Statistical Signal Processing

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Signal processing may broadly be considered to involve the recovery of information from physical observations. The received signals is usually disturbed by random noise due to various reasons. The main aim is to recover the original signal from the noisy observations. Statistics plays an important role for this purpose. In this article we would like to introduce different signal processing models and different statistical issues involved in solving these problems.

Multiple Sinusoidal Model:

This is one of the most important models which has been discussed in the statistical signal processing literature due to its extensive applications in various fields. The multiple sinusoidal model may be expressed as

$$y(t) = \sum_{k=1}^{M} \{A_k \cos(\omega_k t) + B_k \sin \omega_k t)\} + n(t); \qquad t = 1, \cdots, N. \tag{1}$$

Here, for $k = 1, \ldots, M$, $A_k^2 + B_k^2$ is known as the amplitude corresponding to the signal frequency ω_k, $n(t)$'s are error random variables with mean zero and finite variance. The errors need not be independent. The problem is to estimate the unknown parameters $\{(A_k, B_k, \omega_k); k = 1, \ldots, M\}$, given a sample $\{y(t); t = 1, \ldots, N\}$, of size N. Often in practice M is also unknown and one needs to estimate M from the observed data.

This model has been studied quite extensively in the statistical signal processing literature as most of the periodic signals can be approximated by the model (1). The model has been used quite effectively in analyzing periodic data from various fields. For several applications of this model in different areas one is referred to the monographs by Quinn and Hannan (2001) and by Nandi and Kundu (2020). The problem is an extremely challenging problem both

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from the theoretical and computational points of view. Fisher (1929) first introduced this problem to the statistical community. It seems that (1) is a standard non-linear regression model, and the least squares estimators are the natural choice of the estimators in this case. Interestingly, it is observed that finding the least squares estimators, and establishing their properties are far from trivial issues. The standard sufficient conditions needed for the least squares estimators to be consistent and asymptotically normally distributed in case of a non-linear regression model as established by Jennrich (1969) and Wu (1981), do not hold. Special attention is needed in establishing the asymptotic properties of the least squares estimators of the unknown parameters of the model (1), see for example Hannan (1973) and Kundu (1991) in this respect. Moreover, computing the least squares estimators is a challenging problem from the computational point of view due to the presence of several local minima in the least squares surface. The standard Newton-Raphson algorithm or its variants do not often converge even from good starting values. Even if it converges, it may converge to a local minimum rather than a global minimum. Special purpose algorithms have been developed to tackle this problem. Several approximate solutions have been suggested in the literature. Interested readers may read the books by Kay (1987), Bose and Rao (1993), Quinn and Hannan (2001) or Nandi and Kundu (2000) for a comprehensive reviews of the different methods.

Two-Dimensional sinusoidal Model

Similar to one dimensional sinusoidal model (1), two dimensional periodic signals are often being analyzed by the two-dimensional sinusoidal model as given below:

$$y(s,t) = \sum_{k=1}^{M} \{A_k \cos(\omega_k s + \mu_k t) + B_k \sin(\omega_k s + \mu_k t)\} + n(s,t), \quad s = 1, \cdots S, \; t = \cdots, T. \quad (2)$$

Here also, for $k = 1, \ldots, M$, $A_k^2 + B_k^2$ is known as the amplitude corresponding to the frequency pair (ω_k, μ_k), $n(s,t)$'s are the two dimensional noise random variables with mean zero and finite variance. The noise random variables $n(s,t)$ may have spatial dependence also. The problem once again involves the estimation of the signal parameters namely $\{(A_k, B_k, \omega_k, \mu_k); k = 1, \ldots, M\}$ from the noisy data $\{y(s,t); s = 1, \ldots, S, t = 1, \ldots, T\}$.

The model (2) has been used very successfully for analyzing geophysical data, in image restoration, in array processing, in radio astronomy, in synthetic aperture radar imaging, in nuclear magnetic resonance spectroscopy, in medical imaging, in wireless communications, in health assessment of living trees, in source localisation, just to name a few. Estimation of the unknown parameters and establishing their properties are far more challenging than the one dimensional model. Some of the estimation procedures available for the one-dimensional model may be extended to two dimensions. However, several difficulties arise when dealing with high dimensional data. Recently Grover et al. (2022) proposed a very efficient estimation procedures of the unknown parameters of the model (2). There are several open problems in multidimensional frequency estimation, and this continues to be an active area of research.

Multiple Chirp Model

If the signal is nearly periodic several models have been used to analyze nearly periodic signal. Among them, one of the most used one is known as chirp model. The multiple chirp model with an additive noise can be described as follows;

$$y(t) = \sum_{k=1}^{M} \{A_k \cos(\alpha_k t + \beta_k t^2) + B_k \sin(\alpha_k t + \beta_k t^2)\} + n(t); \quad t = 1, \ldots, N. \tag{3}$$

For $k = 1, \ldots, M$, $A_k^2 + B_k^2$ is known as the amplitude corresponds to the frequency α_k and frequency rate β_k, $n(t)$'s are the noise random variables having similar properties as in model (1). The problem remains the same mainly to estimate the unknown parameters from the noisy observations and also to establish their properties.

This model arises in many applications of signal process; one of the most important being the radar problem. For instance, consider a radar illuminating a target. Thus the transmitted signal will undergo a phase shift induced by the distance and relative motion between the target and the receiver. Assuming this motion to be continuous and differentiable, the phase shift can be adequately modelled as $\phi(t) = a_0 + a_1 t + a_2 t^2$, where the parameters a_1 and a_2 are either related to speed and acceleration or range and speed depending on what the radar is intended for, and on that kind of waveform transmitted, see for example Rihaczek (1969).

Djurić and Kay (1990) first consider the estimation of the unknown parameters of the chirp model (3) in presence of independent and identically distributed error components. Since then several methods have been proposed in the literature to estimate the unknown parameters of the chirp model under different model assumptions. Nandi and Kundu (2004) first established the theoretical properties of the least squares estimators of the chirp parameters. A more general two dimensional chirp model has been considered by Lahiri and Kundu (2017). A detailed discussions of the different models, their various applications and properties can be found in a review article by Kundu and Nandi (2021).

Multiple Chirp-Like Model

Recently Grover et al. (2021) introduced a new model as an alternative to the multiple chirp model, and it is named as the multiple chirp-like model. The multiple chirp-like model has the following form;

$$y(t) = \sum_{k=1}^{p}\{A_k \cos(\alpha_k t) + B_k \sin(\alpha_k t)\} + \sum_{k=1}^{q}\{C_k \cos(\beta_k t^2) + D_k \sin(\beta_k t^2)\} + n(t); \quad t = 1, \ldots, N. \tag{4}$$

Here $A_k^2 + B_k^2$ is the amplitude corresponding to the frequency α_k, for $k = 1, \ldots, p$, and $C_k^2 + D_k^2$ is the amplitude corresponding to the frequency rate β_k^2, for $k = 1, \ldots, q$, the noise component $n(t)$ is same as the multiple chirp model (3). The main aim to introduce the chirp-like model (4) is that it behaves very similarly to the chirp model (3). It has been observed that the chirp like model exhibits same type of behavior as the chirp model and is capable of modelling similar physical phenomena. It has been observed by Grover et al. (2021) that it is very difficult to distinguish the signals generated by the two different models. Moreover, to analyze a nearly periodic data, less number of parameters may be required for a chirp-like model than a chirp model. On the other hand it has been shown that analytically as well as computationally the chirp-like model is easier to handle than the chirp model. Grover et al. (2021) established a very efficient estimation procedures of the unknown parameters, and also establish their asymptotic properties. There are several open problems associated with this model, see for example Kundu and Nandi (2021), more work is needed along this direction.

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