A Bivariate Inverse Weibull Distribution and its Application in Complementary Risks Model

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Abstract

In reliability and survival analysis the inverse Weibull distribution has been used quite extensively as a heavy tailed distribution with a non-monotone hazard function. Recently a bivariate inverse Weibull (BIW) distribution has been introduced in the literature, where the marginals have inverse Weibull distributions and it has a singular component. Due to this reason this model cannot be used when there are no ties in the data. In this paper we have introduced an absolutely continuous bivariate inverse Weibull (ACBIW) distribution omitting the singular component from the BIW distribution. A natural application of this model can be seen in the analysis of dependent complementary risks data. We discuss different properties of this model and also address the inferential issues both from the classical and Bayesian approaches. In the classical approach, the maximum likelihood estimators cannot be obtained explicitly and we propose to use the expectation maximization algorithm based on the missing value principle. In the Bayesian analysis, we use a very flexible prior on the unknown model parameters and obtain the Bayes estimates and the associated credible intervals using importance sampling technique. Simulation experiments are performed to see the effectiveness of the proposed methods and two data sets have been analyzed to see how the proposed methods and the model work in practice.

Key Words and Phrases: Marshall-Olkin bivariate distribution; Block and Basu bivariate distribution; Maximum likelihood estimation, EM Algorithm.

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1 Introduction

The Marshall-Olkin [15] bivariate exponential distribution (MOBE) is one of the popular bivariate exponential distributions. The MOBE is a singular distribution with exponentially distributed marginals. The MOBE distribution can be used quite effectively when there are ties in the data set, but if there are no ties, it cannot be used. Block and Basu [2] introduced an absolutely continuous bivariate exponential (BBBE) distribution by removing the singular component from the MOBE distribution and it can be used to analyze a data set when there are no ties. Similar to the BBBE distribution Block and Basu bivariate Weibull distribution (BBBW) also has been studied in the literature and its properties and inferential procedures have been developed, see for example Kundu and Gupta [11] and Pradhan and Kundu [18] and the references cited therein.

Although the Weibull distribution is a very flexible lifetime distribution, it cannot have non-monotone hazard function. In this respect, the inverse Weibull (IW) distribution can be used quite effectively if the data come from a distribution which has a non-monotone hazard function. It is well known that if $0 < \alpha \leq 1$, where α is the shape parameter of an IW distribution as defined in (1), then the mean does not exist, and if $1 < \alpha \leq 2$, then the mean exists but the variance does not exist. Hence, it can be used if the data come from a heavy tailed distribution also with a suitable choice of the shape parameter. Recently, Muhammed [16] and Kundu and Gupta [13] independently introduced a bivariate inverse Weibull (BIW) distribution with the marginals having the IW distributions and discussed different inferential issues of the proposed model. Although, it has been observed that the BIW is a very flexible model, it also has a singular component. Hence, the BIW distribution can be used to analyze a data set if there are ties, similar to the MOBE or Marshall Olkin bivariate Weibull distribution. The main aim of this paper is to introduce an absolutely continuous bivariate inverse Weibull (ACBIW) distribution and use it as a dependent complementary risks model.

In a reliability or in a survival analysis, very often experimental units are exposed to more than one causes of failure and one needs to analyze the effect of one cause in presence of the other causes. In this case one observes the failure time along with the cause of failure. In the statistical literature it is popularly known as the competing risk problem. In a competing risk scenario the failure time T can be written as $T = min(X_1, X_2, \ldots, X_m)$ where X_i denotes the latent failure time due to the i-th risk factor for $i = 1, 2, \ldots, m$. Here X_i's cannot be observed separately, hence, called latent failure times. An extensive amount of work has been done in developing different competing risks models, analyzing their properties and providing various inferential procedures. Interested readers are referred to the book length treatment of Crowder [4]. Recently, a growing interest is on developing different dependent competing risks models, see for example Feizjavadian and Hashemi [7], Shen and Xu [19] and the references cited therein

The duality of the competing risk model is known as the complementary risk model and it was originally introduced by Basu and Ghosh [1]. In this case the failure time T can be written as $T = max(X_1, X_2, \ldots, X_m)$, where X_i's are same as defined before. Although an extensive amount of work has been done in the area of competing risks not much work been done in the area of the complementary risks, although survival or reliability data in presence of complementary risks are observed in several areas such as public health, actuarial sciences, biomedical studies, demography and in industrial reliability, see for example Louzada et al. [14] and Han [8]. It is observed that the proposed ACBIW can be used quite effectively as a dependent complementary risks model.

In this paper first we define the ACBIW distribution and obtain its marginals. We provide different properties of the proposed distribution. The distribution has four parameters and the maximum likelihood (ML) estimators cannot be obtained in closed form. One has to solve a four dimensional optimization problem to compute the ML estimators. To avoid that we have applied the expectation maximization (EM) algorithm, and it is observed that at each 'E'-step the corresponding 'M'-step can be performed by solving a one dimensional optimization problem. We have further considered the Bayesian inference of the unknown parameters based on a very general Gamma-Dirichlet (GD) prior on the scale parameters and an independent log-concave prior on the shape parameter. The Bayes estimators and the associated credible intervals of the unknown parameters are obtained based on importance sampling technique. Extensive simulations have been performed to see the effectiveness of the proposed method and one bivariate data set has been analyzed to show how the model can be used in practice.

We further describe how this model can be used as a dependent complementary risks model. We modify both the EM algorithm and the Bayesian algorithm to analyze dependent complementary risks data. We have analyzed one real complementary risks data set to show how the methods can be used in practice.

The rest of the paper is arranged as follows. In Section 2 we describe the model. Different properties of the ACBIW distribution are discussed in Section 3. In Section 4 we propose the EM algorithm to compute the MLEs and in Section 5 the Bayesian procedure has been provided. Simulation experiments and the analysis of a bivariate data set are provided in Section 6. The application of the ACBIW distribution in complementary risk model is discussed in Section 7 along with a real data analysis. Finally, we conclude the paper in Section 8.

2 Notations and Model Description

2.1 Notation

CDF : Cumulative distribution function.

CRI : Credible interval.

EM : Expectation maximization.

HPD : Highest posterior density.

i.i.d. : Independent and identically distributed.

MLE : Maximum likelihood estimator.

PDF : Probability density function.

$\mathrm{D}(a_1, a_2, \ldots, a_k)$: Multivariate Dirichlet distribution with PDF:

$$f(x_1, \ldots, x_k) = \frac{1}{B(a_1, a_2, \ldots, a_k)} \prod_{i=1}^{k} x_i^{a_i - 1}; \; x_i > 0 \; \forall i = 1, \ldots, k, \; \sum_{i=1}^{k} x_i = 1,$$

$$B(a_1, a_2, \ldots, a_k) = \frac{\prod_{i=1}^{k} \Gamma(a_i)}{\Gamma(a)} \text{ and } \quad a_i > 0, \; \forall i = 1, \ldots, k, \; a = \sum_{i=1}^{k} a_k.$$

$\mathrm{GA}(a, b)$: Gamma distribution with PDF:

$$f_{GA}(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}; \; x > 0, \; a, b > 0.$$

2.2 Model Description

A random variable X is said to follow $\mathrm{IW}(\alpha, \lambda)$ if its CDF is of the form

$$F_{\mathrm{IW}}(x; \alpha, \lambda) = \begin{cases} e^{-\lambda x^{-\alpha}} & \text{if } x > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

and the PDF is as follows

$$f_{\mathrm{IW}}(x; \alpha, \lambda) = \begin{cases} \alpha \lambda x^{-(\alpha+1)} e^{-\lambda x^{-\alpha}} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The BIW can be defined as follows. Supposes U_1, U_2, U_3 are three independent random variables, where $U_i \sim \text{IW}(\alpha, \lambda_i)$, for $i = 1, 2, 3$. If $Y_1 = max(U_1, U_3)$ and $Y_2 = max(U_2, U_3)$, then (Y_1, Y_2) is said to follow a BIW distribution with parameters $\alpha, \lambda_1, \lambda_2, \lambda_3$. The joint CDF of (Y_1, Y_2) where $(Y_1, Y_2) \sim \text{BIW}(\alpha, \lambda_1, \lambda_2, \lambda_3)$ can be written as

$$F_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\lambda_3} F_a(y_1, y_2) + \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3} F_s(y_1, y_2) & \text{if } y_1, y_2 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here,

$$\begin{aligned} F_s(y_1, y_2) &= e^{(\lambda_1+\lambda_2+\lambda_3)z^{-\alpha}} \text{ and} \\ F_a(y_1, y_2) &= \frac{\lambda_1 + \lambda_2 + \lambda_3}{\lambda_1 + \lambda_2} e^{-\lambda_1 y_1^{-\alpha} - \lambda_2 y_2^{-\alpha} - \lambda_3 z^{-\alpha}} - \frac{\lambda_3}{\lambda_1 + \lambda_2} e^{-(\lambda_1+\lambda_2+\lambda_3)z^{-\alpha}}, \end{aligned}$$

where $z = min(y_1, y_2)$.

Therefore $F_{Y_1,Y_2}(\cdot, \cdot)$ consists of $F_s(\cdot)$ and $F_a(\cdot)$ where $F_s(\cdot)$ is the singular part and $F_a(\cdot)$ is the absolute continuous part. The joint PDF of (Y_1, Y_2) can be written as

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\lambda_3} f_a(y_1, y_2) + \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3} f_s(z) & \text{if } y_1, y_2 > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where,

$$f_a(y_1, y_2) = \frac{\lambda_1 + \lambda_2 + \lambda_3}{\lambda_1 + \lambda_2} \begin{cases} f_{\text{IW}}(y_1; \alpha, \lambda_1 + \lambda_3) f_{IW}(y_2, \alpha, \lambda_2) & \text{if } 0 < y_1 < y_2, \\ f_{\text{IW}}(y_1; \alpha, \lambda_1) f_{IW}(y_2, \alpha, \lambda_2 + \lambda_3) & \text{if } 0 < y_2 < y_1, \end{cases}$$

and

$$f_s(y) = f_{\text{IW}}(y; \alpha, \lambda_1 + \lambda_2 + \lambda_3).$$

The ACBIW distribution can be obtained by omitting the singular part of the BIW distribution and it can be defined as follows. (X_1, X_2) is said to follow a ACBIW$(\alpha, \lambda_1, \lambda_2, \lambda_3)$, if the joint CDF is of the form

$$F_{\text{ACBIW}}(x_1, x_2) = \begin{cases} \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1+\lambda_2} e^{-\lambda_1 x_1^{-\alpha} - \lambda_2 x_2^{-\alpha} - \lambda_3 u^{-\alpha}} - \frac{\lambda_3}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2+\lambda_3)u^{-\alpha}} & \text{if } x_1, x_2 > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where, $u = min(x_1, x_2)$.

Therefore, the joint PDF can be written as

$$f_{\text{ACBIW}}(x_1,x_2) = \begin{cases} \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1+\lambda_2} f_{\text{IW}}(x_1;\alpha,\lambda_1+\lambda_3) f_{IW}(x_2,\alpha,\lambda_2) & \text{if } 0 < x_1 < x_2, \\ \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1+\lambda_2} f_{IW}(x_1;\alpha,\lambda_1) f_{IW}(x_2,\alpha,\lambda_2+\lambda_3) & \text{if } 0 < x_2 < x_1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Note that ACBIW has been obtained by removing the singular part from a BIW distribution, and then normalizing it to make a proper distribution function. The main advantage of the ACBIW distribution is that it is an absolutely continuous distribution with respect to two dimensional Lebesgue measure, and it can be used quite effectively to analyze bivariate lifetime data when there are no ties.

The following result provides the shape of the joint PDF of ACBIW distribution.

THEOREM 1.1: Let $(X_1, X_2) \sim \text{ACBIW}(\alpha, \lambda_1, \lambda_2, \lambda_3)$. Then

(a) If $\lambda_1 = \lambda_2$, then $f_{\text{ACBIW}}(x_1, x_2)$ is continuous for $0 < x_1, x_2 < \infty$. Moreover, $f_{\text{ACBIW}}(x_1, x_2)$ is unimodal and the mode is at (x_m, x_m), where $x_m = \{\alpha(2\lambda_1 + \lambda_3)/(2(\alpha + 1))\}^{1/\alpha}$

(b) If $\lambda_1 + \lambda_3 < \lambda_2$, then $f_{\text{ACBIW}}(x_1, x_2)$ is not continuous at $x_1 = x_2$. Further, $f_{\text{ACBIW}}(x_1, x_2)$ is unimodal and the mode is at (x_{1m}, x_{2m}), where $x_{1m} = \{\alpha(\lambda_1 + \lambda_3)/(\alpha + 1))\}^{1/\alpha}$ and $x_{2m} = \{\alpha\lambda_2/(\alpha + 1)\}^{1/\alpha}$.

(c) If $\lambda_2 + \lambda_3 < \lambda_1$, then $f_{\text{ACBIW}}(x_1, x_2)$ is not continuous at $x_1 = x_2$. Further, $f_{\text{ACBIW}}(x_1, x_2)$ is unimodal and the mode is at (x_{1m}, x_{2m}), where $x_{1m} = \{\alpha\lambda_1/(\alpha + 1)\}^{1/\alpha}$ and $x_{2m} = \{\alpha(\lambda_2 + \lambda_3)/(\alpha + 1))\}^{1/\alpha}$ and.

PROOF: It can be easily obtained, see for example Kundu and Gupta [13]. ∎

From Theorem 1.1, it is observed that the joint PDF of a ACBIW is unimodal and it can take various shapes. We have provided the surface plots of the joint PDFs of a ACBIW distribution for different parameter values in Figures 1 - 5. It is observed that it can be heavy tailed also. From the derivation of the ACBIW it is evident that if $(Y_1, Y_2) \sim$ BIW$(\alpha, \lambda_1, \lambda_2, \lambda_3)$ and

$$(X_1, X_2) \stackrel{d}{=} (Y_1, Y_2)|Y_1 \neq Y_2,$$

then $(X_1, X_2) \sim$ ACBIW$(\alpha, \lambda_1, \lambda_2, \lambda_3)$. Here, $\stackrel{d}{=}$ means equal in distribution. Therefore, the following algorithm can be applied to generate a random sample from the ACBIW distribution.

Algorithm 1 :

Step 1: Generate U_1, U_2, U_2 independently where $U_i \sim$ IW(α, λ_i) for $i = 1, 2, 3$.

Step 2: If $U_3 > U_1$ and $U_3 > U_2$ go back to Step 1, otherwise set X_1=max(U_1, U_3) and X_2=max(U_2, U_3).

3 Different Properties of the Marginals

In this section we provide different properties of the marginals of the ACBIW distribution. The following result gives the marginal distributions of a ACBIW distribution.

Theorem 3.1 *If $(X_1, X_2) \sim$ ACBIW$(\alpha, \lambda_1, \lambda_2, \lambda_3)$, then the marginal PDFs of X_1 and X_2 are*

$$f_{X_1}(x_1) = \begin{cases} \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1+\lambda_2} f_{IW}(x_1; \alpha, \lambda_1 + \lambda_3) - \frac{\lambda_3}{\lambda_1+\lambda_2} f_{IW}(x_1; \alpha, \lambda_1 + \lambda_2 + \lambda_3) & \text{if } x_1 > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

and

$$f_{X_2}(x_2) = \begin{cases} \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1+\lambda_2} f_{IW}(x_2; \alpha, \lambda_2 + \lambda_3) - \frac{\lambda_3}{\lambda_1+\lambda_2} f_{IW}(x_2; \alpha, \lambda_1 + \lambda_2 + \lambda_3) & \text{if } x_2 > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

respectively.

Proof: The marginal PDFs can be derived from (3) using simple integration. ∎

Therefore, the marginal distribution functions can be obtained as

$$F_{X_1}(x_1) = \begin{cases} \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_3)x_1^{-\alpha}} - \frac{\lambda_3}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2+\lambda_3)x_1^{-\alpha}} & \text{if } x_1 > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

$$F_{X_2}(x_2) = \begin{cases} \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1+\lambda_2} e^{-(\lambda_2+\lambda_3)x_2^{-\alpha}} - \frac{\lambda_3}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2+\lambda_3)x_2^{-\alpha}} & \text{if } x_2 > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

Note that, the marginal distributions are the generalized mixture of the IW distributions. Therefore, k th moment of the marginals will exist only when $\alpha > k$. It is observed that the PDFs of the marginals are unimodal or a decreasing function depending on the values of α. The hazard functions of the marginals are also either unimodal or a decreasing function.

The next result provides the conditional PDFs and CDFs of the marginals of the ACBIW distribution and it will be needed in the application section 7.

Theorem 3.2 *If* $(X_1, X_2) \sim \text{ACBIW}(\alpha, \lambda_1, \lambda_2, \lambda_3)$, *then the conditional PDF of* $X_1|X_2 = x_2$ *is given by*

$$f_{X_1|X_2=x_2}(x_1) = \begin{cases} \frac{\lambda_2}{e^{-\lambda_3 x_2^{-\alpha}}\left(\lambda_2+\lambda_3(1-e^{-\lambda_1 x_2^{-\alpha}})\right)} f_{IW}(x_1;\alpha,\lambda_1+\lambda_2) & \text{for } 0 < x_1 < x_2, \\ \frac{\lambda_2+\lambda_3}{\lambda_2+\lambda_3(1-e^{-\lambda_1 x_2^{-\alpha}})} f_{IW}(x_1;\alpha,\lambda_1) & \text{for } 0 < x_2 < x_1, \\ 0 & \textit{otherwise,} \end{cases} \tag{8}$$

and the conditional PDF of $X_2|X_1 = x_1$ *is given by*

$$f_{X_2|X_1=x_1}(x_2) = \begin{cases} \frac{\lambda_1+\lambda_3}{\lambda_1+\lambda_3(1-e^{\lambda_2 x_1^{-\alpha}})} f_{IW}(x_2;\alpha,\lambda_2) & \text{for } 0 < x_1 < x_2, \\ \frac{\lambda_1}{e^{-\lambda_3 x_1^{-\alpha}}\left(\lambda_1+\lambda_3(1-e^{-\lambda_2 x_1^{-\alpha}})\right)} f_{IW}(x_2;\alpha,\lambda_2+\lambda_3) & \text{for } 0 < x_2 < x_1, \\ 0 & \textit{otherwise.} \end{cases} \tag{9}$$

PROOF: The results can be derived using (3), (4) and (5). ∎

Some of the following properties will be useful for data analysis purposes and they will be used in Section 6.

Theorem 3.3 *If* $(X_1, X_2) \sim \text{ACBIW}(\alpha, \lambda_1, \lambda_2, \lambda_3)$, *then* *i)* $P(X_1 < X_2) = \frac{\lambda_2}{\lambda_1+\lambda_2}$,

ii) $max(X_1, X_2) \sim \mathrm{IW}(\alpha, \lambda_1 + \lambda_2 + \lambda_3)$,

iii) $X_1|X_1 > X_2 \sim \mathrm{IW}(\alpha, \lambda_1 + \lambda_2 + \lambda_3)$,

iv) $X_2|X_2 > X_1 \sim \mathrm{IW}(\alpha, \lambda_1 + \lambda_2 + \lambda_3)$.

Proof: See in Appendix. ∎

From the above result it is interesting to observe that, though any of the marginals are not IW variables, maximum of the two marginals follows a IW distribution. The marginal distribution of one variable provided it is greater than the other variable follows a IW distribution.

4 Maximum likelihood Estimation

In this section we derive the MLEs of the unknown parameters $\alpha, \lambda_1, \lambda_2, \lambda_3$ of a ACBIW distribution when we have a random sample of size n. The data are of the form $Data = \{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$. We introduce two sets I_1, I_2, where $I_1 = \{i : x_{1i} < x_{2i}\}$ and $I_2 = \{i : x_{1i} > x_{2i}\}$; $|I_1| = n_1, |I_2| = n_2$. Based on the above data the log-likelihood function can be written as

$$\begin{aligned} l(\boldsymbol{\theta}|Data) &= 2n\ln\alpha + n\ln(\lambda_1+\lambda_2+\lambda_3) - n\ln(\lambda_1+\lambda_2) + n_1\ln(\lambda_1+\lambda_3) \\ &\quad + n_1\ln\lambda_2 + n_2\ln\lambda_1 + n_2\ln(\lambda_2+\lambda_3) - (\alpha+1)\sum_{i\in I_1\bigcup I_2}(\ln x_{1i} + \ln x_{2i}) \\ &\quad -(\lambda_1+\lambda_3)\sum_{i\in I_1} x_{1i}^{-\alpha} - \lambda_2\sum_{i\in I_1} x_{2i}^{-\alpha} - \lambda_1\sum_{i\in I_2} x_{1i}^{-\alpha} - (\lambda_2+\lambda_3)\sum_{i\in I_2} x_{2i}^{-\alpha}. \quad (10) \end{aligned}$$

Here $\boldsymbol{\theta} = (\alpha, \lambda_1, \lambda_2, \lambda_3)$. The MLEs of the unknown parameters can be obtained by maximizing (10) with respect to the unknown parameters. Clearly, they cannot be obtained in explicit forms and they have to be obtained by solving a four dimensional optimization problem. To avoid that we propose to use the EM algorithm using the missing information principle. The basic idea behind the proposed EM algorithm is the following. Let us go back to the basic formulation of the ACBIW distributions based on U_1, U_2 and U_3. We will show that along with (X_1, X_2) if we had observed the associated U_1, U_2 and U_3, then the MLEs of the unknown parameters can be obtained by solving a one dimensional optimization problem. Let us assume that along with (X_1, X_2), we observe (δ_1, δ_2), where $\delta_1 = i$ if $X_1 = U_i$ and $\delta_2 = j$ if $X_2 = U_j$, for $j = 1, 2, 3$. It is assumed that we have the complete observation as follows: $\{((x_{1i}, \delta_{1i}), (x_{2i}, \delta_{2i})) : i = 1, \ldots, n\}$. Let us use the following notations.

$$
\begin{aligned}
I_{312} &= \{i : x_{1i} < x_{2i}, \delta_{1i} = 1, \delta_{2i} = 2\} \\
I_{132} &= \{i : x_{1i} < x_{2i}, \delta_{1i} = 3, \delta_{2i} = 2\} \\
I_{321} &= \{i : x_{1i} > x_{2i}, \delta_{1i} = 1, \delta_{2i} = 2\} \\
I_{231} &= \{i : x_{1i} > x_{2i}, \delta_{1i} = 1, \delta_{2i} = 3\}
\end{aligned}
$$

We have $I_1 = I_{312} \bigcup I_{132}$ and $I_2 = I_{321} \bigcup I_{231}$. In the observed data set I_1, $\delta_{2i} = 2$, is known, but δ_{1i} is unknown. Similarly, in I_2, $\delta_{1i} = 1$, is known, but δ_{2i} is unknown. The possible arrangement of the U_i's along with the associated probabilities are provided in Table 1. It will be useful in computing the likelihood function of the complete data set.

Table 1: The possible arrangements of U_1, U_2 and U_3

different arrangements	probabilities	(δ_1, δ_2)	X_1 and X_2	set
$U_3 < U_1 < U_2$	$\frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_2)}$	$(1, 2)$	$X_1 < X_2$	I_{312}
$U_1 < U_3 < U_2$	$\frac{\lambda_2\lambda_3}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_2)}$	$(3, 2)$	$X_1 < X_2$	I_{132}
$U_3 < U_2 < U_1$	$\frac{\lambda_1\lambda_2}{(\lambda_2+\lambda_3)(\lambda_1+\lambda_2)}$	$(1, 2)$	$X_1 > X_2$	I_{321}
$U_2 < U_3 < U_1$	$\frac{\lambda_1\lambda_3}{(\lambda_2+\lambda_3)(\lambda_1+\lambda_2)}$	$(1, 3)$	$X_1 > X_2$	I_{231}

The likelihood contribution of a data point from the complete data, for each set is given below.

(a) From the set I_{312} the contribution is $\alpha\lambda_1 x_{1i}^{-(\alpha+1)} e^{-(\lambda_1+\lambda_3)x_{1i}^{-\alpha}} \alpha\lambda_2 x_{2i}^{-(\alpha+1)} e^{-\lambda_2 x_{2i}^{-\alpha}}$,

(b) From the set I_{132} the contribution is $\alpha\lambda_3 x_{1i}^{-(\alpha+1)} e^{-(\lambda_1+\lambda_3)x_{1i}^{-\alpha}} \alpha\lambda_2 x_{2i}^{-(\alpha+1)} e^{-\lambda_2 x_{2i}^{-\alpha}}$,

(c) From the set I_{321} the contribution is $\alpha\lambda_1\lambda_2 x_{1i}^{-(\alpha+1)} e^{-\lambda_1 x_{1i}^{-\alpha}} \alpha x_{2i}^{-(\alpha+1)} e^{-(\lambda_2+\lambda_3)x_{2i}^{-\alpha}}$,

(d) From the set I_{231} the contribution is $\alpha\lambda_1\lambda_3 x_{1i}^{-(\alpha+1)} e^{-\lambda_1 x_{1i}^{-\alpha}} \alpha x_{2i}^{-(\alpha+1)} e^{-(\lambda_2+\lambda_3)x_{2i}^{-\alpha}}$.

Based on the complete data set $\mathscr{C} = \{((x_{1i}, \delta_{1i}), (x_{2i}, \delta_{2i})) : i = 1, \ldots, n\}$ the log-likelihood function can be obtained as follows.

$$
\begin{aligned}
l(\boldsymbol{\theta}|\mathscr{C}) = & \sum_{i \in I_{312}} (2\ln\alpha + \ln\lambda_1 + \ln\lambda_2) - (\lambda_1+\lambda_3) \sum_{i \in I_{312}} x_{1i}^{-\alpha} - \lambda_2 \sum_{i \in I_{312}} x_{2i}^{-\alpha} \\
& -(\alpha+1) \sum_{i \in I_{312}} (\ln x_{1i} + \ln x_{2i}) \\
& + \sum_{i \in I_{132}} (2\ln\alpha + \ln\lambda_2 + \ln\lambda_3) - (\lambda_1+\lambda_3) \sum_{i \in I_{132}} x_{1i}^{-\alpha} - \lambda_2 \sum_{i \in I_{132}} x_{2i}^{-\alpha} \\
& -(\alpha+1) \sum_{i \in I_{132}} (\ln x_{1i} + \ln x_{2i}) \\
& + \sum_{i \in I_{321}} (2\ln\alpha + \ln\lambda_1 + \ln\lambda_2) - \lambda_1 \sum_{i \in I_{321}} x_{1i}^{-\alpha} - (\lambda_2+\lambda_3) \sum_{i \in I_{321}} x_{2i}^{-\alpha} \\
& -(\alpha+1) \sum_{i \in I_{321}} (\ln x_{1i} + \ln x_{2i}) \\
& + \sum_{i \in I_{231}} (2\ln\alpha + \ln\lambda_1 + \ln\lambda_3) - \lambda_1 \sum_{i \in I_{231}} x_{1i}^{-\alpha} - (\lambda_2+\lambda_3) \sum_{i \in I_{231}} x_{2i}^{-\alpha} \\
& -(\alpha+1) \sum_{i \in I_{231}} (\ln x_{1i} + \ln x_{2i}). \qquad (11)
\end{aligned}
$$

It can be easily seen that for a given α, the maximization of (11) can be obtained in explicit forms in terms of λ_1, λ_2 and λ_3, and the maximization with respect to α can be obtained by solving a one dimensional optimization problem with respect to α. This is the main motivation of the proposed EM algorithm. We need the following result for developing the EM algorithm. They can be obtained from Table 1.

Result 4 :

$$P(\delta_1 = 1, \delta_2 = 2 | X_1 < X_2) = \frac{\lambda_1}{(\lambda_1 + \lambda_3)} = a, P(\delta_1 = 3, \delta_2 = 2 | X_1 < X_2) = \frac{\lambda_3}{(\lambda_1 + \lambda_3)} = 1 - a,$$
$$P(\delta_1 = 1, \delta_2 = 2 | X_1 > X_2) = \frac{\lambda_2}{(\lambda_2 + \lambda_3)} = b, P(\delta_1 = 1, \delta_2 = 3 | X_1 > X_2) = \frac{\lambda_3}{(\lambda_2 + \lambda_3)} = 1 - b.$$

Let us use the following notations. At the r-th stage of the EM algorithm, the estimates of α, λ_1, λ_2 λ_3 will be denoted by $\alpha^{(r)}$, $\lambda_1^{(r)}$, $\lambda_2^{(r)}$, $\lambda_3^{(r)}$, respectively. Similarly, $\lambda^{(r)} = \lambda_1^{(r)} + \lambda_2^{(r)} + \lambda_3^{(r)}$, $a^{(r)} = \lambda_1^{(r)}/(\lambda_1^{(r)} + \lambda_3^{(r)})$ and $b(r) = \lambda_2^{(r)}/(\lambda_2^{(r)} + \lambda_3^{(r)})$ are also defined. Now, using the idea of Dinse [6], see also Kundu [9], the 'pseudo' log-likelihood function at the r-th stage ('E'-step) of the EM algorithm can be written as

$$\begin{aligned} l(\boldsymbol{\theta}|\boldsymbol{\theta}^{(r)}, Data) &= (n_1 a^{(r)} + n_2) \ln \lambda_1 - \lambda_1 \sum_{i \in I_1 \cup I_2} x_{1i}^{-\alpha} + (n_1 + n_2 b^{(r)}) \ln \lambda_2 - \lambda_2 \sum_{i \in I_1 \cup I_2} x_{2i}^{-\alpha} + \\ &\quad (n_1(1 - a^{(r)}) + n_2(1 - b^{(r)})) \ln \lambda_3 - \lambda_3 \left[\sum_{i \in I_1} x_{1i}^{-\alpha} + \sum_{i \in I_2} x_{2i}^{-\alpha} \right] + \\ &\quad 2n \ln \alpha - (\alpha + 1) \sum_{i \in I_1 \cup I_2} (\ln x_{1i} + \ln x_{2i}) . \end{aligned} \tag{12}$$

The 'M'-step can be obtained by maximizing $l(\boldsymbol{\theta}|\boldsymbol{\theta}^{(r)}, Data)$ with respect to the unknown parameters. For fixed α, the function (12) is maximized at $\lambda_1 = \lambda_1^{(r+1)}(\alpha)$, $\lambda_2 = \lambda_2^{(r+1)}(\alpha)$, $\lambda_3 = \lambda_3^{(r+1)}(\alpha)$, where,

$$\lambda_1^{(r+1)}(\alpha) = \frac{n_1 a^{(r)} + n_2}{\sum\limits_{i \in I_1 \bigcup I_2} x_{1i}^{-\alpha}}, \quad \lambda_2^{(r+1)}(\alpha) = \frac{n_1 + n_2 b^{(r)}}{\sum\limits_{i \in I_1 \bigcup I_2} x_{2i}^{-\alpha}}, \quad \lambda_3^{(r+1)}(\alpha) = \frac{n_1(1 - a^{(r)}) + n_2(1 - b^{(r)})}{\sum\limits_{i \in I_1} x_{1i}^{-\alpha} + \sum\limits_{i \in I_2} x_{2i}^{-\alpha}}.$$

The corresponding $\alpha^{(r+1)}$ can be obtained by maximizing pseudo profile log-likelihood function $l(\alpha, \lambda_1^{(r+1)}(\alpha), \lambda_2^{(r+1)}(\alpha), \lambda_3^{(r+1)}(\alpha)|\boldsymbol{\theta}^{(r)}, Data) = h^{(r)}(\alpha)$. Without the additive constant,

$$\begin{aligned} h^{(r)}(\alpha) &= -\Big(n_1 a^{(r)} + n_2\Big) \ln \Big(\sum_{i \in I_1 \bigcup I_2} x_{1i}^{-\alpha} \Big) - \Big(n_1 + n_2 b^{(r)}\Big) \ln \Big(\sum_{i \in I_1 \bigcup I_2} x_{2i}^{-\alpha} \Big) \\ &\quad - \Big(n_1(1 - a^{(r)}) + n_2(1 - b^{(r)})\Big) \ln \Big(\sum_{i \in I_1} x_{1i}^{-\alpha} + \sum_{i \in I_2} x_{2i}^{-\alpha} \Big) \\ &\quad + 2n \ln \alpha - (\alpha + 1) \sum_{i \in I_1 \cup I_2} (\ln x_{1i} + \ln x_{2i}) . \end{aligned}$$

The following result ensures the existence of a unique maximum of the pseudo profile log-likelihood function at every stage of the iteration.

Result 5 : At the r-th stage the pseudo log-likelihood function $h^{(r)}(\alpha)$ is a unimodal function of α.

Proof: See in Appendix. ∎

The maximization of the pseudo log-likelihood function which is a one dimensional optimization problem can be done by the Newton Raphson or bisection method. Once $\alpha^{(r+1)}$ can be obtained, we can compute $\lambda_1^{(r+1)} = \lambda_1^{(r+1)}(\alpha^{(r+1)})$, $\lambda_2^{(r+1)} = \lambda_2^{(r+1)}(\alpha^{(r+1)})$ and $\lambda_3^{(r+1)} = \lambda_3^{(r+1)}(\alpha^{(r+1)})$. The process will be continued till convergence.

5 Bayesian Inference

In this section we provide the Bayesian analysis of the unknown model parameters. The Bayes estimators are derived based on the squared error loss function although any other loss function also can be easily incorporated. The prior assumption and its posterior analysis are discussed below.

5.1 Prior Assumption

Following the approach of Pena and Gupta [17] the following prior assumptions are made on the scale parameters. It is assumed that $\lambda = \lambda_1 + \lambda_2 + \lambda_3$ follows a gamma distribution say GA(a_0, b_0). To incorporate dependence among $\lambda_1, \lambda_2, \lambda_3$, it is assumed that given λ, $(\frac{\lambda_1}{\lambda}, \frac{\lambda_2}{\lambda})$ follows a Dirichlet prior say, $D(a_1, a_2, a_3)$. The joint prior of $(\lambda_1, \lambda_2, \lambda_3)$ can be obtained as

$$\pi_1(\lambda_1, \lambda_2, \lambda_3) \propto (\lambda_1 + \lambda_2 + \lambda_3)^{a_0 - a_1 - a_2 - a_3} \lambda_1^{a_1 - 1} \lambda_2^{a_2 - 1} \lambda_3^{a_3 - 1} e^{-b_0(\lambda_1 + \lambda_2 + \lambda_3)}. \tag{13}$$

The joint prior with hyper parameters a, b, a_1, a_2, a_3 is called the Gamma-Dirichlet prior, denoted by $\text{GD}(a_0, b_0, a_1, a_2, a_3)$ and it is a very flexible prior, see Pena and Gupta [17] for details. It is further assumed that the shape parameter α has a log-concave prior $\pi_2(\alpha)$ with a positive support on $(0,\infty)$ and $\pi_2(\alpha)$ is independent of $\pi_1(\lambda_1,\lambda_2,\lambda_3)$. Hence the joint prior of $\alpha, \lambda_1, \lambda_2, \lambda_3$ is obtained as $\pi(\alpha,\lambda_1,\lambda_2,\lambda_3) = \pi_2(\alpha)\pi_1(\lambda_1,\lambda_2,\lambda_3)$.

5.2 Posterior Analysis

In this section we provide the Bayes estimators based on the squared error loss function and the associated credible intervals of the unknown parameters. The joint posterior density of $\boldsymbol{\theta} = (\alpha, \lambda_1, \lambda_2, \lambda_3)$ can be obtained as

$$\begin{aligned}\pi(\boldsymbol{\theta}|Data) \;\propto\;& \left\{\frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1+\lambda_2}\right\}^n \lambda_1^{n_2+a_1-1}\lambda_2^{n_1+a_2-1}(\lambda_1+\lambda_2)^{n_1}(\lambda_2+\lambda_3)^{n_3} \\ &\times e^{-(\lambda_1+\lambda_3)\sum\limits_{i\in I_1} x_{1i}^{-\alpha}} e^{-\lambda_2\sum\limits_{i\in I_1} x_{2i}^{-\alpha}} e^{-\lambda_1\sum\limits_{i\in I_2} x_{1i}^{-\alpha}} e^{-(\lambda_2+\lambda_3)\sum\limits_{i\in I_2} x_{2i}^{-\alpha}} \\ &\times(\lambda_1+\lambda_2+\lambda_3)^{a_0-a_1-a_2-a_3} e^{-b_0(\lambda_1+\lambda_2+\lambda_3)} \times \alpha^{2n}\pi_2(\alpha)\{\prod_{i\in I_1\bigcup I_2} x_{1i}^{-(\alpha+1)}x_{2i}^{-(\alpha+1)}\}.\end{aligned}$$

The Bayes estimate of any function of $\boldsymbol{\theta}$ say $g(\boldsymbol{\theta})$ can be obtained as

$$E(g(\boldsymbol{\theta})|Data) \;=\; \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty g(\boldsymbol{\theta})\pi(\boldsymbol{\theta}|Data)d\alpha d\lambda_1 d\lambda_2 d\lambda_3,$$

provided it exists. The estimator cannot be obtained in closed form. For further development simplify the posterior density as follows.

$$\pi(\boldsymbol{\theta}|Data) \propto h(\lambda_1,\lambda_2,\lambda_3)\pi_{11}^*(\lambda_1|\alpha, data)\pi_{12}^*(\lambda_2|\alpha, data)\pi_{13}^*(\lambda_3|\alpha, Data)\pi_2^*(\alpha|data)$$

where,

$$\begin{aligned}\pi_{11}^*(\lambda_1|\alpha, Data) \;&\equiv\; \text{GA}(n_2+a_1, b_0+\sum_{i\in I_1\bigcup I_2} x_{1i}^{-\alpha}), \\ \pi_{12}^*(\lambda_2|\alpha, Data) \;&\equiv\; \text{GA}(n_1+a_2, b_0+\sum_{i\in I_1\bigcup I_2} x_{2i,}^{-\alpha}) \\ \pi_{13}^*(\lambda_3|\alpha, Data) \;&\equiv\; \text{GA}(n+a_3, b_0+\sum_{i\in I_1} x_{1i}^{-\alpha}+\sum_{i\in I_2} x_{2i}^{-\alpha}),\end{aligned}$$

$$\pi_2^*(\alpha|Data) \propto \frac{\{\prod_{i\in I_1\bigcup I_2} x_{1i}^{-(\alpha+1)} x_{2i}^{-(\alpha+1)}\}\alpha^{2n}\pi_2(\alpha)}{(b_0+\sum_{i\in I_1\bigcup I_2} x_{1i}^{-\alpha})^{n_2+a_1}(b_0+\sum_{i\in I_1\bigcup I_2} x_{2i}^{-\alpha})^{n_1+a_2}(b_0+\sum_{i\in I_1} x_{1i}^{-\alpha}+\sum_{i\in I_2} x_{2i}^{-\alpha})^{n+a_3}},$$

$$h(\lambda_1,\lambda_2,\lambda_3) = (\lambda_1+\lambda_2+\lambda_3)^{n+a_0-a_1-a_2-a_3}(1+\frac{\lambda_1}{\lambda_3})^{n_1}(1+\frac{\lambda_2}{\lambda_3})^{n_2}(\lambda_1+\lambda_2)^{-n}.$$

As the posterior cannot be obtained in any standard form we rely on the importance sampling technique to compute the Bayes estimates and the associated credible intervals. The following result is needed for further development.

Result 6: When $\pi_2(\alpha)$ is log-concave, $\pi_2^*(\alpha|data)$ is also log-concave.

Proof: It can be obtained similarly as in Kundu [10]. ∎

To apply the importance sampling technique, it is required to generate $\alpha, \lambda_1, \lambda_2, \lambda_2$ from the posterior density. Based on the method by Devroye [5] α can be generated from the log-concave density function $\pi_2^*(\alpha|data)$. Here we follow the simpler method suggested in Kundu [10] to generate α from its log-concave posterior density. Once α is generated we can generate $\lambda_1, \lambda_2, \lambda_3$ from the respective conditional densities. The following algorithm can be used to compute the Bayes estimate and the associated credible intervals.

Algorithm 2:

Step 1: Given data, generate α from $\pi_2^*(\alpha|data)$.

Step 2: For a given α, generate $\lambda_1, \lambda_2, \lambda_3$ from $\pi_{11}^*(\lambda_1|data)$, $\pi_{12}^*(\lambda_2|data)$ and $\pi_{13}^*(\lambda_3|data)$, respectively.

Step 3: Repeat the process say N times to generate $\big((\alpha_1, \lambda_{11}, \lambda_{21}, \lambda_{31}), \ldots, (\alpha_N, \lambda_{1N}, \lambda_{2N}, \lambda_{3N})\big)$.

Step 4: To compute the Bayes estimate of $g(\alpha, \lambda_1, \lambda_2, \lambda_3)$ compute $(g_1, \ldots, g_N)$ and $(h_1, \ldots, h_N)$ where $g_i = g(\alpha_i, \lambda_{1i}, \lambda_{2i}, \lambda_{3i})$ and $h_i = h(\lambda_{1i}, \lambda_{2i}, \lambda_{3i})$ for $i = 1, \ldots, N$.

Step 5: The Bayes estimate of $g(\alpha, \lambda_1, \lambda_2, \lambda_3)$ based on the squared error loss function can be computed as $\frac{\sum_{i=1}^{N} h_i g_i}{\sum_{j=1}^{N} h_j} = \sum_{i=1}^{N} w_i g_i$ where $w_i = \frac{h_i}{\sum_{j=1}^{N} h_j}$.

Step 6: To compute a $100(1-\gamma)\%$ credible interval of $g(\alpha, \lambda_1, \lambda_2, \lambda_3)$, arrange g_i's in an ascending order to obtain $(g_{(1)}, \ldots g_{(N)})$ and record the corresponding w_i as $(w_{[1]}, \ldots, w_{[N]})$. Here $w_{[i]}$ is not ordered and they are assigned to the corresponding ordered $g_{(i)}$'s. A $100(1-\gamma)\%$ credible interval can be obtained as $(g_{(j_1)}, g_{(j_2)})$ where j_1, j_2 such that

$$j_1 < j_2, \quad j_1, j_2 \in \{1, \ldots, N\} \quad \text{and} \quad \sum_{i=j_1}^{j_2} w_{[i]} \leq 1 - \gamma < \sum_{i=j_1}^{j_2+1} w_{[i]}. \tag{14}$$

6 Simulation Results and Data Analysis

6.1 Simulation Results

In this section we present the results based on the simulation experiments. The experiment is performed to check how the different methods perform for different sample sizes and for different parameter values. We consider $n = 20, 30, 40$ for three different sets of parameter values with $\alpha = 0.5, 1, 2$ and $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$. We compute the average estimates (AE) and the corresponding mean squared errors (MSE) of the MLEs derived through EM algorithm as described in Section 4. In Tables 2, 3 and 4 we provide the AEs and MSEs based on 1000 samples. In the EM algorithm we set the initial guesses as the true parameter values and stop the process until the absolute differences of the estimates from two consecutive iteration is less than 10^{-4}. It may be mentioned that in our simulation experiments we have kept λ's to be constant and we have changed α. In fact we have performed some simulations with different values of λ's also, but we have obtained similar results, hence they have not been reported. The effect of change of α is more evident in the simulation experiments. All the computations are performed using R software, and we have attached the codes for convenience.

Based on the observed information matrix we compute the asymptotic confidence intervals and record the average lengths (AL) and coverage percentages (CP) based on 1000 replications in Tables 5-10. In the Bayesian analysis, for the shape parameter α we choose a gamma prior which is a log-concave, i.e. $\pi_2(\alpha) \equiv \text{GA}(c, d)$. The Bayes estimates based on the squared error loss function is computed both for an informative prior (IP) and non-informative prior (NIP). In the informative prior we choose the values of the hyper parameters equating the prior mean with the true parameter values. When $\alpha = 0.5, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$ the hyper parameters are: $a_0 = 1, b_0 = \frac{1}{3}, a_1 = 1, a_2 = 1, a_3 = 1, c = 1, d = 0.5$. When $\alpha = 1, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$ the hyper parameters are: $a_0 = 1, b_0 = \frac{1}{3}, a_1 = 1, a_2 = 1, a_3 = 1, c = 1, d = 1$ and for $\alpha = 2, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$, the hyper-parameters are: $a_0 = 1, b_0 = \frac{1}{3}, a_1 = 1, a_2 = 1, a_3 = 1, c = 2, d = 1$. In the non-informative prior following the idea from Congdon [3] in every set of parameters, the hyper-parameters are as $a = 10^{-5}, b = 10^{-5}, a_1 = 10^{-5}, a_2 = 10^{-5}, a_3 = 10^{-5}, c = 10^{-5}, d = 10^{-5}$. We have considered both the informative and non-informative priors to see the effect of priors on the performance of the Bayes estimators.

Both for the informative and the non-informative priors we compute the AEs and the corresponding MSEs of the Bayes estimators based on 1000 replications are provided in Tables 2, 3 and 4 for different sample sizes and for different sets of parameters. In interval estimation we compute both 90% and 95% symmetric credible intervals and report the ALs and CPs based on 1000 replications in Tables 5-10.

From Tables 2, 3 and 4, it is clear that the performances of the MLEs and both the Bayes estimators are quite satisfactory. It is observed that the MLEs and both the Bayes estimators provide unbiased estimators even for moderate sample sizes. The slight variations which have been observed is mainly due to replication. As sample size increases the MSEs decrease in all the cases. It indicates that all the estimators are consistent estimators. It is further observed that the MSEs of the MLEs are slightly more than the corresponding Bayes estimators. Moreover, the MSEs of the Bayes estimators based on informative priors are slightly smaller than the corresponding non-informative priors as expected.

In interval estimation for small to moderate sample sizes the symmetric credible intervals perform better than the asymptotic intervals in terms of average length and CP. As sample size increases, the CPs of the symmetric credible intervals for the scale parameters go lesser than the nominal level. The asymptotic intervals for the scale parameters perform better than the symmetric credible intervals in terms of CP for large sample. Considering everything we propose to use Bayes estimators with informative priors, if available, otherwise non-informative priors also can be used.

6.2 Data Analysis

In this section we analyze a real data set for illustrative purposes. The data set has been taken from Johnson et al. [20]. This is a bivariate data set (X_1, X_2) on 24 children where X_1 represents the bone mineral density in g/cm^2 for Dominant Ulna and X_2 represents the bone mineral density in g/cm^2 for Ulna bones. The data are as follows.

(0.869, 0.964), (0.602, 0.689), (0.765, 0.738), (0.761, 0.698), (0.551, 0.619), (0.753, 0.515), (0.708,0.787), (0.687 0.715), (0.844 0.656), (0.869 0.789), (0.654, 0.726), (0.692, 0.526), (0.670, 0.580), (0.823,0.773), (0.746, 0.729), (0.656, 0.506), (0.693, 0.740), (0.883, 0.785), (0.577, 0.627), (0.802, 0.769), (0.540, 0.498), (0.804, 0.779), (0.570, 0.634), (0.585, 0.640).

To check whether a IW distribution can fit the maximum of X_1 and X_2, we perform Kolmogorov-Smirnov (K-S) goodness of fit test. The MLEs and the K-S distance between the empirical distribution and the fitted distribution along with the p value are recorded in Table 11. Based on the results in Theorem 3.3 we have fitted also ACBIW distributions both on $X_1|X_1 > X_2$ and $X_2|X_2 > X_1$ and perform the K-S test for both the cases. The results are also provided in Table 11. Based on the results, it is reasonable to assume that the ACBIW may be used to analyze this data set.

Table 11: Goodness of fitting of the real data set

Case	MLE		K-S distance	p value
	shape parameter	scale parameter		
$max(X_1, X_2)$	7.563	10.961	0.110	0.933
$X_1\|X_1 > X_2$	6.602	9.355	0.227	0.461
$X_2\|X_2 > X_1$	9.999	19.067	0.201	0.813

We consider the classical and the Bayes estimations of the parameters α, λ_1, λ_2, λ_3. In the EM algorithm as the initial guesses of the parameters we consider the MLEs for $max(X_1, X_2)$. Here, in Table 11, the MLE of shape parameter is 7.563 and the MLE of the scale parameter is 10.961. Therefore, we set $\alpha^{(0)} = 7$ and $\lambda_1^{(0)} = 3, \lambda_2^{(0)} = 3, \lambda_3^{(0)} = 3$. We start the EM algorithm with these initial guesses and continue until the absolute differences of the estimates in two consecutive iterations is less than 10^{-4}.

The MLEs and the percentile bootstrap confidence intervals are recorded in Table 12. The Bayes estimates are derived for the non-informative prior based on the squared error loss function. We compute the 90% symmetric credible intervals for the unknown model parameters. All these results are provided in Table 12.

Here we have performed the data analysis considering (X_1, X_2) follows a ACBIW distribution. Now we perform K-S tests to check whether the marginal distributions of the ACBIW can fit X_1 and X_2. The p values are 0.647 and 0.449 for X_1 and X_2, respectively. Therefore the assumption of ACBIW on the Data set is quite reasonable.

Table 12: Maximum likelihood estimate and Bayesian estimates for Real Data set

Parameter	MLE	90% Asymptotic CI	Bayes estimate	90% symmetric CRI
α	5.133	(3.331, 6.381)	4.950	(4.745, 5.140)
λ_1	2.142	(0.465, 3.819)	2.147	(1.209, , 3.629)
λ_2	1.434	(0.587, 2.281)	1.409	(0.606, 2.924)
λ_3	3.097	(1.315, 4.879)	2.971	(1.485, , 4.505)

7 Application

In this section we apply the ACBIW distribution as a dependent complementary risks model. It is assumed that n experimental units are put on a life testing experiment and each unit is susceptible to two risk factors with latent failure times X_1 and X_2. It is further assumed that $(X_1, X_2) \sim$ ACBIW$(\alpha, \lambda_1, \lambda_2, \lambda_3)$. For each experimental unit we observe $T = max(X_1, X_2)$. We define a random variable δ with $\delta_i = 1$ if i-th failure occurs due to cause 1 and $\delta_i = 2$ if it is due to cause 2. Therefore we observe the data $\mathscr{D} = \{(t_1, \delta_1), \ldots, (t_n, \delta_n)\}$. The problem is to estimate the unknown parameters $\boldsymbol{\theta} = (\alpha, \lambda_1, \lambda_2, \lambda_3)$ based on the observed data.

The likelihood function of the observation can be written as

$$L(\boldsymbol{\theta}|\mathscr{D}) = \left(\frac{\lambda_1 + \lambda_2 + \lambda_3}{\lambda_1 + \lambda_2}\right)^n \lambda_1^{d_1} \lambda_2^{d_2} \alpha^n \prod_{i=1}^{n} t_i^{-(\alpha+1)} e^{-(\lambda_1+\lambda_2+\lambda_3)\sum_{i=1}^{n} t_i^{-\alpha}}. \tag{15}$$

Here $d_1 = \#\ \{i : \delta_i = 1\}$ and $d_2 = \#\ \{i : \delta_i = 2\}$. Now to compute the MLEs of the unknown parameters, we can modify the EM algorithm described in Section 4, very easily. Using the notation in Section 4, note that if $\delta_i = 1$, then clearly $x_{1i} = t_i$ and $x_{2i} < x_{1i}$, and it is unknown. Moreover, $t_i \in I_2$. Similarly, if $\delta_i = 2$, then clearly $x_{2i} = t_i$ and $x_{1i} < x_{2i}$, and it is unknown. Here $t_i \in I_1$. Therefore, the EM algorithm as described in Section 4, can be modified as follows. In the set I_1, x_{1i}'s are replaced by their expectations, similarly, in the set I_2, the missing x_{2i}'s are replaced by their expectations. These expectations cannot obtained in explicit forms, but they can be obtained in the integration form using Theorem 3.2 as follows:

$$\begin{aligned} E(X_1|X_1 < X_2 = t) &= e^{(\lambda_1+\lambda_3)t^{-\alpha}}(\lambda_1+\lambda_3)^{1/\alpha}\int_{(\lambda_1+\lambda_3)t^{-\alpha}}^{\infty} u^{-1/\alpha}e^{-u}du. \\ E(X_2|X_2 < X_1 = t) &= e^{(\lambda_2+\lambda_3)t^{-\alpha}}(\lambda_2+\lambda_3)^{1/\alpha}\int_{(\lambda_2+\lambda_3)t^{-\alpha}}^{\infty} u^{-1/\alpha}e^{-u}du. \end{aligned}$$

In the Bayesian analysis part, we use the same prior as in Section 6. Based on the likelihood in (15), the joint posterior density function can be written as

$$\pi(\boldsymbol{\theta}|\mathscr{D}) \propto h(\lambda_1,\lambda_2,\lambda_3)\pi_{11}^*(\lambda_1|\alpha,\mathscr{D})\pi_{12}^*(\lambda_2|\alpha,\mathscr{D})\pi_{13}^*(\lambda_3|\alpha,\mathscr{D})\pi_2^*(\alpha|\mathscr{D}) \tag{16}$$

where

$$\begin{aligned} \pi_{11}^*(\lambda_1|\alpha,\mathscr{D}) &\equiv \mathrm{GA}(d_1+a_1, b+\sum_{i=1}^{n} t_i^{-\alpha}), \\ \pi_{12}^*(\lambda_2|\alpha,\mathscr{D}) &\equiv \mathrm{GA}(d_2+a_2, b+\sum_{i=1}^{n} t_i^{-\alpha}), \\ \pi_{13}^*(\lambda_3|\alpha,\mathscr{D}) &\equiv \mathrm{GA}(a_3+n, b+\sum_{i=1}^{n} t_i^{-\alpha}), \\ \pi_2^*(\alpha|\mathscr{D}) &\equiv \frac{\alpha^n\prod_{i=1}^n t_i^{-(\alpha+1)}\pi_2(\alpha)}{(b+\sum_{i=1}^n t_i^{-\alpha})^{2n+a_1+a_2+a_3}}, \\ h(\lambda_1,\lambda_2,\lambda_3) &= \frac{(\lambda_1+\lambda_2+\lambda_3)^{n+a-a_1-a_2-a_3}}{(\lambda_1\lambda_3+\lambda_2\lambda_3)^n}. \end{aligned}$$

To derive the Bayes estimate of any function $g(\alpha,\lambda_1,\lambda_2,\lambda_3)$, based on the squared error loss function, provided it exists, as well as associated credible intervals we follow the Algorithm 2 from Section 6.

A real data set under complementary risk is analyzed for illustrative purposes. The data set are originally taken from Han [8]. Here we see a small unmanned aerial vehicle (SUAV) is equipped with dual propulsion systems and the SUAV drops from its flying altitude only when both the propulsion systems fail. Table 13 indicates the failure time of the SUAV as well as which propulsion system fails later.

Table 13: Real data in presence of complementary risks

Failure Time	Cause of Failure
2.365	1
3.467	2
5.386	2
7.714	2
9.578	1
9.683	2
11.416	1
11.789	1
12.039	2
14.928	1
14.938	2
15.325	2
15.781	2
16.105	1
16.362	2
17.178	2
17.366	1
17.803	1
19.578	2

We ignore the censored data as given in Han [8] and conduct the analysis based on the first $n = 19$ units. To check whether a IW distribution can fit the failure time data (without the cause of failure) we compute the K-S distance between the empirical and the fitted distributions. The MLEs of the shape and scale parameters become 1.427 and 1.991, respectively. The K-S distance and the associated p value become 0.247 and 0.196, respectively. Therefore, based on the p value we say IW distribution can be used to analyze the SUAV failure data. We have calculated the MLEs and the Bayes estimates along with the asymptotic confidence and symmetric credible intervals. The results are presented in Table 14.

Table 14: Maximum likelihood estimate and Bayes estimate for SUAV Data in presence of complementary risks

Parameter	MLE	90% Asymptotic CI	Bayes estimate	90% symmetric CRI
α	1.363	(1.119, 1.607)	1.354	(1.320, 1.546)
λ_1	0.488	(0.325,0.651)	0.480	(0.315, 0.628)
λ_2	0.907	(0.379, 1.435)	0.947	(0.371, 1.065)
λ_3	0.755	(0.300, 1.210)	0.759	(0.563, 1.470)

8 Conclusion

In this paper we have introduced a new absolutely continuous bivariate distribution by omitting the singular part of the BIW distribution introduced by Muhammed [16] and Kundu and Gupta [13]. The proposed ACBIW distribution can be used quite effectively if the marginals have heavy tailed distribution and there are no ties in the data set. We have studied different properties of the model. The MLEs of the unknown parameters cannot be obtained in explicit forms, we have provided a very effective EM algorithm to estimate unknown parameters. It is observed based on extensive simulation studies that the proposed EM algorithm works quite well in practice. We further considered the Bayesian inference of the unknown parameters based on a fairly general set of priors. We have provided a very effective method of generating samples from the joint posterior PDFs, and it can be used to compute Bayes estimates and the associated credible intervals. The model has been used as dependent complementary risks model. In this paper we have mainly concentrated on the bivariate set up, but it can be extended to the multivariate case also. Moreover, in most of the times in reliability and survival analysis, the data are censored. Here, we have considered complete sample only. It will be of interest to study the analysis of this model under different censoring schemes. More work is needed along that direction.

Acknowledgements

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Appendix

Proof of Theorem 3.3:

$$
\begin{aligned}
(i) \qquad P(X_1 < X_2) &= \int_0^{\infty}\int_0^{x_2} f_{ACBIW}(x_1, x_2)dx_1 dx_2 \\
&= \int_0^{\infty}\int_0^{x_2} \frac{(\lambda_1+\lambda_2+\lambda_3)}{(\lambda_1+\lambda_2)} f_{IW}(x_1;\alpha,\lambda_1+\lambda_3) f_{IW}(x_2;\alpha,\lambda_2)dx_1 dx_2 \\
&= \frac{(\lambda_1+\lambda_2+\lambda_3)}{(\lambda_1+\lambda_2)} \int_0^{\infty} f_{IW}(x_2;\alpha,\lambda_2) e^{-(\lambda_1+\lambda_3)x_2^{-\alpha}} dx_2 \\
&= \frac{\lambda_2}{\lambda_1+\lambda_2}.
\end{aligned}
$$

$$
\begin{aligned}
(ii) \quad P(max(X_1, X_2) < x) &= P(X_1 < X_2 < x) + P(X_2 < X_1 < x) \\
&= \frac{(\lambda_1+\lambda_2+\lambda_3)}{(\lambda_1+\lambda_2)} \int_0^x \int_0^{x_2} f_{IW}(x_1; \alpha, \lambda_1+\lambda_3) f_{IW}(x_2; \alpha, \lambda_2) dx_1 dx_2 \\
&\quad + \frac{(\lambda_1+\lambda_2+\lambda_3)}{(\lambda_1+\lambda_2)} \int_0^x \int_0^{x_1} f_{IW}(x_1; \alpha, \lambda_1) f_{IW}(x_2; \alpha, \lambda_2+\lambda_3) dx_2 dx_1 \\
&= \frac{(\lambda_1+\lambda_2+\lambda_3)}{(\lambda_1+\lambda_2)} \Big(\int_0^x f_{IW}(x_2; \alpha, \lambda_2)(e^{-(\lambda_1+\lambda_3)x^{-\alpha}} dx_2 \\
&\quad + \int_0^x f_{IW}(x_1; \alpha, \lambda_1) e^{-(\lambda_2+\lambda_3)x^{-\alpha}} dx_1 \Big) \\
&= \frac{(\lambda_1+\lambda_2+\lambda_3)}{(\lambda_1+\lambda_2)} \Big(\frac{\lambda_2}{(\lambda_1+\lambda_2+\lambda_3)} e^{-(\lambda_1+\lambda_2+\lambda_3)x^{-\alpha}} \\
&\quad + \frac{\lambda_1}{(\lambda_1+\lambda_2+\lambda_3)} e^{-(\lambda_1+\lambda_2+\lambda_3)x^{-\alpha}} \Big) \\
&= e^{-(\lambda_1+\lambda_2+\lambda_3)x^{-\alpha}}.
\end{aligned}
$$

(iii) and (iv) can be proved in similar ways. ∎

Proof of Result 5:

First we will prove that if $y_i > 0$ for $i = 1, \ldots, n$, then $g(\alpha) = -\ln(\sum_{i=1}^n y_i^\alpha)$ is a concave function. It mainly follows from the fact that $\frac{d^2}{d\alpha^2} g(\alpha) < 0$ due to CauchySchwartz inequality. It implies that $h^{(r)}(\alpha)$ is a concave function. Now the result follows as $h^{(r)}(\alpha) \to -\infty$ as $\alpha \to 0$ or $\alpha \to \infty$ ∎

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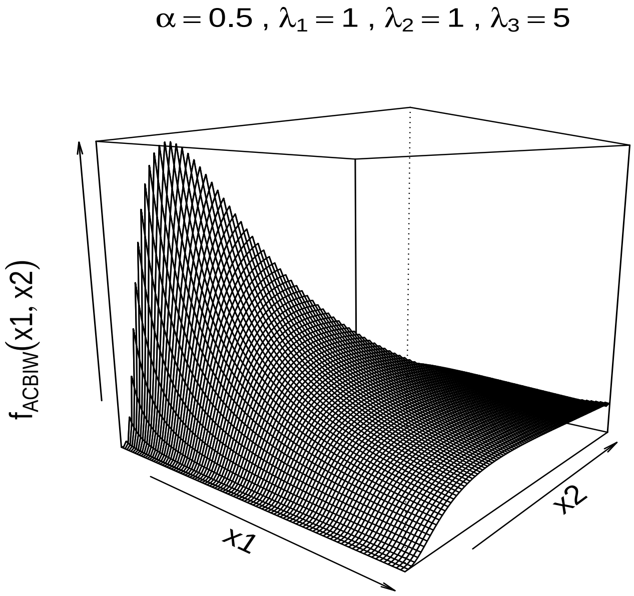

Figure 1: Surface plot of the joint PDF of ACBIW distribution.

Table 2: AE and MSE of the MLEs and Bayes estimators for different values of n with $\alpha = 0.5, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$

n	Parameter	MLE		Bayes (IP)		Bayes (NIP)	
		AE	MSE	AE	MSE	AE	MSE
n=20	α	0.530	0.007	0.516	0.005	0.517	0.005
	λ_1	1.162	0.418	0.956	0.095	0.914	0.115
	λ_2	1.142	0.378	0.956	0.087	0.928	0.113
	λ_3	0.948	0.445	1.165	0.107	1.246	0.163
n=30	α	0.520	0.004	0.507	0.003	0.507	0.003
	λ_1	1.096	0.267	0.928	0.058	0.889	0.072
	λ_2	1.107	0.288	0.939	0.060	0.900	0.075
	λ_3	0.993	0.370	1.181	0.084	1.219	0.117
n=40	α	0.513	0.002	0.503	0.002	0.502	0.002
	λ_1	1.065	0.174	0.894	0.056	0.896	0.058
	λ_2	1.069	0.186	0.846	0.060	0.889	0.063
	λ_3	0.978	0.268	1.213	0.091	1.213	0.099
n=80	α	0.508	0.001	0.498	0.001	0.499	0.001
	λ_1	1.044	0.098	0.888	0.032	0.874	0.036
	λ_2	1.027	0.093	0.895	0.031	0.877	0.036
	λ_3	0.985	0.160	1.186	0.055	1.196	0.063

Table 3: AE and MSE of the MLEs and Bayes estimators for different values of n with $\alpha = 1.0, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$

n	Parameter	MLE		Bayes (IP)		Bayes (NIP)	
		AE	MSE	AE	MSE	AE	MSE
n=20	α	1.060	0.024	1.030	0.021	1.030	0.021
	λ_1	1.162	0.496	0.968	0.098	0.925	0.124
	λ_2	1.159	0.481	0.946	0.082	0.905	0.111
	λ_3	0.957	0.616	1.168	0.096	1.237	0.156
n=30	α	1.039	0.015	1.016	0.012	1.023	0.013
	λ_1	1.095	0.333	0.929	0.064	0.894	0.077
	λ_2	1.089	0.313	0.929	0.055	0.888	0.080
	λ_3	0.998	0.503	1.166	0.076	1.239	0.126
n=40	α	1.026	0.010	1.009	0.008	0.999	0.009
	λ_1	1.102	0.246	0.879	0.055	0.890	0.057
	λ_2	1.076	0.225	0.856	0.058	0.877	0.064
	λ_3	0.953	0.356	1.218	0.095	1.241	0.105
n=80	α	1.014	0.006	0.997	0.004	1.000	0.004
	λ_1	1.043	0.138	0.890	0.031	0.878	0.035
	λ_2	1.043	0.133	0.886	0.032	0.877	0.036
	λ_3	0.968	0.224	1.172	0.051	1.206	0.065

Table 4: AE and MSE of the MLEs and Bayes estimators for different values of n with $\alpha = 2, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$

n	Parameter	MLE		Bayes (IP)		Bayes (NIP)	
		AE	MSE	AE	MSE	AE	MSE
n=20	α	2.940	0.097	2.049	0.073	2.053	0.086
	λ_1	1.097	0.502	0.964	0.095	0.893	0.112
	λ_2	1.118	0.527	0.954	0.094	0.890	0.116
	λ_3	1.027	0.705	1.157	0.102	1.240	0.161
n=30	α	2.076	0.064	2.029	0.047	2.026	0.052
	λ_1	1.061	0.382	0.940	0.063	0.891	0.069
	λ_2	1.065	0.383	0.931	0.062	0.894	0.075
	λ_3	1.645	0.594	1.181	0.086	1.221	0.115
n=40	α	2.062	0.044	2.014	0.032	2.022	0.041
	λ_1	1.080	0.295	0.879	0.053	0.887	0.056
	λ_2	1.084	0.302	0.849	0.058	0.877	0.061
	λ_3	0.991	0.465	1.226	0.101	1.233	0.100
n=80	α	2.032	0.023	2.001	0.017	1.998	0.017
	λ_1	1.025	0.161	0.899	0.032	0.877	0.035
	λ_2	1.035	0.166	0.897	0.031	0.871	0.036
	λ_3	0.992	0.277	1.180	0.054	1.195	0.060

Table 5: AL and CP of 90% Asymptotic CI and 90% Symmetric CRI for different values of n with $\alpha = 0.5, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$

n	Parameter	90% Asymptotic CI		90% Symmetric CRI (IP)		90% Symmetric CRI (NIP)	
		AL	CP	AL	CP	AL	CP
n=20	α	0.262	91.3%	0.202	85.2%	0.207	85.3%
	λ_1	2.824	98.8%	1.102	92.2%	1.132	89.2%
	λ_2	2.855	97.8%	1.109	92.0%	1.116	89.4%
	λ_3	3.047	99.8%	1.232	96.9%	1.288	96.2%
n=30	α	0.209	91.0%	0.165	86.2%	0.168	84.2%
	λ_1	0.428	98.9%	0.905	90.5%	0.909	89.9%
	λ_2	2.420	98.5%	0.886	90.7%	0.903	89.4%
	λ_3	2.642	99.0%	1.019	96.6%	1.049	94.5%
n=40	α	0.176	91.9%	0.141	85.6%	0.142	85.5%
	λ_1	2.088	98.3%	0.780	88.3%	0.782	86.8%
	λ_2	2.089	98.4%	0.755	85.2%	0.768	85.7%
	λ_3	2.400	99.8%	0.917	93.0%	0.909	93.7%
n=80	α	0.122	90.4%	0.098	88.4%	0.098	89.2%
	λ_1	1.555	94.8%	0.542	88.7%	0.548	89.8%
	λ_2	1.556	95.6%	0.543	89.4%	0.550	88.3%
	λ_3	1.925	93.8%	0.633	89.7%	0.651	88.4%

Table 6: AL and CP of 95% Asymptotic CI and 95% Symmetric CRI for different values of n with $\alpha = 0.5, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$

n	Parameter	95% Asymptotic CI		95% Symmetric CRI (IP)		50% Symmetric CRI (NIP)	
		AL	CP	AL	CP	AL	CP
n=20	α	0.312	95.7%	0.245	91.2%	0.242	91.3%
	λ_1	3.129	98.1%	1.275	95.1%	1.285	93.5%
	λ_2	3.186	98.7%	1.257	95.6%	1.280	93.1%
	λ_3	3.530	99.8%	1.382	98.6%	1.446	98.1%
n=30	α	0.246	96.1%	0.98	93.1%	0.199	91.8%
	λ_1	2.645	98.9%	1.019	94.9%	1.038	93.2%
	λ_2	2.650	98.7%	1.030	94.5%	1.036	93.3%
	λ_3	2.993	99.2%	1.139	98.9%	1.172	98.4%
n=40	α	0.211	96.4%	0.169	91.4%	0.169	91.0%
	λ_1	2.376	99.3%	0.877	94.2%	0.887	91.2%
	λ_2	2.356	99.5%	0.894	93.5%	0.889	91.2%
	λ_3	2.766	99.7%	0.987	97.4%	1.007	96.9%
n=80	α	0.146	95.0%	0.117	95.2%	0.118	95.1%
	λ_1	1.812	99.3%	0.619	94.7%	0.626	93.2%
	λ_2	1.811	98.9%	0.628	94.8%	0.623	93.8%
	λ_3	2.187	98.9%	0.708	93.7%	0.713	94.9%

Table 7: AL and CP of 90% Asymptotic CI and 90% Symmetric CRI for different values of n with $\alpha = 1, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$

n	Parameter	90% Asymptotic CI		90% Symmetric CRI (IP)		90% Symmetric CRI (NIP)	
		AL	CP	AL	CP	AL	CP
n=20	α	0.518	91.8%	0.412	86.4%	0.413	84.4%
	λ_1	2.709	97.3%	1.092	92.9 %	1.101	88.8%
	λ_2	2.701	96.1%	1.114	92.1%	1.121	88.6%
	λ_3	3.019	99.7%	1.226	97.0%	1.287	96.3%
n=30	α	0.415	91.3%	0.329	86.9%	0.330	84.6%
	λ_1	2.339	97.5%	0.886	90.2%	0.911	88.7%
	λ_2	2.360	98.2%	0.898	91.2%	0.908	88.5%
	λ_3	2.669	98.2%	1.024	86.6%	1.063	95.%
n=40	α	0.354	91.7%	0.280	85.9%	0.283	85.7%
	λ_1	2.113	98.0%	0.772	88.0%	0.777	87.4%
	λ_2	2.081	98.2%	0.760	85.3%	0.772	85.7%
	λ_3	2.382	98.7%	0.911	93.4%	0.912	92.4%
n=80	α	0.246	91.1%	0.198	88.4%	0.196	87.4%
	λ_1	1.561	94.8%	0.542	89.4%	0.549	88.4%
	λ_2	1.559	98.1%	0.542	88.4%	0.548	89.7%
	λ_3	1.885	99.6%	0.634	87.6%	0.647	88.0%

Table 8: AL and CP of 95% Asymptotic CI and 95% Symmetric CRI for different values of n with $\alpha = 1, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$

n	Parameter	95% Asymptotic CI		95% Symmetric CRI (IP)		95% Symmetric CRI (NIP)	
		AL	CP	AL	CP	AL	CP
n=20	α	0.618	96.0%	0.490	92.2%	0.498	90.2%
	λ_1	3.050	98.2%	1.276	97.4 %	1.306	94.4%
	λ_2	3.079	98.7%	1.286	96.3 %	1.291	94.1%
	λ_3	3.541	99.7%	1.395	99.3%	1.452	98.2%
n=30	α	0.496	96.2%	0.395	91.6%	0.395	92.8%
	λ_1	2.702	99.2%	1.036	96.0%	1.016	94.1%
	λ_2	2.641	98.9%	1.034	96.2%	1.027	91.3%
	λ_3	3.007	99.1%	1.145	98.4%	1.160	97.4%
n=40	α	0.422	95.1%	0.338	93.3%	0.336	91.4%
	λ_1	2.342	98.2%	0.889	93.4%	0.887	91.6%
	λ_2	2.336	98.6%	0.886	93.5%	0.889	93.6%
	λ_3	2.764	99.2%	0.991	98.1%	1.014	96.8%
n=80	α	0.294	96.2%	0.235	94.3%	0.234	94.1%
	λ_1	1.812	99.3%	0.620	94.5%	0.620	95.1%
	λ_2	1.811	98.9%	0.628	94.6%	0.617	94.0%
	λ_3	2.151	99.2%	0.710	95.3%	0.721	95.2%

Table 9: AL and CP of 90% Asymptotic CI and 90% Symmetric CRI for different values of n with $\alpha = 2, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$

n	Parameter	90% Asymptotic CI		90% Symmetric CRI (IP)		90% Symmetric CRI (NIP)	
		AL	CP	AL	CP	AL	CP
n=20	α	1.043	90.3%	0.815	85.7%	0.816	85.1%
	λ_1	2.747	96.7%	1.097	91.6%	1.113	88.9%
	λ_2	2.720	98.3%	1.104	90.6%	1.131	89.3%
	λ_3	3.122	98.8%	1.224	96.0%	1.298	95.4%
n=30	α	0.836	90.3%	0.660	86.0%	0.662	84.9%
	λ_1	2.314	97.2%	0.898	91.7%	0.918	89.5%
	λ_2	2.321	97.8%	0.893	89.8%	0.913	87.8%
	λ_3	2.636	98.6%	1.018	95.0%	1.060	94.3%
n=40	α	0.711	91.8%	0.566	87.9%	0.570	83.1%
	λ_1	2.075	97.8%	0.767	88.3%	0.787	87.7%
	λ_2	2.072	97.6%	0.752	87.8%	0.782	84.8%
	λ_3	2.391	99.2%	0.911	92.7%	0.916	92.9%
n=80	α	0.492	91.0%	0.394	88.9%	0.396	87.8%
	λ_1	1.546	96.4%	0.549	89.5%	0.545	89.6%
	λ_2	1.539	96.1%	0.551	89.5%	0.542	88.7%
	λ_3	1.854	98.2%	0.643	90.2%	0.645	90.8%

Table 10: AL and CP of 95% Asymptotic CI and 95% Symmetric CRI for different values of n with $\alpha = 2, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$

n	Parameter	95% Asymptotic CI		95% Symmetric CRI (IP)		95% Symmetric CRI (NIP)	
		AL	CP	AL	CP	AL	CP
n=20	α	1.246	96.3%	0.976	92.1%	0.981	89.9%
	λ_1	3.098	98.3%	1.269	95.8%	1.275	93.8%
	λ_2	3.106	97.8%	1.251	94.9%	1.303	94.1%
	λ_3	3.514	99.9%	1.371	99.1%	1.438	98.8%
n=30	α	0.998	96.4%	0.784	91.7%	0.785	90.4%
	λ_1	2.648	98.2%	1.028	95.0 %	1.027	92.0%
	λ_2	2.645	98.1%	1.019	94.0%	1.037	94.1%
	λ_3	3.069	98.6%	1.129	97.7%	1.169	98.0%
n=40	α	0.841	96.3%	0.671	91.9%	0.669	90.3%
	λ_1	2.349	98.6%	0.898	94.5%	0.881	91.3%
	λ_2	2.357	98.8%	0.886	94.3%	0.903	92.1%
	λ_3	2.681	99.2%	0.999	97.8%	1.009	96.2%
n=80	α	0.588	94.7%	0.467	94.7%	0.465	94.9%
	λ_1	1.789	98.4%	0.622	94.8%	0.618	95.2%
	λ_2	1.794	98.6%	0.614	94.4%	0.623	94.4%
	λ_3	2.144	99.9%	0.703	94.3%	0.713	94.7%

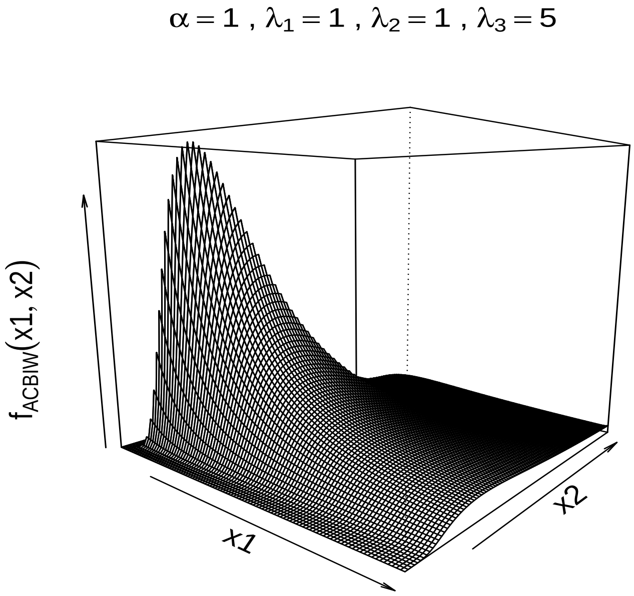

Figure 2: Surface plot of the joint PDF of ACBIW distribution.

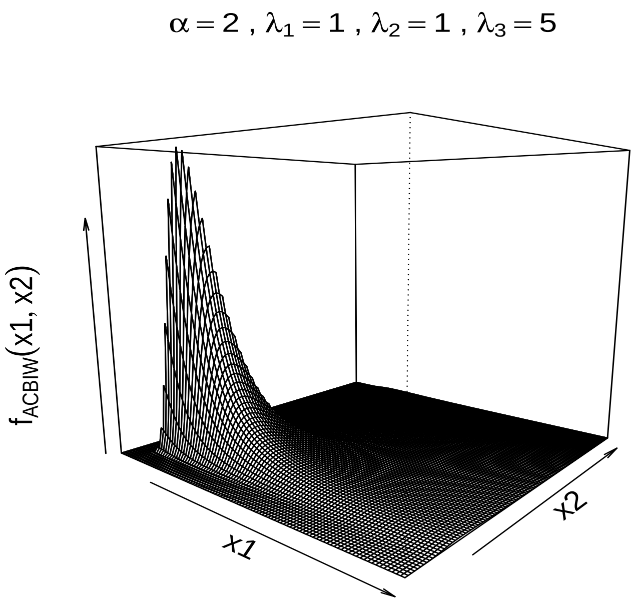

Figure 3: Surface plot of the joint PDF of ACBIW distribution.

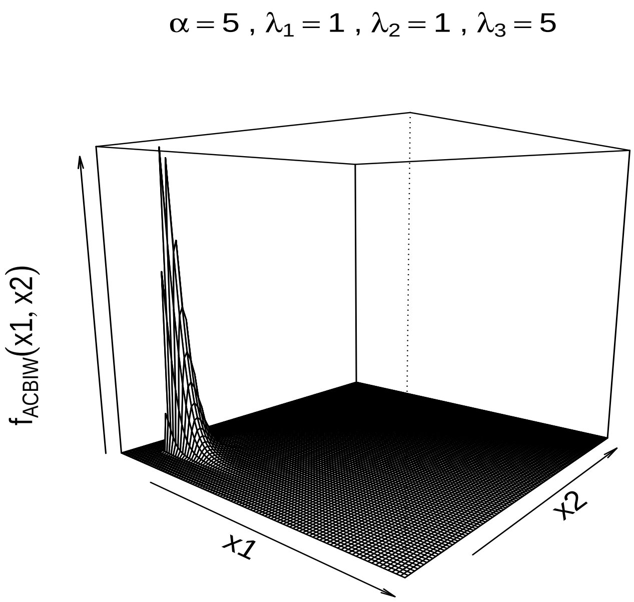

Figure 4: Surface plot of the joint PDF of ACBIW distribution.

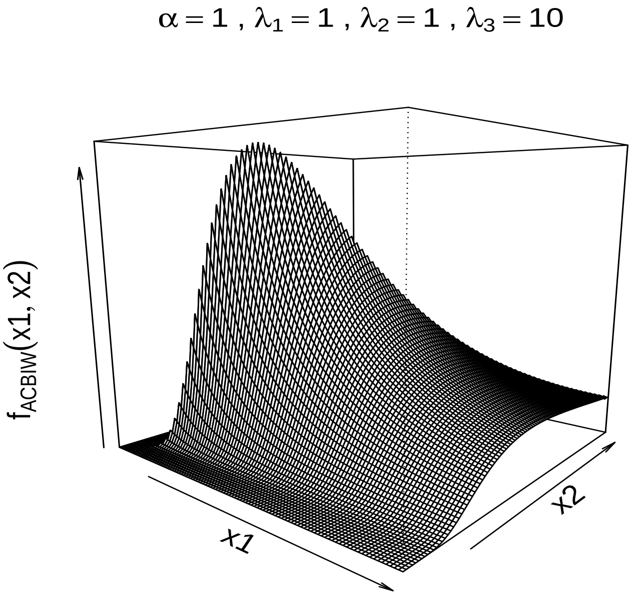

Figure 5: Surface plot of the joint PDF of ACBIW distribution.