partial H(\alpha, \lambda|\mu, x)/\partial\lambda = 0$. Moreover, the uni-modality of the two-dimensional log-PS function can be checked graphically. Thereafter, the partially maximized log-PS function $H^*(\mu, \alpha(\mu), \lambda(\mu))$ may be plotted against varying μ to verify the existence of the global maximum. Then by maximizing the profile log-PS function i.e. $H^*(\mu, \alpha(\mu), \lambda(\mu))$, it is possible to obtain the MPS estimate of μ namely $\widetilde{\mu}$. Once $\widetilde{\mu}$ is obtained the MPS estimates of α and λ can be obtained as $\widetilde{\alpha}(\widetilde{\mu})$ and $\widetilde{\lambda}(\widetilde{\mu})$, respectively. The details are discussed in Section 3.

3 Numerical exploration of the estimation technique

In this study, we restrict our attention to the estimators when the underlying random variate X follows a GED with the shape parameter in the range $(0, 1)$. As we have seen that the MPS estimators can be obtained by solving the three equations (2.3) - (2.5) simultaneously. It cannot be obtained explicitly, and one needs to use some numerical techniques to solve these system of equations. Since it is a three-dimensional problem, it is not very easy to guess the initial values. Due to this reason, we have used the standard profiling method to obtain an effective initial guess which can be used further to obtain the MPS estimators quite efficiently.

The basic idea is very simple. First it is assumed that the location parameter μ is known. For a given μ, we obtain the MPS estimators of α and λ, say $\widetilde{\alpha}(\mu), \widetilde{\lambda}(\mu)$, respectively, by maximizing H with respect to these two parameters. Since it becomes a two-dimensional optimization problem, the initial guesses of α and λ can be obtained from the contour plots of $H(\alpha, \lambda, \mu)$ as a function of α and λ for a fixed μ. Based on these initial guesses the two non-linear equations (2.3) and (2.4) can be solved for different values of μ. Once $\widetilde{\alpha}(\mu)$ and $\widetilde{\lambda}(\mu)$ are obtained, then the MPS estimator of μ can be obtained by maximizing $H(\widetilde{\alpha}(\mu), \widetilde{\lambda}(\mu), \mu)$ with respect to μ. Since it involves solving a one dimensional optimization problem, it can be performed quite easily.

Note that the proposed method can be implemented provided the function $H(\alpha, \lambda, \mu)$ is an uni-modal function. Unfortunately due to the complicated nature of the $H(\alpha, \lambda, \mu)$ function, it is not possible to prove analytically the uni-modality of $H(\alpha, \lambda, \mu)$. But for a given data set, it is possible to check graphically whether the $H(\alpha, \lambda, \mu)$ function is uni-modal or not. To illustrate the above method we have a generated a sample of size 30, from a $GED(0.9, 2.0, 1.5)$. In figure 1 we have provided the contour plots of $-H(\alpha, \lambda, \mu)$ as a function of α and λ for different values μ. It is clear that for fixed μ, $H(\alpha, \lambda, \mu)$ is an uni-modal function of α and λ. Hence, for a given μ, the maximization of $H(\alpha, \lambda, \mu)$ with respect to α and λ can be performed quite conveniently, and $\widetilde{\alpha}(\mu)$ and $\widetilde{\lambda}(\mu)$ can be obtained. In figure 2 we provide the plots of $H(\widetilde{\alpha}(\mu), \widetilde{\lambda}(\mu), \mu)$, as a function μ.

The shaded region of the graph in the first row has been magnified in the second row for a better perception of the maximum. For comparison purposes, we have also graphed the profile log-likelihood function in the second column denoted as $L^*(\mu, \widehat{\alpha}(\mu), \widehat{\lambda}(\mu))$. It is clear that $H(\widetilde{\alpha}(\mu), \widetilde{\lambda}(\mu), \mu)$ is an uni-modal function of μ, where as $L^*(\mu, \widehat{\alpha}(\mu), \widehat{\lambda}(\mu))$ is an increasing function of μ.

Pursuing the afore mentioned process, the partially maximised log-PS profile of a random sample of size 30, generated from $\alpha = 0.9$, $\lambda = 2$, $\mu = 1.5$ is graphed in the first column of figure 2.

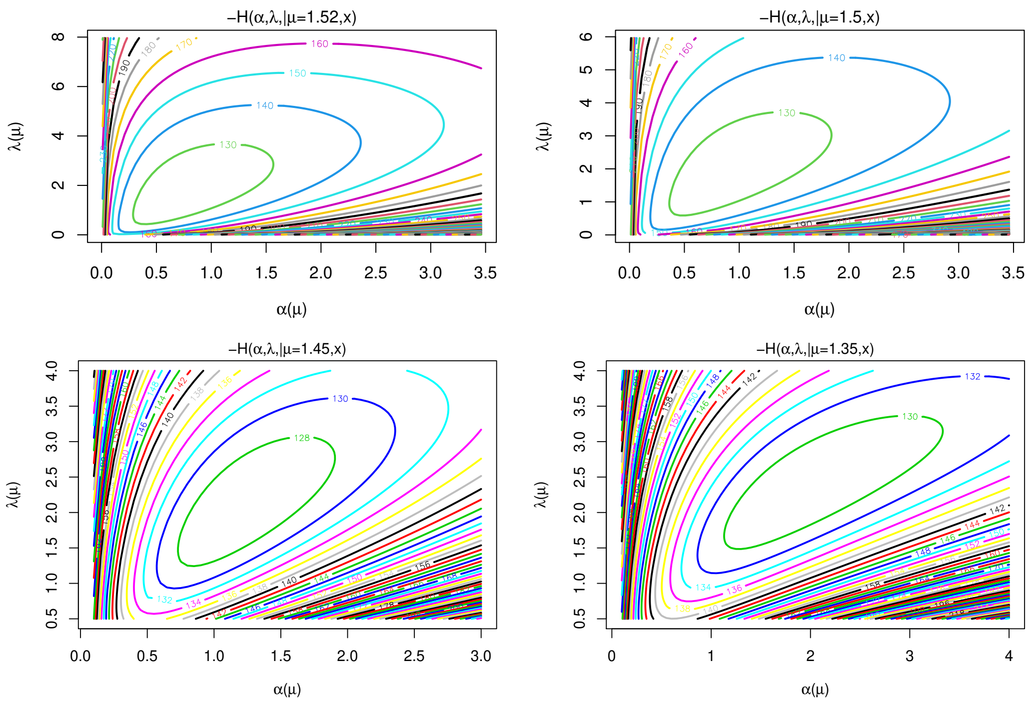

Figure 1: Contour plot of $-H(\alpha, \lambda|\mu)$ based on data generated from $\alpha = 0.9$, $\lambda = 2$ and $\mu = 1.5$

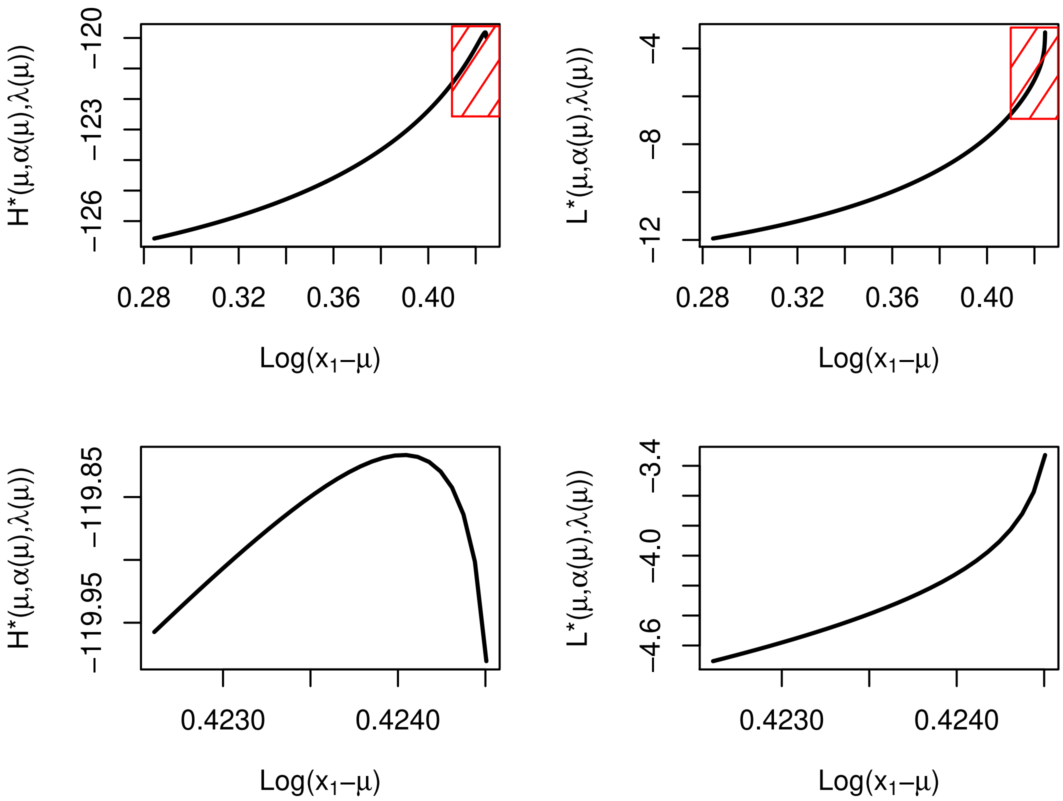

Figure 2: Graphs of partially maximized log-PS and log-likelihood function generated from $GED(\alpha = 0.9, \lambda = 2, \mu = 1.5)$

Optimization Algorithm

For a given sample from density given in eq.1.2, the following steps will furnish the MPS estimates:

Step 1: Obtain an initial estimate of μ, say $\widehat{\mu}_0$, using the observed value of the smallest order statistic.

Step 2: Construct a contour plot of $H(\alpha, \lambda|\widehat{\mu}_0)$ [1] for a given range of (α, λ). Neighbourhood of modified maximum likelihood estimates of (α, λ), elaborated in section 4 may be

[1]Any value smaller than x_1 can also be used as $\widehat{\mu}_0$

used for the purpose.

Step 3: Construct the Profile PS of $H(\widetilde{\alpha}(\mu), \widetilde{\lambda}(\mu), \mu)$ with respect to μ to confirm unimodality of the same.

Step 4: Optimize the partially maximized $H(\widetilde{\alpha}(\mu), \widetilde{\lambda}(\mu), \mu)$ with respect to μ to obtain $\widetilde{\mu}$, wherein $\widehat{\mu}_0$ is used as an initial guess for the adopted optimization routine[2].

Step 5: Optimize[2] $H(\alpha(\widetilde{\mu}), \widetilde{\lambda}(\mu), \widetilde{\mu})$, thereby using $(\widehat{\alpha}_0, \widehat{\lambda}_0)$ obtained from the contour plot in step 2 as the initial guess to obtain the final estimates $\left(\widetilde{\alpha}(\widetilde{\mu}), \widetilde{\lambda}(\widetilde{\mu})\right)$.

4 Asymptotic distribution and simulation study

In this section, we present an analytical study of the asymptotic distribution of the proposed MPS estimators. Theorem 1 (see appendix) provides the asymptotic distribution of the estimators and the simulation study in this section attempts to substantiate the results of the theorem. Since for $\alpha \geq 1$ consistent ML estimators are already confirmed to exists, the simulation study is restricted to $\alpha < 1$, where they are non-existent. The asymptotic distribution of shape and scale parameters is expected to be normal and thus, detailed analysis of the same has been excluded from further demonstration through simulation.

Nonetheless, to provide a clear insight to the distribution of $\tilde{\mu}$, we have graphed $n^{(1/\alpha)}(\tilde{\mu}-\mu)$ for different sample sizes to verify or nullify its underlying distribution to be normal for a particular sample size. Figure 3 provides the empirical density and histogram plots of $n^{(1/\alpha)}(\tilde{\mu}-\mu)$ along-with their mean and variances based on 1500 simulations. Evidently, with increase in sample size n, the means did not stabilize to 0, although the deviations are quite small and reduced further with increased simulations. Another striking feature which can be clearly discerned from the graph is that $n^{(1/\alpha)}(\tilde{\mu}-\mu)$ is not distributed as normal. For small and moderate sample sizes ranging from $30-50$, the quantity under study exhibited a negatively skewed plot with a nearly symmetric curve for sample size 80 and 100, but on increasing the sample size further, the plot manifested a positively skewed

[2] *nlm* in R

nature. Thereafter, normality tests were conducted using K-S statistic and Q-Q plot by assuming the observed mean and variance of $n^{(1/\alpha)}(\tilde{\mu} - \mu)$ as the true mean γ and variance σ^2 respectively for a particular n and these tests were found to corroborate non-normality of $n^{(1/\alpha)}(\tilde{\mu} - \mu)$.

It is worthwhile to mention that $\sqrt{n}(\tilde{\alpha} - \alpha)$ and $\sqrt{n}(\tilde{\lambda} - \lambda)$ were noted to be distributed as normal variates with mean and variance specified in Theorem 1. But, the convergence of α to its asymptotic distribution was observed to be dependent on the sample size as well as the magnitude of α, unlike λ which seemed to be indifferent otherwise. In order to reflect a smooth 45° line in Q-Q plot of $\sqrt{n}(\tilde{\alpha} - \alpha)$, larger sample size was required for higher values of α. However, the sample size required to achieve the asymptotic distribution need not be made indefinitely large since it is very likely to stabilise to a distinct value for all α[3]. In this article, even though we provide a detailed layout of consistency and asymptotic normality of the proposed estimators, yet, several other properties of the MPS estimators remains unexplored and may be studied further.

Since the sample size is believed to have significant effect on the estimators, we compare the proposed estimators based on small, moderate and fairly large ones. For each of these sample sizes, 1000 simulations are conducted for varying choice of $\alpha = (0.50, 0.75)$ with fixed values of $\lambda = 0.8$ and $\mu = -1.5$. Without loss of generality, λ and μ should be set as 1 and 0 respectively. Despite that, any arbitrary set of values of these parameters does not alter their general behaviour and hence may be used as well.

Each simulation was initiated by generating a random sample of a given size using inverse CDF method. To obtain the product of spacings estimate, we employ non-linear optimization technique in **R**[4]. The choice of guess values for α, λ and μ were set as the original values of the parameters from which the sample was generated. Furthermore, for constructing the confidence intervals, based on theorem 1 for MPS estimators, parametric bootstrap technique has been opted.

For the comparative study, the Location-Scale Parameter free (LSPF) method, proposed

[3] The Q-Q plots for this result has been avoided to prevent superfluous length of the article.

[4] (specifically *nlminb* function)

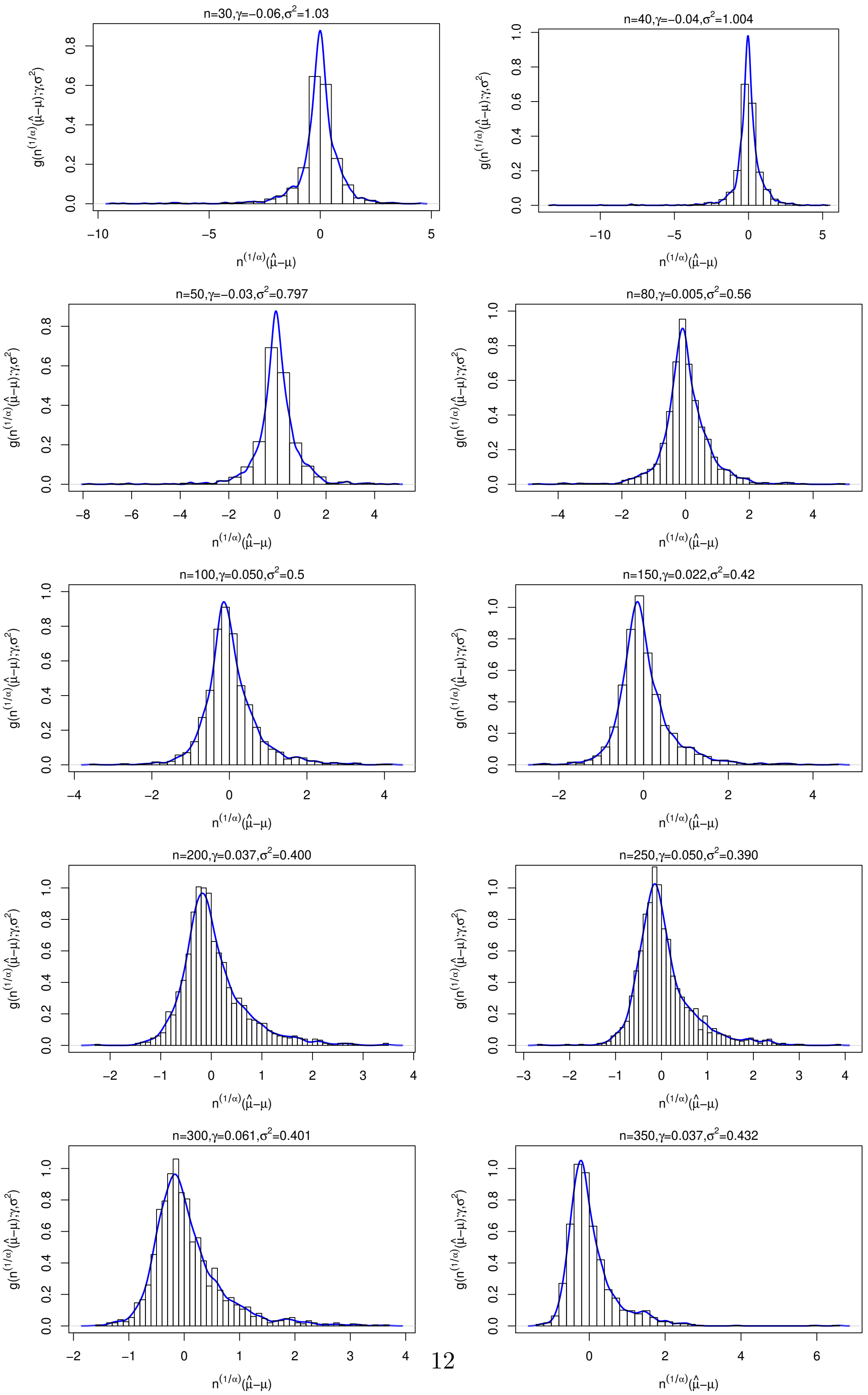

Figure 3: Density and histogram plot of $n^{(1/\alpha)}(\hat{\mu}-\mu)$ based on data generated from $\alpha = 0.9$, $\lambda = 2$ and $\mu = 1.5$

by Nagatsuka and Balakrishnan [2013] and extensively explored for GED by Prajapat et al. [2021] is evaluated and reported. Also, the most popular modified maximum likelihood method (denoted as MMLE) is evaluated, where the lowest order statistic is used as an estimator for the location parameter, thereby, substituting it in the considered density function and successively maximising the two dimensional log-likelihood function of (α, λ), based on $(n-1)$ data points, like a usual maximum likelihood problem. Note that, to construct confidence intervals of LSPF and MMLE, parametric bootstrap method is adopted.

Conspicuously, from tables 1 - 3, some important findings of the simulation study are:

- The estimators considered herein are consistent in terms of the decreasing RMSE with the increase in sample size for a given α, λ, μ.
- In most of the cases, RMSE's of MPS estimators are less than that of the other estimators.
- Almost for all the considered combinations, lengths of confidence intervals of the MPS estimators are smaller than the other considered estimators.
- For nearly all hypothetical combinations of simulation study, even-though the coverage percentages are close to the nominal level (95%) yet, they are lower than the same for some cases.

5 Real data illustration

The proposed methodology is implemented on two real data sets, in addition to the other methods discussed in this article. The first data set represents failure times of the air-conditioning system of an air-plane (reported in [Linhart and Zucchini, 1986, page 69]). This data was found suitable to be analysed using $2-$parameter GED (obtained as a special case of $3-$parameter GED by setting the origin as 0) by Gupta and Kundu [2001a]. ML estimate of the shape parameter was found to be less than 1. It is expected that the MPS estimate of the shape parameter under the assumption of $3-$parameter GED shall be quite

Table 1: Average estimate, RMSE, Lengths of C.I. & CP of $\widetilde{\alpha}$ against varying n and α for $\lambda = 0.8$, $\mu = -1.5$

n		α = 0.50			α = 0.75		
		MPS	LSPF	MMLE	MPS	LSPF	MMLE
20	$\widetilde{\alpha}$	0.4792	0.5886	0.6541	0.8287	0.9478	0.9392
	RMSE	0.2450	0.1817	0.2732	0.1735	0.6998	0.3851
	L	0.8064	1.4229	1.0481	0.6709	3.3057	1.5068
	CP	0.9060	1.0000	0.9600	0.9760	0.8500	0.9900
30	$\widetilde{\alpha}$	0.4679	0.5759	0.6106	0.7235	0.8173	0.8807
	RMSE	0.1252	0.1433	0.1859	0.2609	0.2847	0.2753
	L	0.4586	0.7409	0.7221	1.2004	1.8178	1.0122
	CP	0.9050	1.0000	0.9940	0.8660	0.8400	0.9960
50	$\widetilde{\alpha}$	0.4803	0.5472	0.5626	0.7243	0.7984	0.8287
	RMSE	0.0891	0.1085	0.1152	0.1775	0.1765	0.1735
	L	0.3270	0.3661	0.4587	0.6301	0.8062	0.6709
	CP	0.9040	1.0000	0.9880	0.8700	0.8900	0.9760

Table 2: Average estimate, RMSE, Lengths of C.I. & CP of $\widetilde{\lambda}$ against varying n and α for $\lambda = 0.8$, $\mu = -1.5$

n		α					
		0.50			0.75		
		MPS	LSPF	MMLE	MPS	LSPF	MMLE
20	$\widetilde{\lambda}$	0.7028	1.0830	1.0141	0.8694	1.0372	0.9679
	RMSE	0.3040	0.4949	0.4846	0.1974	0.5068	0.3918
	L	1.1241	1.1803	1.7747	0.7250	1.3051	1.4131
	CP	0.9320	0.8700	0.9360	0.9550	0.8400	0.9710
30	$\widetilde{\lambda}$	0.7511	1.0772	0.9345	0.7371	0.9193	0.9015
	RMSE	0.2828	0.3638	0.3231	0.2578	0.2318	0.2722
	L	0.9200	0.8942	1.2666	0.8011	0.8474	1.0098
	CP	0.8960	0.8300	0.9620	0.8720	0.8700	0.9550
50	$\widetilde{\lambda}$	0.7557	1.0079	0.9014	0.7619	0.9497	0.8694
	RMSE	0.1979	0.2436	0.2452	0.1852	0.2164	0.1974
	L	0.6883	0.5589	0.9082	0.6192	0.5883	0.7250
	CP	0.8960	0.7700	0.9640	0.8820	0.8200	0.9550

Table 3: Average estimate, RMSE, Lengths of C.I. & CP of $\widetilde{\mu}$ against varying n and α for $\lambda = 0.8$, $\mu = -1.5$

n		α					
		0.50			0.75		
		MPS	LSPF	MMLE	MPS	LSPF	MMLE
20	$\widetilde{\mu}$	-1.5011	-1.5066	1.0141	-1.4921	-1.5196	-1.4735
	RMSE	0.0262	0.0154	2.5515	0.0130	0.0525	0.0434
	L	0.0848	0.1367	0.0691	0.0483	0.3122	0.1612
	CP	0.9300	0.9700	0.9480	0.9530	0.9500	0.9530
30	$\widetilde{\mu}$	-1.4993	-1.5029	0.9345	-1.5001	-1.5104	-1.4851
	RMSE	0.0053	0.0066	2.4521	0.0306	0.0244	0.0251
	L	0.0246	0.0528	0.0348	0.1600	0.1771	0.0976
	CP	0.8720	0.9900	0.9820	0.9040	0.9800	0.9620
50	$\widetilde{\mu}$	-1.4997	-1.5006	0.9014	-1.5001	-1.5039	-1.4921
	RMSE	0.0019	0.0032	2.4118	0.0135	0.0109	0.0130
	L	0.0082	0.0149	0.0116	0.0600	0.0712	0.0483
	CP	0.9000	0.9900	0.9880	0.8940	1.0000	0.9530

close to that of its 2−parameter counterpart. Under such situation, employing PS method to obtain the estimates is a necessity as ML method shall quite likely fail. Keeping in mind the tied observations in the data set, PS method is executed in stages as explained earlier in section 2. Adopting the algorithm proposed in subsection 3, $\widehat{\mu}_0 = 1$. Thereafter, contour plots of the partially maximized PS function for a given series of location values in figure 4, demonstrates existence of a unique maxima in all the cases. Subsequently, steps 3 (elaborated in figure 4), 4 and 5 (demonstrated in figure 5) of the algorithm yields $\left(\widetilde{\alpha}(\widetilde{\mu}), \widetilde{\lambda}(\widetilde{\mu})\right)$, which are comprehensively reported in table 4. It is clearly discernible from figure 5, that unique maxima exists for MPS method whereas the ML method fails to provide the same. The point estimates (denoted as $\widetilde{\Theta} = (\widetilde{\alpha}, \widetilde{\lambda}, \widetilde{\mu})$ in the table) of the parameters along with their 95% confidence intervals are reported in table 4.

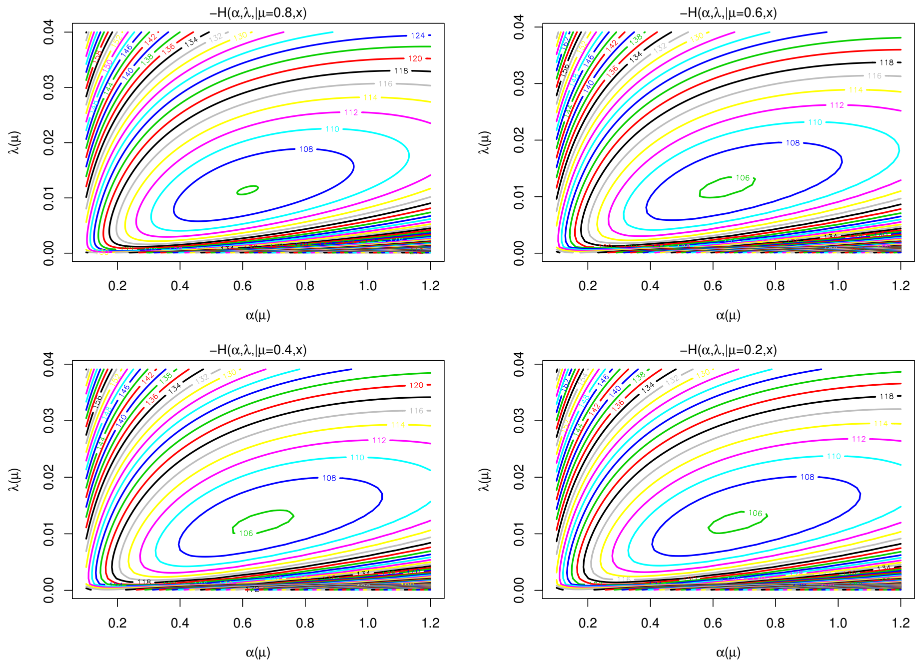

Figure 4: Contour plot of $-H(\alpha, \lambda|\mu)$ for air-condition failure times data

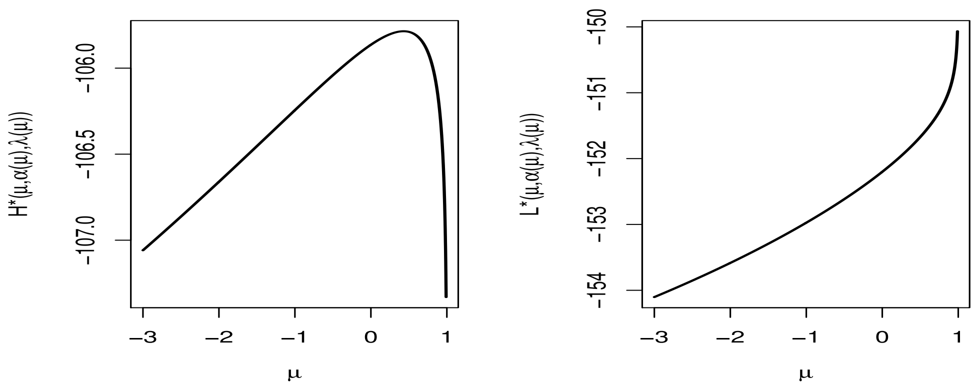

Figure 5: Profile of log-PS and log-likelihood function of air-condition failure times data

Table 4: Point estimates and 95% confidence intervals based on 200 bootstrap samples for air-condition failure data

Method		Shape	Scale	Location
MPS	$\widetilde{\Theta}$	0.656	0.012	0.430
	CI	(0.348, 0.992)	(0.005,0.021)	(0, 4.171)
LSPF	$\widetilde{\Theta}$	0.762	0.014	0.044
	CI	(0.518, 1.447)	(0.009, 0.025)	(0, 3.100)
MMLE	$\widetilde{\Theta}$	0.849	0.015	1.000
	CI	(0.328, 1.180)	(0.003, 0.021)	(0, 2.237)

Furthermore, the Chi-square statistic for the 3− parameter GED is found to have a smaller value ($\chi^2 = 2.44$) as compared to its 2−parameter counterpart[5]. The observed and expected frequencies[6] are given in table 5.

The second data set is the pollution level (measured in number of coliform per 100 ml) recorded for 20 days over a 5-week period in a survey of beach pollution in South Wales. This data set was analysed by Cheng and Amin [1983] using location shifted Weibull distribution.

[5]Gupta and Kundu [2001a] reported $\chi^2 = 3.383$

[6]distribution fitted based on MPS estimates

Table 5: Observed and expected frequencies using GED and EE distribution

Intervals	Observed	GED	EE
0 - 15	11	8.98	7.97
15 - 30	5	4.53	4.91
30 - 60	3	5.71	6.44
60 - 100	6	4.38	4.84
100 -	5	5.53	5.84

The likelihood function failed to furnish a distinguished maxima owing to the underlying shape parameter being less than 1, yet, estimates were successfully derived through MPS. Since Weibull and GED exhibits very marginal differences in model fitting, one may use GED to explore this data set. With the assumption of GED to be the underlying model governing this data, the partially maximized log-likelihood function for a series of location values led to an unbounded situation whereas the partially maximized log-PS function, adjusted against repeated observations, depicted a uni-modal curve as manifested in fig.7 and the contour plot of the same reveals existence of unique mode for varying μ (see fig.6). For computational simplicity, the data was divided by 1000, thereafter, the algorithm proposed in subsection 3 is implemented to the rescaled observations to obtain the corresponding MPS estimates as observed from the overall maxima of $H^*(\mu, \alpha(\mu), \lambda(\mu))$ (reported in table 6), along-with estimates obtained from all the other methods explored in this article.

6 Conclusion

The procedures administered in this article may be used for other similar non-regular distributions as well. The MPS estimators are consistent even for situations where maximum likelihood estimators fails to exist. Under such situations of non-regularity an attempt has been made to derive the asymptotic distribution of the MPS estimators. A detailed discussion to overcome the computational hurdles of implementing MPS technique to any location

Table 6: Point estimates and 95% confidence intervals based on 200 bootstrap samples for beach pollution data

Method		Shape	Scale	Location
MPS	$\widetilde{\Theta}$	0.925	0.143	1.094
	CI	(0.458, 2.365)	(0.100, 0.251)	(0, 2.065)
LSPF	$\widetilde{\Theta}$	1.118	0.170	0.974
	CI	(0.575, 4.080)	(0.090, 0.366)	(0, 2.051)
MMLE	$\widetilde{\Theta}$	1.220	0.181	1.364
	CI	(0, 1.864)	(0.040, 0.272)	(0.165, 1.822)

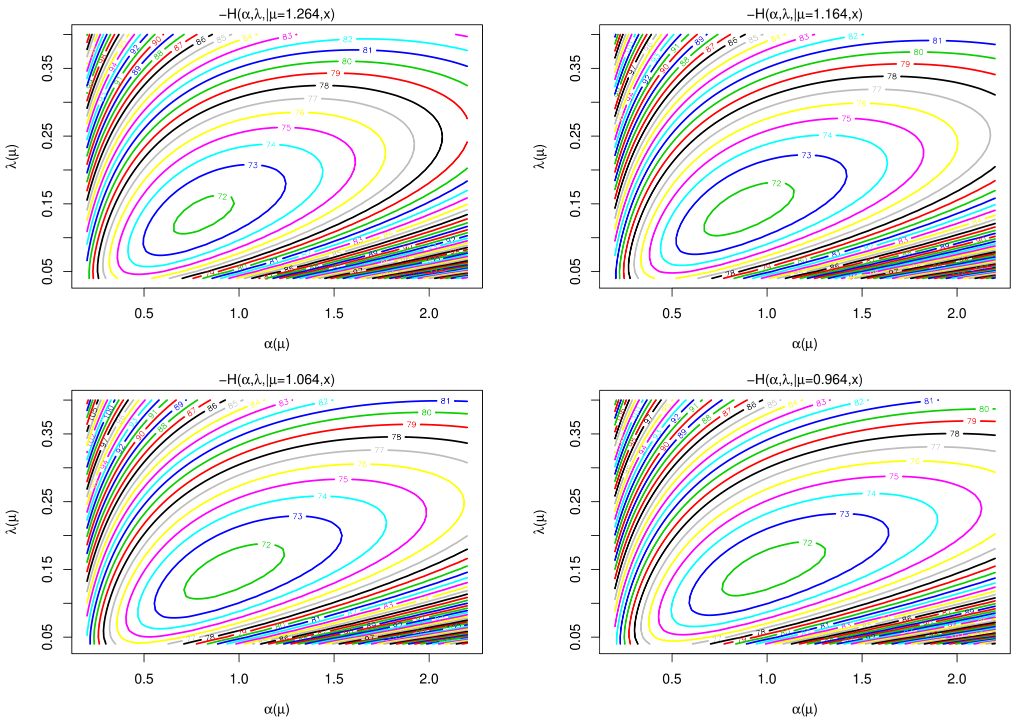

Figure 6: Contour plot of $-H(\alpha, \lambda|\mu)$ for pollution level data

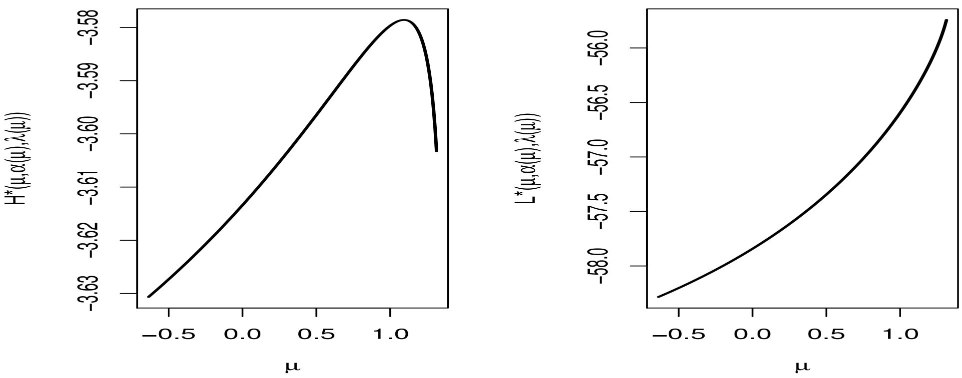

Figure 7: Profile of Likelihood and PS function of flood level data

shifted model has been provided. Since the GED has an explicit CDF, PDF and hazard function, it would be interesting to study this model under various censoring conditions in the restricted parameter space.

Appendix

Note that, to derive exact confidence intervals, the distribution of the MPS estimators are required and because of implicit nature of the MPS estimators, exact distribution cannot be obtained. In addition to the implicit nature, the distribution has finite information matrix provided $\alpha > 2$ and belongs to the non-regular family.

Let $\Theta^0 = (\alpha^0, \lambda^0, \mu^0)^\top$ be the true parameter vector, $\widehat{\Theta}$ be the ML estimator of Θ^0, $\widetilde{\Theta}$ be the MPS estimator of Θ^0 and $L = n^{-1}\sum_{i=1}^{n} \ln(f(x_i; \alpha, \lambda, \mu))$ denote the log-likelihood function. In case of the three-parameter GED, for $\alpha > 2$, the ML estimators exist (may be derived similar to three-parameter Gamma or Weibull in [Bain and Engelhardt, 1991, p. 284],[Johnson et al., 1994, p. 656]), they are consistent and asymptotically normally distributed. The existence of the ML estimators guarantees the existence of the MPS estimators and since Fisher information is finite for the above range of shape parameter, thence asymptotic properties of the MPS estimators can be obtained parallel to the ML asymptotes, i.e. $\widetilde{\Theta} = \widehat{\Theta} + O_p(n^{-1/2})$ (see Cheng and Stephens [1989], Anatolyev and Kosenok

[2005]). To derive the asymptotic properties of the MPS estimators for $0 < \alpha \leq 2$, we adopt the same method as in Smith [1985], Wong and Li [2006]. Before implementing the suggestion by Smith [1985], the following assumptions are to be met after expressing the density function in the given form. Thus, substituting $\lambda = 1$ in 1.2, $f(x; \mu, \alpha)$ may be expressed as:

$$f(x; \mu, \alpha) = (x-\mu)^{\alpha-1} g\,(x-\mu; \alpha) \qquad (\mu < x < \infty) \tag{A1}$$

where, $g\,(x-\mu; \alpha) = \alpha e^{-(x-\mu)} \left\{ \dfrac{1-e^{-(x-\mu)}}{x-\mu} \right\}^{\alpha-1}$.

To validate the assumptions of Smith [1985], let us simplify $g(x-\mu; \alpha)$ by expanding $e^{-(x-\mu)}$ using Taylor's series. Ignoring powers of order greater than 2 for $(x-\mu)$, $g(x-\mu; \alpha)$ reduces to

$$g(x-\mu; \alpha) \simeq \alpha e^{-(x-\mu)} \left\{ 1 - \frac{x-\mu}{2} \right\}^{\alpha-1}; \tag{A2}$$

Note that, the function $g(x-\mu; \alpha)$ converges to α as $x \to \mu$. Assumptions 1 and 2 of Smith [1985] may be validated easily. For assumption 3, we need to derive a suitable $H^*_\eta(x, \alpha)$. Consider, for some $\delta^* \geq 0$ and $\eta > 0$ with $\alpha_1; \alpha_2$ being two distinct values of α,

$$\begin{aligned} &\ln g(x-\mu_1; \alpha_1) - \ln g(x-\mu_2; \alpha_2) \\ &\approxeq \ln \alpha_1 - \ln \alpha_2 + \alpha_1 \ln \left\{ 1 - \frac{x-\mu_1}{2} \right\} - \alpha_2 \ln \left\{ 1 - \frac{x-\mu_2}{2} \right\} - \ln \left\{ 1 - \frac{x-\mu_1}{2} \right\} \\ &+ \ln \left\{ 1 - \frac{x-\mu_2}{2} \right\} + (\mu_1 - \mu_2) \end{aligned} \tag{A3}$$

pertaining to the given conditions $|\mu_1| < \eta$, $|\mu_2| < \eta$, $|\alpha_1 - \alpha_2| < \eta$; expanding $\ln \left\{ 1 - \dfrac{x-\mu_1}{2} \right\}$ and thereby ignoring power greater than 2, $H^*_\eta(x, \alpha)$ may be chosen as ηx. The chosen $H^*_\eta(x, \alpha)$ satisfies the assumption 3. For assumptions 4 and 5, consider $\alpha \in (0, \eta]$ and $\mu \in [-\eta, \eta]$, the same may be established. Furthermore, since $\ln g$ and its partial derivatives are bounded in x, for every α, hence the other assumptions may be also verified.

Theorem 1. *The MPS estimates of the parameters in the considered GED is the solution of the equation $\partial H / \partial \Theta = 0$; where H is defined in eq.2.2.*

(a) *For $\alpha > 2$, the maximum product of spacings estimators $\tilde{\Theta}$ converges to Θ in probability and $\sqrt{n}(\tilde{\Theta} - \Theta)$ is distributed as an asymptotic normal distribution with mean vector* ***0*** *and variance-covariance matrix I^{-1}; where*

$$I = -\frac{1}{n}\begin{bmatrix} E(\frac{\partial^2 L}{\partial \alpha^2}) & E(\frac{\partial^2 L}{\partial \alpha \partial \lambda}) & E(\frac{\partial^2 L}{\partial \alpha \partial \mu}) \\ E(\frac{\partial^2 L}{\partial \lambda \partial \alpha}) & E(\frac{\partial^2 L}{\partial \lambda^2}) & E(\frac{\partial^2 L}{\partial \lambda \partial \mu}) \\ E(\frac{\partial^2 L}{\partial \mu \partial \alpha}) & E(\frac{\partial^2 L}{\partial \mu \partial \lambda}) & E(\frac{\partial^2 L}{\partial \mu^2}) \end{bmatrix} \tag{A4}$$

The explicit expressions involved in the matrix (A4) can be obtained from Gupta and Kundu [1999].

(b) *For $0 < \alpha \leq 2$, $\tilde{\Theta}$ converges to Θ in probability where*

$$\tilde{\mu} - \mu \to O_p(n^{-1/\alpha}) \tag{A5}$$

and $(\tilde{\alpha}, \tilde{\lambda})' = \tilde{\varphi}$, say, is distributed as asymptotic normal, i.e.

$$\sqrt{n}(\tilde{\varphi} - \varphi) \xrightarrow{D} N(0, -E(\partial^2 L/\partial \varphi^2)^{-1}) \tag{A6}$$

Proof : The proof can be obtained along the line of theorem 1 in Smith [1985] and implementing the same as suggested by Cheng and Amin [1983], Wong and Li [2006].

The theorem indicates that for $\alpha > 2$ the MPS estimators exhibit regular behaviour with the asymptotic variances are of the order n^{-1}. However, for $\alpha \leq 2$, the location estimator is hyper-efficient with the asymptotic variance of order less than n^{-1}. The asymptotic distribution of $(\widetilde{\alpha}, \widetilde{\lambda})'$ becomes independent of μ and follows a bivariate normal distribution with variance-covariance matrix obtained from the upper 2×2 sub-matrix of I in (A4), similar to the three-parameter Weibull and gamma distributions.

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