Weighted Least Squares: A Robust Method of Estimation for Sinusoidal Model

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Abstract

In this paper we consider the weighted least squares estimators (WLSEs) of the unknown parameters of a multiple sinusoidal model. Although, the least squares estimators (LSEs) are known to be the most efficient estimators in case of a multiple sinusoidal model, they are quite susceptible in presence of outliers. In presence of outliers, robust estimators like the least absolute deviation estimators (LADEs) or Huber's M-estimators (HMEs) may be used. But implementation of the LADEs and HMEs are quite challenging in case of a sinusoidal model, the problem becomes more severe in case of multiple sinusoidal model. Moreover, to derive the theoretical properties of the robust estimators one needs stronger assumptions on the error random variables than what are needed for the LSEs. The proposed WLSEs are used as robust estimators and they have the following two major advantages. First, they can be implemented very easily in case of multiple sinusoidal model, and their properties can be obtained under the same set of error assumptions as the LSEs. Extensive simulation results suggest that in presence of outliers, the WLSEs behave better than the LSEs, and at par with the LADEs and HEMs. It is observed that the performance of the WLSEs depend on the weight function, and we discuss how to choose a proper weight function for a given data set. We have analyzed one synthetic data set to show how the proposed methods can be implemented in practice.

Key Words and Phrases: sinusoidal model; least squares estimators; weighted least squares estimators; robust estimators; consistency; asymptotic normality.

AMS Subject Classifications: 62F10, 62F03, 62H12.

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1 Introduction

In this paper we have considered the following multicomponent sinusoidal model:

$$y(t) = \sum_{k=1}^{p} \left\{ A_k^0 \cos(\theta_k^0 t) + B_k^0 \sin(\theta_k^0 t) \right\} + \epsilon(t); \qquad t = 1, \ldots, N. \tag{1}$$

Here θ_k^0's are frequencies, A_k^0's and B_k^0's are linear coefficients, $A_k^{0^2} + B_k^{0^2}$ is known as amplitude corresponds to the frequency θ_k^0, $\epsilon(t)$'s are additive error random variables with mean zero, and it has certain correlation structure which will be mentioned later. The integer p is known as the number of sinusoidal component, and at this moment it is assumed to be known. Our problem is to estimate the unknown parameters; $\{(A_k^0, B_k^0, \theta_k^0); k = 1, \ldots, p\}$, based on the sample $\{y(t); t = 1, \ldots, N\}$ under suitable error assumptions on $\epsilon(t)$'s.

This is a very well known model in the statistical signal processing and time series analysis. An extensive amount of work has been done in developing several efficient estimation procedures and deriving their properties under a suitable assumptions on the error random variables, see for example the recent monograph by Nandi and Kundu (2020) who have provided a comprehensive review of different estimation procedures and their theoretical properties when ever they are available. This model can be used very effectively in analyzing any periodic signals with a suitable choice of the unknown parameters including the number of components p. In this paper it is assumed that p is known.

The model (1) is a non-linear regression model, and under the assumptions of independent and identically distributed (i.i.d.) error random variables, the least squares estimators (LSEs) seem to be the most natural choice. Unfortunately, the model does not satisfy the standard sufficient conditions of Jennrich (1969) or Wu (1981) which are needed so that the LSEs become consistent and asymptotically normally distributed. Although, the model does not satisfy the sufficient conditions of Jennrich (1969) or Wu (1981), it can be shown that under a fairly general set of errors assumption, see Assumption 1, the LSEs are consistent and asymptotically normally distributed. The Assumption 1 can be described as follows:

Assumption 1: The error random variables $\epsilon(t)$s have the following structure

$$\epsilon(t) = \sum_{k=-\infty}^{\infty} a(k)e(t-k).$$

Here $e(t)$'s are i.i.d. random variables with mean zero and variance σ^2, and $a(k)$'s are such that

$$\sum_{k=-\infty}^{\infty} |a(k)| < \infty.$$

Although, the LSEs are the most efficient estimators in the sense they have the best convergence rates, it is known that they are not robust. The performances of the LSEs affect significantly in presence of few outliers only. Due to this reason Kim et al. (2000) proposed the least absolute deviation estimators (LADEs) of the unknown of the model (1) as an alternative to the LSEs. They have obtained the consistency and asymptotic normality properties of the LADEs under a fairly restrictive set of assumptions on the error random variables. It is assumed that $\epsilon(t)$'s are i.i.d. absolute continuous random variables with certain restrictions on the probability density functions. Moreover, implementation of the LADEs or HMEs in practice is a computationally challenging problem. It involves solving a $3p$ dimensional optimization problem. The problem becomes more challenging if p is large.

The aim of this paper is the following. In this paper we use the weighted least squares estimators (WLSEs) as robust estimators of the unknown parameters of the model (1) in presence of outliers. At first, we choose the weight function to be a finite degree polynomial, and we provide the WLSEs for one component ($p = 1$) sinusoidal model. The WLSEs can be obtained by solving a one dimensional optimization problem. We derive the consistency and asymptotic normality properties of the WLSEs based on the error Assumption 1. Further we extend the result for general p, and we propose to use the sequential WLSEs in this case. The sequential WLSEs can be obtained by solving p one dimensional optimization problems.

Hence, implementation of the sequential WLSEs becomes quite simple in practice. We derive the consistency and asymptotic normality properties of the sequential WLSEs under the same error Assumption 1.

Extensive simulations have been performed to compare the performances of the WLSEs with the LSEs, LADEs and HMEs. It is observed that in presence of outliers, the WLSEs perform much better than the LSEs and perform at par with the LADEs or HMEs. It is observed that implementation of the LADEs or HMEs in case of large p is very difficult in practice. Finding initial guesses for solving a $3p$ dimensional optimization problem is quite challenging, where as implementation of the sequential WLSEs is quite straight forward. It is observed that the performance of the WLSEs depends on the choice of the weight function. Therefore, one natural question is how to choose a proper weight function. In this paper we have proposed a data driven criteria to choose a proper weight function to compute the WLSEs. One synthetic data set has been analyzed and the results are quite satisfactory.

The rest of the paper is organized as follows. In Section 2 we have provided the WLSEs for one component sinusoidal model and sequential WLSEs for multicomponent sinusoidal model. The consistency and asymptotic normality properties of the WLSEs and sequential WLSEs are provided in Section 3. In Section 4 the simulation results have been provided. The choice of a proper penalty function and the analysis of a synthetic data set have been presented in Section 5. Finally we conclude the paper in Section 6.

2 Weighted Least Squares Estimators

In this section first we present the WLSEs for one component sinusoidal model, and then we provide the sequential WLSEs for multicomponent sinusoidal model. We use the following notations. Consider a polynomial of degree k as follows:

$$w(t) = c_0 + c_1 t + \ldots + c_k t^k; \qquad t \geq 0. \tag{2}$$

Here, $c_0, \ldots, c_k$ are the coefficients of the polynomial $w(t)$, and they satisfy the following conditions

$$\inf_{0 \leq t \leq 1} w(t) \geq \gamma > 0. \tag{3}$$

2.1 One Component Sinusoidal Model

In this section we consider the one component sinusoidal model, i.e.

$$y(t) = A^0 \cos(\theta^0 t) + B^0 \sin(\theta^0 t) + \epsilon(t); \qquad t = 1, \ldots, N. \tag{4}$$

Let us define the weighted residual sum of squares as follows:

$$Q(A, B, \theta) = \sum_{i=1}^{N} w\left(\frac{t}{N}\right) \left(y(t) - A\cos(\theta t) - B\sin(\theta t)\right)^2. \tag{5}$$

The WLSEs of A^0, B^0 and θ^0, say $\widehat{A}$, $\widehat{B}$ and $\widehat{\theta}$, respectively, can be obtained by minimizing $Q(A, B, \theta)$ with respect to A, B and θ. Hence, for fixed θ, the WLSEs of A^0 and B^0, say $\widehat{A}(\theta)$ and $\widehat{B}(\theta)$, respectively, can be obtained as

$$\widehat{A}(\theta) = \frac{1}{a_{11}a_{22} - (a_{12})^2}\left[a_{22}d_1 - a_{12}d_2\right] \quad \text{and} \quad \widehat{B}(\theta) = \frac{1}{a_{11}a_{22} - (a_{12})^2}\left[a_{11}d_2 - a_{12}d_1\right], \tag{6}$$

where

$$a_{11} = \frac{1}{N}\sum_{t=1}^{N} w\left(\frac{t}{N}\right)\cos^2(\theta t), \qquad a_{11} = \frac{1}{N}\sum_{t=1}^{N} w\left(\frac{t}{N}\right)\sin^2(\theta t),$$

$$a_{12} = a_{21} = \frac{1}{N}\sum_{t=1}^{N} w\left(\frac{t}{N}\right)\cos(\theta t)\sin(\theta t),$$

$$d_1 = \frac{1}{N}\sum_{t=1}^{N} w\left(\frac{t}{N}\right) y(t)\cos(\theta t), \qquad d_2 = \frac{1}{N}\sum_{t=1}^{N} w\left(\frac{t}{N}\right) y(t)\sin(\theta t).$$

Once, $\widehat{A}(\theta)$ and $\widehat{B}(\theta)$ are obtained, $\widehat{\theta}$ can be obtained by minimizing

$$Q(\widehat{A}(\theta), \widehat{B}(\theta), \theta) = \sum_{i=1}^{N} w\left(\frac{t}{N}\right)\left(y(t) - \widehat{A}(\theta)\cos(\theta t) - \widehat{B}(\theta)\sin(\theta t)\right)^2, \tag{7}$$

with respect to θ. Now observe that as $N \to \infty$,

$$a_{11} \to \frac{1}{2}\int_0^1 w(t)dt, \quad a_{22} \to \frac{1}{2}\int_0^1 w(t)dt, \quad a_{12} = a_{21} \to 0. \tag{8}$$

Therefore, as $N \to \infty$

$$\widehat{A}(\theta) \to \frac{2d_1}{\int_0^1 w(t)dt} \quad \text{and} \quad \widehat{B}(\theta) \to \frac{2d_2}{\int_0^1 w(t)dt}. \tag{9}$$

The theoretical properties of the WLSEs will be provided in Section 3.

2.2 Multi Component Sinusoidal Model

In this section we provide the estimates of the unknown parameters of the multi component sinusoidal model (1). It is assumed that p is known. The natural extension in case of the multi component model will be to obtain the WLSEs of the unknown parameters by minimizing the following objective function

$$Q_p(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_p) = \sum_{t=1}^{N} w\left(\frac{t}{N}\right)\left(y(t) - \sum_{k=1}^{p}(A_k\cos(\theta_k t) + B_k\sin(\theta_k t))\right)^2, \tag{10}$$

with respect to $\boldsymbol{\theta}_k = (A_k, B_k, \theta_k)^\top$, for $k = 1, \ldots, p$. It can be shown along the same line as the one component model, that the WLSEs of the unknown parameters can be obtained by solving a p dimensional optimization problem, instead of solving a $3p$ dimensional optimization problem. The WLSEs in case of multi component sinusoidal model will be denoted by $(\widehat{A}_k, \widehat{B}_k, \widehat{\theta}_k)^\top$, for $k = 1, \ldots, p$. The theoretical properties of the WLSEs can be established, and it will be presented in Section 3. But the problem can be quite challenging if p is large. Due to this reason, we propose asymptotically equivalent sequential WLSEs, and they can be described as follows.

It is assumed that the data $\{y(t); t = 1, \ldots, N\}$ are coming from the model (1), and without loss of generality it is assumed that

$$A_1^{0^2} + B_1^{0^2} > A_2^{0^2} + B_2^{0^2} > \ldots > A_p^{0^2} + B_p^{0^2} > 0. \tag{11}$$

Now obtain the estimates of $(A_1^0, B_1^0, \theta_1^0)$ by minimizing

$$Q_1(A_1, B_1, \theta_1) = \sum_{t=1}^{N} w\left(\frac{t}{N}\right)\left(y(t) - A_1\cos(\theta_1 t) - B_1\sin(\theta_1 t)\right)^2, \tag{12}$$

with respect to (A_1, B_1, θ_1). Let these estimates be denoted by $\widetilde{A}_1$, $\widetilde{B}_1$ and $\widetilde{\theta}_1$. Now obtain the new data, by removing the effect of the first component as follows:

$$\widetilde{y}(t) = y(t) - \widehat{A}_1\cos(\widetilde{\theta}_1 t) - \widetilde{B}_1\sin(\widetilde{\theta}_1 t); \quad t = 1, \ldots, N. \tag{13}$$

Now obtain the estimates of $(A_2^0, B_2^0, \theta_2^0)$ by minimizing

$$Q_2(A_2, B_2, \theta_2) = \sum_{t=1}^{N} w\left(\frac{t}{N}\right)\left(\widetilde{y}(t) - A_2\cos(\theta_2 t) - B_2\sin(\theta_2 t)\right)^2, \tag{14}$$

with respect to (A_2, B_2, θ_2). Continue the process p times, to obtain estimates of the unknown parameters. We call these estimates as the sequential WLSEs. It is important to mention that the WLSEs and the sequential WLSEs have the same rate of convergence. The theoretical properties of the estimators will be presented in Section 3. The sequential WLSEs in case of multi component sinusoidal model will be denoted by $(\widetilde{A}_k, \widetilde{B}_k, \widetilde{\theta}_k)$, for $k = 1, \ldots, p$.

3 Theoretical Properties of the WLSEs

3.1 One Component Sinusoidal Model

In this section first we consider the theoretical properties of the one component sinusoidal model (4). We have the following consistent and asymptotic normality properties of the WLSEs of A^0, B^0 and θ^0.

Theorem 1: It is assumed that $0 < \theta^0 < \pi$, there exists a $M < \infty$, such that $0 < |A^0| < M$, $0 < |B^0| < M$, and $\{\epsilon(t)\}$ satisfies Assumption 1, then $\widehat{A}$, $\widehat{B}$ and $\widehat{\theta}$, the WLSEs of A^0, B^0 and θ^0, respectively are strongly consistent.

Proof: See in the Appendix A. ∎

We need the following notations for further development. For $p = 0, 1, 2, \ldots$,

$$u_p = \int_0^1 t^p w(t) dt \qquad v_p = \int_0^1 t^p w^2(t) dt.$$

The 3×3 diagonal matrix $\boldsymbol{D}$ is

$$\boldsymbol{D} = \text{diag}\left\{N^{1/2}, N^{1/2}, N^{3/2}\right\},$$

and the matrices

$$\boldsymbol{\Sigma} = \begin{bmatrix} v_0 & 0 & B^0 v_1 \\ 0 & v_0 & -A^0 v_1 \\ B^0 v_1 & -A^0 v_1 & (A^{0^2} + B^{0^2}) v_2 \end{bmatrix} \qquad \boldsymbol{\Gamma} = \begin{bmatrix} u_0 & 0 & B^0 u_1 \\ 0 & u_0 & -A^0 u_1 \\ B^0 u_1 & -A^0 u_1 & (A^{0^2} + B^{0^2}) u_2 \end{bmatrix}. \quad (15)$$

Theorem 2: Under the same assumptions as in Theorem 1, and if the matrices $\boldsymbol{\Sigma}$ and $\boldsymbol{\Gamma}$ as defined in (15) are of full rank, then

$$\left((\widehat{A} - A^0), (\widehat{B} - B^0), (\widehat{\theta} - \theta^0)\right) \boldsymbol{D} \xrightarrow{d} N_3((\mathbf{0}, 2\sigma^2 \xi \boldsymbol{\Gamma}^{-1} \boldsymbol{\Sigma} \boldsymbol{\Gamma}^{-1}).$$

Here, $\xrightarrow{d}$ means convergence in distribution, $N_3(\mathbf{0}, \boldsymbol{A})$ indicates a 3-variate normal distribution with the mean vector $\mathbf{0}$, and the dispersion matrix $\boldsymbol{A}$, and

$$\xi = \left[\sum_{k=-\infty}^{\infty} a(k) \cos(\theta^0 k)\right]^2 + \left[\sum_{k=-\infty}^{\infty} a(k) \sin(\theta^0 k)\right]^2 = \left|\sum_{k=-\infty}^{\infty} a(k) e^{i\theta^0 k}\right|^2.$$

Proof: See in the Appendix B. ∎

Note that when $w(t) = 1$, for $0 \le t \le 1$, then the WLSEs become the LSEs, and in this case $u_p = v_p = \dfrac{1}{p+1}$, for $p = 0, 1, 2, \ldots$. Hence,

$$\boldsymbol{\Sigma} = \boldsymbol{\Gamma} = \begin{bmatrix} 1 & 0 & \frac{1}{2} B^0 \\ 0 & 1 & -\frac{1}{2} A^0 \\ \frac{1}{2} B^0 & -\frac{1}{2} A^0 & \frac{1}{3}(A^{0^2} + B^{0^2}) \end{bmatrix}.$$

3.2 Multi Component Sinusoidal Model

In this section we present the theoretical properties of the multi component WLSE. The proofs can be obtained exactly along the same line as the one component model, hence the details are avoided.

Theorem 3: It is assumed that $0 < \theta_1^0, \ldots, \theta_p^0 < \pi$, there exists a $M < \infty$, such that $0 < |A_1^0|, \ldots, |A_p|^0, |B_1^0|, \ldots, |B_p|^0 < M$, and $\{\epsilon(t)\}$ satisfies Assumption 1, then $\widehat{A}_k$, $\widehat{B}_k$ and $\widehat{\theta}_k$, the WLSEs of A_k^0, B_k^0 and θ_k^0, respectively, for $k = 1, \ldots, p$, are strongly consistent.

Theorem 4: Under the same assumptions as in Theorem 3, and if the matrices $\mathbf{\Sigma}$ and $\mathbf{\Gamma}$ as defined in (15) are of full rank, then

$$\left((\widehat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_1^0)\boldsymbol{D}, \ldots, (\widehat{\boldsymbol{\theta}}_p - \boldsymbol{\theta}_p^0)\boldsymbol{D}\right) \xrightarrow{d} N_{3p}((\mathbf{0}, 2\sigma^2\xi\boldsymbol{H}).$$

Here, $\boldsymbol{H}$ is a $3p \times 3p$ block matrix with the k-th block as $\mathbf{\Gamma}_k^{-1}\mathbf{\Sigma}_k\mathbf{\Gamma}_k^{-1}$, where $\mathbf{\Gamma}_k$ and $\mathbf{\Sigma}_k$ can be obtained from $\mathbf{\Gamma}$ and $\mathbf{\Sigma}$, respectively, by replacing A^0, B^0, θ^0 with A_k^0, B_k^0 and θ_k^0, for $k = 1, \ldots, p$.

Theorem 5: Under the same set of assumptions as in Theorem 3, $\widetilde{A}_1$, $\widetilde{B}_1$ and $\widetilde{\theta}_1$, the sequential WLSEs of A_1^0, B_1^0 and θ_1^0, respectively, are strongly consistent.

Proof: See in the Appendix B.

Comments: It should be mentioned that all the above results can be easily extended even when $w(t)$ is a continuous function in $[0, 1]$, and it satisfies the condition (3).

4 Simulation Results

We have performed some simulation experiments and presented the results in this section. The main aim of this section is to see how the weighted (sequential weighted) least squares estimators behave compared to the least squares (sequential least squares) estimators, least absolute deviation (sequential least absolute deviation) estimators and Huber's (sequential Huber's) M-estimators in terms of biases and mean squared errors in presence of outliers. We have considered the following models,

Model 1:

$$y(t) = A^0 \cos(\theta^0 t) + B^0 \sin(\theta^0 t) + \epsilon(t), \tag{16}$$

and

Model 2:

$$y(t) = A_1^0 \cos(\theta_1^0 t) + B_1^0 \sin(\theta_1^0 t) + A_2^0 \cos(\theta_2^0 t) + B_2^0 \sin(\theta_2^0 t) + \epsilon(t). \tag{17}$$

In case of Model 1, we have taken $A^0 = B^0 = 1$, and two different θ^0 values namely $\theta^0 = 0.15\pi$ and $\theta^0 = 0.75\pi$. In case of Model 2, we have considered $A_1^0 = B_1^0 = 2.0$, $\theta_1^0 = 1.25\pi$ and $A_2^0 = B_2^0 = 1.0$, $\theta_2^0 = 0.15\pi$. We have considered two different forms of $\epsilon(t)$'s.

Structure 1: $\epsilon(t) = e(t)$.

Structure 2: $\epsilon(t) = e(t) + 0.5e(t-1)$.

Here $e(t)'s$ are i.i.d. normal random variables with mean 0, and variance $\sigma^2 = 1$. It is assumed that 10%, 20%, 30% or 40% outliers are present in the sample, and in those cases $e(t)$'s are i.i.d. normal random variables with mean 0, and variance $\sigma_{out}^2 = 5$. Moreover, it is assumed that the outliers are present in the middle of the data set. It may be mentioned there is nothing special about the choice of these parameter values and the error distributions. Similar phenomena have been observed for other choices also.

We have considered the LSEs, LADEs, HMEs and also two different WLSEs. Note that to compute the LADEs and HMEs we have used Nelder-Mead algorithm, and it invloves solving three and six dimensional optmization problems for Model 1 and Model 2, respestively. The weight functions of the two WLSEs are as follows for $0 \le t \le 1$:

$$w_1(t) = 1 + \left(t - \frac{1}{2}\right)^2 \qquad \text{and} \quad w_2(t) = \begin{cases} 2 - 4t & \text{if} \quad 0 \le t \le \frac{1}{4} \\ 1 & \text{if} \quad \frac{1}{4} \le t \le \frac{3}{4} \\ 4t - 2 & \text{if} \quad \frac{3}{4} \le t \le 1. \end{cases}$$

Note that $w_2(t)$ is not a polynomial, but it is a continuous function in $[0, 1]$, and it satisfies (3). The main idea of choosing these two weight functions is that they put less weight in the middle of the data set compared to the edges, as it has been assumed that the outliers are in this area. We have generated samples and we have computed the LSEs, LADEs, HMEs and the two WLSEs. We have reported the average estimates and the associated mean squared errors (MSEs) based on 10,000 replications. It may be mentioned that in case of Model 1, the LSEs and the two WLSEs are obtained by solving a one-dimensional optimization problem, whereas the LADEs and HMEs are obtained by solving three-dimensional optimization problem each. In case of Model 2, we have adopted the sequential procedure for all the cases. Hence, the sequential LSEs and the two sequential WLSEs are obtained by solving two one-dimensional optimization problems, whereas the sequential LADEs and HMEs are obtained by solving two three-dimensional optimization problems. The results for Model 1 are presented in Tables 1 to 4, whereas the results for Model 2 are presented in Tables 5 to 8.

Some of the points are quite clear from the simulation results. In each case as the sample size increases the biases and the MSEs decrease for all the methods. It is also observed in all the cases considered in presence of outliers among the five estimates the MSEs are minimum either the WLSEs based on weight function w_1 or w_2. In most of the cases it is minimum when the weight function is w_2. Based on the simulation experiments it can be said that in presence of outliers there exists weight functions such that the weighted LSEs based on these weight functions behave better than the ordinary LSEs, LADEs or HMEs. On the other hand the computations of WLSEs are much less involved compared to the some of the standard robust estimators like LADEs or HMEs. It is clear that the choice of a weight function is an important problem, and that will be discussed in the next section.

5 Weight Function

5.1 How to Choose the Weight Function?

Now one of the natural question is how to choose the weight function in practice. It has been observed in the previous section that with the proper choice of a weight function, the WLSEs perform better than the LSEs, LADEs and HMEs in presence of outliers. Hence, it is an important problem to choose a proper weight function. In this section we suggest the following heuristic method to choose a proper weight function from a class of weight functions.

In many situations a practitioner might have some idea about the portion of the data set where outlier might occur, i.e. whether it is in the beginning, middle or end portion. In that case the weight function can be chosen to put less weight in that particular portion. If it is not known at all, then we choose a class of weight functions with different shapes, i.e. some weight function puts less weight in the middle, some puts less weight at the sides, or a mixture of these. Now compute the weighted least squares estimates based on all these weight functions. In this class it is recommended that we should keep the constant weight function also which generates the LSEs. Now choose that WLSEs which has the smallest weighted residual sum of squares. We recommend that one should keep around 5-6 weight functions of different shapes, and obtain the best weighted least squares estimators among these class of estimators in terms of minimum weighted residual sum of squares. We explain it in details, in the following synthetic data analysis.

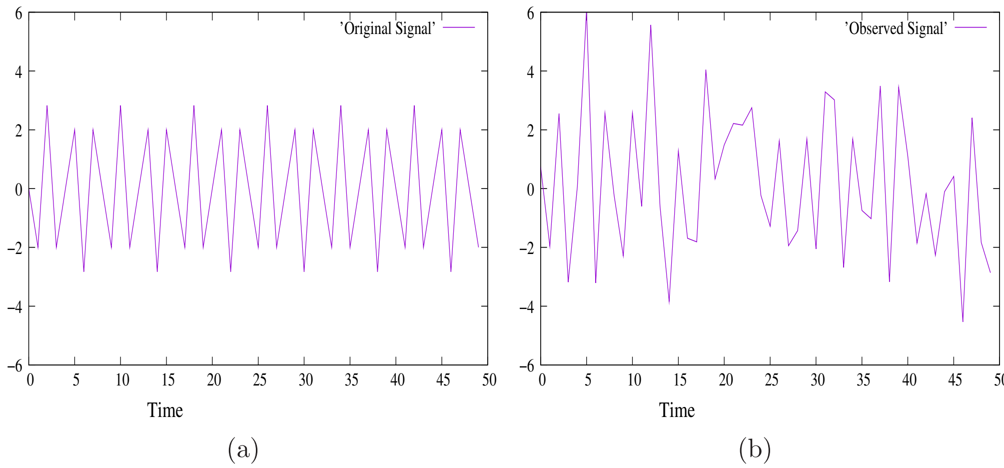

Figure 1: (a) Original Signal without any noise. (b) Observed Signal with noise generated from (19).

5.2 Analysis of A Synthetic Data

In this section we have performed the analysis of a synthetic data mainly to show how the proposed method can be used in practice mainly when outliers are present. We have considered the following model:

$$y(t) = 5.0\cos(0.75\pi t) + 5.0\sin(0.75\pi t) + \epsilon(t); \quad t = 1, \ldots, 50. \tag{18}$$

It is assumed that $\epsilon(t)$'s are i.i.d. random variables, and it has the following structure

$$\epsilon(t) = 0.8N(0, 1) + 0.2N(0, 10), \tag{19}$$

i.e. there are 20% outliers present in the data. The original signal and the observed signal are presented in Figure 1.

We have considered the following weight functions with different shapes;

$$(i)\ w(x) = 1 + (x - 0.5)^2, \quad (ii)\ w(x) = 1 + (x - 0.25)^2,$$

$$(iii)\ w(x) = \begin{cases} 2 - 4x & \text{if} \quad 0 < x < 0.25 \\ 1 & \text{if} \quad 0.25 \le x \le 0.75 \\ 4x - 2 & \text{if} \quad 0.75 < x \le 1, \end{cases}$$

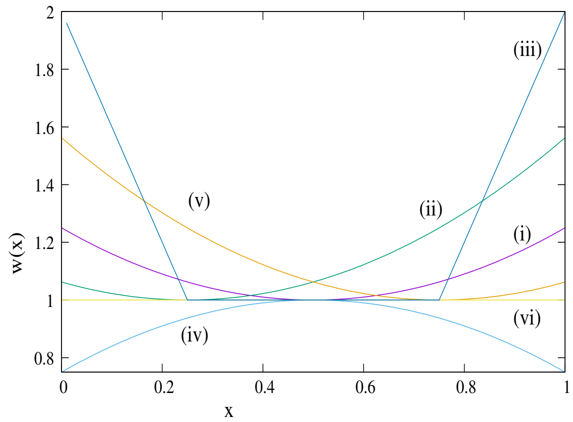

Figure 2: Different weight functions.

$$(iv)\ w(x) = 1 - (x-0.5)^2, \quad (v) = 1 + (x-0.75)^2, \quad (vi)\ w(x) = 1.$$

The plots of different weight functions are provided in Figure 2. The LSEs also can be obtained as a special case. We have computed the WLSEs of A, B and ω, the normalized weighted residual sum of squares (NWRS) and the squared error distance (SED) between the original signal and the fitted signal based on all these six weight functions. The results are reported in Table 9. Based on this result it is clear that the WLSEs based on the weight function (iii) provides the minimum NWRS. Moreover, in this case the SED is also the minimum, i.e. it provides the 'best' estimate to the original signal in this class in terms of the minimum SED.

6 Conclusions

In this paper we have considered the WLSEs in case one component and multicomponent sinusoidal models. We have established the strong consistency and the asymptotic normality results of the WLSEs under a fairly general error structure. Based on extensive simulation experiments it is observed that the proposed WLSEs behave at par if not better compared to some of the well known robust estimators like LADEs or HMEs, although the implementation of the proposed estimators is more convenient than any of these robust estimators. The advantage of the proposed estimators is more prominent in case of multiple sinusoidal model. We have discussed some heuristic method to choose a proper weight function also. Through out this article it has been assumed that the number of components p is known. No attempt has been made to estimate p. It may be mentioned that estimation of p is a challenging problem even when there is no outlier, see for example Chapter 5 of Nandi and Kundu [6]. An extensive amount of work has been done to develop efficient estimation procedure of p. The problem will be more challenging in presence of outlier. It will be interesting to develop a robust procedure to estimate p also. Most likely some kind of information theoretic criteria may be used for this purpose. More work is needed in this direction.

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Appendix A

To prove Theorem 1, we need the following lemma.

Lemma 1: Let $\{X(t)\}$ be a stationary sequence which satisfies Assumption 1, then for $m = 0, 1, 2, \ldots$.

$$\lim_{N\to\infty} \sup_{\theta} \left| \frac{1}{N^{m+1}} \sum_{t=1}^{N} X(t) t^m \cos(t\theta) \right| = 0 \qquad a.s.$$

The same result holds when $\cos(t\theta)$ is replaced by $\sin(t\theta)$.

Proof For m = 0,1,2, see Lemma 1 and its associated corollaries of Kundu (1997). For general k, the results can be obtained along the same line. ∎

Lemma 2: Let $\{X(t)\}$ be a stationary sequence which satisfies Assumption 1, and $w(t)$ is of the form (2), then

$$\lim_{N\to\infty} \sup_{\theta} \left| \frac{1}{N} \sum_{t=1}^{N} X(t) w\left(\frac{t}{N}\right) \cos(t\theta) \right| = 0 \qquad a.s.$$

The same result holds when $\cos(t\theta)$ is replaced by $\sin(t\theta)$.

Proof Using Lemma 1, it can be obtained. ∎

Lemma 3: For any given $\delta > 0$

$$\liminf_{N\to\infty} \inf_{\left|\boldsymbol{\theta}^0 - \boldsymbol{\theta}\right| > \delta} \frac{1}{N}\{Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\} > 0,$$

here $\boldsymbol{\theta} = (A, B, \theta)$, $\boldsymbol{\theta}^0 = (A^0, B^0, \theta^0)$ and $Q(\boldsymbol{\theta})$ is same as defined in (5).

Proof: Let us denote

$$f_t(\boldsymbol{\theta}) = A\cos(\theta t) + B\sin(\theta t); \quad t = 1, \ldots, N.$$

Consider

$$\begin{aligned}
\frac{1}{N} Q(\boldsymbol{\theta}) &= \frac{1}{N} \sum_{t=1}^{N} w\left(\frac{t}{N}\right) (y(t) - f_t(\boldsymbol{\theta}))^2 \\
&= \frac{1}{N} \sum_{t=1}^{N} w\left(\frac{t}{N}\right) \left(f_t(\boldsymbol{\theta}^0) - f_t(\boldsymbol{\theta}) + X(t)\right)^2 \\
&= \frac{1}{N} \sum_{t=1}^{N} w\left(\frac{t}{N}\right) \left(f_t(\boldsymbol{\theta}^0) - f_t(\boldsymbol{\theta})\right)^2 + \frac{1}{N} \sum_{t=1}^{N} w\left(\frac{t}{N}\right) X^2(t) \\
&\quad \frac{2}{N} \sum_{t=1}^{N} w\left(\frac{t}{N}\right) \left(f_t(\boldsymbol{\theta}^0) - f_t(\boldsymbol{\theta})\right) X(t).
\end{aligned}$$

Hence, because of Lemma 2, and the condition on the weight function (3),

$$\liminf_{N\to\infty} \inf_{\left|\boldsymbol{\theta}^0 - \boldsymbol{\theta}\right| > \delta} \frac{1}{N}\{Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\} \geq \liminf_{N\to\infty} \inf_{\left|\boldsymbol{\theta}^0 - \boldsymbol{\theta}\right| > \delta} \frac{\gamma}{N} \sum_{t=1}^{N} \left(f_t(\boldsymbol{\theta}^0) - f_t(\boldsymbol{\theta})\right)^2.$$

Consider the following sets:

$$\Gamma_1 = \{\boldsymbol{\theta} : -M < A, B < M, 0 < \theta < \pi, |A - A^0| > \delta\}$$

$$\begin{aligned}\Gamma_2 &= \{\boldsymbol{\theta} : -M < A, B < M, 0 < \theta < \pi, |B - B^0| > \delta\} \\ \Gamma_3 &= \{\boldsymbol{\theta} : -M < A, B < M, 0 < \theta < \pi, |\theta - \theta^0| > \delta\}.\end{aligned}$$

Since,

$$\{\boldsymbol{\theta} : \left|\boldsymbol{\theta}^0 - \boldsymbol{\theta}\right| > \delta\} \subset \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,$$

$$\liminf_{N\to\infty} \inf_{\left|\boldsymbol{\theta}^0-\boldsymbol{\theta}\right|>\delta} \frac{\gamma}{N}\sum_{t=1}^{N}\left(f_t(\boldsymbol{\theta}^0) - f_t(\boldsymbol{\theta})\right)^2 \geq \liminf_{N\to\infty} \inf_{\left\{\boldsymbol{\theta}\in\Gamma_1\cup\Gamma_2\cup\Gamma_3\right\}} \frac{\gamma}{N}\sum_{t=1}^{N}\left(f_t(\boldsymbol{\theta}^0) - f_t(\boldsymbol{\theta})\right)^2.$$

Observe that

$$\liminf_{N\to\infty} \inf_{\left\{\boldsymbol{\theta}\in\Gamma_1\right\}} \frac{\gamma}{N}\sum_{t=1}^{N}\left(f_t(\boldsymbol{\theta}^0) - f_t(\boldsymbol{\theta})\right)^2 = \liminf_{N\to\infty} \frac{\gamma|A-A^0|^2}{N}\sum_{i=1}^{N}\cos^2(\theta t) = \frac{\gamma|A-A^0|^2}{2} > 0.$$

Similarly, it follows that

$$\begin{aligned}&\liminf_{N\to\infty} \inf_{\left\{\boldsymbol{\theta}\in\Gamma_2\right\}} \frac{\gamma}{N}\sum_{t=1}^{N}\left(f_t(\boldsymbol{\theta}^0) - f_t(\boldsymbol{\theta})\right)^2 > 0 \quad \text{and} \\ &\liminf_{N\to\infty} \inf_{\left\{\boldsymbol{\theta}\in\Gamma_3\right\}} \frac{\gamma}{N}\sum_{t=1}^{N}\left(f_t(\boldsymbol{\theta}^0) - f_t(\boldsymbol{\theta})\right)^2 > 0.\end{aligned}$$

Hence, the result follows. ∎

Proof of Theorem 1: In this proof only let us denote $\widehat{\boldsymbol{\theta}}_N$, the WLSE of $\boldsymbol{\theta}^0$, just to emphasize that it is a function of N. Let us assume that $\widehat{\boldsymbol{\theta}}_N$ is not a consistent estimator of $\boldsymbol{\theta}^0$. Then, there exists a subsequence $\{N_k\}$, such that $\widehat{\boldsymbol{\theta}}_{N_k} \to \boldsymbol{\theta}' \neq \boldsymbol{\theta}^0$. Since for all $k \geq 1$,

$$\frac{1}{N_k} Q\left(\widehat{\boldsymbol{\theta}}_{N_k}\right) \leq \frac{1}{N_k} Q\left(\boldsymbol{\theta}^0\right),$$

there exists a $\delta > 0$, such that

$$\liminf_{N\to\infty} \inf_{\left|\boldsymbol{\theta}^0-\boldsymbol{\theta}\right|>\delta} \frac{1}{N}\{Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\} \leq 0.$$

This contradicts Lemma 3, hence the result is proved. ∎

We need the following results for further development.

Lemma 4: If $\theta \in (0, \pi)$, then for $k = 0, 1, \ldots$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \cos(\theta t) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \sin(\theta t) = 0,$$

$$\lim_{N\to\infty} \frac{1}{N^{k+1}} \sum_{t=1}^{N} n^k \cos^2(\theta t) = \frac{1}{2(k+1)},$$

$$\lim_{N\to\infty} \frac{1}{N^{k+1}} \sum_{t=1}^{N} n^k \sin^2(\theta t) = \frac{1}{2(k+1)},$$

$$\lim_{N\to\infty} \frac{1}{N^{k+1}} \sum_{t=1}^{N} n^k \cos(\theta t) \sin(\theta t) = 0.$$

Proof: See Mangulis (1965)

Proof of Theorem 2: To prove this result, let us use the following notations:

$$Q'(\boldsymbol{\theta}) = \left(\frac{\partial Q(\boldsymbol{\theta})}{\partial A}, \frac{\partial Q(\boldsymbol{\theta})}{\partial B}, \frac{\partial Q(\boldsymbol{\theta})}{\partial \theta} \right)$$

and $Q''(\boldsymbol{\theta})$ is a 3×3 matrix contains the double derivative of $Q(\boldsymbol{\theta})$. Now using the Taylor series expansion, we obtain

$$Q'(\widehat{\boldsymbol{\theta}}) - Q'(\boldsymbol{\theta}^0) = (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0) Q''(\overline{\boldsymbol{\theta}}), \tag{20}$$

here $\overline{\boldsymbol{\theta}}$ is a point on the line joining $\widehat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}^0$. Since, $Q'(\widehat{\boldsymbol{\theta}}) = \mathbf{0}$, (20) can be written as

$$-Q'(\boldsymbol{\theta}^0)\boldsymbol{D}^{-1} \left[\boldsymbol{D}^{-1} Q''(\overline{\boldsymbol{\theta}})\boldsymbol{D}^{-1}\right]^{-1} = (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)\boldsymbol{D}. \tag{21}$$

Now using the Central Limit Theorem of linear process, see for example Feller (1976) and Lemma 4, it can be shown that

$$Q'(\boldsymbol{\theta}^0)\boldsymbol{D}^{-1} \xrightarrow{d} N_3(\mathbf{0}, 2\sigma^2\xi\boldsymbol{\Sigma}),$$

and using Lemma 4, it can be shown that

$$\lim_{N\to\infty} \boldsymbol{D}^{-1} Q''(\overline{\boldsymbol{\theta}})\boldsymbol{D}^{-1} = \lim_{N\to\infty} \boldsymbol{D}^{-1} Q''(\boldsymbol{\theta}^0)\boldsymbol{D}^{-1} = \boldsymbol{\Gamma},$$

hence the result follows. ∎

Appendix B

To prove Theorem 5, we need the following Lemma.

Lemma 5: Let us denote

$$S_{1c} = \{\boldsymbol{\theta} : \boldsymbol{\theta} = (A, B, \theta)^{\top}, |\boldsymbol{\theta}^0 - \boldsymbol{\theta}| > 3c\}.$$

If there exists a $c > 0$, such that

$$\underline{\lim} \inf_{\boldsymbol{\theta} \in S_{1c}} \frac{1}{N} [Q_1(\boldsymbol{\theta}) - Q_1(\boldsymbol{\theta}_1^0)] > 0, \qquad a.s, \tag{22}$$

then $\widetilde{\boldsymbol{\theta}}$ is a strongly consistent estimator of $\boldsymbol{\theta}_1^0$.

Proof: It follows by using simple contradiction argument. ∎

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Outliers	n	WLSE(w_1)	LSE	LAD	WLSE(w_2)	HME
10%	25	0.4782	0.4780	0.4801	0.4773	0.4769
		(7.87E-03)	(7.82E-03)	(1.07E-02)	(7.77E-03)	(7.77E-03)
	50	0.4718	0.4715	0.4717	0.4712	0.4714
		(6.07E-04)	(4.47E-04)	(6.41E-04)	(1.52E-04)	(3.39E-04)
	75	0.4713	0.4714	0.4715	0.4712	0.4713
		(4.11E-05)	(3.09E-04)	(1.60E-04)	(3.38E-05)	(3.47E-05)
	100	0.4712	0.4712	0.4712	0.4712	0.4713
		(1.25E-05)	(1.23E-05)	(1.80E-05)	(1.32E-05)	(1.38E-05)
20%	25	0.4831	0.4848	0.4841	0.4814	0.4809
		(1.36E-02)	(1.37E-02)	(1.49E-02)	(1.12E-02)	(1.12E-02)
	50	0.4721	0.4725	0.4719	0.4709	0.4718
		(1.69E-03)	(1.56E-03)	(9.59E-04)	(3.68E-04)	(9.03E-04)
	75	0.4713	0.4714	0.4716	0.4713	0.4713
		(3.43E-05)	(5.55E-05)	(2.49E-04)	(1.194E-04)	(3.55E-05)
	100	0.4713	0.4713	0.4713	0.4713	0.4713
		(5.53E-05)	(1.30E-04)	(1.08E-04)	(1.33E-05)	(1.39E-05)
30%	25	0.4894	0.4927	0.4905	0.4850	0.4840
		(1.90E-02)	(2.07E-02)	(2.08E-02)	(1.46E-02)	(1.47E-02)
	50	0.4730	0.4736	0.4717	0.4714	0.4720
		(2.22E-03)	(3.24E-03)	(1.16E-03)	(7.42E-04)	(1.38E-03)
	75	0.4722	0.4719	0.4719	0.4716	(0.4715)
		(8.51E-04)	(3.76E-04)	(4.38E-04)	(2.79E-04)	(3.24E-04)
	100	0.4713	0.4713	0.4713	0.4713	0.4713
		(5.60E-05)	(8.91E-05)	(1.10E-04)	(1.39E-05)	(1.46E-05)
40%	25	0.5091	0.5092	0.5055	0.4970	0.4984
		(3.55E-02)	(3.51E-02)	(3.39E-02)	(2.63E-02)	(2.78E-02)
	50	0.4763	0.4777	0.4748	0.4731	0.4740
		(4.69E-03)	(6.57E-03)	(3.76E-03)	(2.35E-03)	(2.92E-03)
	75	0.4722	0.4723	0.4722	0.4719	0.4715
		(6.86E-04)	(6.92E-04)	(6.96E-04)	(4.61E-04)	(4.71E-04)
	100	0.4716	0.4715	0.4715	0.4716	0.4715
		(2.62E-04)	(4.35E-04)	(1.78E-04)	(1.22E-04)	(2.78E-04)

Table 1: The average estimates and the associated mean squares errors of LSEs, WLSEs and LADEs of $\theta^0 = 0.15\pi = 0.4712$ for Model 1, Structure 1 error and when there are 10%, 20%, 30% and 40% outliers are present.

Out-liers	n	WLSE(w_1)	LSE	LAD	WLSE(w_2)	HME
10%	25	2.3742 (2.94E-02)	2.3731 (3.23E-02)	2.3792 (4.15E-02)	2.3791 (2.73E-02)	2.3712 (2.77E-02)
	50	2.3601 (6.81E-03)	2.3604 (7.82E-03)	2.3597 (5.82E-03)	2.3580 (2.96E-03)	2.3583 (3.64E-03)
	75	2.3563 (1.09E-04)	2.3563 (2.78E-04)	2.3567 (5.85E-04)	2.3561 (3.10E-05)	2.3564 (1.79E-04)
	100	2.3563 (9.18E-05)	2.3562 (1.22E-04)	2.3562 (1.69E-05)	2.3564 (2.63E-04)	2.3563 (1.12E-04)
20%	25	2.3828 (6.65E-02)	2.3788 (6.81-02)	2.3906 (7.19E-02)	2.3884 (6.02E-02)	2.3835 (6.03E-02)
	50	2.3613 (9.92E-03)	2.3648 (1.65E-02)	2.3617 (9.13E-03)	2.3607 (6.73E-03)	2.3604 (7.00E-03)
	75	2.3567 (5.95E-04)	2.3570 (1.21E-03)	2.3570 (1.07E-03)	2.3563 (2.78E-04)	2.3568 (1.14E-03)
	100	2.3562 (1.28E-05)	2.3564 (3.10E-04)	2.3566 (5.46E-04)	2.3564 (2.64E-04)	2.3562 (1.40E-05)
30%	25	2.3868 (9.22E-02)	2.3884 (10.37E-02)	2.4025 (9.52E-02)	2.3925 (8.34E-02)	2.3921 (8.80E-02)
	50	2.3681 (2.34E-02)	2.3696 (2.57E-02)	2.3651 (1.47E-02)	2.3649 (1.37E-02)	2.3649 (1.38E-02)
	75	2.3581 (3.07E-03)	2.3584 (3.82E-03)	2.3581 (2.71E-03)	2.3573 (2.24E-03)	2.3574 (2.72E-03)
	100	2.3565 (3.38E-04)	2.3566 (5.51E-04)	2.3566 (5.48E-04)	2.3566 (3.38E-04)	2.3564 (3.38E-04)
40%	25	2.3995 (0.1486)	2.3965 (0.1525)	2.4080 (0.1338)	2.3980 (0.1092)	2.4026 (0.1161)
	50	2.3769 (3.94E-02)	2.3783 (4.67E-02)	2.3694 (2.40E-02)	2.3694 (2.21E-02)	2.3708 (2.43E-02)
	75	2.3594 (5.37E-03)	2.3607 (9.24E-03)	2.3604 (6.31E-03)	2.3583 (3.97E-03)	2.3578 (3.96E-03)
	100	2.3580 (2.70E-03)	2.3579 (2.54E-03)	2.3570 (1.24E-03)	2.3568 (9.59E-04)	2.3568 (9.86E-04)

Table 2: The average estimates and the associated mean squares errors of LSEs, WLSEs and LADEs of $\theta^0 = 0.75\pi = 2.3562$ for Model 1, Structure 1 error and when there are 10%, 20%, 30% and 40% outliers are present.

Outliers	n	WLSE(w_1)	LSE	LAD	WLSE(w_2)	HME
10%	25	0.4807	0.4839	0.4867	0.4807	0.4828
		(1.28E-02)	(1.46E-02)	(1.76E-02)	(1.24E-02)	(1.37E-02)
	50	0.4723	0.4727	0.4725	0.4724	0.4715
		(1.13E-03)	(1.53E-03)	(1.40E-03)	(1.12E-03)	(1.12E-03)
	75	0.4716	0.4714	0.4716	0.4716	0.4716
		(5.33E-04)	(6.18E-04)	(1.64E-04)	(2.50E-04)	(3.36E-04)
	100	0.4712	0.4715	0.4713	0.4714	0.4711
		(1.61E-05)	(2.32E-04)	(3.80E-05)	(1.44E-04)	(1.80E-05)
20%	25	0.4910	0.4954	0.4925	0.4882	0.4891
		(2.12E-02)	(2.26E-02)	(2.21E-02)	(1.99E-02)	(2.00E-02)
	50	0.4734	0.4757	0.4733	0.4730	0.4725
		(3.16E-03)	(5.06E-03)	(3.00E-03)	(2.82E-03)	(3.01E-03)
	75	0.4720	0.4720	0.4719	0.4720	0.4718
		(7.08E-04)	(8.74E-04)	(8.68E-04)	(6.60E-04)	(6.98E-04)
	100	0.4717	0.4717	0.4714	0.4714	0.4711
		(4.23E-04)	(3.95E-04)	(2.81E-04)	(2.08E-04)	(2.13E-04)
30%	25	0.5035	0.5080	0.5058	0.4959	0.4889
		(3.22E-02)	(3.48E-02)	(3.23E-02)	(2.57E-02)	(2.77E-02)
	50	0.4747	0.4756	0.4726	0.4728	0.4729
		(4.53E-03)	(5.84E-03)	(3.38E-03)	(2.81E-03)	(2.85E-03)
	75	0.4740	0.4742	0.4728	0.4729	0.4726
		(2.38E-03)	(2.14E-03)	(1.20E-03)	(1.32E-03)	(1.35E-03)
	100	0.4722	0.4725	0.4718	0.4716	0.4714
		(7.80E-04)	(1.03E-03)	(5.47E-04)	(2.84E-04)	(2.88E-04)
40%	25	0.5281	0.5370	0.5279	0.5121	0.5215
		(5.72E-02)	(6.32E-02)	(5.60E-02)	(4.20E-02)	(5.20E-02)
	50	0.4811	0.4832	0.4782	0.4768	0.4776
		(9.15E-03)	(1.64E-02)	(7.55E-03)	(6.51E-03)	(6.74E-03)
	75	0.4757	0.4754	0.4739	0.4739	0.4738
		(4.03E-03)	(2.97E-03)	(2.06E-03)	(1.83E-03)	(1.83E-03)
	100	0.4733	0.4731	0.4720	0.4719	0.4716
		(1.68E-03)	(1.47E-03)	(5.97E-04)	(5.53E-04)	(5.58E-04)

Table 3: The average estimates and the associated mean squares errors of LSEs, WLSEs and LADEs of $\theta^0 = 0.15\pi = 0.4712$ for Model 1, Structure 2 error and when there are 10%, 20%, 30% and 40% outliers are present.

Out-liers	n	WLSE(w_1)	LSE	LAD	WLSE(w_2)	HME
10%	25	2.3811	2.3816	2.3952	2.3905	2.3888
		(6.40E-02)	(8.04E-02)	(8.62E-02)	(6.92E-02)	(6.45E-02)
	50	2.3615	2.3657	2.3636	2.3628	2.3605
		(1.02E-02)	(1.78E-02)	(1.28E-02)	(1.10E-02)	(1.09E-02)
	75	2.3574	2.3565	2.3570	2.3566	2.3563
		(1.70E-03)	(2.51E-03)	(1.37E-03)	(8.03E-04)	(8.13E-04)
	100	2.3564	2.3564	2.3564	2.3562	2.3563
		(2.86E-04)	(1.11E-04)	(2.68E-04)	(1.66E-05)	(2.69E-04)
20%	25	2.3894	2.3807	2.4011	2.3912	2.3958
		(9.56E-02)	(1.08E-01)	(1.09E-01)	(8.64E-02)	(8.61E-02)
	50	2.3667	2.3699	2.3675	2.3671	2.3644
		(2.16E-02)	(3.00E-02)	(2.30E-02)	(2.15E-02)	(2.15E-02)
	75	2.3599	2.3589	2.3577	2.3570	2.3573
		(4.67E-03)	(4.22E-03)	(2.50E-03)	(1.32E-03)	(1.61E-03)
	100	2.3564	2.3566	2.3565	2.3564	2.3565
		(2.74E-04)	(4.99E-04)	(2.38E-04)	(2.64E-04)	(5.22E-04)
30%	25	2.3967	2.3926	2.4059	2.3948	2.3989
		(1.41E-01)	(1.68E-01)	(1.35E-01)	(1.18E-01)	(1.19E-01)
	50	2.3766	2.3766	2.3734	2.3728	2.3707
		(4.05E-02)	(4.39E-02)	(3.52E-02)	(2.98E-02)	(2.99E-02)
	75	2.3620	2.3617	2.3588	2.3581	2.3590
		(9.38E-03)	(1.01E-02)	(3.42E-03)	(3.11E-03)	(3.80E-03)
	100	2.3579	2.3571	2.3577	2.3573	2.3564
		(2.78E-03)	(2.43E-03)	(2.13E-03)	(1.53E-03)	(1.67E-03)
40%	25	2.3972	2.3927	2.4133	2.4098	2.4083
		(0.2232)	(0.2539)	(0.2217)	(0.1812)	(0.1990)
	50	2.3779	2.3860	2.3878	2.3794	2.3783
		(5.87E-02)	(7.70E-02)	(6.52E-02)	(5.07E-02)	(5.19E-02)
	75	2.3651	2.3635	2.3617	2.3596	2.3607
		(1.48E-02)	(1.59E-02)	(9.09E-03)	(6.29E-03)	(7.57E-03)
	100	2.3591	2.3608	2.3593	2.3581	2.3573
		(5.21E-03)	(7.06E-03)	(4.48E-03)	(2.96E-03)	(2.97E-03)

Table 4: The average estimates and the associated mean squares errors of LSEs, WLSEs and LADEs of $\theta^0 = 0.75\pi = 2.3562$ for Model 1, Structure 2 error and when there are 10%, 20%, 30% and 40% outliers are present.

Outliers	n	WLSE(w_1)	LSE	LAD	WLSE(w_2)	HME
10%	25	3.9308	3.9313	3.9295	3.9311	3.9312
		(1.96E-03)	(3.34E-03)	(1.57E-03)	(6.73E-04)	(1.54E-03)
	50	3.9283	3.9283	3.9282	3.9287	3.9286
		(2.67E-05)	(2.72E-05)	(3.94E-05)	(3.06E-05)	(3.90E-05)
	75	3.9267	3.9267	3.9266	3.9266	3.9266
		(6.67E-06)	(7.48E-06)	(9.93E-06)	(7.24E-06)	(9.02E-06)
	100	3.9272	3.9272	3.9271	3.9273	3.9272
		(2.49E-06)	(4.38E-06)	(3.72E-06)	(2.75E-06)	(3.99E-06)
20%	25	3.9314	3.9314	3.9305	3.9315	3.9316
		(2.66E-03)	(3.88E-03)	(1.60E-03)	(1.41E-03)	(1.57E-03)
	50	3.9283	3.9282	3.9282	3.9287	3.9286
		(2.83E-05)	(3.84E-05)	(4.22E-05)	(3.09E-05)	(4.08E-05)
	75	3.9267	3.9267	3.9267	3.9266	3.9267
		(7.01E-06)	(7.92E-06)	(1.07E-05)	(7.38E-06)	(9.57E-06)
	100	3.9272	3.9272	3.9271	3.9273	3.9272
		(2.62E-06)	(3.61E-06)	(3.88E-06)	(2.80E-06)	(3.49E-06)
30%	25	3.9326	3.9323	3.9301	3.9323	3.9318
		(5.24E-03)	(5.91E-03)	(4.42E-03)	(3.67E-03)	(3.87E-03)
	50	3.9284	3.9283	3.9283	3.9287	3.9287
		(3.00E-05)	(3.09E-05)	(4.62E-05)	(3.22E-05)	(4.32E-05)
	75	3.9267	3.9267	3.9267	3.9266	3.9267
		(7.60E-06)	(7.63E-06)	(1.16E-05)	(6.19E-06)	(1.00E-05)
	100	3.9272	3.9272	3.9272	3.9273	3.9273
		(2.88E-06)	(2.90E-06)	(4.12E-06)	(2.96E-06)	(3.79E-06)
40%	25	3.9390	3.9373	3.9342	3.9348	3.9341
		(1.99E-02)	(2.20E-02)	(1.83E-02)	(1.18E-02)	(1.32E-02)
	50	3.9284	3.9283	3.9281	3.9287	3.9287
		(3.44E-05)	(3.64E-05)	(5.07E-05)	(3.49E-05)	(4.86E-05)
	75	3.9267	3.9267	3.9267	3.9266	3.9267
		(8.68E-06)	(8.89E-06)	(1.23E-05)	(8.27E-06)	(1.11E-05)
	100	3.9272	3.9273	3.9272	3.9273	3.9273
		(3.33E-06)	(3.39E-06)	(4.62E-06)	(3.20E-06)	(4.18E-06)

Table 5: The average estimates and the associated mean squares errors of LSEs, WLSEs and LADEs of $\theta_1^0 = 1.25\pi = 3.9269$ for Model 2, error follows Structure 1 and when there are 10%, 20%, 30% and 40% outliers are present.

Outliers	n	WLSE(w_1)	LSE	LAD	WLSE(w_2)	HME
10%	25	0.4782	0.4793	0.4955	0.4777	0.4789
		(8.00E-03)	(9.67E-03)	(2.57E-02)	(8.44E-03)	(9.99E-03)
	50	0.4721	0.4717	0.4722	0.4717	0.4717
		(6.08E-04)	(4.46E-04)	(1.05E-03)	(1.52E-04)	3.93E-04
	75	0.4713	0.4714	0.4714	0.4713	0.4713
		(4.14E-05)	(3.11E-05)	(4.69E-05)	(3.42E-05)	(3.48E-05)
	100	0.4712	0.4712	0.4712	0.4713	0.4713
		(1.25E-05)	(2.17E-05)	(1.77E-05)	(1.32E-05)	(1.37E-05)
20%	25	0.4835	0.4856	0.5049	0.4816	0.4820
		(1.45E-02)	(1.53E-02)	(3.42E-02)	(1.16E-02)	(1.28E-02)
	50	0.4725	0.4729	0.4733	0.4715	0.4720
		(1.79E-03)	(1.76E-03)	(2.19E-03)	(3.86E-04)	(8.45E-04)
	75	0.4713	0.4714	0.4714	0.4714	0.4713
		(3.45E-05)	(3.57E-05)	(5.46E-05)	(4.21E-05)	(4.38E-05)
	100	0.4713	0.4713	0.4713	0.4713	0.4712
		(5.54E-05)	(1.29E-04)	(1.05E-04)	(1.33E-05)	(1.45E-05)
30%	25	0.4916	0.4931	0.5140	0.4862	0.4886
		(2.15E-02)	(2.15E-02)	(4.11E-02)	(1.69E-02)	(1.81E-02)
	50	0.4735	0.4740	0.4740	0.4720	0.4724
		(2.34E-03)	(2.44E-03)	(3.13E-03)	(9.47E-04)	(1.62E-03)
	75	0.4723	0.4719	0.4723	0.4717	0.4714
		(9.17E-04)	(8.18E-04)	(7.45E-04)	(2.80E-04)	(2.89E-04)
	100	0.4713	0.4713	0.4713	0.4713	0.4714
		(5.61E-05)	(8.89E-05)	(1.00E-04)	(1.39E-05)	(1.84E-04)
40%	25	0.5129	0.5146	0.5481	0.4987	0.5099
		(4.08E-02)	(4.30E-02)	(7.29E-02)	(2.82E-02)	(3.98E-02)
	50	0.4767	0.4782	0.4796	0.4737	0.4750
		(4.90E-03)	(6.51E-03)	(7.57E-03)	(2.25E-03)	(3.55E-03)
	75	0.4724	0.4723	0.4732	0.4720	0.4716
		(8.07E-04)	(5.88E-04)	(1.70E-03)	(5.63E-04)	(5.73E-04)
	100	0.4716	0.4715	0.4718	0.4717	0.4716
		(2.89E-04)	(3.35E-04)	(5.47E-04)	(4.42E-04)	(3.38E-04)

Table 6: The average estimates and the associated mean squares errors of LSEs, WLSEs and LADEs of $\theta_2^0 = 0.15\pi = 0.4712$ for Model 2, error follows Structure 1 and when there are 10%, 20%, 30% and 40% outliers are present.

Outliers	n	WLSE(w_1)	LSE	LAD	WLSE(w_2)	HME
10%	25	3.9306	3.9313	3.9307	3.9314	3.9306
		(1.61E-03)	(3.34E-03)	(6.33E-03)	(2.86E-03)	(2.16E-03)
	50	3.9284	3.9281	3.9280	3.9286	3.9286
		(3.41E-05)	(4.42E-05)	(4.58E-05)	(3.71E-05)	(4.88E-05)
	75	3.9267	3.9268	3.9266	3.9266	3.9267
		(8.57E-06)	(9.56E-06)	(1.20E-05)	(9.06E-06)	(1.20E-05)
	100	3.9272	3.9272	3.9272	3.9273	3.9272
		(3.04E-06)	(5.03E-06)	(4.47E-06)	(3.50E-06)	(4.30E-06)
20%	25	3.9327	3.9327	3.9328	3.9322	3.9330
		(5.74E-03)	(6.07E-03)	(1.06E-02)	(4.37E-03)	(4.70E-03)
	50	3.9283	3.9282	3.9280	3.9286	3.9286
		(3.55E-05)	(3.59E-05)	(4.88E-05)	(3.82E-05)	(5.06E-05)
	75	3.9267	3.9268	3.9266	3.9266	3.9267
		(9.07E-06)	(1.05E-05)	(1.27E-05)	(9.32E-06)	(1.25E-05)
	100	3.9272	3.9272	3.9272	3.9273	3.9272
		(3.29E-06)	(3.32E-06)	(4.69E-06)	(3.58E-06)	(4.50E-06)
30%	25	3.9339	3.9341	3.9334	3.9334	3.9349
		(1.14E-02)	(1.21E-02)	(1.23E-02)	(7.14E-03)	(1.09E-02)
	50	3.9283	3.9282	3.9281	3.9286	3.9286
		(3.80E-05)	(4.19E-05)	(5.28E-05)	(3.97E-05)	(5.30E-05)
	75	3.9266	3.9267	3.9266	3.9266	3.9267
		(9.77E-06)	(1.18E-05)	(1.33E-05)	(9.83E-06)	(1.31E-05)
	100	3.9273	3.9272	3.9272	3.9273	3.9272
		(3.57E-06)	(3.76E-06)	(4.95E-06)	(3.76E-06)	(4.77E-06)
40%	25	3.9404	3.9364	3.9352	3.9390	3.9365
		(3.17E-02)	(2.31E-02)	(3.14E-02)	(2.06E-02)	(3.39E-02)
	50	3.9285	3.9284	3.9281	3.9286	3.9286
		(3.98E-04)	(4.03E-04)	(5.96E-05)	(4.36E-05)	(6.20E-05)
	75	3.9267	3.9267	3.9266	3.9266	3.9267
		(1.11E-04)	(1.14E-05)	(1.48E-05)	(1.05E-05)	(1.47E-05)
	100	3.9273	3.9272	3.9272	3.9273	3.9273
		(9.04E-06)	(4.34E-06)	(5.53E-06)	(4.01E-06)	(5.34E-06)

Table 7: The average estimates and the associated mean squares errors of LSEs, WLSEs and LADEs of $\theta_1^0 = 1.25\pi = 3.9269$ for Model 2, error follows Structure 2 and when there are 10%, 20%, 30% and 40% outliers are present.

Outliers	n	WLSE(w_1)	LSE	LAD	WLSE(w_2)	HME
10%	25	0.4810 (1.24E-02)	0.4837 (1.52E-02)	0.5130 (4.19E-02)	0.4803 (1.26E-02)	0.4853 (1.63E-02)
	50	0.4726 (1.13E-03)	0.4728 (1.36E-03)	0.4741 (2.60E-03)	0.4727 (1.04E-04)	0.4720 (1.05E-04)
	75	0.4719 (7.49E-04)	0.4714 (1.19E-03)	0.4720 (6.42E-04)	0.4716 (2.51E-04)	0.4718 (5.12E-04)
	100	0.4713 (1.61E-05)	0.4715 (2.33E-04)	0.4715 (2.45E-04)	0.4714 (1.43E-04)	0.4711 (1.43E-04)
20%	25	0.4925 (2.39E-02)	0.4972 (2.73E-02)	0.5215 (4.98E-02)	0.4884 (2.06E-02)	0.4994 (2.47E-02)
	50	0.4737 (3.04E-03)	0.4758 (5.05E-03)	0.4774 (6.28E-03)	0.4735 (2.71E-03)	0.4737 (2.82E-03)
	75	0.4720 (5.17E-04)	0.4720 (7.75E-04)	0.4724 (8.03E-04)	0.4720 (5.65E-04)	0.4720 (5.75E-04)
	100	0.4717 (3.78E-04)	0.4717 (3.96E-04)	0.4717 (4.92E-04)	0.4715 (2.09E-04)	0.4711 (2.10E-04)
30%	25	0.5070 (3.69E-02)	0.5107 (3.87E-02)	0.5475 (7.35E-02)	0.4991 (2.96E-02)	0.5070 (3.53E-02)
	50	0.4750 (4.33E-03)	0.4757 (5.82E-03)	0.4785 (8.17E-03)	0.4732 (2.69E-03)	0.4737 (2.87E-03)
	75	0.4741 (2.42E-03)	0.4742 (2.24E-03)	0.4743 (2.50E-03)	0.4729 (1.26E-03)	0.4730 (1.29E-03)
	100	0.4721 (7.81E-04)	0.4725 (1.05E-03)	0.4721 (8.42E-04)	0.4716 (3.17E-04)	0.4713 (3.86E-04)
40%	25	0.5348 (6.40E-02)	0.5395 (6.64E-02)	0.5909 (1.17E-01)	0.5179 (4.93E-02)	0.5401 (6.69E-02)
	50	0.4817 (9.60E-03)	0.4838 (1.16E-02)	0.4890 (1.81E-02)	0.4772 (6.30E-03)	0.4793 (6.69E-03)
	75	0.4758 (4.18E-03)	0.4756 (3.13E-03)	0.4762 (4.04E-03)	0.4743 (2.06E-03)	0.4745 (2.45E-03)
	100	0.4734 (1.78E-03)	0.4730 (1.37E-03)	0.4737 (2.36E-03)	0.4719 (5.83E-04)	0.4718 (5.94E-04)

Table 8: The average estimates and the associated mean squares errors of LSEs, WLSEs and LADEs of $\theta_2^0 = 0.15\pi = 0.4712$ for Model 2, error follows Structure 2 and when there are 10%, 20%, 30% and 40% outliers are present.

Weight	$\widehat{A}$	$\widehat{B}$	$\widehat{\omega}$	WRSS	Diff
(i)	1.9929	1.7418	2.3549	139.1043	1.1099
(ii)	1.9752	1.7361	2.3548	135.9698	1.2356
(iii)	1.9325	1.9384	2.3575	131.9607	0.5022
(iv)	2.0234	1.5836	2.3526	195.9873	2.3396
(v)	2.0081	1.7446	2.3548	142.4155	1.0269
(vi)	2.0032	1.6739	2.3539	140.7699	1.5809

Table 9: WLSEs, normalized residual sums of squares and squared error distance of the fitted and the originals based on different weight functions