Substructural Logics - Proof Theory

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Calcutta Logic Circle, September 2009

Cut elimination

Cut elimination is one of most important tools in proof-theoretic approach. Cut elimination for a sequent system **L** means:

> If a sequent is provable in **L** then it is also provable in **L** without using cut rule.

Though cut elimination holds only for a limited number of sequent systems, it holds for most of sequent systems for basic substructural logics discussed so far.

While cut-free proofs may be "longer" than proofs with cut, they have "good" properties like subformula property, i.e.

Any cut-free proof of a given sequent $\Gamma \Rightarrow \theta$ contains only sequents that consist of subformulas of some formulas in $\Gamma \Rightarrow \theta$.

Thus, we can infer many of important logical consequences by analyzing structures of cut-free proofs — Proof Theory

Basic ideas of cut elimination

How cuts can be eliminated (e.g. in **FL**)?

Using double induction on:

- the rank: the total number of sequents over the cut,
- the grade: the number of logical connectives in the cut formula.

It consists of two kinds of procedures. By applying them repeatedly, we can eventually reduce the cut to the one, at least one of whose upper sequents is an initial sequent. Such a cut can be easily eliminable.

I. Decrease the rank

1. Pushing a cut up:

$$\dfrac{\dfrac{\beta, \Gamma \Rightarrow \alpha}{\beta \wedge \gamma, \Gamma \Rightarrow \alpha} \quad \alpha, \Delta \Rightarrow \delta}{\beta \wedge \gamma, \Gamma, \Delta \Rightarrow \delta} \ (cut)$$

This can be replaced by the following (of a smaller rank).

$$\dfrac{\dfrac{\beta, \Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \delta}{\beta, \Gamma, \Delta \Rightarrow \delta} \ (cut)}{\beta \wedge \gamma, \Gamma, \Delta \Rightarrow \delta}$$

II. Decrease the grade

2. Replacing the cut of a smaller grade:

$$\dfrac{\dfrac{\Gamma \Rightarrow \beta \quad \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \beta \wedge \gamma} \qquad \dfrac{\beta, \Delta \Rightarrow \delta}{\beta \wedge \gamma, \Delta \Rightarrow \delta}}{\Gamma, \Delta \Rightarrow \delta} \; (cut)$$

This can be replaced by the following.

$$\dfrac{\Gamma \Rightarrow \beta \quad \beta, \Delta \Rightarrow \delta}{\Gamma, \Delta \Rightarrow \delta} \; (cut)$$

Cut and contraction rule

The presence of contraction rule causes some difficulties in pushing up a cut. To avoid them, Gentzen introduced a generalized form of cut rule, called mix rule and replaced cuts by mixes. Then he showed mix elimination theorem.

This replacement makes the definition of the rank much more complicated.

For further discussions, see: H. O., Proof-theoretic methods in nonclassical logics – an introduction, 1998

Consequences of cut elimination

Cut elimination holds for $\mathbf{FL}$, $\mathbf{FL_e}$, $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$ and $\mathbf{FL_{ec}}$.

But it doesn't hold for $\mathbf{FL_c}$.

1. Decidability

All of these substructural logics are decidable.

2. Craig's interpolation theorem – Maehara's method

If $\alpha \rightarrow \beta$ is provable then there exists a formula γ such that a) both $\alpha \rightarrow \gamma$ and $\gamma \rightarrow \beta$ are provable, and b) $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$.

3. Maksimova's principle of variable separation

4. Variable sharing property for logics without weakening.

5. Disjunction property for logics without right contraction rule

If $\alpha \vee \beta$ is provable then either α or β is provable.

Decision procedure: the details

For a given sequent $\Gamma \Rightarrow \alpha$, we will search for a cut-free proof of it by the decompositions, i.e. applications of rules of **FL** in the reverse direction.

- Sometimes backtracking process becomes necesssary in searching proofs, as there are several choices of applicable rules.
- Since decomposed sequents are always simpler than the original one, every decomposition will eventually terminate.
- When no further decompositions are applicable, check whether each sequent at the top is an initial one or not.

This proof-search algorithm terminates eventually, since there are only finite number of possible decompositions.

- If all of them are initial sequents then this gives us a required cut-free proof of $\Gamma \Rightarrow \alpha$.
- On the other hand, if every such trial fails, then the sequent is not provable.

Thus

FL is decidable.

Similar arguments work also for (even predicate) $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$. Thus, their predicate extensions (with function symbols) are decidable.

On the other hand, the presence of contraction rule causes some difficulties in searching proofs, since upper sequents are not always simpler than the lower one.

- For **LJ**, these difficulties are avoided by considering only reduced sequents (by G. Gentzen).
- For $\mathbf{FL_{ec}}$ we need Curry's lemma and Kripke's lemma to overcome difficulties. In fact, this decision procedure is of high computational complexity.
- Moreover, the combination of contraction rule with distributive law is even worse. In fact, it gives us the undecidability (by A. Urquhart).

Deducibility

For a set of formulas Σ and a formula α, α is deducible from Σ in **FL** ($\Sigma \vdash_{\mathbf{FL}} \alpha$) iff:

the sequent $\Rightarrow \alpha$ is provable in **FL** when adding sequents $\Rightarrow \gamma$ (for each $\gamma \in \Sigma$) as extra initial sequents.

Obviously, the provability of a formula α is equivalent to its deducibility from the empty set. i.e.

$$\vdash_{\mathbf{Int}} \alpha \quad \text{iff} \quad \emptyset \vdash_{\mathbf{Int}} \alpha.$$

But the deducibility is different from the provability. For example, while $\alpha \Rightarrow \alpha^2$ is not provable in **FL**, $\alpha \vdash_{\mathbf{FL}} \alpha^2$ holds as;

$$\frac{\Rightarrow \alpha \quad \Rightarrow \alpha}{\Rightarrow \alpha \cdot \alpha} \ (\Rightarrow \cdot)$$

Can the deducibility relation be reduced to the provability?

Yes, for both classical and intuitionistic logics. In fact, the following deduction theorem (DT) holds for them:

$$\Sigma, \alpha \vdash \beta \text{ iff } \Sigma \vdash (\alpha \to \beta).$$

By this, the decidability of the deducibility in classical and intuitionistic logics follows from that of the provability.

Parameterized local DT

This is not always the case. Still, the following parameterized local deduction theorem (PLDT) holds for **FL**. (cf. Czelakowski-Dziobiak)

$\Sigma, \alpha \vdash_{\mathbf{FL}} \beta$ iff there exist iterated conjugates δ_i of α ($i \leq m$ for some m) such that $\Sigma \vdash_{\mathbf{FL}} (\prod \delta_i)\backslash\beta$.

Here, each iterated conjugate of α is obtained from α by applying the left-conjugate $\lambda_\theta(\alpha) = (\theta\backslash\alpha\theta) \wedge 1$ and/or the right-conjugate $\rho_\phi(\alpha) = (\phi\alpha/\phi) \wedge 1$ with some parameters $\theta, \phi, \ldots$, repeatedly.

The proof goes similarly to that of DT for **LJ**. Though **FL** has no structural rules, we can simulate both weakening and exchange rules (but not contraction).

- if $\Gamma, \Delta \Rightarrow \theta$ is provable then $\Gamma, \psi \wedge 1, \Delta \Rightarrow \theta$ is provable,
- if $\Gamma, \alpha, \beta, \Delta \Rightarrow \theta$ is provable then both $\Gamma, \beta, \lambda_\beta(\alpha), \Delta \Rightarrow \theta$ and $\Gamma, \rho_\alpha(\beta), \alpha, \Delta \Rightarrow \theta$ are provable.

Note that

$$\beta(\lambda_\beta(\alpha)) \Leftrightarrow \beta((\beta\backslash\alpha\beta) \wedge 1) \Rightarrow \beta(\beta\backslash\alpha\beta) \Rightarrow \alpha\beta.$$

Local deduction theorem

In a system with exchange rule, conjugates are not necessary. Thus, PLDT can be simplified into the following local deduction theorem.

$\Sigma, \alpha \vdash_{\mathbf{FL_e}} \beta$ iff $\Sigma \vdash_{\mathbf{FL_e}} (\alpha \wedge 1)^m \to \beta$ for some m.

It is still local, as we cannot always determine such an m from given Σ, α, β. In fact,

- The provability problem in $\mathbf{FL_e}$ is decidable. (by cut elimination)
- The deducibility problem in $\mathbf{FL_e}$ is undecidable (essentially by Lincoln, Mitchell, Scedrov & Shankar).

Appendix: An alternative definition of substructural logics

By identifying a logic with a set of formulas provable in it, we can define a substructural logics over **FL** as follows:

A set of formula Σ is deductively closed w.r. to $\vdash_{\mathbf{FL}}$, if $\Delta \vdash_{\mathbf{FL}} \beta$ for a subset Δ of Σ then $\beta \in \Sigma$.

A set of formulas **L** is a substructural logic iff

- it is deductively closed w.r. to $\vdash_{\mathbf{FL}}$,
- it is closed under substitution.

As a consequence of our PLDT, this is restated alternatively as follows:

A set of formulas **L** is a substructural logic iff

- every formula provable in **FL** belongs to **L**,
- if φ and $\varphi\backslash\psi$ are in **L**, then $\psi \in \mathbf{L}$,
- if $\varphi \in \mathbf{L}$ then $\varphi \wedge 1 \in \mathbf{L}$,
- if $\varphi \in \mathbf{L}$ and γ is a formula, then both $\gamma\backslash\varphi\gamma$ and $\gamma\varphi/\gamma$ are in **L**,
- it is closed under substitution.

Appendix: Craig's Interpolation Property

A logic **L** has the Craig's interpolation property (CIP), if for all formulas φ, ψ such that $\varphi \to \psi$ is provable in **L**, there exists a formula α such that

- both $\varphi \to \alpha$ and $\alpha \to \psi$ are provable in **L**,
- $Var(\alpha) \subseteq Var(\varphi) \cap Var(\psi)$.

Note that when **L** is noncommutative, we need to replace $\to$ by $\backslash$.

Maehara's method

S. Maehara gives a way of showing CIP as a consequence of cut elimination. Here is an outline of the method e.g. for $\mathbf{FL_{ew}}$. We show the CIP of the following form.

If $\Gamma \Rightarrow \psi$ is provable in $\mathbf{FL_{ew}}$, then there exists a formula α, called an interpolant, such that

- both $\Gamma \Rightarrow \alpha$ and $\alpha \Rightarrow \psi$ are provable in $\mathbf{FL_{ew}}$,
- $Var(\alpha) \subseteq Var(\Gamma) \cap Var(\psi)$.

By cut elimination for $\mathbf{FL_{ew}}$, there is a cut-free proof Π of $\Gamma \Rightarrow \psi$.

Take an arbitrary sequent $\Delta \Rightarrow \beta$ in Π, and let $\langle \Delta_1, \Delta_2 \rangle$ be an arbitrary partition of Δ (i.e. the multiset union of Δ_1 and Δ_2 is equal to Δ). Then, we show the following by induction on the length of a proof of $\Delta \Rightarrow \beta$ in Π.

There exists a formula γ such that

- both $\Delta_1 \Rightarrow \gamma$ and $\gamma, \Delta_2 \Rightarrow \beta$ are provable in $\mathbf{FL_{ew}}$,
- $Var(\gamma) \subseteq Var(\Delta_1) \cap (Var(\Delta_2) \cup Var(\beta))$.

Then the CIP follows immediately. Maehara's method gives us an interpolant in a constructive way, when a cut-free proof is given. (CIP holds for $\mathbf{FL}$, $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$ and $\mathbf{FL_{ec}}$).

Deductive interpolation property

A substructural logic **L** has the strong deductive interpolation property (strong DIP), if for every set of formulas $\Gamma \cup \Sigma \cup \{\varphi\}$ such that $\Gamma, \Sigma \vdash_L \varphi$, there exists a set of formulas Δ such that

- $\Gamma \vdash_L \psi$ for all $\psi \in \Delta$ and $\Delta, \Sigma \vdash_L \varphi$,
- $Var(\Delta) \subseteq Var(\Gamma) \cap Var(\Sigma \cup \{\varphi\})$.

When Σ is empty, it is called the DIP.

For each logic over $\mathbf{FL_e}$, CIP implies DIP, and DIP is equivalent to SDIP.