A NOVEL TWO-SAMPLE JOINT UNIFIED HYBRID CENSORING SCHEME WITH THE APPLICATION OF INSULATING FLUID DATA

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Abstract

In life testing tests, a variety of censoring schemes are employed, primarily Type-I and Type-II censoring schemes and their modified forms. In life testing experiments, the majority of the tests are based on a single sample. When conducting comparative life tests of products from different production lines within the same facility, joint censoring schemes are quite useful. In reliability and survival analysis, joint censoring schemes have gained considerable interest over the last decade in comparing several lifetime experiments. In this article, a novel joint unified hybrid censoring scheme has been proposed for two sample populations. Based on the assumptions that the lifetime distributions of the two populations follow the Weibull distribution with the same shape parameter and different scale parameters, we provide the maximum likelihood estimators of the unknown parameters. The asymptotic confidence intervals for the parameters have been constructed using the observed Fisher information matrix. Further, the Bayes estimates have been derived using informative gamma priors under symmetric and asymmetric loss functions. To obtain the Bayes estimates and associated highest posterior density credible intervals, the Markov chain Monte Carlo method has been employed. A Monte Carlo simulation study has been carried out to compare the performance of the proposed estimates. Finally, a real-life data set has been analyzed to demonstrate the utility of the presented techniques in investigating such phenomena.

Keywords: Joint unified hybrid censoring scheme; Maximum likelihood estimation; Bayesian estimation; MCMC; HPD credible interval; Weibull distribution.

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1 Introduction

In modern industrial systems, a corporation that is flexible may modify its capacity and resources to produce different products in response to changing market conditions and diverse client demands. In the several fields of industries the product comes from more than one production lines, which is required to work the comparative life tests. This problem requires sampling products from the different production lines. When a researcher wants to do a life-test quality comparison study on products created by different production lines within the same institution, joint censoring procedures are especially useful. The customer satisfaction is one of the first objectives of a company. To be more accurate, suppose manufactured products are taken from two production lines with the same conditions. Then study looked at the two-sample formulation of quality check to save time and money. Two sample joint censoring schemes are becoming very popular to deal with monitoring and controlling quality of the products in industrial process control. In recent years, different type of joint censoring schemes have been proposed in statistical literature for monitoring the data, for example see, Balakrishnan and Rasouli [1], Rasouli and Balakrishnan [2], Balakrishnan et al. [3], Su and Zhu [4], Ashour and Abo-Kasem [5], Mondal and Kundu [6], Mondal and Kundu [7], Abo-Kasem and Elshahhat [8] and the references cited therein.

Several types of monitoring data are available. One is the censoring scheme, which is a popular problem in life testing experiments. When conducting trials on reliability and life testing, the investigators want to monitor the breakdown times of the units being tested that are put under evaluation. An experiment has been terminated before recording all the failure times because of time constraints, financial constraints, or other unavoidable factors. The results of these kinds of tests are referred to as censored data. Type I censoring and Type II censoring are the two most prevalent censoring techniques taken into account in life testing. In case of Type I censoring scheme, a small number of failures may be observed. However, in Type II censoring the time period may become large to get the intended quantity of failures. To overcome such drawbacks, a mixture of Type I and Type II censoring scheme was proposed by Epstein [9], known as Type I hybrid censoring scheme (HCS). Type I HCS also have the disadvantage as similar as Type I censoring scheme. To overcome this, Childs et al. [10] proposed Type II HCS to get sufficient number of failures. The generalized hybrid censoring schemes have been proposed by Chandrasekar et al. [11]. Unified hybrid censoring scheme (UHCS) was proposed by Balakrishnan et al. [12] which ensures us to get at least k number of failures.

Most of the work under censoring schemes has been focused on how to cope with a scenario where just a single sample is taken into account. However, in many circumstances, a comparison between two samples can be found necessary for researchers. Therefore, it is highly advantageous to compare the lifetime distribution of products originating from distinct units that are created on two distinct production lines in an identical facility by using a joint censoring scheme. In the case of the joint censoring scheme (JCS), two samples of products have been put on an experiment at the same time where m and n number of products have been taken from population 1 (Pop-1) and population 2 (Pop-2), respectively. The lifetime experiment may be called off by the experimenter after getting a prefixed number of failures because of time and cost constraints. Balakrishnan and Rasouli [1] proposed Type II JCS and obtained the exact inference based on two exponentially distributed populations. Rasouli and Balakrishnan [2] discussed exact inference for two exponentially distributed populations under the Type II joint progressive censoring scheme (JPCS). Balakrishnan et al. [3] discussed exact inference under JPCS for several exponential populations. Su and Zhu [4] discussed point and interval estimates for two populations based on the Type I joint generalized hybrid censoring scheme (JGHCS). Ashour and Abo-Kasem [5] discussed statistical inference for two exponential distributions based on a Type I joint progressive hybrid censoring scheme (JPHCS). Mondal and Kundu [6] discussed Type II JPCS for the exponential distribution. Mondal and Kundu [7] discussed the statistical inference for two Weibull populations under the Type II balanced joint progressive censoring scheme (BJPCS). Abo-Kasem and Elshahhat [8] discussed statistical inference for two Weibull populations under Type II JGHCS.

Since UHCS ensures us to get at least k number of failures, it is more efficient than the other hybrid censoring schemes. Similarly, joint unified hybrid censoring scheme (JUHCS) will be more efficient than the other joint hybrid censoring schemes. To the best of our knowledge, JUHCS has not been proposed yet in the statistical literature. To present two sample joint unified hybrid censoring schemes (JUHCS), we first need to understand how UHCS works. Let us consider n number of elements tested in an experiment. Here, k and m ($k < m \leq n$) are two prefixed integers, and T_1 and T_2 ($T_1 < T_2$) are considered as pre-fixed time constraints in this scheme. Based on UHCS, six different cases are observed as follows:

Case A: If $0 < X_{k:n} < X_{m:n} < T_1 < T_2$, then $T^* = T_1$,

Case B: If $0 < X_{k:n} < T_1 < X_{m:n} < T_2$, then $T^* = X_{m:n}$,

Case C: If $0 < X_{k:n} < T_1 < T_2 < X_{m:n}$, then $T^* = T_2$,

Case D: If $0 < T_1 < X_{k:n} < X_{m:n} < T_2$, then $T^* = X_{m:n}$,

Case E: If $0 < T_1 < X_{k:n} < T_2 < X_{m:n}$, then $T^* = T_2$,

Case F: If $0 < T_1 < T_2 < X_{k:n} < X_{m:n}$, then $T^* = X_{k:n}$,

where $X_{i:n}$ denotes the time of i-th failure unit and T^* denotes the termination time of the experiment. Note that,

- UHCS $\rightarrow$ Generalized Type I HCS, as $T_1 \rightarrow 0$.
- UHCS $\rightarrow$ Generalized Type II HCS, as $k \rightarrow 0$.
- UHCS $\rightarrow$ Type I HCS, as $T_1 \rightarrow 0$ and $k \rightarrow 0$.
- UHCS $\rightarrow$ Type II HCS, as $T_2 \rightarrow \infty$ and $k \rightarrow n$.

In recent years, substantial work on statistical inference based on UHCS has been done by several researchers in the statistical literature. One may refer to Panahi and Sayyareh [13], Ateya [14], Panahi [15], Sen et al. [16], Dutta and Kayal [17], and Dutta et al. [18]. Another advantage of proposing the joint unified hybrid censoring scheme (JUHCS) is that it will take a smaller expected test time to observe that k no. of failures compared to UHCS which is shown in Section 6.

The Weibull distribution has gained acceptance as an efficient model in life-testing analysis because of the variability in the form of the hazard rate function and the different shapes of the probability density function (PDF). Tadikamalla [19] discussed the use of Weibull distribution in inventory control to approximate the lead time demand in several cases of distributed demands coupled with distributed lead times. Velazquez et al. [20] used a windmill as an application to the Weibull distribution to estimate the volume of the water pumping. Azad et al. [21] used the Weibull distribution to examine wind speed data from three possible locations in Australia. Mahmood et al. [22] used the Weibull distribution to assess the wind speed data characteristics at the Al-Salman site in Iraq. Motivated by the applicability of Weibull distribution in real life scenarios, it has been chosen in this current study.

Motivated by the existing joint censoring models in literature a novel joint censoring scheme, named JUHCS has been proposed in this paper. The main highlights of the work is given below:

- The primary goal of this study is to derive point and interval estimates of the parameters when data has been observed from two Weibull populations with common shapes and different scale parameters under JUHCS based on both frequentist and Bayesian approaches.

- It has been observed that the maximum likelihood estimate (MLE) of the unknown parameters can not be obtained explicitly. An iterative method is applied to derive the MLEs.

- Approximate confidence intervals (ACIs) for the unknown model parameters have been constructed based on the asymptotic normality properties of the MLEs.

- Bayes estimates of the unknown model parameters have been obtained under squared error loss function (SELF) and LINEX loss function (LLF) with dependent and independent prior assumptions. Further, the Markov chain Monte Carlo (MCMC) method has been employed to obtain Bayes estimates and the highest posterior density (HPD) credible intervals.

- The performance of the proposed estimates has been compared using a Monte Carlo simulation study. Mean squared error (MSE) and absolute bias (AB) have been used to evaluate the precision of various point estimators.

- A comparison of expected test time of JUHCS and UHCS has been incorporated.

- The data set, reported by Nelson [23], which is associated with breakdown times of an insulating fluid due to high voltage stress has been analyzed for illustrative purposes.

The rest of the article has been continued with the following structure. In Section 2, the model has been presented together with the underlying assumptions. The MLEs of the unknown model parameters and the associated ACIs have been provided in Section 3. In Section 4, the Bayes estimates and the associated HPD credible intervals are constructed. Section 5 deals with an extensive Monte Carlo simulation study to compare the performance of the proposed estimates. A comparison of expected test time is done in section 6. A real-life data set has been analyzed for illustrative purposes in Section 7. Finally Section 8 ends with some conclusive remarks.

2 Model description

Assume that two samples are considered from two populations of engineering components, and are included in a life-testing trial having two simultaneously working lines, say L-I and L-II. Further, assume that n_1 components are placed in the first line and n_2 are placed in the second line, such that $n_1 + n_2 = N$. The lifetimes of the components in L-I (L-II) are independent and identically distributed with PDF $f_1(\cdot)(f_2(\cdot))$ and cumulative distribution function (CDF) $F_1(\cdot)(F_2(\cdot))$. Consider $k,\ m \in \{1,\ 2,\ \cdots,\ N\}$ and $T_1,\ T_2 \in (0,\ \infty)$ such that $k < r$ and $0 < T_1 < T_2$ as some prefixed numbers determined by the product's reliability information. Let the first failure take place at time $U_{(1)}$. This failure may occur either from L-I or from L-II. Further, assume that the second failure occurs at time $U_{(2)}$. Like the first failure, the second failure may occur either from L-I or from L-II, and so on. The schematic diagram of the JUHCS scheme for Case A is given in Figure 1. For JUHCS, six different cases are given below:

$$\text{Case A: If } 0 < U_{(k)} < U_{(m)} < T_1 < T_2, \text{ then } \mathbf{U} = \{U_{(1)}, \cdots, U_{(D_1)}\} \text{ and } T^* = T_1,$$

$$\text{Case B: If } 0 < U_{(k)} < T_1 < U_{(m)} < T_2, \text{ then } \mathbf{U} = \{U_{(1)}, \cdots, U_{(m)}\} \text{ and } T^* = U_{(m)},$$

$$\text{Case C: If } 0 < U_{(k)} < T_1 < T_2 < U_{(m)}, \text{ then } \mathbf{U} = \{U_{(1)}, \cdots, U_{(D_2)}\} \text{ and } T^* = T_2,$$

$$\text{Case D: If } 0 < T_1 < U_{(k)} < U_{(m)} < T_2, \text{ then } \mathbf{U} = \{U_{(1)}, \cdots, U_{(m)}\} \text{ and } T^* = U_{(m)},$$

$$\text{Case E: If } 0 < T_1 < U_{(k)} < T_2 < U_{(m)}, \text{ then } \mathbf{U} = \{U_{(1)}, \cdots, U_{(D_2)}\} \text{ and } T^* = T_2,$$

$$\text{Case F: If } 0 < T_1 < T_2 < U_{(k)} < U_{(m)}, \text{ then } \mathbf{U} = \{U_{(1)}, \cdots, U_{(k)}\} \text{ and } T^* = U_{(k)}.$$

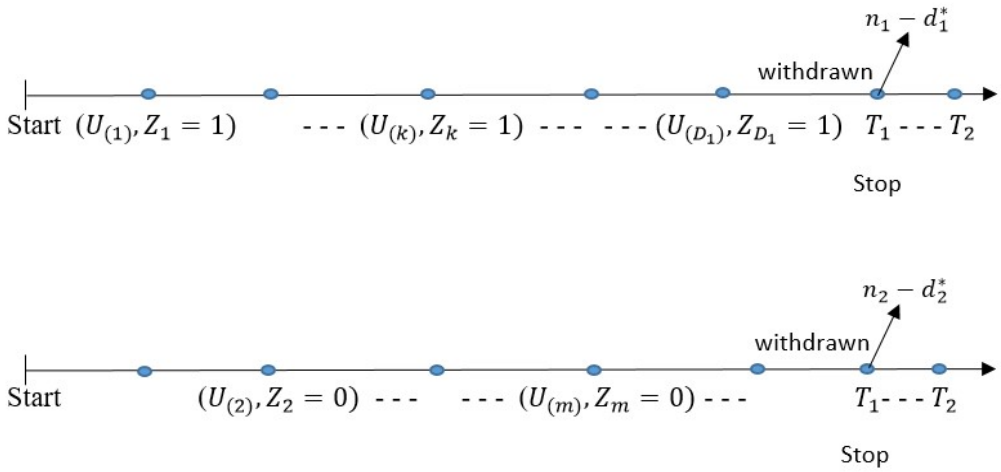

Figure 1: Schematic diagram of JUHCS for Case A.

where T^* is the termination time. Now, define

$$Z_i = \begin{cases} 1, & \text{if the i-th failure occurs from the first sample,} \\ 0, & \text{if the i-th failure occurs from the second sample.} \end{cases}$$

Thus, the random observations associated with JUHCS can be considered as $(\mathbf{Z}, \mathbf{U})$, where $\mathbf{Z} = (Z_1, \cdots, Z_{D^*})$, $\mathbf{U} = (U_{(1)}, \cdots, U_{(D^*)})$ and

$$D^* = \begin{cases} D_1, & \text{Case A,} \\ m, & \text{Cases B and D,} \\ D_2, & \text{Cases C and E,} \\ k, & \text{Case F.} \end{cases}$$

Here in JUHCS data $\mathbf{T}^*$ and $\mathbf{D}^*$ are also random quantities. Further, denote $d_1^* = \sum_{i=1}^{D^*} Z_i$ and $d_2^* = D^* - d_1^*$ for the number of failures occur from the first sample and second sample, respectively. Clearly, $D^* = d_1^* + d_2^*$. Thus, based on the above-described settings, the joint likelihood function can be written as

$$L \propto \prod_{i=1}^{D^*} \left[f_1(u_{(i)};\theta_1)\right]^{Z_i} \left[f_2(u_{(i)};\theta_2)\right]^{1-Z_i} \left[1 - F_1(T^*;\theta_1)\right]^{n_1-d_1^*} \left[1 - F_2(T^*;\theta_1)\right]^{n_2-d_2^*}, \tag{2.1}$$

where $w_{(i)}$ is the realization of $W_{(i)}$ and

$$T^* = \begin{cases} T_1, & \text{Case A,} \\ u_{(m)}, & \text{Cases B and D,} \\ T_2, & \text{Cases C and E,} \\ u_{(k)}, & \text{Case F.} \end{cases}$$

Using this above-described model in (2.1), various outcomes may be achieved as special instances, including the following:

- When $T_1 \to 0$, JUHCS $\to$ Type-I JGHCS. For more details, see Su and Zhu [4].
- When $k \to 0$, JUHCS $\to$ Type-II JGHCS. For more details, see Abo-Kasen and Elshahhat [8].
- When $T_1 \to 0$ and $k \to 0$, JUHCS $\to$ joint Type-I HCS. For more details, see Su [24].
- When $T_2 \to \infty$ and $k \to n$, JUHCS $\to$ joint Type-II HCS. For more details, see Su [24].
- When $T_1 \to 0$, $T_2 \to \infty$ and $k \to 0$, JUHCS $\to$ Type-I JCS.
- When $T_1 \to 0$, $T_2 \to \infty$ and $k \to n$, JUHCS $\to$ Type-II JCS. For more details, see Balakrishnan et al. [25].

Due to its flexible hazard function, the two-parameter Weibull distribution is often employed in reliability and survival analysis. This model has seen wide application for simulating a wide variety of lifetime models. For more details, one may refer to Dutta and Kayal [17]. Since comparable test facilities frequently display the same overall Weibull shape due to shared intrinsic failure causes, we assume a common shape parameter (see, Zhu [26]). Similar to Mondal and Kundu [27], it has been considered that the observed failures from L-I and L-II are following two different Weibull populations

$W(\gamma, \alpha_1)$ and $W(\gamma, \alpha_2)$, respectively, where γ is the common shape and α_1 and α_2 are different scale parameters. Then the corresponding CDF and PDF are given as

$$F_1(x; \gamma, \alpha_1) = 1 - \exp(-\alpha_1 x^\gamma),\ x > 0, \gamma, \alpha_1 > 0, \tag{2.2}$$

$$f_1(x; \gamma, \alpha_1) = \alpha_1 \gamma x^{\gamma-1} \exp(-\alpha_1 x^\gamma),\ x > 0, \gamma, \alpha_1 > 0, \tag{2.3}$$

and

$$F_2(x; \gamma, \alpha_2) = 1 - \exp(-\alpha_2 x^\gamma),\ x > 0, \gamma, \alpha_2 > 0, \tag{2.4}$$

$$f_2(x; \gamma, \alpha_2) = \alpha_2 \gamma x^{\gamma-1} \exp(-\alpha_2 x^\gamma),\ x > 0, \gamma, \alpha_2 > 0, \tag{2.5}$$

respectively.

3 Maximum likelihood estimation

In this section, the MLEs have been obtained under JUHCS for the unknown model parameters. Substituting (2.2) – (2.4) into (2.1), the likelihood function can be expressed as

$$\begin{aligned} L(\alpha_1, \alpha_2, \gamma|\mathbf{u}) \propto \alpha_1^{d_1^*} \alpha_2^{d_2^*} \gamma^{D^*} \prod_{i=1}^{d_1^*} u_{1i}^{(\gamma-1)} \prod_{i=1}^{d_2^*} u_{2i}^{(\gamma-1)} \exp\left(-\alpha_1 \sum_{i=1}^{d_1^*} u_{1i}^\gamma - \alpha_2 \sum_{i=1}^{d_2^*} u_{2i}^\gamma\right) \\ \times \exp\bigg(- (\alpha_1(n_1 - d_1^*) + \alpha_2(n_2 - d_2^*))T^*\bigg), \end{aligned} \tag{3.1}$$

where $\mathbf{u} = (u_1, \ \cdots, \ u_{D^*})$, u_{1i} is the observed failure from L-I and u_{2i} is the observed failure from L-II under JUHCS. Then the associated log-likelihood function can be written as

$$l^* = \log L \propto \ d_1^* \log \alpha_1 + d_2^* \log \alpha_2 + D^* \log \gamma + (\gamma - 1) \sum_{i=1}^{D^*} \log u_i - \alpha_1 \Phi_1(\gamma) - \alpha_2 \Phi_2(\gamma), \tag{3.2}$$

where $\Phi_1(\gamma) = \sum_{i=1}^{d_1^*} u_{1i}^{\gamma} + (n_1 - d_1^*)$ and $\Phi_2(\gamma) = \sum_{i=1}^{d_2^*} u_{2i}^{\gamma} + (n_2 - d_2^*)$. Differentiating (3.2) with respect to α_1 and α_2 and equating to zero, we get

$$\hat{\alpha}_1 = \frac{d_1^*}{\Phi_1(\gamma)} \quad \text{and} \quad \hat{\alpha}_2 = \frac{d_2^*}{\Phi_2(\gamma)}. \tag{3.3}$$

From (3.3), it has been observed that if d_1^* or d_2^* becomes zero then the MLE of $\hat{\alpha}_1$ or $\hat{\alpha}_2$ does not exist. Hence the MLEs of α_1 and α_2 for any given γ exist if $0 \leq d_1^* \leq D^* - 1$. Now, substituting $\hat{\alpha}_1(\gamma)$ and $\hat{\alpha}_2(\gamma)$ in (3.2), we get

$$l(\gamma) \propto D^* \log \gamma + \sum_{i=1}^{2} d_i^* \log \left(\frac{d_i^*}{\Phi_i(\gamma)} \right) + (\gamma - 1) \sum_{i=1}^{D^*} \log u_i - D^*, \tag{3.4}$$

which is informative in determining the MLE of γ, denoted as $\hat{\gamma}$.

Theorem 3.1. *$\hat{\gamma}$ exists uniquely when $d_i^* \geq 1$, for $i = 1,\ 2$ and can be obtained from $l'(\gamma) = 0$, where*

$$l'(\gamma) = \frac{D^*}{\gamma} + \sum_{i=1}^{D^*} \log u_i - \sum_{i=1}^{2} d_i^* \frac{\Phi_i'(\gamma)}{\Phi_i(\gamma)}, \tag{3.5}$$

where $\Phi_i'(\gamma) = \sum_{j=1}^{d_i^*} u_{1j}^{\gamma} \log u_{1j}$.

Proof. See Appendix A.1. □

It has been noticed that $\hat{\gamma}$ can not be expressed explicitly. Hence, an iterative method is required to compute $\hat{\gamma}$. Using the MLE $\hat{\gamma}$, the MLEs $\hat{\alpha}_1$ and $\hat{\alpha}_2$ can be obtained from (3.3).

3.1 Asymptotic confidence interval

In this subsection, the $100(1-\gamma)\%$ ACIs of unknown parameters α_1, α_2 and γ are constructed using asymptotic normality properties of the associated MLEs (see Bartlett [28]). To obtain the ACIs, the variance-covariance matrix is required and can be obtained using the observed Fisher information matrix (FIM). Under some mild regularity conditions, the asymptotic distribution of the MLEs $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\gamma})$ can be obtained as

$$(\hat{\alpha}_1,\ \hat{\alpha}_2,\ \hat{\gamma}) - (\alpha_1,\ \alpha_2,\ \gamma) \sim N\Big(0, I^{-1}(\hat{\alpha}_1,\ \hat{\alpha}_2,\ \hat{\gamma})\Big),$$

where

$$\widehat{I}^{-1}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\gamma}) = \begin{bmatrix} -l^*_{11} & -l^*_{12} & -l^*_{13} \\ -l^*_{21} & -l^*_{22} & -l^*_{23} \\ -l^*_{31} & -l^*_{32} & -l^*_{33} \end{bmatrix}^{-1}_{\Theta=\widehat{\Theta}} = \begin{bmatrix} Var(\hat{\alpha}_1) & Cov(\hat{\alpha}_1, \hat{\alpha}_2) & Cov(\hat{\alpha}_1, \hat{\gamma}) \\ Cov(\hat{\alpha}_2, \hat{\alpha}_1) & Var(\hat{\alpha}_2) & Cov(\hat{\alpha}_2, \widehat{\alpha}) \\ Cov(\hat{\gamma}, \hat{\alpha}_1) & Cov(\hat{\gamma}, \hat{\alpha}_2) & Var(\hat{\gamma}) \end{bmatrix}, \tag{3.6}$$

is the inverse of the observed FIM, $\Theta = (\theta_1,\ \theta_2,\ \theta_3) = (\alpha_1,\ \alpha_2,\ \gamma)$ and $l^*_{ij} = \frac{\partial^2 l^*}{\partial\theta_i \partial\theta_j}$, for $i,\ j = 1,\ 2,\ 3$. Here,

$$\begin{aligned} l^*_{11} &= \frac{\partial^2 l^*}{\partial\alpha_1^2} = -\frac{d^*_1}{\alpha_1^2},\ l_{12} = l_{21} = \frac{\partial^2 l^*}{\partial\alpha_1\partial\alpha_2} = 0,\ l_{13} = l_{31} = \frac{\partial^2 l^*}{\partial\alpha_1\partial\gamma} = -\Phi'_1(\gamma), \\ l^*_{22} &= \frac{\partial^2 l^*}{\partial\alpha_2^2} = -\frac{d^*_2}{\alpha_2^2},\ l_{23} = l_{32} = \frac{\partial^2 l^*}{\partial\alpha_2\partial\gamma} = -\Phi'_2(\gamma), \\ l^*_{33} &= \frac{\partial^2 l^*}{\partial\gamma^2} = -\frac{D^*}{\gamma^2} - \alpha_1\Phi''_1(\gamma) - \alpha_2\Phi''_2(\gamma), \end{aligned}$$

where $\Phi''_i(\gamma) = \sum_{j=1}^{d^*_i} u^{\gamma}_{1j} (\log u_{1j})^2$. Thus, the $100(1-\gamma)\%$ ACIs for α_1, α_2 and γ are given by

$$\left(\hat{\alpha}_1\ \pm z_{\frac{\beta}{2}}\sqrt{Var(\hat{\alpha}_1)}\right),\quad \left(\hat{\alpha}_2\ \pm z_{\frac{\beta}{2}}\sqrt{Var(\hat{\alpha}_2)}\right),\ \text{and}\ \left(\widehat{\gamma}\ \pm z_{\frac{\beta}{2}}\sqrt{Var(\hat{\gamma})}\right),$$

where $z_{\frac{\beta}{2}}$ denotes the upper $\frac{\beta}{2}$-th percentile point of a standard normal distribution.

It has been demonstrated that the MLEs exist uniquely under a very wide range of conditions. Even though the MLEs' results are quite good, it might be challenging to find the corresponding confidence intervals. Therefore, it appears that in this instance, the Bayesian inference makes sense.

4 Bayesian estimation

Due to the ability to assimilate prior information, Bayesian approaches are frequently applied in reliability and survival analysis. In this section, Bayesian estimates and the associated HPD credible intervals of α_1, α_2, and γ are derived based on JUHCS under SELF and LLF. Here, the SELF is symmetric in nature and LLF is asymmetric. If σ be an estimator of η, then SELF and LLF are given by

$$L_{SE}(\eta,\ \sigma) = (\sigma - \eta)^2, \tag{4.1}$$

and

$$L_{LI}(\eta,\ \sigma) = e^{q(\sigma-\eta)} - q(\sigma - \eta) - 1,\ q \neq 0, \tag{4.2}$$

respectively. Both underestimation and overestimation are given equal importance by SELF (see James et al. [29]). It is also known as the balanced loss function certain times. There are several circumstances in which the symmetric loss function is not a suitable tool. For instance, overestimating a satellite's lifetime is more dangerous than underestimating it. Furthermore, underestimating a river's water level during the rainy season is more dangerous than overestimating it. In these circumstances, asymmetric loss function has been employed. The LLF, which Varian [30] introduced is such an asymmetric loss function. The Bayes estimates of η under SELF and LLF are obtained as

$$\hat{\eta}_{SE} = E_\eta(\eta|\mathbf{u}), \tag{4.3}$$

$$\hat{\eta}_{LI} = -\frac{1}{q}\log E_\eta(e^{-q\eta}|\mathbf{u}),\ q \neq 0. \tag{4.4}$$

4.1 When the Scale Parameters are Independent

As part of the Bayesian technique, the initial step is to calculate prior distributions for the parameters that are not known, which represent opinionated prior conclusions about the parameters. Bayesian inference relies heavily on this step. It is challenging to determine the joint conjugate prior when all the parameters are unknown. Furthermore, since none of the elements in the associated expected Fisher information matrix have closed forms, Jeffrey's prior cannot be specified in our situation. It is not clear to say that one prior is superior to another, as noted by Arnold and Press [31]. Additionally, they recommended choosing to focus just on a certain flexible family of prior distributions. As a result, we may select the distribution family that appears to meet our situation the best. The density function may be transformed to take on many forms using the flexibility of the gamma distribution. In the interval $(0, \infty)$, it exhibits a log-concave density function. As a particular instance of the gamma prior, Jeffrey's prior can be acquired. Due to the versatility of gamma distribution, it is assumed that the model parameters α_1, α_2, and γ follow independent gamma priors denoted by $G(a_1,\ b_1)$, $G(a_2,\ b_2)$ and $G(a_3,\ b_3)$, respectively, where $a_i, b_i > 0$ are the hyperparameters for $i = 1,\ 2,\ 3$. Then the join prior density function can be expressed as

$$\pi(\alpha_1,\ \alpha_2,\ \gamma) \propto \alpha_1^{a_1-1} \alpha_2^{a_2-1} \gamma^{a_3-1} \exp\left[-(b_1\alpha_1 + b_2\alpha_2 + b_3\gamma)\right]. \tag{4.5}$$

Combining (3.1) and (4.5), the joint posterior density function for α_1, α_2, and γ is obtained as

$$\begin{aligned}\pi(\alpha_1,\ \alpha_2,\ \gamma|\mathbf{u}) \propto\ & \alpha_1^{d_1^*+a_1-1} \alpha_2^{d_2^*+a_2-1} \gamma^{D^*+a_3-1} \exp\left[-\alpha_1\left(b_1 + (n_1 - d_1^*)T^* + \sum_{i=1}^{d_1^*} u_{1i}^{\gamma}\right)\right] \\ & \times \exp\left[-\alpha_2\left(b_2 + (n_2 - d_2^*)T^* + \sum_{i=1}^{d_2^*} u_{2i}^{\gamma}\right)\right] \exp\left[-\gamma\left(b_3 - \sum_{i=1}^{d_1^*} \log u_{1i} - \sum_{i=1}^{d_2^*} \log u_{2i}\right)\right].\end{aligned} \tag{4.6}$$

Thus, under SELF and LLF, the Bayes estimates of any function of α_1, α_2 and γ, say $\psi(\alpha_1, \alpha_2, \gamma)$ are obtained as follows

$$\widehat{\psi}_{SE} = \frac{\int_0^\infty \int_0^\infty \int_0^\infty \psi(\alpha_1, \alpha_2, \gamma)\pi(\alpha_1, \alpha_2, \gamma|\mathbf{u})\ d\alpha_1\ d\alpha_2\ d\gamma}{\int_0^\infty \int_0^\infty \int_0^\infty \pi(\alpha_1, \alpha_2, \gamma|\mathbf{u})\ d\alpha_1\ d\alpha_2\ d\gamma} \tag{4.7}$$

and

$$\widehat{\psi}_{LI} = -\left(\frac{1}{q}\right) \log\left[\frac{\int_0^\infty \int_0^\infty \int_0^\infty \exp(-q\psi(\alpha_1, \alpha_2, \gamma))\pi(\alpha_1, \alpha_2, \gamma|\mathbf{u})\ d\alpha_1\ d\alpha_2\ d\gamma}{\int_0^\infty \int_0^\infty \int_0^\infty \pi(\alpha_1, \alpha_2, \gamma|\mathbf{u})\ d\alpha_1\ d\alpha_2\ d\gamma}\right], \tag{4.8}$$

respectively. From (4.7) and (4.8), the Bayes estimates of $\psi(\alpha_1, \alpha_2, \gamma)$ can not be obtained explicitly. In this context, an approximation technique is employed to derive the Bayesian estimators.

4.1.1 Gibbs sampling and posterior analysis

From (4.6), it has been noticed that the marginal posterior density of γ follows gamma distribution such that

$$\pi(\gamma|\mathbf{u}) \propto G\left(D^* + a_3,\ b_3 - \sum_{i=1}^{d_1^*} \log u_{1i} - \sum_{i=1}^{d_2^*} \log u_{2i}\right). \tag{4.9}$$

Again, from (4.6), the conditional posterior density functions of α_1 and α_2 for given $(\gamma,\ \mathbf{u})$, can be expressed as

$$\pi(\alpha_1|\gamma,\ \mathbf{u}) \propto G\left(d_1^* + a_1,\ b_1 + (n_1 - d_1^*)T^* + \sum_{i=1}^{d_1^*} u_{1i}^{\gamma}\right) \tag{4.10}$$

and

$$\pi(\alpha_2|\gamma,\ \mathbf{u}) \propto G\left(d_2^* + a_2,\ b_2 + (n_2 - d_2^*)T^* + \sum_{i=1}^{d_2^*} u_{2i}^{\gamma}\right), \tag{4.11}$$

respectively. Since $\pi(\gamma|\mathbf{u})$ is log-concave, Gibbs sampling method, proposed by Devroye [32], is used to generate samples from (4.9). Then, the Bayes estimates, and the associated HPD credible intervals of α_1, α_2 and γ have been computed using these samples. These samples are generated using the algorithm which is given as follows:

Algorithm

Step I: Generate the samples of γ from (4.9).

Step II: Generate α_1 and α_2 from (4.10) and (4.11), respectively using the generated values of γ,.

Step III: Repeat Steps 1 and 2 upto B times to generate the samples $\{\alpha_1^{(1)},\ \cdots,\ \alpha_1^{(B)}\}$, $\{\alpha_2^{(1)},\ \cdots,\ \alpha_2^{(B)}\}$, and $\{\gamma^{(1)},\ \cdots,\ \gamma^{(B)}\}$.

The Bayes estimates of any function the parameters, say $\psi(\alpha_1,\ \alpha_2,\ \gamma)$, under SELF and LLF are obtained as

$$\psi_{SE} = \frac{1}{B}\sum_{i=1}^{B}\psi(\alpha_1^{(i)},\ \alpha_2^{(i)},\ \gamma^{(i)}),\quad \text{and} \quad \psi_{LI} = -\frac{1}{q}\log\left(\sum_{i=1}^{B}\frac{\exp\big[-q\psi(\alpha_1^{(i)},\ \alpha_2^{(i)},\ \gamma^{(i)})\big]}{B}\right),$$

respectively. Replacing $\psi(\alpha_1,\ \alpha_2,\ \gamma)$ with any parameters, the Bayes estimates of the parameters can be obtained.

In order to construct HPD credible intervals of the parameters α_1, α_2 and γ, MCMC samples are rearranged in an increasing order. After rearranging, the samples become $\{\alpha_1^1,\ \cdots,\ \alpha_1^B\}$, $\{\alpha_2^1,\ \cdots,\ \alpha_2^B\}$, and $\{\gamma^1,\ \cdots,\ \gamma^B\}$. Further, the $100(1-\beta)\%$ HPD credible interval for γ is obtained as

$$\left(\gamma^{i^*},\ \gamma^{i^*+B-[B\beta+1]}\right),\quad \text{for} \quad i = 1,\ \cdots,\ [\beta B], \tag{4.12}$$

where $[\cdot]$ depicts the greatest integer value and i^* satisfies the following condition

$$\gamma^{i^*+B-[B\beta+1]} - \gamma^{i^*} = \min\left(\gamma^{i+B-[B\beta+1]} - \gamma^{i}\right),\ \text{for}\ i = 1,\ \cdots,\ [\beta B].$$

One can easily derive the HPD credible intervals of α_1 and α_2 in a similar manner. It is recalled that they become non-informative if the hyperparameters go towards zero. On the other hand, the marginal posterior densities have not been altered. As a result, the process for sampling that was explained before may be used to successfully get the appropriate Bayes estimates.

4.2 When the Scale Parameters are Dependent

In this section, we will assume that the scale parameter α_1 and α_2 are dependent random variables. To incorporate dependence among α_1 and α_2 the following prior assumptions are made by Pena and Gupta [33] on the scale parameters. It is assumed that $\alpha = \alpha_1 + \alpha_2$ follows a gamma distribution say $G(a_0, b_0)$ and also it is assumed that given α, $(\frac{\alpha_1}{\alpha},\ \frac{\alpha_2}{\alpha})$ follows a Dirichlet prior say, $D(a_1, a_2)$.

$$\pi_1(\alpha_1,\ \alpha_2) \propto (\alpha_1+\alpha_2)^{a_0-a_1-a_2}\alpha_1^{a_1-1}\alpha_2^{a_2-1}e^{-b_0(\alpha_1+\alpha_2)} \tag{4.13}$$

The joint prior of $(\alpha_1,\ \alpha_2)$ with hyper-parameters $a_0,\ b_0,\ a_1,\ a_2$ is called Gamma-Dirichlet prior, denoted by $GD(a_0,\ b_0,\ a_1,\ a_2)$ and it is very flexible prior, see Pena and Gupta [33] for more details. We also assumed that the shape parameter γ has a log-concave prior $\pi_2(\gamma)$ with a positive support on $(0,\infty)$. Note that $\pi_2(\gamma)$ is independent of $\pi_1(\alpha_1,\ \alpha_2)$. Then the joint prior of $(\alpha_1,\ \alpha_2,\ \gamma)$ is given by

$$\pi(\alpha_1,\ \alpha_2,\ \gamma) = \pi_1(\alpha_1,\ \alpha_2)\pi_2(\gamma)$$

Then the posterior distribution is given by

$$\begin{aligned}\pi(\Theta|Data) \propto\ & \alpha_1^{d_1^*+a_1-1}\alpha_2^{d_2^*+a_2-1}\gamma^{D^*}(\alpha_1+\alpha_2)^{a_0-a_1-a_2}\exp\left[-\alpha_1\left(b_0+(n_1-d_1^*)T^*+\sum_{i=1}^{d_1^*}u_{1i}^{\gamma}\right)\right] \\ & \times \exp\left[-\alpha_2\left(b_0+(n_2-d_2^*)T^*+\sum_{i=1}^{d_2^*}u_{2i}^{\gamma}\right)\right]\pi_2(\gamma)\prod_{i=1}^{d_1^*}u_{1i}^{(\gamma-1)}\prod_{i=1}^{d_2^*}u_{2i}^{(\gamma-1)}. \end{aligned} \tag{4.14}$$

The Bayes estimate of function of Θ say $g(\Theta)$ can be obtained as

$$E(g(\Theta)|Data) = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} g(\Theta)\pi(\Theta|Data)\ d\gamma d\alpha_1 d\alpha_2,$$

provided it exits. The estimator cannot be obtained in closed form so we have to use the simulation approach. Since the joint posterior density can be written as

$$\pi(\Theta|Data) \propto h(\alpha_1,\ \alpha_2)\pi_{11}^*(\alpha_1|\gamma,\ Data)\pi_{12}^*(\alpha_2|\gamma,`Data)\pi_2^*(\gamma|Data),$$

where

$$\pi_{11}^*(\alpha_1|\gamma,\ Data) \equiv G(d_1^*+a_1,\ b_0+\sum_{i=1}^{d_1^*}u_{1i}^{\gamma}+(n_1-d_1^*)T^*)$$

$$\pi_{12}^*(\alpha_2|\gamma,\ Data) \equiv G(d_2^*+a_2,\ b_0+\sum_{i=1}^{d_2^*}u_{2i}^{\gamma}+(n_2-d_2^*)T^*)$$

$$\pi_2^*(\gamma|Data) \propto \gamma^{D^*}\prod_{i=1}^{d_1^*}u_{1i}^{(\gamma-1)}\prod_{i=1}^{d_2^*}u_{2i}^{(\gamma-1)}\pi_2(\gamma)$$

$$h(\alpha_1,\ \alpha_2) = (\alpha_1+\alpha_2)^{a_0-a_1-a_2}$$

As the posterior cannot be obtained in any standard form we rely on the importance sampling technique to compute the Bayes estimates and associated credible intervals. Note that when $\pi_2(\gamma)$ is log-concave $\pi_2^*(\gamma|Data)$ is also log-concave.

To apply the importance sampling, it is required to generate $\gamma,\ \alpha_1,\ \alpha_2$ from the posterior density. Here, we follow the method suggested by Kundu [34] to generate γ from its log-concave posterior density. Once γ is generated we can generate $\alpha_1,\ \alpha_2$ from respective conditional densities.

- Given data, generate γ from $\pi_2^*(\gamma|Data)$.
- For given γ, generate $\alpha_1,\ \alpha_2$ from $\pi_{11}^*(\alpha_1|\gamma,\ Data)$ and $\pi_{12}^*(\alpha_2|\gamma,\ Data)$.
- Repeat the process N times to generate $(\gamma_1,\ \alpha_{11},\ \alpha_{12}), \cdots, (\gamma_N,\ \alpha_{1N},\ \alpha_{2N})$.
- Compute $(g_1,\ g_1,\ \cdots,\ g_N)$ and $(h_1,\ h_2,\ \cdots,\ h_N)$

$$g_i = g(\gamma_i,\ \alpha_{1i},\ \alpha_{2i}) \text{ and } h_i = h(\alpha_{1i},\ \alpha_{2i}) \quad i = 1,\ 2,\ \cdots,\ N.$$

- The Compute estimate of $g(\gamma,\ \alpha_1,\ \alpha_2)$ based on the squared error loss function can be computed as

$$\frac{\sum_{i=1}^{N} h_i g_i}{\sum_{j=1}^{N} h_j} = \sum_{i=1}^{N} w_i g_i,$$

  where $w_i = \frac{h_i}{\sum_{j=1}^{N} h_j}$.

- To compute a $100(1-\beta)\%$ credible interval of $g(\gamma,\ \alpha_1,\ \alpha_2)$ arrange g_i's in an ascending order to obtain $(g_{(1)}, \ldots, g_{(N)})$ and corresponding w_i as $(w_{[1]},\ \cdots,\ w_{[N]})$. Here $w_{[i]}$ is not ordered and they are assigned to the corresponding ordered $g_{(i)}'$s. A $100(1-\beta)\%$ credible interval can be obtained as $(g_{(j_1)},\ g_{(j_2)})$ where $j_1,\ j_2$ such that

$$j_1 < j_2;\ j_1,\ j_2 \in \{1,\ 2,\ \cdots,\ N\} \text{ and } \sum_{i=j_1}^{j_2} w_{[i]} \le 1-\beta < \sum_{i=j_1}^{j_2+1} w_{[i]}$$

5 Simulation study

A comprehensive Monte Carlo simulation study which is used to assess the effectiveness of the proposed statistical inference procedures under JUHCS in this section. The performances of the point estimates of the parameters are compared in terms of the associated ABs and MSEs. The interval estimates are compared based on average lengths (ALs) and coverage probabilities (CPs). Two different sets of true values of the parameters $(\alpha_1,\ \alpha_2,\ \gamma)$ have been considered as $(1.2,\ 0.8,\ 1.5)$ and $(1.5,\ 1.0,\ 0.8)$ without loss of generality. An extensive simulation study has been carried out with $N = 15,000$ samples generated from Weibull based on joint unified hybrid censoring with different values of n_1, n_2, r, and k, with $(T_1,\ T_2) = (0.7,\ 1.4)$, and $(0.8,\ 1.6)$. All these computations have been done by using R and MATLAB software.

In classical estimation, MLEs for the model parameters have been obtained by using the Newton-Raphson iterative method. In Bayesian estimation, we have obtained under consideration scale parameters that are independent and dependent. To obtain the Bayes estimates informative gamma priors have been considered with positive hyperparameters. These hyperparameters have been elicited by using the method discussed in Dutta and Kayal [35]. To obtain the Bayes estimates SELF and LLF have been considered. In the case of LLF, $q = 0.5$ has been taken into account. The necessary numerical computations have been carried out by using R statistical software. The values of the ABs and MSEs associated with MLEs and Bayes estimates for the parameters α_1, α_2, and γ. To compute the Bayes estimates, $5,000$ MCMC samples are generated using Gibb's sampling method. Further, 95% confidence and HPD credible intervals have been constructed. The ABs and MSEs of MLEs and Bayes estimates have been reported in Tables 1 and 3. The ALs and CPs associated with interval estimates have been tabulated in Tables 2 and 4. From these tables the following conclusions are made:

- For fixed $(n_1,\ n_2)$, r, and $(T_1,\ T_2)$, when k increases, ABs and MSEs for α_1, α_2, and γ decrease.
- For fixed $(n_1,\ n_2)$, and $(T_1,\ T_2)$, when r increases, ABs and MSEs for α_1, α_2, and γ decrease.
- For fixed $(T_1,\ T_2)$, when $(n_1,\ n_2)$ increase, ABs and MSEs for α_1, α_2, and γ decrease.
- For fixed $(n_1,\ n_2)$, when $(T_1,\ T_2)$ increases, ABs and MSEs for α_1, α_2, and γ decrease.

- For fixed $(n_1,\ n_2)$, $(T_1,\ T_2)$, r and k, the Bayes estimates under LLF outperform than other estimates with regard to AB and MSE.

- For fixed $(n_1,\ n_2)$, r, and $(T_1,\ T_2)$, when k increases, the ALs for interval estimates decrease, and CPs increase.

- For fixed $(n_1,\ n_2)$, and $(T_1,\ T_2)$, when r increases, the ALs for interval estimates decrease, and CPs increase.

- For fixed $(T_1,\ T_2)$, when $(n_1,\ n_2)$ increases, the ALs for interval estimates decrease, and CPs increase.

- For fixed $(n_1,\ n_2)$, when $(T_1,\ T_2)$ increases, the ALs for interval estimates decrease, and CPs increase.

- For fixed $(n_1,\ n_2)$, $(T_1,\ T_2)$, r, and k, HPD credible interval performs better than ACIs in terms of AL and CP.

- For any fixed $(n_1,\ n_2)$, $(T_1,\ T_2)$, r and k, Bayes estimates under consideration of dependent scale parameters perform better than other estimates based on ABs and MSEs for point estimates, AL and CPs for interval estimates.

As a summary of the observations, reliability practitioners are recommended to use Bayes estimates when scale parameters are dependent under LLF and HPD credible intervals for Weibull distributions for joint unified hybrid censored samples.

6 Expected Test Time

In this instance, it is challenging to calculate the precise formula for the expected test time (ETT), hence we have computed the empirical ETTs. Corresponding to every choice of (n_1, n_2), r and k, we have generated $N = 15000$ samples based on JUHCS and also for UHCS based on different values of $(n_1 + n_2, r)$ and k. The empirical ETT is then given by $\frac{1}{N}\sum_{i=1}^{N} T_i^*$, where T_i^* is the termination time of the i-th sample drawn. All the results are tabulated in Table 5. It is clear from the table that JUHCS has an expected test time smaller than the UHCS.

Table 1: ABs and MSEs (in parenthesis) of the estimates of α_1, α_2, and γ when $\Theta = (1.2,\ 0.8,\ 1.5)$ with different values of r and k with different values of $(T_1,\ T_2)$ when scale parameters are independent.

							Independent						Dependent					
				MLE			SELF			LLF			SELF			LLF		
$(T_1,\ T_2)$	$(n_1,\ n_2)$	r	k	α_1	α_2	γ	α_1	α_2	γ	α_1	α_2	γ	α_1	α_2	γ	α_1	α_2	γ
(0.7, 1.4)	(25, 25)	30	20	0.1219	0.1035	0.1016	0.0758	0.0695	0.0654	0.0734	0.0667	0.0637	0.0715	0.0638	0.0611	0.0701	0.0614	0.0598
				(0.0852)	(0.0695)	(0.0622)	(0.0365)	(0.0284)	(0.0253)	(0.0342)	(0.0276)	(0.0241)	(0.0337)	(0.0251)	(0.0229)	(0.0328)	(0.0244)	(0.0223)
			25	0.1186	0.1014	0.1003	0.0727	0.0667	0.0636	0.0710	0.0639	0.0622	0.0698	0.0617	0.0598	0.0682	0.0605	0.0592
				(0.0831)	(0.0678)	(0.0614)	(0.0353)	(0.0276)	(0.0248)	(0.0337)	(0.0270)	(0.0235)	(0.0334)	(0.0242)	(0.0223)	(0.0331)	(0.0240)	(0.0218)
		40	30	0.1029	0.0951	0.0942	0.0698	0.0640	0.0623	0.0687	0.0617	0.0610	0.0673	0.0602	0.0590	0.0661	0.0597	0.0583
				(0.0802)	(0.0653)	(0.0602)	(0.0346)	(0.0269)	(0.0241)	(0.0333)	(0.0266)	(0.0231)	(0.0330)	(0.0239)	(0.0220)	(0.0327)	(0.0236)	(0.0215)
			35	0.1008	0.0939	0.0924	0.0682	0.0629	0.0611	0.0673	0.0605	0.0601	0.0658	0.0596	0.0583	0.0652	0.0590	0.0576
				(0.0783)	(0.0645)	(0.0596)	(0.0336)	(0.0263)	(0.0237)	(0.0330)	(0.0261)	(0.0227)	(0.0325)	(0.0233)	(0.0217)	(0.0321)	(0.0232)	(0.0211)
	(35, 35)	50	40	0.0964	0.0871	0.0859	0.0610	0.0588	0.0574	0.0605	0.0581	0.0570	0.0598	0.0572	0.0566	0.0587	0.0567	0.0560
				(0.0515)	(0.0338)	(0.0415)	(0.0335)	(0.0213)	(0.0237)	(0.0324)	(0.0206)	(0.0228)	(0.0320)	(0.0210)	(0.0223)	(0.0313)	(0.0204)	(0.0217)
			45	0.0935	0.0856	0.0838	0.0592	0.0577	0.0567	0.0589	0.0573	0.0562	0.0574	0.0569	0.0560	0.0569	0.0561	0.0551
				(0.0504)	(0.0333)	(0.0403)	(0.0329)	(0.0208)	(0.0230)	(0.0317)	(0.0198)	(0.0222)	(0.0311)	(0.0195)	(0.0216)	(0.0305)	(0.0192)	(0.0207)
		60	50	0.0924	0.0850	0.0834	0.0587	0.0573	0.0560	0.0580	0.0568	0.0556	0.0570	0.0564	0.0553	0.0563	0.0557	0.0545
				(0.0496)	(0.0328)	(0.0392)	(0.0323)	(0.0203)	(0.0225)	(0.0311)	(0.0195)	(0.0217)	(0.0306)	(0.0191)	(0.0210)	(0.0299)	(0.0190)	(0.0202)
			55	0.0911	0.0843	0.0825	0.0576	0.0569	0.0555	0.0572	0.0563	0.0551	0.0567	0.0560	0.0550	0.0558	0.0553	0.0541
				(0.0487)	(0.0322)	(0.0385)	(0.0320)	(0.0199)	(0.0219)	(0.0305)	(0.0191)	(0.0213)	(0.0301)	(0.0189)	(0.0207)	(0.0296)	(0.0187)	(0.0195)
	(50, 50)	80	70	0.0895	0.0835	0.0820	0.0572	0.0564	0.0551	0.0565	0.0557	0.0548	0.0560	0.0557	0.0546	0.0552	0.0548	0.0537
				(0.0476)	(0.0315)	(0.0380)	(0.0315)	(0.0196)	(0.0215)	(0.0300)	(0.0189)	(0.0209)	(0.0297)	(0.0186)	(0.0201)	(0.0293)	(0.0184)	(0.0192)
			75	0.0890	0.0830	0.0815	0.0566	0.0560	0.0544	0.0560	0.0554	0.0544	0.0554	0.0553	0.0542	0.0548	0.0543	0.0532
				(0.0443)	(0.0280)	(0.0367)	(0.0264)	(0.0168)	(0.0205)	(0.0251)	(0.0160)	(0.0203)	(0.0247)	(0.0159)	(0.0197)	(0.0239)	(0.0156)	(0.0190)
		90	80	0.0871	0.0822	0.0803	0.0560	0.0555	0.0541	0.0553	0.0550	0.0540	0.0550	0.0548	0.0536	0.0542	0.0539	0.0527
				(0.0430)	(0.0271)	(0.0355)	(0.0254)	(0.0158)	(0.0196)	(0.0245)	(0.0153)	(0.0193)	(0.0243)	(0.0153)	(0.0190)	(0.0241)	(0.0152)	(0.0186)
			85	0.0859	0.0813	0.0792	0.0553	0.0549	0.0532	0.0547	0.0545	0.0480	0.0546	0.0545	0.0531	0.0538	0.0536	0.0522
				(0.0410)	(0.0259)	(0.0343)	(0.0239)	(0.0148)	(0.0192)	(0.0232)	(0.0146)	(0.0186)	(0.0231)	(0.0145)	(0.0183)	(0.0226)	(0.0144)	(0.0181)
(0.8, 1.6)	(25, 25)	30	20	0.1152	0.0989	0.0995	0.0695	0.0674	0.0638	0.0677	0.0666	0.0627	0.0670	0.0658	0.0623	0.0658	0.0652	0.0620
				(0.0759)	(0.0627)	(0.0610)	(0.0346)	(0.0268)	(0.0243)	(0.0335)	(0.0263)	(0.0239)	(0.0331)	(0.0247)	(0.0225)	(0.0324)	(0.0241)	(0.0220)
			25	0.1140	0.0980	0.0984	0.0688	0.0668	0.0629	0.0671	0.0661	0.0567	0.0666	0.0655	0.0618	0.0654	0.0649	0.0615
				(0.0751)	(0.0622)	(0.0605)	(0.0340)	(0.0264)	(0.0240)	(0.0330)	(0.0259)	(0.0236)	(0.0326)	(0.0245)	(0.0221)	(0.0321)	(0.0255)	(0.0216)
		40	30	0.1017	0.0942	0.0931	0.0691	0.0634	0.0614	0.0681	0.0612	0.0550	0.0669	0.0599	0.0585	0.0657	0.0594	0.0578
				(0.0744)	(0.0617)	(0.0596)	(0.0328)	(0.0260)	(0.0235)	(0.0325)	(0.0255)	(0.0232)	(0.0323)	(0.0250)	(0.0227)	(0.0319)	(0.0252)	(0.0211)
			35	0.0974	0.0918	0.0905	0.0661	0.0605	0.0598	0.0652	0.0599	0.0593	0.0650	0.0591	0.0580	0.0646	0.0585	0.0573
				(0.0767)	(0.0633)	(0.0585)	(0.0319)	(0.0260)	(0.0233)	(0.0315)	(0.0253)	(0.0225)	(0.0312)	(0.0230)	(0.0215)	(0.0307)	(0.0229)	(0.0209)
	(35, 35)	50	40	0.0945	0.0839	0.0837	0.0599	0.0574	0.0565	0.0590	0.0559	0.0553	0.0587	0.0556	0.0551	0.0581	0.0553	0.0548
				(0.0503)	(0.0329)	(0.0404)	(0.0330)	(0.0209)	(0.0231)	(0.0319)	(0.0202)	(0.0223)	(0.0316)	(0.0200)	(0.0220)	(0.0309)	(0.0199)	(0.0214)
			45	0.0908	0.0837	0.0815	0.0573	0.0559	0.0552	0.0569	0.0553	0.0549	0.0566	0.0550	0.0546	0.0564	0.0547	0.0542
				(0.0495)	(0.0315)	(0.0392)	(0.0314)	(0.0198)	(0.0221)	(0.0310)	(0.0195)	(0.0218)	(0.0306)	(0.0193)	(0.0213)	(0.0301)	(0.0190)	(0.0205)
		60	50	0.0909	0.0831	0.0822	0.0580	0.0565	0.0551	0.0575	0.0561	0.0545	0.0567	0.0559	0.0543	0.0560	0.0553	0.0539
				(0.0480)	(0.0317)	(0.0383)	(0.0315)	(0.0197)	(0.0218)	(0.0307)	(0.0192)	(0.0213)	(0.0301)	(0.0189)	(0.0208)	(0.0296)	(0.0187)	(0.0199)
			55	0.0896	0.0824	0.0809	0.0568	0.0557	0.0549	0.0564	0.0552	0.0546	0.0560	0.0549	0.0543	0.0555	0.0547	0.0539
				(0.0472)	(0.0311)	(0.0374)	(0.0318)	(0.0197)	(0.0216)	(0.0301)	(0.0189)	(0.0210)	(0.0298)	(0.0187)	(0.0203)	(0.0293)	(0.0185)	(0.0192)
	(50, 50)	80	70	0.0876	0.0823	0.0811	0.0566	0.0560	0.0545	0.0557	0.0553	0.0543	0.0553	0.0551	0.0540	0.0550	0.0547	0.0535
				(0.0462)	(0.0303)	(0.0365)	(0.0306)	(0.0190)	(0.0209)	(0.0298)	(0.0186)	(0.0202)	(0.0295)	(0.0183)	(0.0198)	(0.0290)	(0.0181)	(0.0189)
			75	0.0876	0.0819	0.0802	0.0560	0.0552	0.0537	0.0555	0.0550	0.0535	0.0550	0.0549	0.0532	0.0546	0.0541	0.0529
				(0.0427)	(0.0267)	(0.0359)	(0.0257)	(0.0163)	(0.0200)	(0.0247)	(0.0158)	(0.0198)	(0.0243)	(0.0155)	(0.0195)	(0.0236)	(0.0153)	(0.0186)
		90	80	0.0856	0.0810	0.0791	0.0551	0.0546	0.0532	0.0545	0.0542	0.0527	0.0541	0.0539	0.0525	0.0539	0.0536	0.0522
				(0.0412)	(0.0263)	(0.0346)	(0.0250)	(0.0156)	(0.0193)	(0.0242)	(0.0150)	(0.0189)	(0.0240)	(0.0151)	(0.0186)	(0.0237)	(0.0148)	(0.0183)
			85	0.0847	0.0804	0.0782	0.0544	0.0541	0.0526	0.0540	0.0538	0.0522	0.0535	0.0534	0.0520	0.0531	0.0530	0.0517
				(0.0399)	(0.0250)	(0.0332)	(0.0235)	(0.0145)	(0.0187)	(0.0229)	(0.0143)	(0.0184)	(0.0227)	(0.0141)	(0.0180)	(0.0222)	(0.0139)	(0.0177)

Table 2: ALs and CPs (in parenthesis) of the 95% ACIs and HPD credible intervals of α_1, α_2, and γ when $\Theta = (1.2,\ 0.8,\ 1.5)$ with different values of r and k with different values of $(T_1,\ T_2)$ when scale parameters are independent.

							HPD					
				ACI			Independent			Dependent		
$(T_1,\ T_2)$	$(n_1,\ n_2)$	r	k	α_1	α_2	γ	α_1	α_2	γ	α_1	α_2	γ
(0.7, 1.4)	(25, 25)	30	20	1.0992	0.8046	0.9326	0.8775	0.6418	0.6651	0.8693	0.6150	0.6295
				(0.9285)	(0.9023)	(0.9361)	(0.9413)	(0.9396)	(0.9412)	(0.9419)	(0.9304)	(0.9421)
			25	1.0896	0.7984	0.9080	0.8733	0.6394	0.6577	0.8563	0.6182	0.6304
				(0.9323)	(0.9052)	(0.9407)	(0.9430)	(0.9411)	(0.9427)	(0.9439)	(0.9426)	(0.9433)
		40	30	0.9134	0.6476	0.7380	0.7901	0.5579	0.5954	0.7583	0.5384	0.5721
				(0.9355)	(0.9147)	(0.9422)	(0.9433)	(0.9420)	(0.9432)	(0.9442)	(0.9434)	(0.9438)
			35	0.9064	0.6462	0.7320	0.7837	0.5514	0.5886	0.7439	0.5282	0.5605
				(0.9372)	(0.9209)	(0.9425)	(0.9442)	(0.9429)	(0.9436)	(0.9447)	(0.9435)	(0.9439)
	(35, 35)	50	40	0.7744	0.5692	0.6448	0.7004	0.5044	0.5323	0.6542	0.4859	0.5005
				(0.9403)	(0.9238)	(0.9437)	(0.9458)	(0.9436)	(0.9445)	(0.9462)	(0.9445)	(0.9449)
			45	0.7632	0.5654	0.6308	0.6973	0.5018	0.5280	0.6694	0.4795	0.4974
				(0.9421)	(0.9257)	(0.9445)	(0.9462)	(0.9441)	(0.9454)	(0.9472)	(0.9447)	(0.9461)
		60	50	0.7157	0.5285	0.5636	0.6631	0.4766	0.4887	0.6354	0.4501	0.4670
				(0.9429)	(0.9295)	(0.9441)	(0.9470)	(0.9445)	(0.9462)	(0.9476)	(0.9450)	(0.9466)
			55	0.6989	0.4823	0.5597	0.6539	0.4155	0.4893	0.6293	0.3972	0.4612
				(0.9439)	(0.9311)	(0.9451)	(0.9475)	(0.9452)	(0.9469)	(0.9481)	(0.9459)	(0.9471)
	(50, 50)	80	70	0.6863	0.4732	0.5508	0.6382	0.4071	0.4827	0.6055	0.3882	0.4505
				(0.9448)	(0.9332)	(0.9453)	(0.9479)	(0.9463)	(0.9476)	(0.9482)	(0.9468)	(0.9481)
			75	0.6727	0.4659	0.5421	0.6239	0.4026	0.4744	0.5942	0.3896	0.4404
				(0.9451)	(0.9352)	(0.9453)	(0.9481)	(0.9468)	(0.9480)	(0.9483)	(0.9473)	(0.9482)
		90	80	0.6554	0.4570	0.5364	0.6167	0.3984	0.4703	0.5895	0.3795	0.4448
				(0.9451)	(0.9370)	(0.9456)	(0.9484)	(0.9474)	(0.9481)	(0.9486)	(0.9475)	(0.9484)
			85	0.6358	0.4432	0.5276	0.6052	0.3943	0.4673	0.5688	0.3796	0.4500
				(0.9459)	(0.9382)	(0.9463)	(0.9488)	(0.9475)	(0.9489)	(0.9490)	(0.9481)	(0.9491)
(0.8, 1.6)	(25, 25)	30	20	0.9964	0.7968	0.8786	0.8560	0.6253	0.6534	0.8391	0.5980	0.6201
				(0.9322)	(0.9079)	(0.9391)	(0.9427)	(0.9416)	(0.9433)	(0.9434)	(0.9428)	(0.9437)
			25	0.9732	0.7904	0.8548	0.8535	0.6220	0.6478	0.8437	0.5905	0.6072
				(0.9335)	(0.9121)	(0.9410)	(0.9434)	(0.9424)	(0.9437)	(0.9439)	(0.9433)	(0.9440)
		40	30	0.8582	0.6344	0.7188	0.7699	0.5431	0.5779	0.7412	0.5188	0.5367
				(0.9362)	(0.9183)	(0.9420)	(0.9438)	(0.9433)	(0.9442)	(0.9444)	(0.9441)	(0.9448)
			35	0.8528	0.6292	0.7126	0.7624	0.5389	0.5742	0.7154	0.5153	0.5282
				(0.9381)	(0.9219)	(0.9429)	(0.9444)	(0.9440)	(0.9446)	(0.9449)	(0.9451)	(0.9452)
	(35, 35)	50	40	0.7538	0.5614	0.6225	0.6775	0.4858	0.5158	0.6494	0.4562	0.4880
				(0.9396)	(0.9268)	(0.9439)	(0.9450)	(0.9441)	(0.9451)	(0.9453)	(0.9452)	(0.9457)
			45	0.7456	0.5508	0.6171	0.6735	0.4813	0.5091	0.6455	0.4611	0.4702
				(0.9410)	(0.9319)	(0.9443)	(0.9457)	(0.9448)	(0.9458)	(0.9462)	(0.9453)	(0.9461)
		60	50	0.7044	0.4742	0.5466	0.6522	0.4660	0.4784	0.6239	0.4401	0.4561
				(0.9423)	(0.9334)	(0.9450)	(0.9464)	(0.9453)	(0.9463)	(0.9468)	(0.9459)	(0.9467)
			55	0.6942	0.4710	0.5484	0.6421	0.4063	0.4810	0.6088	0.3857	0.4471
				(0.9430)	(0.9345)	(0.9455)	(0.9472)	(0.9458)	(0.9470)	(0.9481)	(0.9462)	(0.9473)
	(50, 50)	80	70	0.6510	0.4454	0.5266	0.6146	0.3945	0.4559	0.5912	0.3797	0.4389
				(0.9435)	(0.9359)	(0.9458)	(0.9474)	(0.9461)	(0.9477)	(0.9478)	(0.9469)	(0.9482)
			75	0.6464	0.4408	0.5236	0.6063	0.3894	0.4522	0.5763	0.3753	0.4264
				(0.9443)	(0.9375)	(0.9464)	(0.9480)	(0.9465)	(0.9481)	(0.9483)	(0.9472)	(0.9486)
		90	80	0.6406	0.4350	0.5202	0.6008	0.3876	0.4503	0.5757	0.3659	0.4082
				(0.9453)	(0.9410)	(0.9469)	(0.9486)	(0.9477)	(0.9483)	(0.9490)	(0.9480)	(0.9487)
			85	0.6360	0.4323	0.5176	0.5977	0.3841	0.4472	0.5570	0.3645	0.4170
				(0.9458)	(0.9425)	(0.9478)	(0.9493)	(0.9482)	(0.9492)	(0.9495)	(0.9484)	(0.9498)

Table 3: ABs and MSEs (in parenthesis) of the estimates of α_1, α_2, and γ when $\Theta = (1.5,\ 1.0,\ 0.8)$ with different values of r and k with different values of $(T_1,\ T_2)$ when scale parameters are independent.

							Independent						Dependent					
				MLE			SELF			LLF			SELF			LLF		
$(T_1,\ T_2)$	$(n_1,\ n_2)$	r	k	α_1	α_2	γ	α_1	α_2	γ	α_1	α_2	γ	α_1	α_2	γ	α_1	α_2	γ
(0.7, 1.4)	(25, 25)	30	20	0.1246	0.1048	0.1031	0.0771	0.0706	0.0665	0.0745	0.0676	0.0644	0.0724	0.0643	0.0618	0.0708	0.0619	0.0603
				(0.0879)	(0.0711)	(0.0638)	(0.0380)	(0.0296)	(0.0267)	(0.0361)	(0.0289)	(0.0260)	(0.0353)	(0.0272)	(0.0243)	(0.0342)	(0.0256)	(0.0235)
			25	0.1213	0.1027	0.1018	0.0740	0.0678	0.0647	0.0721	0.0648	0.0629	0.0707	0.0622	0.0605	0.0689	0.0610	0.0597
				(0.0864)	(0.0695)	(0.0623)	(0.0365)	(0.0279)	(0.0253)	(0.0342)	(0.0273)	(0.0251)	(0.0337)	(0.0263)	(0.0234)	(0.0328)	(0.0243)	(0.0227)
		40	30	0.1056	0.0964	0.0957	0.0711	0.0651	0.0634	0.0698	0.0626	0.0617	0.0682	0.0607	0.0597	0.0668	0.0602	0.0588
				(0.0831)	(0.0670)	(0.0604)	(0.0353)	(0.0261)	(0.0242)	(0.0328)	(0.0262)	(0.0239)	(0.0325)	(0.0252)	(0.0218)	(0.0309)	(0.0228)	(0.0215)
			35	0.1035	0.0952	0.0939	0.0695	0.0640	0.0622	0.0684	0.0614	0.0608	0.0667	0.0601	0.0590	0.0659	0.0595	0.0581
				(0.0815)	(0.0656)	(0.0593)	(0.0326)	(0.0248)	(0.0233)	(0.0313)	(0.0249)	(0.0227)	(0.0313)	(0.0241)	(0.0208)	(0.0296)	(0.0211)	(0.0207)
	(35, 35)	50	40	0.0991	0.0884	0.0874	0.0623	0.0599	0.0585	0.0616	0.0590	0.0577	0.0607	0.0577	0.0573	0.0594	0.0572	0.0565
				(0.0592)	(0.0361)	(0.0439)	(0.0352)	(0.0226)	(0.0265)	(0.0346)	(0.0217)	(0.0241)	(0.0333)	(0.0213)	(0.0232)	(0.0325)	(0.0210)	(0.0225)
			45	0.0962	0.0869	0.0853	0.0605	0.0588	0.0578	0.0600	0.0582	0.0569	0.0583	0.0574	0.0567	0.0576	0.0566	0.0556
				(0.0566)	(0.0355)	(0.0427)	(0.0340)	(0.0221)	(0.0258)	(0.0327)	(0.0212)	(0.0233)	(0.0319)	(0.0208)	(0.0225)	(0.0312)	(0.0203)	(0.0219)
		60	50	0.0951	0.0863	0.0849	0.0600	0.0584	0.0571	0.0591	0.0577	0.0563	0.0579	0.0569	0.0560	0.0570	0.0562	0.0550
				(0.0535)	(0.0339)	(0.0409)	(0.0333)	(0.0212)	(0.0241)	(0.0318)	(0.0206)	(0.0222)	(0.0302)	(0.0201)	(0.0214)	(0.0299)	(0.0198)	(0.0209)
			55	0.0938	0.0856	0.0840	0.0589	0.0580	0.0566	0.0583	0.0572	0.0558	0.0576	0.0565	0.0557	0.0565	0.0558	0.0546
				(0.0511)	(0.0330)	(0.0394)	(0.0325)	(0.0203)	(0.0228)	(0.0309)	(0.0198)	(0.0214)	(0.0299)	(0.0197)	(0.0209)	(0.0297)	(0.0192)	(0.0199)
	(50, 50)	80	70	0.0922	0.0848	0.0835	0.0585	0.0575	0.0562	0.0576	0.0566	0.0555	0.0569	0.0562	0.0553	0.0559	0.0553	0.0542
				(0.0463)	(0.0305)	(0.0365)	(0.0298)	(0.0184)	(0.0205)	(0.0281)	(0.0178)	(0.0201)	(0.0267)	(0.0172)	(0.0198)	(0.0272)	(0.0169)	(0.0189)
			75	0.0917	0.0843	0.0830	0.0579	0.0571	0.0555	0.0571	0.0563	0.0551	0.0563	0.0558	0.0549	0.0555	0.0548	0.0537
				(0.0452)	(0.0296)	(0.0360)	(0.0279)	(0.0184)	(0.0201)	(0.0274)	(0.0175)	(0.0196)	(0.0261)	(0.0168)	(0.0194)	(0.0265)	(0.0165)	(0.0184)
		90	80	0.0898	0.0835	0.0818	0.0573	0.0566	0.0552	0.0564	0.0559	0.0547	0.0559	0.0553	0.0543	0.0549	0.0544	0.0532
				(0.0443)	(0.0291)	(0.0352)	(0.0274)	(0.0180)	(0.0192)	(0.0268)	(0.0171)	(0.0190)	(0.0255)	(0.0164)	(0.0190)	(0.0260)	(0.0161)	(0.0180)
			85	0.0886	0.0826	0.0807	0.0566	0.0560	0.0543	0.0558	0.0554	0.0487	0.0555	0.0550	0.0538	0.0545	0.0541	0.0527
				(0.0432)	(0.0286)	(0.0345)	(0.0268)	(0.0177)	(0.0188)	(0.0261)	(0.0168)	(0.0185)	(0.0250)	(0.0161)	(0.0184)	(0.0249)	(0.0159)	(0.0177)
(0.8, 1.6)	(25, 25)	30	20	0.1179	0.1002	0.1010	0.0708	0.0685	0.0649	0.0688	0.0675	0.0634	0.0679	0.0663	0.0630	0.0665	0.0657	0.0625
				(0.0792)	(0.0641)	(0.0634)	(0.0357)	(0.0277)	(0.0257)	(0.0349)	(0.0272)	(0.0248)	(0.0343)	(0.0258)	(0.0236)	(0.0336)	(0.0252)	(0.0229)
			25	0.1167	0.0993	0.0999	0.0701	0.0679	0.0640	0.0682	0.0670	0.0574	0.0675	0.0660	0.0625	0.0661	0.0654	0.0620
				(0.0775)	(0.0631)	(0.0620)	(0.0361)	(0.0279)	(0.0251)	(0.0339)	(0.0270)	(0.0248)	(0.0333)	(0.0261)	(0.0232)	(0.0325)	(0.0241)	(0.0225)
		40	30	0.1044	0.0955	0.0946	0.0704	0.0645	0.0625	0.0692	0.0621	0.0557	0.0678	0.0604	0.0592	0.0664	0.0599	0.0583
				(0.0739)	(0.0611)	(0.0598)	(0.0350)	(0.0259)	(0.0240)	(0.0325)	(0.0258)	(0.0236)	(0.0321)	(0.0250)	(0.0215)	(0.0305)	(0.0225)	(0.0213)
			35	0.1001	0.0931	0.0920	0.0674	0.0616	0.0609	0.0663	0.0608	0.0601	0.0659	0.0596	0.0587	0.0653	0.0590	0.0578
				(0.0721)	(0.0602)	(0.0590)	(0.0322)	(0.0241)	(0.0231)	(0.0309)	(0.0249)	(0.0225)	(0.0310)	(0.0239)	(0.0207)	(0.0296)	(0.0209)	(0.0205)
	(35, 35)	50	40	0.0972	0.0852	0.0852	0.0612	0.0585	0.0576	0.0601	0.0568	0.0560	0.0596	0.0561	0.0558	0.0588	0.0558	0.0553
				(0.0576)	(0.0352)	(0.0431)	(0.0346)	(0.0222)	(0.0261)	(0.0342)	(0.0214)	(0.0236)	(0.0327)	(0.0209)	(0.0228)	(0.0320)	(0.0207)	(0.0221)
			45	0.0935	0.0850	0.0830	0.0586	0.0570	0.0563	0.0580	0.0562	0.0556	0.0575	0.0555	0.0553	0.0571	0.0552	0.0547
				(0.0558)	(0.0350)	(0.0421)	(0.0335)	(0.0217)	(0.0253)	(0.0322)	(0.0209)	(0.0229)	(0.0314)	(0.0205)	(0.0221)	(0.0308)	(0.0199)	(0.0215)
		60	50	0.0936	0.0844	0.0837	0.0593	0.0576	0.0562	0.0586	0.0570	0.0552	0.0576	0.0564	0.0550	0.0567	0.0558	0.0544
				(0.0527)	(0.0331)	(0.0398)	(0.0328)	(0.0209)	(0.0238)	(0.0313)	(0.0203)	(0.0220)	(0.0299)	(0.0199)	(0.0212)	(0.0295)	(0.0196)	(0.0206)
			55	0.0923	0.0837	0.0824	0.0581	0.0568	0.0560	0.0575	0.0561	0.0553	0.0569	0.0554	0.0550	0.0562	0.0552	0.0544
				(0.0496)	(0.0319)	(0.0383)	(0.0319)	(0.0199)	(0.0224)	(0.0302)	(0.0195)	(0.0209)	(0.0293)	(0.0193)	(0.0205)	(0.0292)	(0.0190)	(0.0195)
	(50, 50)	80	70	0.0903	0.0836	0.0826	0.0579	0.0571	0.0556	0.0568	0.0562	0.0550	0.0562	0.0556	0.0547	0.0557	0.0552	0.0540
				(0.0452)	(0.0298)	(0.0354)	(0.0293)	(0.0182)	(0.0201)	(0.0277)	(0.0176)	(0.0198)	(0.0263)	(0.0170)	(0.0195)	(0.0269)	(0.0167)	(0.0186)
			75	0.0903	0.0832	0.0817	0.0573	0.0563	0.0548	0.0566	0.0559	0.0542	0.0559	0.0554	0.0539	0.0553	0.0546	0.0534
				(0.0443)	(0.0293)	(0.0357)	(0.0273)	(0.0182)	(0.0198)	(0.0271)	(0.0173)	(0.0193)	(0.0259)	(0.0165)	(0.0191)	(0.0261)	(0.0162)	(0.0180)
		90	80	0.0883	0.0823	0.0806	0.0564	0.0557	0.0543	0.0556	0.0551	0.0534	0.0550	0.0544	0.0532	0.0546	0.0541	0.0527
				(0.0432)	(0.0287)	(0.0345)	(0.0271)	(0.0177)	(0.0188)	(0.0263)	(0.0167)	(0.0187)	(0.0251)	(0.0162)	(0.0187)	(0.0256)	(0.0159)	(0.0177)
			85	0.0874	0.0817	0.0797	0.0557	0.0552	0.0537	0.0551	0.0547	0.0529	0.0544	0.0539	0.0527	0.0538	0.0535	0.0522
				(0.0419)	(0.0281)	(0.0340)	(0.0266)	(0.0175)	(0.0185)	(0.0257)	(0.0165)	(0.0182)	(0.0247)	(0.0159)	(0.0181)	(0.0245)	(0.0155)	(0.0173)

Table 4: ALs and CPs (in parenthesis) of the 95% ACIs and HPD credible intervals of α_1, α_2, and γ when $\Theta = (1.5,\ 1.0,\ 0.8)$ with different values of r and k with different values of $(T_1,\ T_2)$ when scale parameters are independent.

							HPD					
				ACI			Independent			Dependent		
$(T_1,\ T_2)$	$(n_1,\ n_2)$	r	k	α_1	α_2	γ	α_1	α_2	γ	α_1	α_2	γ
(0.7, 1.4)	(25, 25)	30	20	1.4764	0.9966	1.0924	1.0321	0.7680	0.8046	1.0215	0.7611	0.7947
				(0.9321)	(0.9152)	(0.9385)	(0.9439)	(0.9402)	(0.9428)	(0.9447)	(0.9414)	(0.9435)
			25	1.4020	0.9164	1.0238	0.9495	0.7218	0.7667	0.9241	0.7198	0.7595
				(0.9344)	(0.9187)	(0.9403)	(0.9451)	(0.9423)	(0.9440)	(0.9460)	(0.9441)	(0.9449)
		40	30	1.3950	0.9068	1.0124	0.9184	0.7032	0.7485	0.9021	0.6933	0.7416
				(0.9368)	(0.9194)	(0.9402)	(0.9448)	(0.9422)	(0.9443)	(0.9452)	(0.9431)	(0.9450)
			35	1.3242	0.8892	0.9416	0.8890	0.6940	0.7368	0.8763	0.6898	0.7262
				(0.9392)	(0.9217)	(0.9434)	(0.9466)	(0.9439)	(0.9458)	(0.9472)	(0.9445)	(0.9461)
	(35, 35)	50	40	1.2718	0.8596	0.8866	0.8568	0.6561	0.7091	0.8405	0.6538	0.6994
				(0.9362)	(0.9204)	(0.9406)	(0.9449)	(0.9423)	(0.9437)	(0.9455)	(0.9431)	(0.9442)
			45	1.2558	0.8496	0.8754	0.8507	0.6506	0.7009	0.8411	0.6479	0.6930
				(0.9375)	(0.9247)	(0.9417)	(0.9458)	(0.9431)	(0.9450)	(0.9462)	(0.9437)	(0.9455)
		60	50	1.1708	0.8256	0.8572	0.8391	0.6357	0.6805	0.8170	0.6298	0.6716
				(0.9398)	(0.9270)	(0.9436)	(0.9467)	(0.9444)	(0.9459)	(0.9470)	(0.9449)	(0.9462)
			55	1.1462	0.8200	0.8488	0.8266	0.6274	0.6663	0.8108	0.6180	0.6604
				(0.9417)	(0.9305)	(0.9452)	(0.9478)	(0.9459)	(0.9466)	(0.9482)	(0.9465)	(0.9469)
	(50, 50)	80	70	1.1040	0.7946	0.8224	0.8111	0.6052	0.6467	0.8003	0.6007	0.6401
				(0.9421)	(0.9329)	(0.9458)	(0.9483)	(0.9466)	(0.9471)	(0.9487)	(0.9470)	(0.9476)
			75	1.0833	0.7858	0.8162	0.7993	0.5921	0.6301	0.7899	0.5894	0.6225
				(0.9429)	(0.9352)	(0.9460)	(0.9487)	(0.9470)	(0.9476)	(0.9489)	(0.9474)	(0.9479)
		90	80	1.0728	0.7776	0.8050	0.7970	0.5961	0.6385	0.7894	0.5915	0.6306
				(0.9437)	(0.9375)	(0.9465)	(0.9491)	(0.9478)	(0.9482)	(0.9494)	(0.9484)	(0.9489)
			85	1.0566	0.7612	0.7918	0.7883	0.5722	0.6200	0.7694	0.5713	0.6112
				(0.9440)	(0.9387)	(0.9469)	(0.9503)	(0.9484)	(0.9488)	(0.9505)	(0.9489)	(0.9492)
(0.8, 1.6)	(25, 25)	30	20	1.1856	0.9344	1.0320	0.9480	0.7516	0.7877	0.9182	0.7479	0.7814
				(0.9368)	(0.9189)	(0.9403)	(0.9442)	(0.9427)	(0.9436)	(0.9453)	(0.9434)	(0.9441)
			25	1.1078	0.8812	0.9548	0.9034	0.7128	0.7587	0.8959	0.7042	0.7431
				(0.9389)	(0.9206)	(0.9415)	(0.9450)	(0.9431)	(0.9441)	(0.9455)	(0.9438)	(0.9447)
		40	30	1.0632	0.8680	0.9114	0.8900	0.6964	0.7397	0.8763	0.6922	0.7289
				(0.9396)	(0.9237)	(0.9419)	(0.9458)	(0.9436)	(0.9447)	(0.9461)	(0.9440)	(0.9449)
			35	1.0422	0.8588	0.8944	0.8821	0.6876	0.7355	0.8623	0.6768	0.7206
				(0.9411)	(0.9268)	(0.9425)	(0.9467)	(0.9444)	(0.9455)	(0.9470)	(0.9448)	(0.9459)
	(35, 35)	50	40	0.9946	0.8402	0.8762	0.8601	0.6741	0.7200	0.8454	0.6699	0.7060
				(0.9419)	(0.9295)	(0.9434)	(0.9474)	(0.9453)	(0.9462)	(0.9476)	(0.9459)	(0.9465)
			45	0.9864	0.8276	0.8650	0.8515	0.6662	0.7087	0.8410	0.6610	0.7002
				(0.9422)	(0.9335)	(0.9440)	(0.9479)	(0.9457)	(0.9468)	(0.9484)	(0.9460)	(0.9472)
		60	50	0.8044	0.7191	0.7541	0.7441	0.5641	0.6102	0.7384	0.5628	0.6023
				(0.9430)	(0.9374)	(0.9446)	(0.9481)	(0.9463)	(0.9472)	(0.9486)	(0.9468)	(0.9476)
			55	0.8624	0.8206	0.8456	0.8514	0.6606	0.6956	0.8269	0.6485	0.6874
				(0.9433)	(0.9392)	(0.9450)	(0.9485)	(0.9465)	(0.9476)	(0.9488)	(0.9469)	(0.9480)
	(50, 50)	80	70	0.8167	0.8072	0.8383	0.8416	0.6434	0.6841	0.8205	0.6385	0.6779
				(0.9438)	(0.9405)	(0.9454)	(0.9489)	(0.9470)	(0.9479)	(0.9491)	(0.9474)	(0.9485)
			75	0.8052	0.8026	0.8328	0.8351	0.6387	0.6806	0.8124	0.6314	0.6635
				(0.9442)	(0.9411)	(0.9458)	(0.9492)	(0.9475)	(0.9483)	(0.9495)	(0.9481)	(0.9487)
		90	80	0.7934	0.7958	0.8242	0.8293	0.6268	0.6669	0.8083	0.6161	0.6606
				(0.9447)	(0.9420)	(0.9463)	(0.9496)	(0.9479)	(0.9485)	(0.9498)	(0.9486)	(0.9489)
			85	0.7863	0.7890	0.8164	0.8208	0.6220	0.6608	0.8049	0.6207	0.6506
				(0.9457)	(0.9436)	(0.9472)	(0.9504)	(0.9489)	(0.9494)	(0.9506)	(0.9489)	(0.9496)

Table 5: ETT under UHCS and JUHCS when $\Theta = (1.5,\ 1.0,\ 0.8)$ with different values of r and k with different values of $(T_1,\ T_2)$ when scale parameters are independent.

$(T_1,\ T_2)$	$(n_1,\ n_2)$	r	k	UHCS	JUHCS
(0.7, 1.4)	(25, 25)	30	20	1.1912	1.1369
			25	1.2889	1.1395
		40	30	1.6768	1.5518
			35	1.6912	1.5547
	(35, 35)	50	40	1.5110	1.3559
			45	1.5121	1.3582
		60	50	1.7790	1.7250
			55	1.7802	1.7254
	(50, 50)	80	70	1.7153	1.5664
			75	1.7176	1.5693
		90	80	1.8326	1.6915
			85	1.8542	1.7205
(0.8, 1.6)	(25, 25)	30	20	1.2182	1.1666
			25	1.2505	1.1645
		40	30	1.7123	1.5534
			35	1.7247	1.5543
	(35, 35)	50	40	1.5028	1.3587
			45	1.5169	1.3590
		60	50	1.8994	1.7392
			55	1.9013	1.7402
	(50, 50)	80	70	1.8999	1.5695
			75	1.9014	1.5704
		90	80	2.0235	1.9187
			85	2.0244	1.9197

7 Insulating fluid data analysis

In order to demonstrate the applicability of the proposed censoring scheme in real-life situations, two samples of size 10 from two different groups have been considered. This data set, reported by Nelson [23], is associated with the breakdown times of an insulating fluid due to high voltage stress. The breakdown time data set has been tabulated in Table 6. Many Authors discussed this data, see, for example, Shafay et al. [36].

Table 6: Breakdown times of two groups from given real data set.

Group	Data
G_1	1.99, 0.64, 2.15, 1.08, 2.57, 0.93, 4.75, 0.82, 2.06, 0.49
G_2	8.11, 3.17, 5.55, 0.80, 0.20, 1.13, 6.63, 1.08, 2.44, 0.78

Let us consider, the combined samples from these two groups G_1 and G_2 as $\{u_{(1)}, \cdots, u_{(N)}\}$ with $N = n_1 + n_2$, where $n_1 = n_2 = 10$. Then, five different JUHCS samples have been generated by using the data given in Table 6, for different values of (m, k) when $(T_1, T_2) = (1.25, 2.5)$. All these JUHCS samples and their associated termination time T^* are mentioned below in the Table 7.

Table 7: JUHCS samples generated from given real data set.

Sample	m	k	Data	D^*	T^*
I	8	4	{0.20, 0.49, 0.64, 0.78, 0.80, 0.82, 0.93, 1.08, 1.08, 1.13}	10	1.25
II	12	6	{0.20, 0.49, 0.64, 0.78, 0.80, 0.82, 0.93, 1.08, 1.08, 1.13, 1.99, 2.06}	12	2.06
III	14	11	{0.20, 0.49, 0.64, 0.78, 0.80, 0.82, 0.93, 1.08, 1.08, 1.13, 1.99, 2.06, 2.15, 2.44}	14	2.44
IV	16	8	{0.20, 0.49, 0.64, 0.78, 0.80, 0.82, 0.93, 1.08, 1.08, 1.13, 1.99, 2.06, 2.15, 2.44}	14	2.50
V	18	15	{0.20, 0.49, 0.64, 0.78, 0.80, 0.82, 0.93, 1.08, 1.08, 1.13, 1.99, 2.06, 2.15, 2.44, 2.57}	15	2.57

Before conducting the data analysis, the goodness-of-fit test of this data for the Weibull distribution is performed using the Kolmogorov-Smirnov (K-S) test. The K-S test statistic and the corresponding p-values (in parentheses) for L-I and L-II are obtained as 0.1726 (0.8792) and 0.2092 (0.7005), respectively. To assess the goodness-of-fit graphically, the empirical CDF (ECDF), probability-probability (P-P), and quantile-quantile (Q-Q) plots for L-I and L-II are displayed in Figure 2 and Figure 3, respectively. Based on the K-S tests and the graphs in the figures, this data set can fit reasonably using the Weibull distribution.

Based on the five JUHCS data sets, the point and interval estimates of the model parameters are obtained and tabulated in Table 8 and Table 9, respectively. In order to obtain Bayes estimates, non-

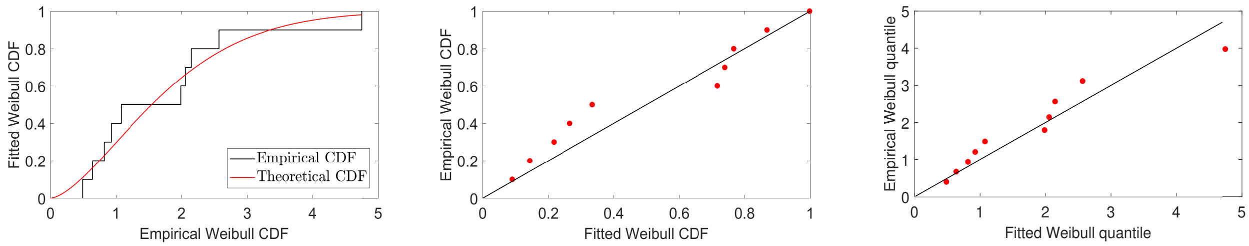

Figure 2: (a) ECDF, (b) P-P, and (c) Q-Q plots for the Weibull distribution with real data I for L-I.

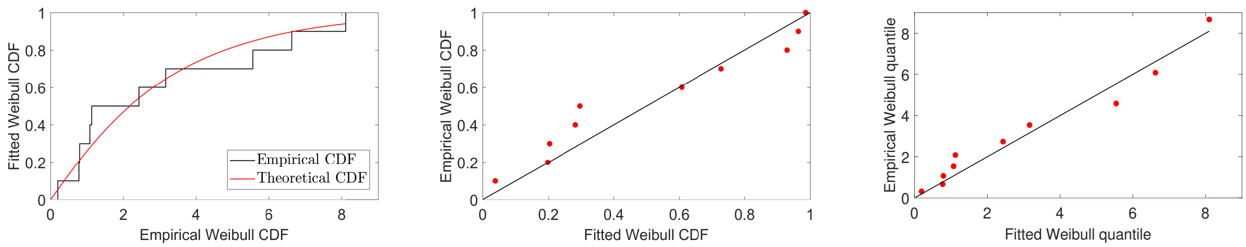

Figure 3: (a) ECDF, (b) P-P, and (c) Q-Q plots for the Weibull distribution with real data I for L-II.

Table 8: Average values and SEs (in parenthesis) of the MLE and Bayes estimate of the parameters based on the JUHCS samples.

	MLE			SELF			LLF		
Sample	α_1	α_2	γ	α_1	α_2	γ	α_1	α_2	γ
I	0.4381	0.4219	2.1830	0.4683	0.4647	1.4156	0.4581	0.4541	1.3540
	(0.0395)	(0.0371)	(0.4111)	(0.0225)	(0.0206)	(0.2795)	(0.0214)	(0.0202)	(0.2562)
II	0.3959	0.2778	1.4409	0.4956	0.3576	1.0149	0.4859	0.3497	0.9959
	(0.0284)	(0.0185)	(0.1353)	(0.0149)	(0.0121)	(0.0629)	(0.0138)	(0.0108)	(0.0593)
III	0.4047	0.2683	1.4689	0.3808	0.3194	0.9475	0.4107	0.2988	0.9302
	(0.0278)	(0.0164)	(0.1159)	(0.0133)	(0.0118)	(0.0581)	(0.0124)	(0.0103)	(0.0575)
IV	0.4058	0.2670	1.4434	0.3781	0.3892	0.9081	0.3754	0.3840	0.9203
	(0.0267)	(0.0161)	(0.1118)	(0.0142)	(0.0119)	(0.0602)	(0.0133)	(0.0114)	(0.0589)
V	0.4268	0.2445	1.5226	0.3729	0.2942	0.8827	0.3602	0.3868	0.8709
	(0.0290)	(0.0141)	(0.1139)	(0.0131)	(0.0110)	(0.0573)	(0.0122)	(0.0104)	(0.0534)

Table 9: 95% ACIs and HPD credible intervals and their lengths (in parenthesis) of the parameters based on the JUHCS samples.

	ACI			HPD		
Sample	α_1	α_2	γ	α_1	α_2	γ
I	(0.0482, 0.8281)	(0.0440, 0.7996)	(0.9263, 3.4397)	(0.2307, 0.6488)	(0.2606, 0.6765)	(0.7029, 2.2686)
	(0.7799)	(0.7556)	(2.5134)	(0.4181)	(0.4159)	(1.5657)
II	(0.0654, 0.7263)	(0.0102, 0.5454)	(0.7197, 2.1620)	(0.2369, 0.6028)	(0.2729, 0.6179)	(0.4256, 1.5291)
	(0.6609)	(0.5352)	(1.4423)	(0.3659)	(0.3450)	(1.1035)
III	(0.0775, 0.7019)	(0.0171, 0.5195)	(0.8015, 2.1363)	(0.2021, 0.5910)	(0.2240, 0.5611)	(0.6703, 1.2515)
	(0.6244)	(0.5024)	(1.3348)	(0.3889)	(0.3371)	(0.8812)
IV	(0.0781, 0.7334)	(0.0167, 0.5172)	(0.7882, 2.0987)	(0.2142, 0.5895)	(0.2387, 0.5638)	(0.4754, 1.3984)
	(0.6554)	(0.5005)	(1.3105)	(0.3753)	(0.3251)	(0.9240)
V	(0.0912, 0.7624)	(0.0117, 0.4773)	(0.8610, 2.1842)	(0.2335, 0.5726)	(0.2410, 0.5539)	(0.5104, 1.2021)
	(0.6712)	(0.4656)	(1.3232)	(0.3391)	(0.3129)	(0.6917)

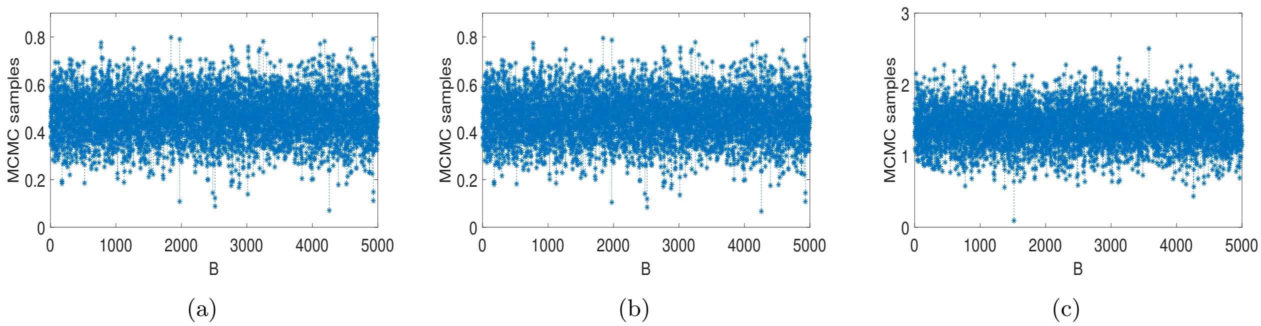

Figure 4: Trace plots of MCMC samples of (a) α_1, (b) α_2, and (c) γ for Weibull distributions under JUHCS data of sample I.

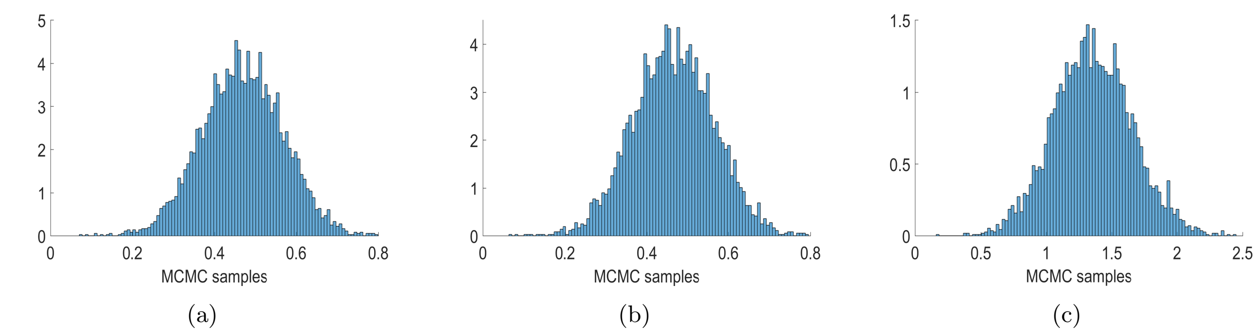

Figure 5: Histogram plots of MCMC samples of (a) α_1, (b) α_2, and (c) γ for Weibull distributions under JUHCS data of sample I.

informative priors have been considered making the hyper-parameters $a = b = c = d = 0$. The MLEs and Bayes estimates in Table 8 are remarkably comparable. Bayes estimates under LLF perform better than other estimates based on standard errors (SEs). Based on the widths of the 95% interval estimates, HPD intervals are narrower than the ACIs in Table 9. The trace plots of the MCMC samples and the associated histograms corresponding to sample I are plotted in Figures 4 and 5. These figures show that the MCMC samples converge to some finite values.

8 Conclusions

Findings

In this study, a novel joint censoring scheme has been proposed. Based on the suggested design, the MLEs, and the Bayes estimates under two different loss functions for two Weibull populations with repeated shapes and different scale parameters have been established. The dependent and independent prior distributions are considered to see the effect on the Bayes estimates. Then the ACIs of the model parameters are obtained using asymptotic normality properties of the MLEs. Additionally, the HPD credible intervals are also constructed. An extensive Monte Carlo simulation study is assessed to compare the performance of the proposed estimates. The results indicated that the Bayes estimates, together with their corresponding credible intervals, outperform the other estimators in a very satisfactory manner. A comparison of expected test time is done between UHCS and JUHCS, where JUHCS has an expected test time smaller than the UHCS. Further breakdown time of the insulating fluid data set has been analyzed to demonstrate all of the inferential conclusions made.

Research limitations

In case of joint censoring, when the data set is heavily censored, model identifiability becomes a major problem, leading to higher variance and decreased estimation precision. Since the current approach is novel joint censoring, it has similar limitations.

Recommendations for future research

Finally, as a matter for future research, it could be crucial to take into account the issue of predicting the failure times of units set up on the suggested censoring scheme for two populations. Following Zhu et al. ([37] and [38]) one may consider the study of joint unified hybrid censoring if two scale parameters are equal under the Weibull model.

Data availability

Dataset is available within the article.

Declaration of competing interest

We have declared that there is no potential conflict of interest in the article.

Funding Information

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References

[1] N. Balakrishnan and A. Rasouli, "Exact likelihood inference for two exponential populations under joint Type II censoring," *Computational Statistics & Data Analysis*, vol. 52, no. 5, pp. 2725–2738, 2008.

[2] A. Rasouli and N. Balakrishnan, "Exact likelihood inference for two exponential populations under joint progressive type II censoring," *Communications in Statistics—Theory and Methods*, vol. 39, no. 12, pp. 2172–2191, 2010.

[3] N. Balakrishnan, F. Su, and K.-Y. Liu, "Exact likelihood inference for k exponential populations under joint progressive type II censoring," *Communications in Statistics-Simulation and Computation*, vol. 44, no. 4, pp. 902–923, 2015.

[4] F. Su and X. Zhu, "Exact likelihood inference for two exponential populations based on a joint

generalized type I hybrid censored sample," *Journal of Statistical Computation and Simulation*, vol. 86, no. 7, pp. 1342–1362, 2016.

[5] S. Ashour and O. Abo-Kasem, "Statistical inference for two exponential populations under joint progressive type I censored scheme," *Communications in Statistics-Theory and Methods*, vol. 46, no. 7, pp. 3479–3488, 2017.

[6] S. Mondal and D. Kundu, "Exact inference on multiple exponential populations under a joint type II progressive censoring scheme," *Statistics*, vol. 53, no. 6, pp. 1329–1356, 2019.

[7] S. Mondal and D. Kundu, "Inference on Weibull parameters under a balanced two-sample type II progressive censoring scheme," *Quality and Reliability Engineering International*, vol. 36, no. 1, pp. 1–17, 2020.

[8] O. Abo-Kasem and A. Elshahhat, "A new two sample generalized type II hybrid censoring scheme," *American Journal of Mathematical and Management Sciences*, vol. 41, no. 2, pp. 170–184, 2022.

[9] B. Epstein, "Truncated life tests in the exponential case," *The Annals of Mathematical Statistics*, pp. 555–564, 1954.

[10] A. Childs, B. Chandrasekar, N. Balakrishnan, and D. Kundu, "Exact likelihood inference based on type I and type II hybrid censored samples from the exponential distribution," *Annals of the Institute of Statistical Mathematics*, vol. 55, pp. 319–330, 2003.

[11] B. Chandrasekar, A. Childs, and N. Balakrishnan, "Exact likelihood inference for the exponential distribution under generalized type I and type II hybrid censoring," *Naval Research Logistics (NRL)*, vol. 51, no. 7, pp. 994–1004, 2004.

[12] N. Balakrishnan, Q. Xie, and D. Kundu, "Exact inference for a simple step-stress model from the exponential distribution under time constraint," *Annals of the Institute of Statistical Mathematics*, vol. 61, pp. 251–274, 2009.

[13] H. Panahi and A. Sayyareh, "Estimation and prediction for a unified hybrid censored Burr type XII distribution," *Journal of Statistical Computation and Simulation*, vol. 86, no. 1, pp. 55–73, 2016.

[14] S. F. Ateya, "Estimation under inverse Weibull distribution based on Balakrishnan's unified hybrid censored scheme," *Communications in Statistics-Simulation and Computation*, vol. 46, no. 5,

pp. 3645–3666, 2017.

[15] H. Panahi, “Estimation of the burr type III distribution with application in unified hybrid censored sample of fracture toughness,” *Journal of applied Statistics*, vol. 44, no. 14, pp. 2575–2592, 2017.

[16] T. Sen, R. Bhattacharya, B. Pradhan, and Y. M. Tripathi, “Statistical inference and Bayesian optimal life-testing plans under type II unified hybrid censoring scheme,” *Quality and Reliability Engineering International*, vol. 37, no. 1, pp. 78–89, 2021.

[17] S. Dutta and S. Kayal, “Bayesian and non-Bayesian inference of Weibull lifetime model based on partially observed competing risks data under unified hybrid censoring scheme,” *Quality and Reliability Engineering International*, vol. 38, no. 7, pp. 3867–3891, 2022.

[18] S. Dutta, H. K. T. Ng, and S. Kayal, “Inference for a general family of inverted exponentiated distributions under unified hybrid censoring with partially observed competing risks data,” *Journal of Computational and Applied Mathematics*, vol. 422, p. 114934, 2023.

[19] P. R. Tadikamalla, “Applications of the Weibull distribution in inventory control,” *Journal of the Operational Research Society*, vol. 29, no. 1, pp. 77–83, 1978.

[20] M. T. Velázquez, J. H. Rodríguez, M. V. Del Carmen, F. F. Murrieta, G. T. Eslava, *et al.*, “Application of the Weibull distribution to estimate the volume of water pumping by a windmill,” *Journal of Power and Energy Engineering*, vol. 4, no. 09, p. 36, 2016.

[21] K. Azad, M. Rasul, P. Halder, and J. Sutariya, “Assessment of wind energy prospect by Weibull distribution for prospective wind sites in Australia,” *Energy Procedia*, vol. 160, pp. 348–355, 2019.

[22] F. H. Mahmood, A. K. Resen, and A. B. Khamees, “Wind characteristic analysis based on Weibull distribution of Al-Salman site, Iraq,” *Energy reports*, vol. 6, pp. 79–87, 2020.

[23] W. B. Nelson, *Applied life data analysis*. John Wiley & Sons, 2005.

[24] F. Su, *Exact likelihood inference for multiple exponential populations under joint censoring*. PhD thesis, 2013.

[25] N. Balakrishnan, A. Rasouli, and N. Sanjari Farsipour, “Exact likelihood inference based on an unified hybrid censored sample from the exponential distribution,” *Journal of Statistical Computation and Simulation*, vol. 78, no. 5, pp. 475–488, 2008.

[26] T. Zhu, “Reliability estimation for two-parameter Weibull distribution under block censoring,”

Reliability Engineering & System Safety, vol. 203, p. 107071, 2020.

[27] S. Mondal and D. Kundu, "Point and interval estimation of Weibull parameters based on joint progressively censored data," *Sankhya b*, vol. 81, pp. 1–25, 2019.

[28] M. S. Bartlett, "Approximate confidence intervals. II. More than one unknown parameter," *Biometrika*, vol. 40, no. 3/4, pp. 306–317, 1953.

[29] W. James, C. Stein, *et al.*, "Estimation with quadratic loss," in *Proceedings of the fourth Berkeley symposium on mathematical statistics and probability*, vol. 1, pp. 361–379, University of California Press, 1961.

[30] H. R. Varian, "A Bayesian approach to real estate assessment," *Studies in Bayesian econometrics and statistics in honor of Leonard J. Savage*, 1975.

[31] B. C. Arnold and S. J. Press, "Bayesian inference for Pareto populations," *Journal of Econometrics*, vol. 21, no. 3, pp. 287–306, 1983.

[32] L. Devroye, "A simple algorithm for generating random variates with a log-concave density," *Computing*, vol. 33, no. 3-4, pp. 247–257, 1984.

[33] E. A. Peña and A. K. Gupta, "Bayes estimation for the Marshall–Olkin exponential distribution," *Journal of the Royal Statistical Society Series B: Statistical Methodology*, vol. 52, no. 2, pp. 379–389, 1990.

[34] D. Kundu, "Bayesian inference and life testing plan for the Weibull distribution in presence of progressive censoring," *Technometrics*, vol. 50, no. 2, pp. 144–154, 2008.

[35] S. Dutta and S. Kayal, "Estimation and prediction for Burr type III distribution based on unified progressive hybrid censoring scheme," *Journal of Applied Statistics*, pp. 1–33, 2022.

[36] A. Shafay, N. Balakrishnan, and Y. Abdel-Aty, "Bayesian inference based on a jointly type II censored sample from two exponential populations," *Journal of Statistical Computation and Simulation*, vol. 84, no. 11, pp. 2427–2440, 2014.

[37] X. Zhu, N. Balakrishnan, and Y. Zhou, "Exact likelihood-ratio tests for a simple step-stress cumulative exposure model with censored exponential data," *Methodology and Computing in Applied Probability*, vol. 22, pp. 497–509, 2020.

[38] X. Zhu, N. Balakrishnan, and H.-Y. So, "Exact one-and two-sample likelihood ratio tests based on

time-constrained life-tests from exponential distributions," *Methodology and Computing in Applied Probability*, pp. 1–20, 2022.

Appendix

Appendix A.1 Proof of Theorem 3.1: Differentiating $l(\gamma)$ given in (3.4) with respect to γ and equating to zero, we get

$$\frac{1}{D^*}\sum_{i=1}^{D^*}\log u_i + \frac{1}{\gamma} = \frac{1}{D^*}\sum_{i=1}^{2} d_i^* \frac{\Phi_i'(\gamma)}{\Phi_i(\gamma)}.$$

Let us consider, $\Psi_1(\gamma) = \frac{1}{D^*}\sum\limits_{i=1}^{D^*}\log u_i + \frac{1}{\gamma}$ and $\Psi_2(\gamma) = \frac{1}{D^*}\sum\limits_{i=1}^{2} d_i^* \frac{\Phi_i'(\gamma)}{\Phi_i(\gamma)}$. Here, $\Psi_1'(\gamma) = -\frac{1}{\gamma^2} < 0$, which shows that $\Psi_1(\gamma)$ decreases in the range $\left(\frac{1}{D^*}\sum\limits_{i=1}^{D^*}\log u_i, \infty\right)$. For $\Psi_2(\gamma)$, it has been obtained that

$$\Psi_2'(\gamma) = \frac{1}{D^*}\sum_{i=1}^{2}\left[\left(\sum_{j=1}^{d_i^*} u_{ij}^{\gamma}(\log u_{ij})^2\right)\left(\sum_{j=1}^{d_i^*} u_{ij}^{\gamma} + (n_i - d_i^*)\right) - \left(\sum_{j=1}^{d_i^*} u_{ij}^{\gamma}\log u_{ij}\right)^2\right].$$

Using Cauchy-Schwarz inequality, it has been obtained that $\Psi_2'(\gamma) > 0$, which represents that $\Psi_2^(\gamma)$ is increasing in γ with

$$\lim_{\gamma\to 0}\Psi_2(\gamma) = \frac{1}{D^*}\sum_{i=1}^{2}\left[\frac{d_i^*}{n_i - d_i^*}\sum_{j=1}^{d_i^*}\log u_{ij}\right], \quad \text{and} \quad \lim_{\gamma\to\infty}\Psi_2(\gamma) = \frac{1}{D^*}\sum_{i=1}^{2}\sum_{j=1}^{d_i^*}\log u_{ij}.$$

Thus the curves $\Psi_1(\gamma)$ and $\Psi_2(\gamma)$ have unique intersection, which yields that the MLE of γ exists and is unique.