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Statistical Inference of Jointly Type-II Lifetime Samples under Weibull Competing Risks Models

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Abstract: In this paper, we develop statistical inference of competing risks samples which are collected under a joint Type-II censoring scheme of products with Weibull lifetime distributions. These inferences are drawn from two independent fatal risks and come from two different lines of production with the same facility. The model parameters and the parameters of life (reliability and hazard rate functions) are estimated using maximum likelihood (ML), bootstrap and Bayes methods. Additionally, we constructed asymptotic ML confidence intervals, bootstrap confidence intervals and Bayes credible intervals. Furthermore, the theoretical results are assessed and compared through Monte Carlo simulations. Finally, one real data set is analyzed under the proposed model for illustrative purpose.

Keywords: Bayes estimation; bootstrap confidence intervals; competing risks model; joint Type-II censoring scheme; maximum likelihood; Weibull distribution



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1. Introduction

The lifetime data which are available from life products are either in the form of complete data or censored data under some restrictions of time and cost. The word complete failure times (data) appear when we observe the time-to-failure of all units under the test, while the word censored failure times appear when we observe some failure times of units, not all times under the test at a fixed period of time. Different schemes of censoring are available in the literature among which the earliest schemes are Type-I and Type-II censoring schemes. The first scheme has a random number of failures and fixed test time. Additionally, the second scheme has a prefixed number of failures and a random test time. When the products or units comes from different lines of production under the same facility, then the censoring scheme is called a joint censoring scheme. The joint censoring scheme can be combined with Type-I censoring to present a joint Type-I censoring scheme. However, for Type-II censoring, the scheme is called a joint Type-II censoring scheme. Hence, the comparative life tests are applied on life products to measure the relative merits of two competing durations of life products. For products produced from two different lines of production under the same facility, two independent sets are randomly selected to subject to lifetime testing. For the consideration of time and cost, researchers terminate the experiment after a pre-fixed time or number of failures, and the failure time and the corresponding unit type are recorded. This type of data and their statistical inferences were studied earlier by [1–6]. Additionally, this problem in modern times has been studied by [7–9] and recently, this problem is studied by [10–12]. Recently, the joint censoring scheme presented as a

balanced combination with progressive Type-II censoring, see [13]. The joint balanced scheme is easier to handle than ordinary joint censoring schemes. The joint family of exponential distributions and its statistical properties under the balanced scheme are discussed by [13]. Additionally, under joint Weibull lifetime distributions, this study is developed by [14]. Recently, in [15] more inferences procedures for Weibull parameters are discussed under a balanced two-sample Type-II progressive censoring scheme.

With the consideration that with the joint Type-II censoring scheme, a sample of size $M + N$ is selected from two lines of production Ψ_1 and Ψ_2 , M from the line Ψ_1 with i.i.d. lifetimes $W_1, W_2, \dots, W_M$. Additionally, N from the line Ψ_2 with i.i.d. lifetimes $Y_1, Y_2, \dots, Y_N$. The random variable vectors W and Y are distributed with a population have probability density functions (PDF) and cumulative distribution functions (CDF) given, respectively, by $f_l(\cdot)$ and $F_l(\cdot)$, $l = 1, 2$. Then, the ordered lifetimes $\{T_1, T_2, \dots, T_m\}$ constructed from the sample $\{W_1, W_2, \dots, W_{M_m}, Y_1, Y_2, \dots, Y_{N_m}\}$ where $m = M_m + N_m$ defines the joint Type-II censoring sample. Hence, for each random lifetime sample under the joint Type-II censoring scheme is described with failure time and the corresponding type (T, ω) . Then, $T = \{(T_1, \omega_1), (T_2, \omega_2), \dots, (T_m, \omega_m)\}$, where $1 \leq m \leq M + N$ and the value of ω_i take the value (1 or 0) and depends on W or Y failure, respectively. Let, $m_1 = \sum_{i=1}^m \omega_i$ denote the number of unit fails from the line Ψ_1 and $m_2 = \sum_{i=1}^m (1 - \omega_i)$ the number of unit fails from the line Ψ_2 . Therefore, to give an observed joint Type-II censoring sample $t = ((t_1, \omega_1), (t_2, \omega_2), \dots, (t_m, \omega_m))$, the joint likelihood density function of t is given by

$$f_{1,2,\dots,m}(t|\Theta) = \frac{M!N! \left(S_1^{M-m_1}(t_m) S_2^{N-m_2}(t_m) \right)}{(M - m_1)!(N - m_2)!} \prod_{i=1}^m (f_1(t_i))^{\omega_i} (f_2(t_i))^{1-\omega_i}, \quad (1)$$

where Θ is the parameter vector and $S_l(\cdot)$, $i = 1, 2$ is presented to survival functions.

In reliability study or life testing experiments, the failure time under different causes of failure is known as a competing risks model. In this model, we aim to measure the risk of one cause of failure with respect to other causes. The problem of competing risks has been studied by different authors—[16] discussed this problem in exponential populations. Different properties of a competing risks model were studied by [17–20]. Recently, the properties of a competing risks model under accelerated life tests were discussed by [21–23]. For other similar results see [24–26].

The plan of the competing risks model under the Type-II censoring scheme can be described for a given random sample of size n and prior integer m . Under the assumption that only two independent causes of failure are available, the time-to-failure and the corresponding cause of failure are recorded, say $T = \{(T_i, \rho_i)\}$ $i = 1, 2, \dots, m$. Hence, the ordered random sample $T = \{(T_1, \rho_1), (T_2, \rho_2), \dots, (T_m, \rho_m)\}$ define the Type-II competing risks sample. Additionally, the joint density function of the observed sample $t = ((t_1, \rho_1), (t_2, \rho_2), \dots, (t_m, \rho_m))$ is given by

$$f_{1,2,\dots,m}(t|\Theta) = \frac{n!}{(n - m)!} \{S_1(t_m) S_2(t_m)\}^{(n-m)} \prod_{i=1}^m \{f_1(t_i) S_2(t_i)\}^{I(\rho_i=1)} \{f_2(t_i) S_1(t_i)\}^{I(\rho_i=2)}, \quad (2)$$

where

$$I(\rho_i = k) = \begin{cases} 1, & \rho_i = k \\ 0, & \rho_i \neq k \end{cases}, \quad k = 1, 2, \quad (3)$$

and

$$0 < t_1 < t_2 < \dots < t_m < \infty.$$

In real life analysis, the properties of increasing or decreasing hazard rate function of the two-parameter Weibull distribution makes it a widely used distribution to analyze life-time data. In this paper, we combine the competing risks model with Weibull joint lifetime samples from two lines of production competing in duration. This model is formulated, taking into consideration only two causes of failure and two different lines of production in the same facility. The failure time is obtained under the Type-II censoring scheme and the corresponding joint Type-II competing risk sample is used to build the likelihood function.

The model parameters as well as the parameters of life are estimated with different methods of estimation, namely the maximum likelihood, bootstrap, and Bayes estimation methods. Additionally, interval estimation with asymptotic confidence intervals, two bootstrap confidence intervals and credible intervals are constructed. Different tools are used to present a quality assessment of these estimators. For the point estimation we computed means and mean squared errors (MSEs). Additionally, for interval estimation, the interval length (IL) and coverage probability (CP) were obtained. Moreover, we analyzed one real data set to illustrate our intention.

The paper is structured as follows: in Section 2, general assumptions and modelling are given. Estimation based on MLE and asymptotic confidence intervals is presented in Section 3. Bootstrap-p and bootstrap-t confidence intervals are presented in Section 4. Bayes estimation is presented in Section 5. Assessment and comparison of the numerical results is conducted with a simulation study in Section 6. A real example is used and analyzed in Section 7. The conclusion about the main results discussed in this paper is given in Section 8.

2. Modelling

Suppose a random sample of size $(M + N)$ of units is selected from the two independent production lines Ψ_1 and Ψ_2 , M units are chosen from the line Ψ_1 and N units are chosen from the line Ψ_2 and subject to a life testing experiment. The prior integer m is needed for statistical inferences to be proposed. When the first failure T_1 is observed, the type ω_1 (means either from Ψ_1 or Ψ_2) and cause of the failure ρ_1 (meaning from either cause 1 or 2 of failure) are determined, then we can say that (T_1, ω_1, ρ_1) is recorded. When the second failure T_2 is observed with ω_2 and ρ_2 , we can say that (T_2, ω_2, ρ_2) is recorded. The experiment continues until the final failure T_m is observed with ω_m and ρ_m , then (T_m, ω_m, ρ_m) is recorded. Then, the random variable set $T = \{(T_1, \omega_1, \rho_1), (T_2, \omega_2, \rho_2), \dots, (T_m, \omega_m, \rho_m)\}$ with $1 \leq m \leq M + N$ is called a Type-II joint competing risks sample (JCRS). Suppose, $m_1 = \sum_{i=1}^M \omega_i$ denotes the number of unit failures from the line Ψ_1 , $m_2 = \sum_{i=1}^m (1 - \omega_i)$ denotes the number of units failures from the line Ψ_2 , $n_{1j} = \sum_{i=1}^m \omega_i * I(\rho_i = j)$, $j = 1, 2$ denotes the number of units failures from the line Ψ_1 and the cause j and $n_{2j} = \sum_{i=1}^m (1 - \omega_i) * I(\rho_i = j)$ denotes the number of units failures from the line Ψ_2 and the cause j , $j = 1, 2$.

The joint density function of the observed sample $\underline{t} = ((t_1, \omega_1, \rho_1), (t_2, \omega_2, \rho_2), \dots, (t_m, \omega_m, \rho_m))$ is given by

$$f_{1,2,\dots,m}(t|\Theta) = \frac{M!N!}{(M-m_1)!(N-m_2)!} \left(\prod_{i=1}^m \left[(f_{11}(t_i)S_{12}(t_i))^{I(\rho_i=1)} (f_{12}(t_i)S_{11}(t_i))^{I(\rho_i=2)} \right]^{\omega_i} \right. \\ \left. \times \left[(f_{21}(t_i)S_{22}(t_i))^{I(\rho_i=1)} (f_{22}(t_i)S_{21}(t_i))^{I(\rho_i=2)} \right]^{1-\omega_i} \right) \\ \times [S_{11}(t_m)S_{12}(t_m)]^{M-m_1} [S_{21}(t_m)S_{22}(t_m)]^{N-m_2}. \quad (4)$$

where $I(\rho_i = j)$ is given by (3). Under consideration of two independent Weibull distributions, we report the following remarks.

Remark 1

- (1) The failure time T_{ik} , $i = 1, 2, \dots, m$ is defined as $T_{ik} = \min\{T_{ik1}, T_{ik2}\}$ where T_{ikj} denotes failure time from the line k under cause j .
- (2) The random variable T_{ikj} is distributed with Weibull lifetime distribution with a distribution function (CDF) given by

$$F_{kj}(t) = 1 - \exp(-\beta_{kj}t^{\alpha_k}), t > 0, \alpha_k, \beta_{kj} > 0, k, j = 1, 2. \quad (5)$$

The corresponding probability density function (PDF), the reliability $S(\cdot)$ and hazard rate functions $h(\cdot)$, respectively, are given by

$$f_{kj}(t) = \alpha_k \beta_{kj} t^{\alpha_k - 1} \exp(-\beta_{kj} t^{\alpha_k}), \quad (6)$$

$$S_{kj}(t) = \exp(-\beta_{kj} t^{\alpha_k}), \quad (7)$$

Moreover,

$$h_{kj}(t) = \alpha_k \beta_{kj} t^{\alpha_k - 1}. \quad (8)$$

where k represents the type of the failed unit and j represents the cause of the failed unit.

- (1) The latent failure time (time-to-failure) is distributed as the Weibull distribution with a $\beta_{k1} + \beta_{k2}$ scale and α_k shape parameters.
- (2) The discrete random variables n_{k1} and n_{k2} which describe the number of unit failures the under first and second causes of failure are distributed with binomial distributions with sample size m_k with a probability of success of $\frac{\beta_{k1}}{\beta_{k1} + \beta_{k2}}$ and $\frac{\beta_{k2}}{\beta_{k1} + \beta_{k2}}$, respectively.

3. Maximum Likelihood Estimation

In this section, we discuss estimation of the unknown model parameters as well as reliability and hazard rate functions when the units fail under two independent causes of failure and the Weibull failure time. Additionally, the failure times and the corresponding unit type and cause of failure are recorded. The point and interval estimates of the model parameters are obtained by giving Type-II JCRS $T = \{(T_1, \omega_1, \rho_1), (T_2, \omega_2, \rho_2), \dots, (T_m, \omega_m, \rho_m)\}$. Hence, the likelihood function (4) with distribution (5) is reduced to

$$\begin{aligned} L(\underline{\alpha}, \underline{\beta} | \underline{t}) &\propto \alpha_1^{m_1} \alpha_2^{m_2} \beta_{11}^{n_{11}} \beta_{12}^{n_{12}} \beta_{21}^{n_{21}} \beta_{22}^{n_{22}} \exp \left\{ (\alpha_1 - 1) \sum_{i=1}^m \omega_i \log t_i - (\beta_{11} + \beta_{12}) \sum_{i=1}^m \omega_i t_i^{\alpha_1} \right. \\ &\quad \left. + (\alpha_2 - 1) \sum_{i=1}^m (1 - \omega_i) \log t_i - (\beta_{21} + \beta_{22}) \sum_{i=1}^m (1 - \omega_i) t_i^{\alpha_2} \right. \\ &\quad \left. - (M - m_1)(\beta_{11} + \beta_{12}) t_m^{\alpha_1} - (N - m_2)(\beta_{21} + \beta_{22}) t_m^{\alpha_2} \right\}, \end{aligned} \quad (9)$$

where $\underline{\alpha} = \{\alpha_1, \alpha_2\}$ and $\underline{\beta} = \{\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}\}$. Additionally, the natural logarithm of the likelihood function is reduced to

$$\begin{aligned} \ln L(\underline{\alpha}, \underline{\beta} | \underline{t}) &= m_1 \log \alpha_1 + m_2 \log \alpha_2 + n_{11} \log \beta_{11} + n_{12} \log \beta_{12} + n_{21} \log \beta_{21} + n_{22} \log \beta_{22} \\ &\quad + (\alpha_1 - 1) \sum_{i=1}^m \omega_i \log t_i - (\beta_{11} + \beta_{12}) \sum_{i=1}^m \omega_i t_i^{\alpha_1} \\ &\quad + (\alpha_2 - 1) \sum_{i=1}^m (1 - \omega_i) \log t_i - (\beta_{21} + \beta_{22}) \sum_{i=1}^m (1 - \omega_i) t_i^{\alpha_2} \\ &\quad - (M - m_1)(\beta_{11} + \beta_{12}) t_m^{\alpha_1} - (N - m_2)(\beta_{21} + \beta_{22}) t_m^{\alpha_2}. \end{aligned} \quad (10)$$

3.1. MLEs

The likelihood equations are obtained by taking the first partial derivatives of the log-likelihood function (10) with respect to the model parameters. Hence, the conditional estimators under ML estimation of the parameters β_{kj} , for given $\alpha_k > 0$ and numbers $n_{kj} > 0, k, j = 1, 2$, can be written as

$$\hat{\beta}_{1j}(\alpha_1) = \frac{n_{1j}}{\sum_{i=1}^m \omega_i t_i^{\alpha_1} + (M - m_1) t_m^{\alpha_1}}, \quad j = 1, 2, \quad (11)$$

and

$$\hat{\beta}_{2j}(\alpha_2) = \frac{n_{2j}}{\sum_{i=1}^m (1 - \omega_i) t_i^{\alpha_2} + (N - m_2) t_m^{\alpha_2}}, \quad j = 1, 2. \quad (12)$$

The likelihood equations obtained from partial derivatives of the log-likelihood function (10) with respect to α_k

$$\frac{\partial(\alpha, \beta | t)}{\partial \alpha_k} = 0, \quad k = 1, 2,$$

are reduced to

$$\frac{m_1}{\alpha_1} + \sum_{i=1}^m \omega_i \log t_i - (\beta_{11} + \beta_{12}) \sum_{i=1}^m \omega_i t_i^{\alpha_1} \log t_i - (M - m_1)(\beta_{11} + \beta_{12}) t_m^{\alpha_1} \log t_m = 0. \quad (13)$$

and

$$\begin{aligned} \frac{m_2}{\alpha_2} + \sum_{i=1}^m (1 - \omega_i) \log t_i - (\beta_{21} + \beta_{22}) \sum_{i=1}^m (1 - \omega_i) t_i^{\alpha_2} \log t_i \\ - (N - m_2)(\beta_{21} + \beta_{22}) t_m^{\alpha_2} \log t_m = 0. \end{aligned} \quad (14)$$

The ML estimate of α_k is not in a closed form and, thus, we need an iteration method such as the Newton Raphson or fixed-point iterative method. The following theorem can be applied to obtain the MLE of α_k as follows.

Theorem 1. The conditional ML estimators of parameters α_k for given numbers $n_{kj} > 0, k, j = 1, 2$, are given by

$$g(\alpha_k) = \alpha_k, \quad k = 1, 2, \quad (15)$$

where

$$g(\alpha_k) = \begin{cases} \frac{m_1}{(n_{11} + n_{12}) \left(\frac{\sum_{i=1}^m \omega_i t_i^{\alpha_1} \log t_i + (M - m_1) t_m^{\alpha_1} \log t_m}{\sum_{i=1}^m \omega_i t_i^{\alpha_1} + (M - m_1) t_m^{\alpha_1}} \right) - \sum_{i=1}^m \omega_i \log t_i}, & \text{at } k = 1 \\ \frac{m_2}{(n_{21} + n_{22}) \left(\frac{\sum_{i=1}^m (1 - \omega_i) t_i^{\alpha_2} \log t_i + (N - m_2) t_m^{\alpha_2} \log t_m}{\sum_{i=1}^m (1 - \omega_i) t_i^{\alpha_2} + (N - m_2) t_m^{\alpha_2}} \right) - \sum_{i=1}^m (1 - \omega_i) \log t_i}, & \text{at } k = 2 \end{cases} \quad (16)$$

Proof The proof is immediately obtained after substituting β_{kj} from (11) and (12) into (13) and (14). $\square$

The iteration method which is applied to obtain the ML estimate of α_k needs an initial value of iteration and the best method can be obtained from the profile log-likelihood function and is given by

$$\begin{aligned} Z(\alpha_1, \alpha_2) = \sum_{k=1}^2 \left\{ m_k \log \alpha_k + n_{1k} \log \left[\frac{n_{1k}}{D_1} \right] + n_{2k} \log \left[\frac{n_{2k}}{D_2} \right] - \frac{n_{1k}}{D_1} \sum_{i=1}^m \omega_i t_i^{\alpha_1} \right. \\ \left. - \frac{n_{2k}}{D_2} \sum_{i=1}^m (1 - \omega_i) t_i^{\alpha_2} - \frac{n_{1k}}{D_1} (M - m_1) t_m^{\alpha_1} - \frac{n_{2k}}{D_2} (N - m_2) t_m^{\alpha_2} \right\} \\ + (\alpha_1 - 1) \sum_{i=1}^m \omega_i \log t_i + (\alpha_2 - 1) \sum_{i=1}^m (1 - \omega_i) \log t_i \end{aligned} \quad (17)$$

where

$$D_1 = \sum_{i=1}^m \omega_i t_i^{\alpha_1} + (M - m_1) t_m^{\alpha_1} \text{ and } D_2 = \sum_{i=1}^m (1 - \omega_i) t_i^{\alpha_2} + (N - m_2) t_m^{\alpha_2}. \quad (18)$$

In fixed point method, iteration is stopped after $|\alpha_k^{i+1} - \alpha_k^i|$ is sufficiently small. The corresponding ML estimate of β_{11} , β_{12} , β_{21} and β_{22} , say $\hat{\beta}_{11}$, $\hat{\beta}_{12}$, $\hat{\beta}_{21}$ and $\hat{\beta}_{22}$, can be

obtained from (11) and (12). The maximum likelihood estimate of reliability and hazed rate function can be obtained from

$$\hat{S}_{kj}(t) = \exp(-\hat{\beta}_{kj}t^{\hat{\alpha}_k}),$$

and

$$\hat{h}_{kj}(t) = \hat{\alpha}_k \hat{\beta}_{kj} t^{\hat{\alpha}_k - 1}.$$

Remark 2 The two equations (11) and (12) showed that, under consideration of $n_{1j} = 0$ and $n_{2j} = 0$, then $\hat{\beta}_{1j}$ and $\hat{\beta}_{2j}$, $j = 1, 2$ do not exist, respectively. The exact distributions of estimators $\hat{\beta}_{1j}$ and $\hat{\beta}_{2j}$ are defined as a mixture of discrete and continuous distributions, hence as given in [27], this is difficult to obtain.

3.2. Interval Estimation

The second partial derivative of the joint log-likelihood function given by (10) is reduced to

$$\frac{\partial^2 \downarrow(\underline{\alpha}, \underline{\beta} | \underline{t})}{\partial \beta_{kj}^2} = \frac{-n_{kj}}{\beta_{kj}^2} \Big|_{k,j=1,2} \quad (19)$$

$$\frac{\partial^2 \downarrow(\underline{\alpha}, \underline{\beta} | \underline{t})}{\partial \alpha_1^2} = \frac{-m_1}{\alpha_1^2} - (\beta_{11} + \beta_{12}) \sum_{i=1}^m \omega_i t_i^{\alpha_1} \log^2 t_i - (M - m_1)(\beta_{11} + \beta_{12}) t_m^{\alpha_1} \log^2 t_m, \quad (20)$$

$$\frac{\partial^2 \downarrow(\underline{\alpha}, \underline{\beta} | \underline{t})}{\partial \alpha_2^2} = \frac{-m_2}{\alpha_2^2} - (\beta_{21} + \beta_{22}) \sum_{i=1}^m (1 - \omega_i) t_i^{\alpha_2} \log^2 t_i - (N - m_2)(\beta_{21} + \beta_{22}) t_m^{\alpha_2} \log^2 t_m, \quad (21)$$

$$\frac{\partial^2 \downarrow(\underline{\alpha}, \underline{\beta} | \underline{t})}{\partial \beta_{ij} \partial \beta_{kl}} = 0, \text{ For each } ij \neq kl, \quad (22)$$

$$\frac{\partial^2 \downarrow(\underline{\alpha}, \underline{\beta} | \underline{t})}{\partial \alpha_1 \partial \beta_{1j}} = \frac{\partial^2 \downarrow(\underline{\alpha}, \underline{\beta} | \underline{t})}{\partial \beta_{1j} \partial \alpha_1} = - \sum_{i=1}^m \omega_i t_i^{\alpha_1} \log t_i - (M - m_1) t_m^{\alpha_1} \log t_m, \quad j = 1, 2, \quad (23)$$

$$\frac{\partial^2 \downarrow(\underline{\alpha}, \underline{\beta} | \underline{t})}{\partial \alpha_2 \partial \beta_{2j}} = \frac{\partial^2 \downarrow(\underline{\alpha}, \underline{\beta} | \underline{t})}{\partial \beta_{2j} \partial \alpha_2} = - \sum_{i=1}^m (1 - \omega_i) t_i^{\alpha_2} \log t_i - (N - m_2) t_m^{\alpha_2} \log t_m, \quad j = 1, 2, \quad (24)$$

and

$$\frac{\partial^2 \downarrow(\underline{\alpha}, \underline{\beta} | \underline{t})}{\partial \alpha_1 \partial \beta_{2j}} = \frac{\partial^2 \downarrow(\underline{\alpha}, \underline{\beta} | \underline{t})}{\partial \beta_{2j} \partial \alpha_1} = \frac{\partial^2 \downarrow(\underline{\alpha}, \underline{\beta} | \underline{t})}{\partial \alpha_2 \partial \beta_{1j}} = \frac{\partial^2 \downarrow(\underline{\alpha}, \underline{\beta} | \underline{t})}{\partial \beta_{1j} \partial \alpha_2} = \frac{\partial^2 \downarrow(\underline{\alpha}, \underline{\beta} | \underline{t})}{\partial \alpha_1 \partial \alpha_2} = \frac{\partial^2 \downarrow(\underline{\alpha}, \underline{\beta} | \underline{t})}{\partial \alpha_2 \partial \alpha_1} = 0, \quad (25)$$

Then, the minus expectation of (19) to (25) is taken to represent the Fisher information matrix of the parameters $\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{21}$ and β_{22} as $\Sigma(\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})$, where

$$\Sigma(\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}) = \begin{pmatrix} I_{11} & 0 & I_{13} & I_{14} & 0 & 0 \\ 0 & I_{22} & 0 & 0 & I_{25} & I_{26} \\ I_{31} & 0 & I_{33} & 0 & 0 & 0 \\ I_{41} & 0 & 0 & I_{44} & 0 & 0 \\ 0 & I_{52} & 0 & 0 & I_{55} & 0 \\ 0 & I_{62} & 0 & 0 & 0 & I_{66} \end{pmatrix}, \quad (26)$$

where

$$I_{ij} = -E \left(\frac{\partial^2 \downarrow(\underline{\alpha}, \underline{\beta} | \underline{t})}{\partial \Theta_i \partial \Theta_l} \right), \quad i, l = 1, 2, 3, 4, 5, 6, \quad (27)$$

and $\Theta = (\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})$ is the parameter vector. The approximate information matrix is used to obtain the second derivatives of the log-likelihood at the estimated parameter values. Additionally, the inverse approximate information matrix under non-zero values of the elements of the diagonal exists. Hence, the approximate

$(1 - 2\gamma)\%$ confidence intervals of the parameters $\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{21}$ and β_{22} under conditions $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_{11}, \hat{\beta}_{12}, \hat{\beta}_{21}, \hat{\beta}_{22})$ are approximately normal distributions with mean $(\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})$ and covariance matrix $\Sigma(\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})$ and are presented by:

$$\begin{cases} \hat{\alpha}_1 \mp z_\gamma v_{11} \text{ and } \hat{\alpha}_2 \mp z_\gamma v_{22} \\ \hat{\beta}_{11} \mp z_\gamma v_{33}, \hat{\beta}_{12} \mp z_\gamma v_{44}, \hat{\beta}_{21} \mp z_\gamma v_{55} \text{ and } \hat{\beta}_{22} \mp z_\gamma v_{66} \end{cases} \quad (28)$$

with significant level 2γ , z_γ , which defines the tabulated value of the standard normal distribution. In addition $v_{ii}|_{i=1,2,\dots,6}$ are the diagonals of the information matrix.

4. Bootstrap Confidence Intervals

In reliability analysis or life testing experiments, the bootstrap technique, which is basically a re-sampling method, can be used to solve the problem of parameters estimation. Additionally, this technique can be applied in computation of bias and variance of an estimator as well as to calibrate hypothesis tests. In this regard, [27,28] presented the parametric and nonparametric forms of bootstrap techniques; for more details see [29]. In this section, we apply the parametric bootstrap technique in the estimation problem, see [30,31]. In the following algorithms, the bootstrap confidence intervals (boot- p) are described as follows

- (1) For given M, N, m and the original data set $\underline{t} = ((t_1, \omega_1, \rho_1), (t_2, \omega_2, \rho_2), \dots, (t_m, \omega_m, \rho_m))$ compute integer numbers m_k and n_{kj} , $k, j = 1, 2$. Then, under ML procedures, report the estimates $\hat{\underline{\alpha}} = \{\hat{\alpha}_1, \hat{\alpha}_2\}$ and $\hat{\underline{\beta}} = \{\hat{\beta}_{11}, \hat{\beta}_{12}, \hat{\beta}_{21}, \hat{\beta}_{22}\}$.
- (2) Generate a sample of size M from Weibull distribution with a $\hat{\beta}_{11} + \hat{\beta}_{12}$ scale parameter and a α_1 shape parameter. Additionally, generate a sample of size N from Weibull distribution with a $\hat{\beta}_{21} + \hat{\beta}_{22}$ scale parameter and α_2 shape parameter.
- (3) From the joint sample of size $M + N$, choose the smallest m simulated lifetime to present as a Type-II JCRS, say a bootstrap random sample $\underline{t}^* = \{t_1^*, t_2^*, \dots, t_m^*\}$.
- (4) From the Type-II JCRS $\underline{t}^* = \{t_1^*, t_2^*, \dots, t_m^*\}$, compute m_1^* and m_2^* .
- (5) The integers n_{kj}^* , $k, j = 1, 2$ are generated from the two-parameter binomial distribution m_k^* and $\frac{\hat{\beta}_{kj}}{\hat{\beta}_{k1} + \hat{\beta}_{k2}}$, $k, j = 1, 2$.
- (6) From steps 3, 4 and 5, compute the bootstrap estimate $\hat{\underline{\alpha}}^* = \{\hat{\alpha}_1^*, \hat{\alpha}_2^*\}$ and $\hat{\underline{\beta}}^* = \{\hat{\beta}_{11}^*, \hat{\beta}_{12}^*, \hat{\beta}_{21}^*, \hat{\beta}_{22}^*\}$.
- (7) Repeat Step 2 to 6 S times.
- (8) If Θ is the parameter vectors, then $\Theta^* = \{\hat{\alpha}_1^*, \hat{\alpha}_2^*, \hat{\beta}_{11}^*, \hat{\beta}_{12}^*, \hat{\beta}_{21}^*, \hat{\beta}_{22}^*\}$.

The bootstrap sample Θ^* is arrange in ascending order to form

$$\Theta^* = \left\{ \hat{\alpha}_1^{*(i)}, \hat{\alpha}_2^{*(i)}, \hat{\beta}_{11}^{*(i)}, \hat{\beta}_{12}^{*(i)}, \hat{\beta}_{21}^{*(i)}, \hat{\beta}_{22}^{*(i)} \right\} \Big|_{i=1,2,\dots,S} \quad (29)$$

The maximum likelihood estimate of reliability and the hazed rate function can be obtained from (7) and (8).

The point bootstrap estimate of model parameters is given by

$$\hat{\Theta}_{l-\text{boot}} = \frac{1}{S} \sum_{i=1}^S \Theta_l^{*(i)}. \quad (30)$$

and the corresponding bootstrap estimate of parameters of life is given by

$$\hat{S}_{kj}(t) = \exp\left(-\hat{\beta}_{kj}^* t^{\hat{\alpha}_k^*}\right), \quad (31)$$

and

$$\hat{h}_{kj}(t) = \hat{\alpha}_k^* \hat{\beta}_{kj}^* t^{\hat{\alpha}_k^*-1}. \quad (32)$$

Percentile bootstrap confidence interval (Boot- p)

Let, $\Psi(x) = P(\Theta_l^* < x)$ define the cumulative distribution of ordering samples Θ_l^* , $l = 1, 2, 3, 4, 5, 6$. Hence, the $100(1 - 2\gamma)\%$ Boot- p confidence interval is given by

$$\left(\Theta_{l-\text{boot}(\gamma)}^*, \Theta_{l-\text{boot}(1-\gamma)}^* \right), \quad (33)$$

where $\Theta_{l-\text{boot}}^* = \Psi^{-1}(x)$.

5. Bayesian Estimation

Under a given joint Type-II competing risks sample, we discuss the Bayes point and credible interval estimators of the unknown parameter vector $\Theta = (\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})$. However, in the Bayesian approach it is necessary to formulate the prior considerations of unknown parameters, and gamma prior distributions represent a suitable assumption of this problem due to their flexibility in different shapes. Then, for the prior distribution parameters $\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{21}$ and β_{22} have independent gamma priors with densities defined by

$$\pi_i^*(\Theta_i) = \frac{b_i^{a_i}}{\Gamma(a_i)} \Theta_i^{a_i-1} \exp(-b_i \Theta_i), \quad \Theta_i > 0, (a_i, b_i > 0), i = 1, 2, 3, 4, 5, 6, \quad (34)$$

where $\Theta = (\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})$ and the joint prior density of Θ are defined by

$$\pi^*(\Theta) = \prod_{i=1}^6 \frac{b_i^{a_i}}{\Gamma(a_i)} \Theta_i^{a_i-1} \exp(-b_i \Theta_i). \quad (35)$$

The prior density (35) and likelihood function (9) are used to present the joint posterior density $\pi(\Theta)$ as follows

$$\begin{aligned} \pi(\Theta|t) &\propto \alpha_1^{m_1+a_1-1} \alpha_2^{m_2+a_2-1} \beta_{11}^{n_{11}+a_3-1} \beta_{12}^{n_{12}+a_4-1} \beta_{21}^{n_{21}+a_5-1} \beta_{22}^{n_{22}+a_6-1} \\ &\times \exp\{-b_1\alpha_1 - b_2\alpha_2 - b_3\beta_{11} - b_4\beta_{12} - b_5\beta_{21} - b_6\beta_{22} \\ &+ (\alpha_1 - 1) \sum_{i=1}^m \omega_i \log t_i - (\beta_{11} + \beta_{12}) \sum_{i=1}^m \omega_i t_i^{\alpha_1} \\ &+ (\alpha_2 - 1) \sum_{i=1}^m (1 - \omega_i) \log t_i - (\beta_{21} + \beta_{22}) \sum_{i=1}^m (1 - \omega_i) t_i^{\alpha_2} \\ &- (M - m_1)(\beta_{11} + \beta_{12}) t_m^{\alpha_1} - (N - m_2)(\beta_{21} + \beta_{22}) t_m^{\alpha_2}\} \end{aligned} \quad (36)$$

From the joint posterior distribution (36), we obtained the full conditional PDFs as follows

$$\begin{aligned} \pi_1(\alpha_1|\alpha_2, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, t) &\propto \alpha_1^{m_1+a_1-1} \exp\left\{-b_1\alpha_1(\alpha_1 - 1) \sum_{i=1}^m \omega_i \log t_i - (\beta_{11} + \beta_{12}) \right. \\ &\times \left. \sum_{i=1}^m \omega_i t_i^{\alpha_1} - (M - m_1)(\beta_{11} + \beta_{12}) t_m^{\alpha_1}\right\}, \end{aligned} \quad (37)$$

$$\begin{aligned} \pi_2(\alpha_2|\alpha_1, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, t) &\propto \alpha_2^{m_2+a_2-1} \exp\left\{-b_2\alpha_2 + (\alpha_2 - 1) \sum_{i=1}^m (1 - \omega_i) \log t_i \right. \\ &\times \left. - (\beta_{21} + \beta_{22}) \sum_{i=1}^m (1 - \omega_i) t_i^{\alpha_2} - (N - m_2)(\beta_{21} + \beta_{22}) t_m^{\alpha_2}\right\}, \end{aligned} \quad (38)$$

$$\pi_3(\beta_{11}|\alpha_1, \alpha_2, \beta_{12}, \beta_{21}, \beta_{22}, t) \propto \beta_{11}^{n_{11}+a_3-1} \exp\left\{-b_3\beta_{11} - \beta_{11} \sum_{i=1}^m \omega_i t_i^{\alpha_1} - (M - m_1)\beta_{11} t_m^{\alpha_1}\right\}, \quad (39)$$

$$\pi_4(\beta_{12}|\alpha_1, \alpha_2, \beta_{11}, \beta_{21}, \beta_{22}, t) \propto \beta_{12}^{n_{12}+a_4-1} \exp\left\{-b_4\beta_{12} - \beta_{12} \sum_{i=1}^m \omega_i t_i^{\alpha_1} - (M - m_1)\beta_{12} t_m^{\alpha_1}\right\}, \quad (40)$$

$$\pi_5(\beta_{21}|\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{22}, t) \propto \exp\left\{-b_5\beta_{21} - \beta_{21} \sum_{i=1}^m (1 - \omega_i)t_i^{\alpha_2} - (N - m_2)\beta_{21}t_m^{\alpha_2}\right\} \times \beta_{21}^{n_{21}+a_5-1}, \quad (41)$$

and

$$\pi_6(\beta_{22}|\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{21}, t) \propto \exp\left\{-b_6\beta_{22} - \beta_{22} \sum_{i=1}^m (1 - \omega_i)t_i^{\alpha_2} - (N - m_2)\beta_{22}t_m^{\alpha_2}\right\} \times \beta_{22}^{n_{22}+a_6-1}. \quad (42)$$

Using the Markov chain Monte Carlo (MCMC) technique, the full conditional distributions obtained from the joint posterior distribution can be adopted to estimate the parameters, reliability and hazard rate functions. Different types of MCMC schemes are available in the literature, and the problem to determine a suitable one is discussed in this section. The conditional posterior distributions out of (39) to (42) follows gamma distributions, but the plot of two functions presented by (37) and (38) is similar to the normal distribution. Therefore, we use the algorithms of Gibbs sampling and generally Metropolis Hasting (MH) under Gibbs; for more detail see [32]. The algorithms are described as follows MH under Gibbs algorithms

- (1) Begin with indicator $I = 1$ and initial parameter vector $\Theta^{(0)} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_{11}, \hat{\beta}_{12}, \hat{\beta}_{21}, \hat{\beta}_{22})$.
- (2) From gamma distribution, generate $\beta_{kj}^{(I)}$.
- (3) MH algorithms with $N(\alpha_k^{(k-1)}, v_{kk})$, $k = 1, 2$, proposal distribution generates $\alpha_k^{(I)}$.
- (4) Report the values $S_{kj}^{(I)}(t) = \exp\left(-\beta_{kj}^{(I)}t^{\alpha_k^{(I)}}\right)$, and $h_{kj}^{(I)}(t) = \alpha_k^{(I)}\beta_{kj}^{(I)}t^{\alpha_k^{(I)}-1}$.
- (5) Report the simulated parameters vector $\Theta^{(s)} = (\alpha_1^{(I)}, \alpha_2^{(I)}, \beta_{11}^{(I)}, \beta_{12}^{(I)}, \beta_{21}^{(I)}, \beta_{22}^{(I)}, S_{kj}^{(I)}(t), h_{kj}^{(I)}(t))$
- (6) Steps from 2 to 5 are repeated S times.

Bayes estimation under the MCMC method requires some measurements reported on the generation method and determining the number needed to reach the stationary distribution (burn-in) denoted by S^* . Then, the Bayes estimates are given by

$$\hat{\Theta}_{IB} = E_{\pi}(\Theta|t) = \frac{1}{S - S^*} \sum_{i=S^*+1}^S \Theta_l^{(i)}, \quad l = 1, 2, 3, 4, 5, 6, 7, 8$$

Additionally, the posterior variance of function Θ is given by

$$\widehat{V}(\Theta_l|t) = \frac{1}{S - S^*} \sum_{i=S^*+1}^S \left(\Theta_l^{(i)} - \hat{\Theta}_{IB}\right)^2.$$

The credible intervals are obtained by ordering the simulated vector Θ_l . Then, the $100(1 - 2\gamma)\%$ credible interval of function Θ_l is given by

$$\left(\Theta_{l\gamma(S-S^*)}, \Theta_{l(1-\gamma)(S-S^*)}\right).$$

6. Simulation Studies

The developed results of the proposed model are tested in this section through a Monte Carlo simulation study. Therefore, we measure the change of sample size $m + n$ and affect sample size m as well as the parameter vector $\Theta = (\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})$, see [33]. Then, we adopted two sets of the parameter values $\Theta_1 = (0.6, 0.4, 1.0, 1.5, 1.2, 2.0)$ and $\Theta_2 = (1.2, 1.4, 1.6, 2.0, 1.8, 2.4)$. A different combination of (M, N) reported in Tables 1–4 of simulation results. The simulation study is conducted with respect to 1000 simulated data sets. To satisfy that, the true parameter value in the range of prior distribution, we select the prior parameters to satisfy the property that $E(\Theta_i) \cong a_i/b_i$.

Table 1. Mean and MSEs of the ML, boot and Bayes (P^0, P^1) estimates $\Theta_1 = \{0.6, 0.4, 1.0, 1.5, 1.2, 2.0\}$.

(M, N, m)		α_1		α_2		β_{11}		β_{12}		β_{21}		β_{22}	
(30,30,25)	ML	0.654	0.154	0.513	0.132	1.425	1.254	1.754	1.425	1.421	1.305	2.514	2.147
	Boot	0.681	0.187	0.538	0.165	1.401	1.272	1.791	1.466	1.451	1.337	2.554	2.140
	Bayes0	0.646	0.118	0.494	0.111	1.342	1.182	1.721	1.400	1.411	1.289	2.498	2.109
	Bayes1	0.632	0.099	0.452	0.097	1.245	0.998	1.695	1.234	1.365	1.085	2.352	1.821
(30,30,40)	ML	0.635	0.110	0.495	0.103	1.366	1.211	1.715	1.401	1.402	0.997	2.421	2.103
	Boot	0.601	0.142	0.488	0.128	1.381	1.231	1.701	1.440	1.415	1.007	2.424	2.128
	Bayes0	0.632	0.101	0.475	0.099	1.325	1.101	1.702	1.398	1.398	0.980	2.403	2.097
	Bayes1	0.614	0.094	0.434	0.089	1.211	1.000	1.670	1.199	1.332	0.8058	2.315	1.754
(60,30,40)	ML	0.628	0.103	0.497	0.101	1.354	1.199	1.707	1.385	1.407	0.982	2.418	2.105
	Boot	0.614	0.125	0.494	0.127	1.366	1.280	1.714	1.420	1.400	0.999	2.425	2.141
	Bayes0	0.615	0.099	0.482	0.097	1.323	1.187	1.708	1.362	1.391	0.977	2.397	2.088
	Bayes1	0.610	0.095	0.431	0.091	1.207	1.003	1.665	1.194	1.325	0.814	2.314	1.742
(60,60,50)	ML	0.594	0.088	0.428	0.089	1.215	0.876	1.642	1.054	1.365	0.912	2.225	1.821
	Boot	0.607	0.128	0.414	0.179	1.227	0.899	1.625	1.081	1.388	0.941	2.214	1.856
	Bayes0	0.573	0.082	0.414	0.082	1.211	0.862	1.635	1.028	1.324	0.901	2.214	1.785
	Bayes1	0.562	0.059	0.409	0.073	1.189	0.756	1.611	0.987	1.314	0.701	2.198	1.452

Table 2. Mean interval length and CP of ML, boot and Bayes (P^0, P^1) estimates $\Theta_1 = \{0.6, 0.4, 1.0, 1.5, 1.2, 2.0\}$.

(M, N, m)		α_1		α_2		β_{11}		β_{12}		β_{21}		β_{22}	
(30,30,25)	ML		CP	ML	CP	ML	CP	ML	CP	ML	CP	ML	CP
	ML	1.721	0.88	1.521	0.88	2.323	0.87	3.214	0.88	1.823	0.89	3.854	0.88
	Boot	1.788	0.89	1.562	0.89	2.354	0.89	3.264	0.90	1.871	0.89	3.894	0.89
	Bayes0	1.701	0.89	1.503	0.89	2.285	0.88	3.245	0.88	1.811	0.89	3.811	0.88
	Bayes1	1.622	0.88	1.452	0.90	2.099	0.88	3.122	0.89	1.745	0.90	3.625	0.89
(30,30,40)	ML	1.690	0.89	1.501	0.89	2.301	0.89	3.180	0.90	1.801	0.89	3.803	0.90
	Boot	1.727	0.90	1.541	0.89	2.325	0.90	3.221	0.90	1.837	0.90	3.828	0.89
	Bayes0	1.672	0.89	1.487	0.89	2.252	0.90	3.165	0.90	1.795	0.90	3.798	0.90
	Bayes1	1.600	0.90	1.411	0.91	2.021	0.90	3.100	0.91	1.714	0.91	3.614	0.91
(60,30,40)	ML	1.671	0.90	1.482	0.90	2.287	0.90	3.120	0.91	1.782	0.90	3.798	0.91
	Boot	1.694	0.90	1.491	0.91	2.311	0.89	3.151	0.90	1.811	0.91	3.821	0.93
	Bayes0	1.654	0.90	1.445	0.91	2.214	0.89	3.124	0.91	1.745	0.91	3.782	0.90
	Bayes1	1.583	0.92	1.387	0.92	2.003	0.93	3.009	0.93	1.701	0.94	3.602	0.93
(60,60,50)	ML	1.625	0.91	1.444	0.92	2.264	0.90	3.099	0.92	1.771	0.90	3.767	0.90
	Boot	1.651	0.90	1.471	0.91	2.285	0.93	3.140	0.93	1.799	0.92	3.789	0.91
	Bayes0	1.618	0.92	1.432	0.93	2.211	0.91	3.068	0.94	1.750	0.93	3.745	0.92
	Bayes1	1.521	0.91	1.325	0.96	1.987	0.92	2.874	0.92	1.658	0.96	3.587	0.93

The tools that were used to test the point estimate are the mean estimate (ME) and the corresponding mean squared error (MSE). However, the interval estimate test under the mean interval length (ML) and probability coverage (PC). For the prior information, we adopted non-informative and informative prior information. Therefore, we proposed P^0 ($(a_i, b_i) = (0.0001, 0.0001)$, $i = 1, 2, \dots, 6$) for the non-informative prior and P^1 (a non-zero value of (a_i, b_i) , $i = 1, 2, \dots, 6$) for the informative prior). The important sample iteration runs 11,000 iterations discarding the first 1000 values as run-in. The results of the simulation study are reported in Tables 1–4. All of the computations were performed using Mathematica Version 10. The simulation study is constructed mainly to compare the MLEs, bootstrap and Bayes estimators and to explore their effects on different parameter values.

Table 3. Mean and MSEs of ML, boot and Bayes (P^0, P^1) estimates $\Theta_2 = \{1.2, 1.4, 1.6, 2.0, 1.8, 2.4\}$.

(M, N, m)			α_1		α_2		β_{11}		β_{12}		β_{21}		β_{22}	
(30,30,25)	ME		MSE	ME	MSE	ME	MSE	ME	MSE	ME	MSE	ME	MSE	
	ML	1.454	0.451	1.624	0.541	1.987	1.325	2.389	1.741	2.142	1.524	2.611	2.302	
	Boot	1.471	0.469	1.651	0.566	1.999	1.342	2.410	1.759	2.157	1.539	2.623	2.331	
	Bayes0	1.422	0.436	1.609	0.527	1.924	1.302	2.355	1.711	2.215	1.507	2.598	2.298	
	Bayes1	1.388	0.254	1.598	0.418	1.824	1.200	2.265	1.621	2.108	2.412	2.511	2.099	
(30,30,40)	ML	1.441	0.422	1.611	0.509	1.918	1.282	2.342	1.688	2.111	1.501	2.590	2.287	
	Boot	1.462	0.440	1.629	0.537	1.945	1.299	2.361	1.705	2.134	1.528	2.621	2.298	
	Bayes0	1.408		1.615	0.502	1.909	1.274	2.327	1.691	2.207	1.692	2.582	2.265	
	Bayes1	1.362	0.223	1.576	0.391	1.812	1.187	2.225	1.594	2.065	2.375	2.499	2.056	
(60,30,40)	ML	1.447	0.418	1.603	0.501	1.911	1.277	2.335	1.675	2.114	1.498	2.571	2.279	
	Boot	1.468	0.434	1.621	0.527	1.934	1.292	2.361	1.689	2.132	1.515	2.594	2.301	
	Bayes0	1.403	0.414	1.602	0.505	1.910	1.271	2.328	1.688	2.199	1.688	2.565	2.261	
	Bayes1	1.354	0.220	1.557	0.393	1.804	1.181	2.214	1.591	2.049	2.369	2.491	2.047	
(60,60,50)	ML	1.412	0.380	1.598	0.470	1.875	1.133	2.286	1.601	2.045	1.414	2.514	2.154	
	Boot	1.441	0.397	1.614	0.487	1.882	1.154	2.311	1.627	2.064	1.431	2.528	2.167	
	Bayes0	1.385	0.371	1.577	0.452	1.852	1.105	2.274	1.592	2.022	1.402	2.515	2.122	
	Bayes1	1.311	0.189	1.501	0.311	1.714	0.999	2.187	1.503	1.985	2.311	2.425	1.892	

Table 4. Mean length and CP of of ML and Bayes (pP^0, P^1) estimates $\Theta_2 = (1.2, 1.4, 1.6, 2.0, 1.8, 2.4)$.

(M, N, m)	α_1		α_2		β_{11}		β_{12}		β_{21}		β_{22}		
(30,30,25)	ML		CP	ML	CP	ML	CP	ML	CP	ML	CP	ML	CP
	ML	2.854	0.89	3.142	0.89	3.321	0.87	3.582	0.88	3.421	0.89	4.214	0.90
	Boot	2.887	0.88	3.171	0.90	3.354	0.89	3.599	0.89	3.464	0.89	4.252	0.89
	Bayes0	2.822	0.89	3.124	0.90	3.302	0.89	3.549	0.90	3.399	0.89	4.198	0.88
	Bayes1	2.645	0.89	3.002	0.90	3.224	0.89	3.424	0.89	3.288	0.89	4.025	0.89
(30,30,40)	ML	2.782	0.90	3.100	0.90	3.287	0.90	3.502	0.90	3.385	0.90	4.178	0.90
	Boot	2.797	0.90	3.132	0.91	3.298	0.91	3.524	0.89	3.411	0.92	4.159	0.91
	Bayes0	2.744	0.90	3.088	0.91	3.271	0.91	3.487	0.91	3.369	0.91	4.154	0.92
	Bayes1	2.600	0.91	2.974	0.91	3.187	0.96	3.390	0.92	3.215	0.93	4.001	0.93
(60,30,40)	ML	2.777	0.91	3.114	0.90	3.269	0.91	3.498	0.92	3.368	0.90	4.159	0.91
	Boot	2.794	0.91	3.145	0.92	3.287	0.90	3.524	0.91	3.389	0.94	4.172	0.93
	Bayes0	2.735	0.90	3.082	0.93	3.254	0.92	3.469	0.91	3.364	0.94	4.151	0.91
	Bayes1	2.597	0.92	2.969	0.92	3.181	0.93	3.379	0.93	3.201	0.93	3.987	0.94
(60,60,50)	ML	2.725	0.91	3.088	0.92	3.214	0.93	3.415	0.93	3.311	0.90	4.098	0.93
	Boot	2.754	0.93	3.124	0.90	3.247	0.91	3.445	0.94	3.327	0.89	4.111	0.92
	Bayes0	2.704	0.94	3.001	0.91	3.201	0.92	3.403	0.91	3.314	0.92	4.075	0.93
	Bayes1	2.521	0.94	2.913	0.95	3.125	0.96	3.307	0.94	3.175	0.96	3.914	0.92

The numerical computations which were presented by Monte Carlo simulation study show that the proposed model and the corresponding methods of estimation of the ML, bootstrap and Bayes approach work well in all cases. Therefore, the proposed model is more acceptable. The results obtained from the simulation study of the current estimation agree with the results obtained by [34] if the model reduced the exponential distribution. Additionally, the results presented in Tables 1–4 show the following points.

- The results are improving for increasing values of sample size (M, N).
- The results under ML and Bayes with non-informative prior information are closed.
- The joint censoring scheme in comparative life testing under Type-II censoring scheme presents a suitable scheme.
- Bayesian estimation under the informative prior is better than MLEs and bootstrap.
- The results for the selected two sets of parameters are more acceptable.

7. Real Data Analysis

In this section, the proposed model is used to analyze the real data set given in [35]; several authors used the Weibull lifetime population for analysis of this data set; for more details see [36]. These data was obtained from the laboratory experiment on two groups of radiation male mice are reported for two causes of failure—thymine lymphoma as cause one and other causes of failure as cause two. The failure time and the corresponding causes of failure are reported in Tables 5 and 6. These data under the joint censoring scheme with competing risks model are analyzed in [37] under the Burr XII distribution, and the results agree with the results obtained in the current paper. For $m = 60$, the joint Type-II JCRS with the type and cause of failure is reported in Table 7, after dividing by 1000 for simplicity. From Table 7, with the effective sample size $m = 60$, we obtained the numbers $(m_1, m_2) = (37, 23)$ and $(n_{11}, n_{12}, n_{21}, n_{22}) = (19, 18, 19, 4)$. The random sample is selected from the population with $(M, N) = (61, 67)$ units from the two lines Ψ_1 and Ψ_2 . Under the Type-II censoring scheme with $m = 60$, we have $n_{11} = 19$ failure times from the line Ψ_1 and the first cause of death, $n_{12} = 18$ failure times from the line Ψ_1 and the second cause of death, $n_{21} = 19$ failure times from the line Ψ_2 and the first cause of death and $n_{22} = 4$ failure times from the line Ψ_2 and the second cause of death. Under Type-II JCRS, the profile log-likelihood function (17) is plotted in Figure 1. Then, the iteration is running with an initial guess $(\alpha_1, \alpha_2) = (1.8, 2.0)$. For Bayes estimation, we adopted a non-informative prior with $a_i = b_i = 0.0001$, $i = 1, 2, \dots, 6$. For the MCMC approach in the Bayes method, we run the 11,000 chain with the first 1000 values as burn-in. The MCMC approach that describes the empirical posterior distribution is shown by Figures 2–7. Hence, the results of the point and interval MLEs and different Bayes estimates under squared error loss function are computed and presented in Table 8. Also, the MLEs, bootstrap and Bayes estimates of the reliability and hazard rate functions presented in Table 9.

Table 5. Autopsy data for 99 germ-free male mice (RFM) conventional male mice which received a radiation dose or 300r at age 5–6 weeks.

Thymic Lymphoma	159	189	191	198	200	207	220	235	245	250	256
	261	265	266	280	343	356	383	403	414	428	432
Other causes	40	42	51	62	163	179	206	222	228	252	249
	282	324	333	341	366	385	407	420	431	441	461
	462	482	517	517	524	564	567	586	619	620	621
	622	647	651	686	761	763					

Table 6. Autopsy data for 99 RFM conventional male mice which received a radiation dose or 300 r at age 5–6 weeks.

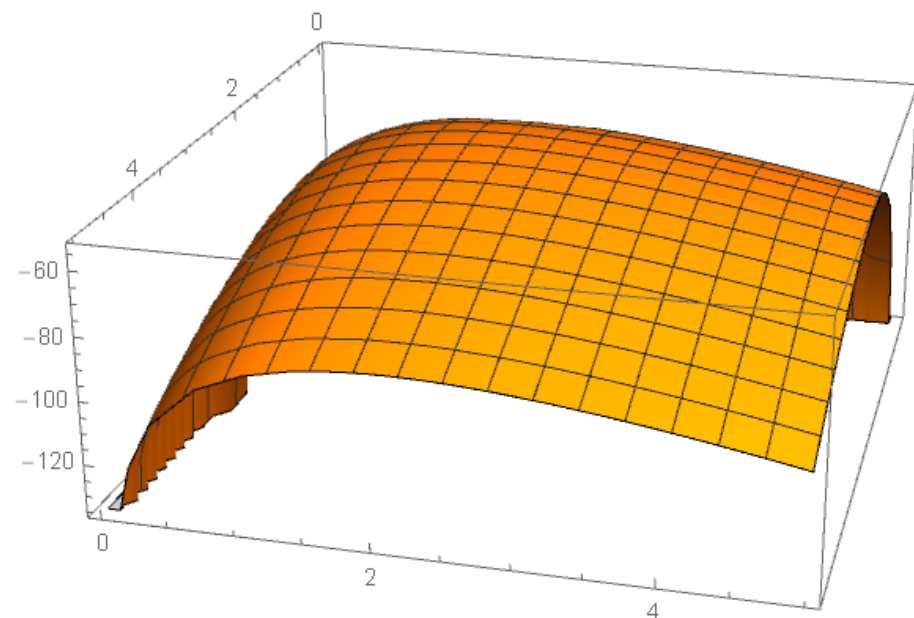
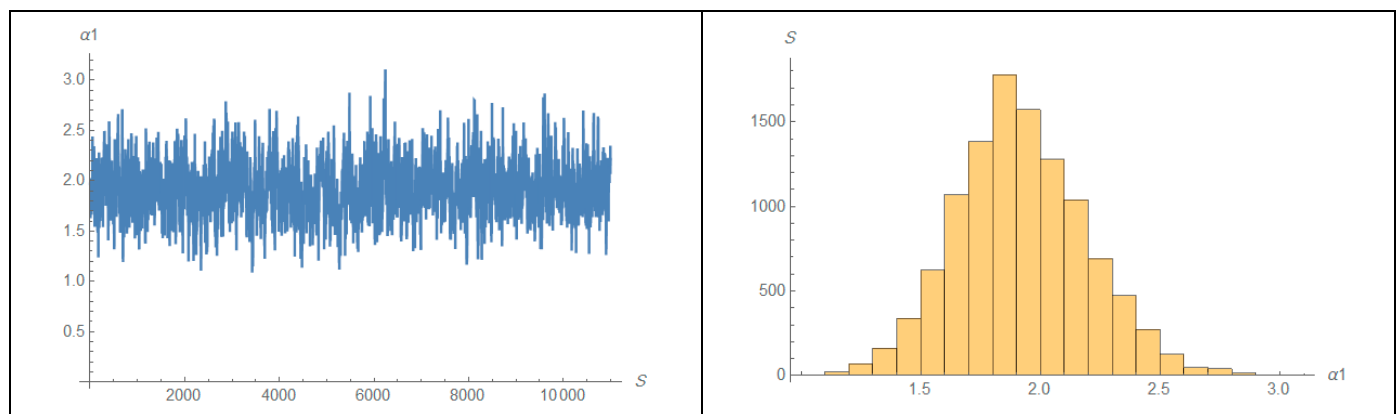
Thymic Lymphoma	158	192	193	194	195	202	212	215	229	230	237
	240	244	247	259	300	301	321	337	415	434	444
	485	529	537	624	707	800					
Other causes	136	246	255	376	421	565	616	617	652	655	658
	660	662	675	681	734	736	737	757	769	777	800
	807	825	855	857	864	868	870	870	873	882	895
	910	934	942	1015	1019						

The numerical computations presented with a real data set show:

- The proposed censoring scheme of Type-II JCRS contributes well to solving the problem of analysis of a real data set distributed under consideration that, the unit life has Weibull lifetime distribution.
- The MLE under the fixed-point iteration as well as the parametric bootstrap technique function well.
- Figures 2–7 show the convergence happens in generation from the posterior distribution under the MCMC method.

Table 7. Type-II JCRS from real data with $m = 60$.

t_i	0.040	0.042	0.051	0.062	0.136	0.158	0.159	0.163	0.179	0.189	0.191	0.192	0.193	0.194
ω_i	1	1	1	1	2	2	1	1	1	1	1	2	2	2
ρ_i	0	0	0	0	0	1	1	0	0	1	1	1	1	1
t_i	0.195	0.198	0.200	0.202	0.206	0.207	0.212	0.215	0.220	0.222	0.228	0.229	0.230	0.235
ω_i	2	1	1	2	1	1	2	2	1	1	1	2	2	1
ρ_i	1	1	1	1	0	1	1	1	1	0	0	1	1	1
t_i	0.237	0.240	0.244	0.245	0.246	0.247	0.249	0.250	0.252	0.255	0.256	0.259	0.261	0.265
ω_i	2	2	2	1	2	2	1	1	1	2	1	2	1	1
ρ_i	1	1	1	1	0	1	0	1	0	0	1	1	1	1
t_i	0.266	0.280	0.282	0.300	0.301	0.321	0.324	0.333	0.337	0.341	0.343	0.356	0.366	0.376
ω_i	1	1	1	2	2	2	1	1	2	1	1	1	1	2
ρ_i	1	1	0	1	1	1	0	0	1	0	1	1	0	0
t_i	0.383	0.385	0.403	0.407										
ω_i	1	1	1	1										
ρ_i	1	0	1	0										

**Figure 1.** The profile log-likelihood of α_1 and α_2 .**Figure 2.** Simulation MCMC-generated number and histogram of the parameter α_1 .

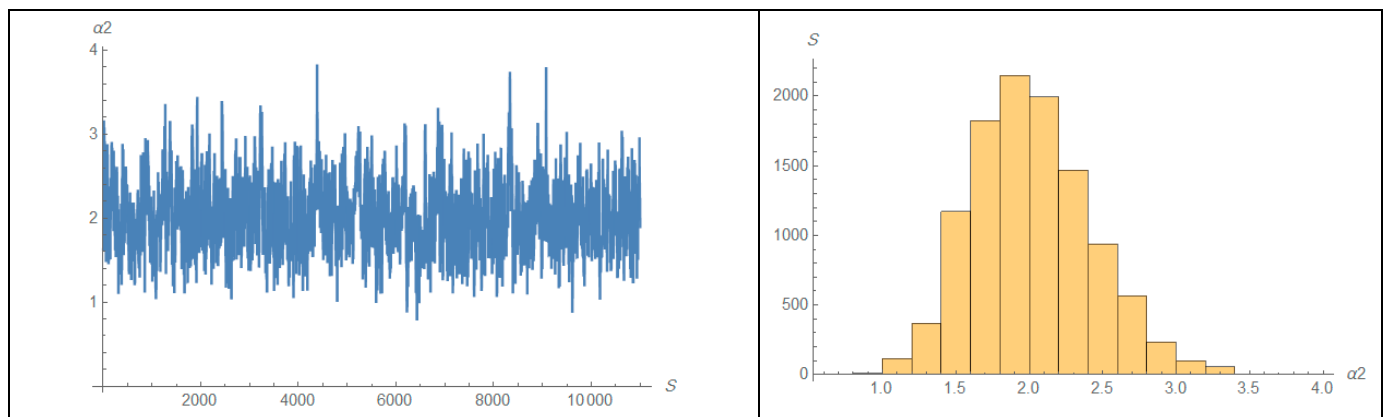


Figure 3. Simulation MCMC-generated number and histogram of the parameter α_2 .

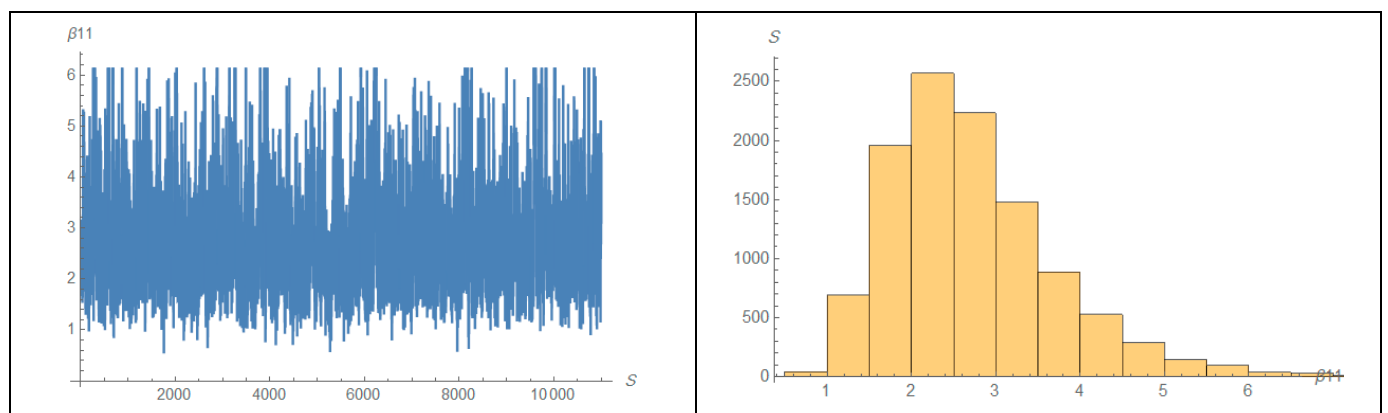


Figure 4. Simulation MCMC-generated number and histogram of the parameter β_{11} .

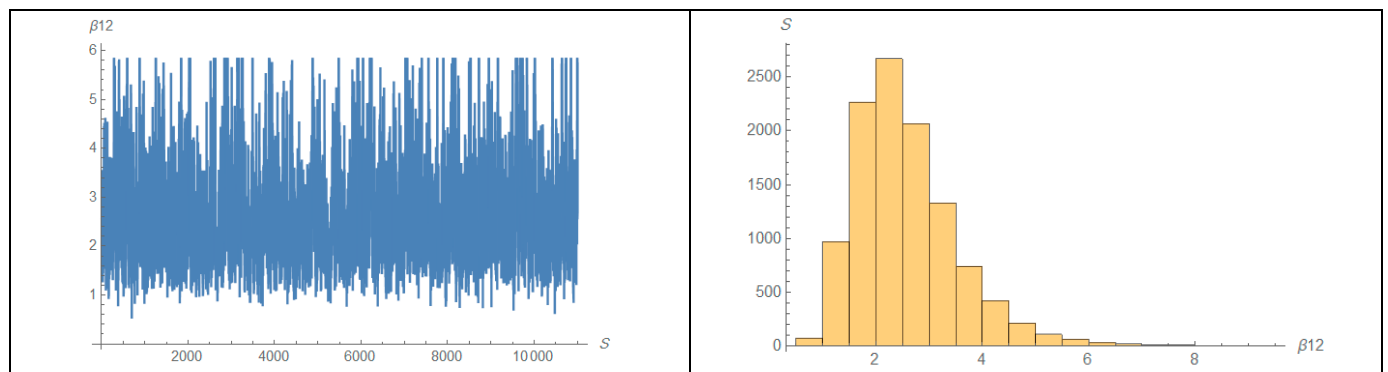


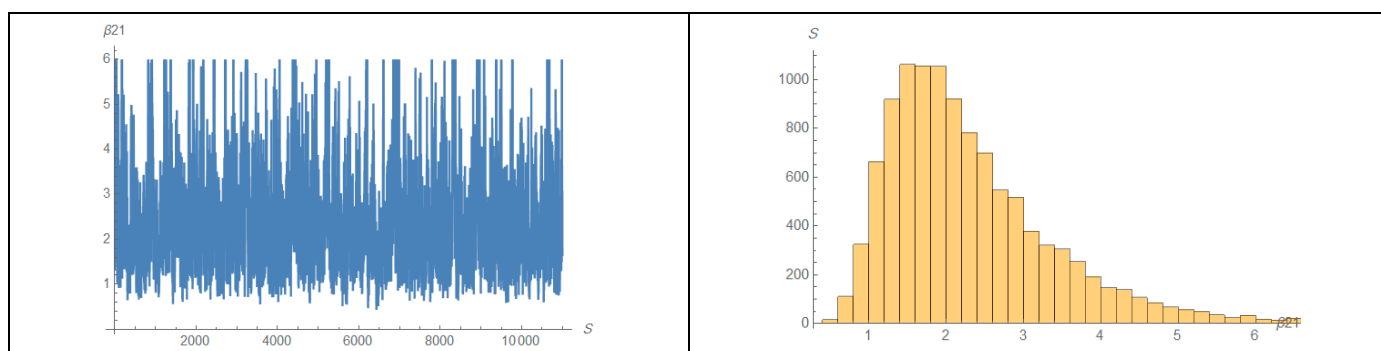
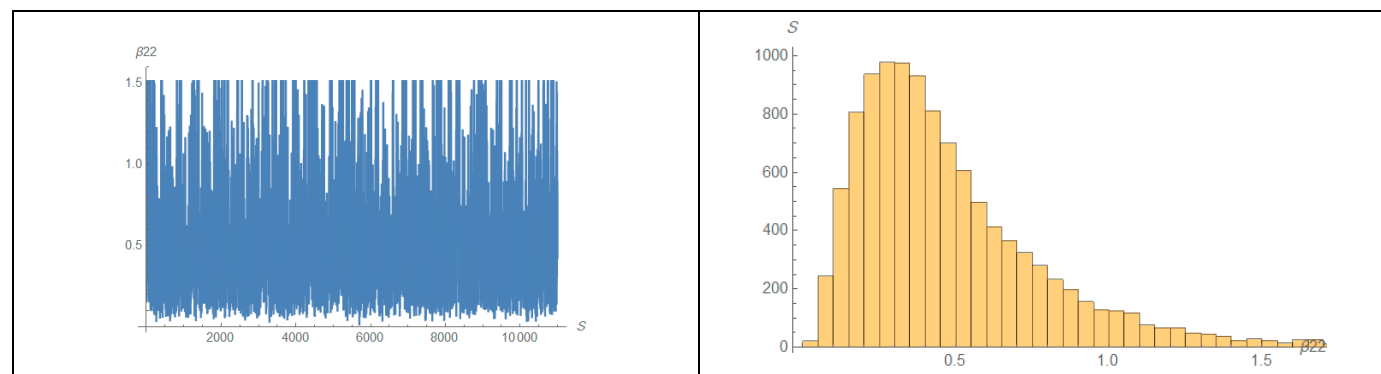
Figure 5. Simulation MCMC-generated number and histogram of the parameter β_{12} .

Table 8. Classical and Bayes estimate of the point and the corresponding 95% confidence interval.

Pa.	(.) _{ML}	(.) _{Boot}	(.) _B	95% ACI	Lenth	95% Boot	Lenth	95% B	Lenth
α_1	1.997	1.842	1.919	(1.395, 2.600)	1.205	(1.315, 2.784)	1.469	(1.405, 2.478)	1.073
α_2	2.111	2.001	2.016	(1.147, 2.847)	1.7	(1.245, 3.042)	1.797	(1.303, 2.894)	1.591
β_{11}	2.905	2.645	2.716	(0.674, 5.133)	4.46	(0.562, 5.422)	4.859	(1.267, 5.150)	3.881
β_{12}	2.751	2.444	2.573	(0.618, 4.883)	4.265	(0.633, 4.994)	4.361	(1.195, 4.946)	3.751
β_{21}	2.436	2.245	2.323	(0.130, 4.741)	4.611	(0.152, 4.975)	4.823	(0.902, 5.198)	4.296
β_{22}	0.513	0.4	0.492	(-0.147, 1.172)	1.319	(0.231, 1.412)	1.181	(0.102, 1.377)	1.275

Table 9. The reliability $S(t)$ and hazard rat function $h(t)$ at the time $t = 0.1$.

Parameters	Reliability			Hazard Failure Rate		
	(.) _{ML}	(.) _{Boot}	(.) _B	(.) _{ML}	(.) _{Boot}	(.) _B
α_1, β_{11}	0.971	0.963	0.975	0.584	0.701	0.586
α_1, β_{12}	0.973	0.966	0.979	0.553	0.648	0.551
α_2, β_{21}	0.981	0.978	0.979	0.398	0.448	0.399
α_2, β_{22}	0.996	0.996	0.991	0.084	0.080	0.081

**Figure 6.** Simulation MCMC-generated number and histogram of the parameter β_{21} .**Figure 7.** Simulation MCMC-generated number and histogram of the parameter β_{22} .

8. Conclusions

The statistical inference under Type II JCS of Weibull competing risk models is considered in this paper. The unknown parameters of the proposed model are estimated using classical ML and bootstrap methods and the non-informative and informative Bayes method. In the Bayesian approach, we adopt squared error and loss function. The asymptotic confidence intervals and Bayes credible intervals are also discussed. We analyze one real data set and conduct a Monte Carlo simulation study to assess the performance of the proposed methods. From the numerical result, we observe that the MLEs and non-informative Bayes estimates provide similar results. The Bayes estimates under informative prior perform better than their counterparts. It was also observed that, when the effective sample size m increases, the MSEs of the estimates and the confidence intervals decrease. Finally, we can conclude that the proposed model and methods of estimation work well and the results can be utilized in the field of comparative life testing, especially when units fail with several causes of failure.

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