Substructural Logics - Prologue

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Introduction

- Give a brief survey of the study of Substructural Logics, which is an attempt to understand various nonclassical logics in a uniform way.
- Show how deeply logic and algebra are connected to each other.

N. Galatos, P. Jipsen, T. Kowalski, HO: Residuated Lattices: an algebraic glimpse at substructural logics, Studies in Logic, vol.151, Elsevier, April, 2007

Two main directions in nonclassical logics:

- Logics with additional operators
  modal logics, temporal logics, epistemic logics etc.
- Logics with nonclassical implications

Constructive reasoning

Mathematical arguments are often infinitary and non-constructive.

From intuitionists' viewpoint:

mathematical arguments must be constructive.

- To infer $\alpha \to \beta$, it is required to have an algorithm for constructing a proof of β from any given proof of α,
- To infer $\alpha \vee \beta$, it is required to tell which of α and β holds, and also to have the justification.

Thus, in constructive reasoning, both the law of double negation $\neg\neg\alpha \to \alpha$ and the law of excluded middle $\alpha \vee \neg\alpha$ are rejected.

Relevant reasoning

The implication in classical logic is material implication, i.e. $\alpha \to \beta$ is identified with $\neg\alpha \vee \beta$.

Thus, both $(\alpha \wedge \neg\alpha) \to \beta$ and $\beta \to (\alpha \to \alpha)$ are classically valid (as both $\neg(\alpha \wedge \neg\alpha)$ and $\neg\alpha \vee \alpha$ are true), but their validity will be counterintuitive.

Relevant logicians try to formalize relevant implication, which expresses “implication” used in our daily reasoning.

For instance, relevant implication must satisfy:

Relevance principle:

If α (relevantly) implies β, there must be some ”connections” between α and β. (Without such a connection, why does β follow from α?)

Many-valued logics

In 1920s, J. Łukasiewicz introduced both $n+1$-valued logic (for each $n > 0$) with the set of truth values $\{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$, and also infinite-valued logic with the unit interval $[0, 1]$ as the set of truth values.

The truth table of each connective is defined as follows:

$$
\begin{aligned}
a \wedge b &= \min\{a, b\} & \qquad a \vee b &= \max\{a, b\} \\
\neg a &= 1 - a & a \to b &= \min\{1, 1 - a + b\} \\
& & &= \begin{cases} 1 & a \leq b \\ 1 - a + b & a > b \end{cases}
\end{aligned}
$$

Fuzzy logics

P. Hájek discusses fuzzy logics based on triangular norms (t-norms).

A binary operation on $[0,1]$ is a t-norm if it is associative, commutative and monotone with the unit element 1.

A t-norm $\cdot$ is left-continuous if $x \cdot \sup Z = \sup(x \cdot Z)$ for each $x \in [0,1]$ and each $Z \subseteq [0,1]$. For each left-continuous t-norm $\cdot$, define an implication $\to$ by

$$a \to b = \sup\{z : a \cdot z \leq b\}$$

- Are there something common among these logics?
- Is it possible to discuss them within a uniform framework?

Substructural Logics

We will explain what are substructural logics. Usually, they are introduced as sequent systems.

Note

When Y. Komori and I tried to find a sequent system for the (dual) of implicational BCK, we found the important role of "structural rules" of Gentzen's sequent systems.

Sequent system LJ

A sequent is an expression of the following form with $m \geq 0$.

$$\alpha_1, \ldots, \alpha_m \Rightarrow \beta$$

Intuitively, it means "β *follows from assumptions* $\alpha_1, \ldots, \alpha_m$".

Each sequent system consists of initial sequents (axioms) and rules that determine *correct* sequents in the system.

The sequent system **LJ** for intuitionistic logic introduced by Gentzen consists of initial sequents, i.e. sequents of the form $\alpha \Rightarrow \alpha$, and the following three kinds of rules.

- Rules for logical connectives
- Cut
- Structural rules

Intuitive algebraic meaning

An algebraic interpretation of sequents in **LJ** is given by using Heyting algebras, so as to satisfy:

A sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable in **LJ** iff

$\mathbf{A} \models \alpha_1 \wedge \ldots \wedge \alpha_m \leq \beta$ for every Heyting algebra $\mathbf{A}$.

Rules for $\vee$ and $\wedge$

Capital Greek letters denote finite sequences of formulas.

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \varphi \quad \Gamma, \beta, \Delta \Rightarrow \varphi}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \varphi} \ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} \ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta} \ (\Rightarrow \vee 2)$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \varphi} \ (\wedge 1 \Rightarrow) \qquad \frac{\Gamma, \beta, \Delta \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \varphi} \ (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta} \ (\Rightarrow \wedge)$$

Rules for implication, and Cut rule

- Rules for implication

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \varphi}{\Gamma, \alpha \to \beta, \Delta \Rightarrow \varphi} \ (\to\Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \ (\Rightarrow\to)$$

- Cut

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \alpha, \Xi \Rightarrow \varphi}{\Sigma, \Gamma, \Xi \Rightarrow \varphi} \ (\mathrm{cut})$$

In algebraic terms, $x \leq a$ and $y \wedge a \wedge z \leq d$ imply $y \wedge x \wedge z \leq d$.

Negation

Negation is defined by using a constant 0, and define the negation $\neg\alpha$ of a formula α by

$$\neg\alpha = \alpha \to 0.$$

For 0, we assume the initial sequent $0 \Rightarrow$, and the following rule:

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (\text{0 weakening})$$

0 means empty formula in the right-hand side.

Structural rules

Structural rules control the meaning of commas in sequents. (i) together with (o) is called (w) (weakening rules).

(e) exchange rule (commutativity): $\dfrac{\Gamma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \varphi}$

(c) contraction rule (square-increasing): $\dfrac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}$

(i) left weakening rule (integrality): $\dfrac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}$

(o) right weakening rule (minimality of 0): $\dfrac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}$

Note

Before the discovery of substructural logics, structural rules are regarded simply as auxiliary rules.

Structural rules will change the meaning of commas

a) Exchange rule allows us to use assumptions in an arbitrary order:

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \varphi}$$

b) Without contraction rule, every (occurrence of each) assumption is used at most once in deriving a conclusion:

$$\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}$$

c) Without weakening rule (i), every assumption is used at least once in deriving a conclusion:

$$\frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}$$

Examples of proofs

- A proof of $\Rightarrow \alpha \to (\beta \to \alpha)$

$$\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha, \beta \Rightarrow \alpha}\ (\mathit{weak})}{\alpha \Rightarrow \beta \to \alpha}\ (\Rightarrow\to)}{\Rightarrow \alpha \to (\beta \to \alpha)}\ (\Rightarrow\to)$$

- A proof of distributive law in **LJ**

$$
\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha, \beta \Rightarrow \alpha}\ (\mathit{weak}) \quad \dfrac{\beta \Rightarrow \beta}{\alpha, \beta \Rightarrow \beta}\ (\mathit{weak})}{\alpha, \beta \Rightarrow \alpha \wedge \beta}}{\alpha, \beta \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)} \quad \dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha, \gamma \Rightarrow \alpha}\ (\mathit{weak}) \quad \dfrac{\gamma \Rightarrow \gamma}{\alpha, \gamma \Rightarrow \gamma}\ (\mathit{weak})}{\alpha, \gamma \Rightarrow \alpha \wedge \gamma}}{\alpha, \gamma \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}}{\dfrac{\dfrac{\dfrac{\alpha, \beta \vee \gamma \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}{\alpha \wedge (\beta \vee \gamma), \beta \vee \gamma \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}}{\alpha \wedge (\beta \vee \gamma), \alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}}{\alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}\ (\mathit{cont})}
$$

Basic systems of substructural logics

We introduce several sequent systems of basic substructural logics. They are obtained from **LJ** for intuitionistic logic by deleting some or all of structural rules (and then sometimes adding the law of double nagation)

- **FL** — deleting all structural rules from **LJ**
- $\mathbf{FL_e}$ — **FL**$+$ exchange
- $\mathbf{FL_c}$ — **FL**$+$ contraction
- $\mathbf{FL_{ew}}$ — **FL**$+$ exchange $+$ weakening
- $\mathbf{CFL_e}$ — $\mathbf{FL_e} + \neg\neg\alpha \to \alpha$

Note

- I introduced a series of logics over **FL** at Heyting '88 Conference at Bulgaria.
- At Tübingen Conference in October, 1990, the name "substructural logics" was introduced.

Non commutative case

When we have no exchange rule, it is natural to introduce two kinds of "implication", left-residuation $\backslash$ and right residuation $/$, with the following rules.

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\Rightarrow \backslash) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta, \Sigma \Rightarrow \theta}{\Delta, \Gamma, \alpha\backslash\beta, \Sigma \Rightarrow \theta}\ (\backslash \Rightarrow)$$

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (\Rightarrow /) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta, \Sigma \Rightarrow \theta}{\Delta, \beta/\alpha, \Gamma, \Sigma \Rightarrow \theta}\ (/ \Rightarrow)$$

We can introduce two kinds of "negation" by

$$\sim \alpha = \alpha\backslash 0 \textit{ and } -\alpha = 0/\alpha.$$

An example of proofs in **FL**

$$
\dfrac{\alpha \Rightarrow \alpha \qquad \dfrac{\beta \Rightarrow \beta \quad \gamma \Rightarrow \gamma}{\beta/\gamma, \gamma \Rightarrow \beta}}{\dfrac{\alpha, \alpha\backslash(\beta/\gamma), \gamma \Rightarrow \beta}{\dfrac{\alpha\backslash(\beta/\gamma), \gamma \Rightarrow \alpha\backslash\beta}{\alpha\backslash(\beta/\gamma) \Rightarrow (\alpha\backslash\beta)/\gamma}}}
$$

Substructural logics in general

Substructural logics are axiomatic extensions of **FL**.

- Lambek calculus — logic without structural rules, i.e. **FL**

  Calculus for categorial grammer introduced by Ajdukiewicz and Bar-Hillel (J. Lambek, 1958), which was rediscovered in early 80s (J. van Benthem and W. Buszkowski).

- Relevant logics — logics without weakening rules

  A. Anderson, N. Belnap Jr., R.K. Meyer, M. Dunn, A. Urquhart etc.

- Logics without contraction rule

  V. Grishin (middle of 1970), H.O. & Y. Komori (1985).

- Linear logic — logic only with exchange rule, **MALL** $=$ $\mathbf{FL_e}$ $+$ double negation

  J.-Y. Girard (1987)

- Relevant logic **R** is $\mathbf{FL_{ec}}$ + double negation + distributive law
- Both fuzzy logics and Łukasiewicz's many-valued logics are extensions of $\mathbf{FL_{ew}}$

Commas

By using contraction and (left) weakening, we can show that :

a sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable in **LJ** iff

$\alpha_1 \wedge \ldots \wedge \alpha_m \Rightarrow \beta$ is provable in **LJ**.

Thus, commas of **LJ** can be understood as conjunctions.

But this is not always the case. We introduce a logical connective $\cdot$, called the fusion or the multiplicative conjunction, which can represent a comma always.

Rules for $\cdot$ are given as:

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta} \ (\Rightarrow \cdot) \qquad \frac{\alpha, \beta, \Gamma \Rightarrow \gamma}{\alpha \cdot \beta, \Gamma \Rightarrow \gamma} \ (\cdot \Rightarrow)$$

Comma and fusion

Then we have the following:

- $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable iff $\alpha_1 \cdot \ldots \cdot \alpha_m \Rightarrow \beta$ is provable,
- $\alpha \cdot \beta \Rightarrow \gamma$ is provable iff $\alpha \Rightarrow \beta \rightarrow \gamma$ is provable.

Implications as residuals of fusion

Also, we can show the following equivalences which say that Implications are residuals of fusion.

> $\alpha, \beta \Rightarrow \varphi$ *is provable iff* $\beta \Rightarrow \alpha\backslash\varphi$ *is provable iff* $\alpha \Rightarrow \varphi/\beta$ *is provable.*

For our algebraic understanding of sequents, we will introduce a constant 1 and assume the initial sequent $\Rightarrow 1$, and the following rule:

$$\frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, 1, \Delta \Rightarrow \varphi}\ (1\ \text{weakening})$$

1 means empty formula in the left-hand side.

Appendix: Intuitive meaning of fusion

Let α: *one pays 1500 yen.*
β: *one can get a hardcover.*
γ: *one can have lunch.*

Assume that

1) one (fixed) hardcover costs 1500 yen,
2) lunch at a Japanese restaurant costs 1500 yen.

Thus, we can assume both $\alpha \Rightarrow \beta$ and $\alpha \Rightarrow \gamma$ are provable. Then

(1) $\alpha \cdot \alpha \Rightarrow \beta \cdot \gamma$ *is provable,*
(2) $\alpha \Rightarrow \beta \cdot \gamma$ *is not always provable,*
(3) $\alpha \Rightarrow \beta \wedge \gamma$ *is provable.*

What are differences among them?

(1) $\alpha \cdot \alpha \Rightarrow \beta \cdot \gamma$ *is provable,*
(2) $\alpha \Rightarrow \beta \cdot \gamma$ *is not always provable,*
(3) $\alpha \Rightarrow \beta \wedge \gamma$ *is provable.*

(1) if one pays 1500 plus 1500 yen, i.e. 3000 yen, then one can have both a hardcover and a lunch.

(2) 1500 yen is not enough to have both of them.

(3) if one pays 1500 yen then one can get a hardcover and also can have lunch, "but not both".

Then, what is a difference between conjunction and disjunction?

Note

Substructural logics have been considered as resource-sensitive logics, i.e., logics sensitive to numbers and the order of assumptions.

This sounds reasonable. But, in what respects are Łukasiewicz' many-valued logics resource-sensitive?

Appendix: Hilbert-style system for $\mathbf{FL_{ew}}$

It has *modus ponens* with the following axiom schemata.

- $\alpha \to (\beta \to \alpha)$,
- $(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$,
- $0 \to \alpha$ and $(\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$,
- $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma)$,
- $\alpha \to (\alpha \vee \beta)$ and $\beta \to (\alpha \vee \beta)$,
- $((\gamma \to \alpha) \wedge (\gamma \to \beta)) \to (\gamma \to (\alpha \wedge \beta))$,
- $(\alpha \wedge \beta) \to \alpha$ and $(\alpha \wedge \beta) \to \beta$,
- $\alpha \to (\beta \to (\alpha \wedge \beta))$,
- $\alpha \to (\beta \to (\alpha \cdot \beta))$,
- $(\alpha \to (\beta \to \gamma)) \to ((\alpha \cdot \beta) \to \gamma)$.

- $(\alpha \to (\alpha \to \gamma)) \to (\alpha \to \gamma)$

Appendix: Natural deduction system

Additive connectives vs Multiplicative connectives

$$\dfrac{\dfrac{\underline{\Sigma}}{\vdots}\ \alpha \qquad \dfrac{\underline{\Sigma}}{\vdots}\ \beta}{\alpha \wedge \beta} \qquad\qquad \dfrac{\dfrac{\underline{\Gamma}}{\vdots}\ \alpha \qquad \dfrac{\underline{\Delta}}{\vdots}\ \beta}{\alpha \cdot \beta}$$

$$\Sigma \Rightarrow \alpha \wedge \beta \qquad\qquad \Gamma, \Delta \Rightarrow \alpha \cdot \beta$$