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Substructural Logics and Residuated Lattices — an Introduction

Abstract. This is an introductory survey of substructural logics and of residuated lattices which are algebraic structures for substructural logics. Our survey starts from sequent systems for basic substructural logics and develops the proof theory of them. Then, residuated lattices are introduced as algebraic structures for substructural logics, and some recent developments of their algebraic study are presented. Based on these facts, we conclude at the end that substructural logics are logics of residuated structures, and in this way explain why sequent systems are suitable for formalizing substructural logics.

Keywords: substructural logics, sequent systems, residuated structures, varieties of residuated lattices

1. Introduction

Substructural logics are logics lacking some or all of the structural rules when they are formalized in sequent systems. They cover many of the well-known nonclassical logics, e.g. Lambek calculus for categorial grammar (with no structural rules), linear logic (with only the exchange rule), BCK-logic and Łukasiewicz's many-valued logics (lacking the contraction rule), and relevant logics (lacking the weakening rule). The purpose of the study of substructural logics is to introduce a uniform framework in which various kinds of nonclassical logics that originated from different motivations can be discussed together, and to find common features among them, taking structural rules for a clue.

This is a brief introductory survey of substructural logics and residuated lattices. Residuated lattices are structures that have been studied by algebraists since the 1930s, but the study has been revived recently as a study of mathematical structures for substructural logics. In the present paper, we will mainly concentrate on examining basic ideas of substructural logics in relation to residuated lattices, and will clarify what substructural logics are and why formalizations in sequent systems are essential. The paper is far from a comprehensive survey and will not touch on technical details in

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most cases. An algebraic study of substructural logics in the present paper will be developed in depth and in details in the book [18] in preparation by the author with P. Jipsen and T. Kowalski. For general information on substructural logics see [10], which is the first book on this topic, and also [38, 39].

Since basic substructural logics are formalized as sequent systems, our survey starts from these sequent systems and their proof theory. Then we will discuss the algebraic study of substructural logics based on residuated lattices, which has made rapid progress in recent years. It turns out that universal algebra offers useful and powerful tools in developing the study. These algebraic apsects of substructural logics give us a wider and deeper understanding of them.

For some major reference books about algebraic studies of particular logics in the scope of the present paper, see for instance, [41] for linear logic, [15] for fuzzy logics, [9] for many-valued logics, and [1] and [2] for relevant logics.

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2. Substructural Logics

In this section we introduce "basic" substructural logics as sequent systems. They are obtained from the sequent system **LK** for classical logic or **LJ** for intuitionistic logic by restricting their structural rules. In particular the roles of the structural rules will be examined.

2.1. Gentzen's sequent systems and structural rules

We begin with some explanations of sequent systems **LK** and **LJ** for classical propositional logic and intuitionistic logic, respectively, which were introduced by G. Gentzen [12] in the middle of the 1930s.

Here we consider the language $\mathcal{L}$ of **LK** and **LJ** which consists of logical connectives $\wedge, \vee, \rightarrow$ and $\neg$. A *sequent* of **LK** is an expression of the form $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$, with $m, n \geq 0$, whose intuitive meaning is that "$\beta_1 \vee \ldots \vee \beta_n$ *follows from assumptions* $\alpha_1, \ldots, \alpha_m$". In this sequent, $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_n$ are called the *antecedent* and the *succedent*, respcetively. In the following, Greek capital letters Σ, Λ, Γ etc. denote (finite, possibly empty) sequences of formulas. Initial sequents of **LK** are sequents of the form $\alpha \Rightarrow \alpha$. The rules of inference of **LK** given below can be divided into three categories, i.e. structural rules, cut rule and rules for logical connectives.

Structural rules: These rules determine the meaning of *commas* in sequents. We will discuss this in detail later.

Weakening rules:

$$\frac{\Gamma, \Sigma \Rightarrow \Delta}{\Gamma, \alpha, \Sigma \Rightarrow \Delta}\ (w \Rightarrow) \qquad\qquad \frac{\Gamma \Rightarrow \Lambda, \Theta}{\Gamma \Rightarrow \Lambda, \alpha, \Theta}\ (\Rightarrow w)$$

Contraction rules:

$$\frac{\Gamma, \alpha, \alpha, \Sigma \Rightarrow \Delta}{\Gamma, \alpha, \Sigma \Rightarrow \Delta}\ (c \Rightarrow) \qquad\qquad \frac{\Gamma \Rightarrow \Lambda, \alpha, \alpha, \Theta}{\Gamma \Rightarrow \Lambda, \alpha, \Theta}\ (\Rightarrow c)$$

Exchange rules:

$$\frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \Delta}{\Gamma, \beta, \alpha, \Sigma \Rightarrow \Delta}\ (e \Rightarrow) \qquad\qquad \frac{\Gamma \Rightarrow \Lambda, \alpha, \beta, \Theta}{\Gamma \Rightarrow \Lambda, \beta, \alpha, \Theta}\ (\Rightarrow e).$$

Cut rule: Usually the cut rule is regarded as one of the structural rules. But, for convenience's sake, we separate the cut rule here from the other structural rules.

$$\frac{\Gamma \Rightarrow \alpha, \Theta \quad \Sigma, \alpha, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \Pi \Rightarrow \Delta, \Theta}$$

Rules for logical connectives: Similarly to the structural rules, there exist right and left rules for each logical connective. Taken together, they describe the role of each connective.

$$\frac{\Gamma \Rightarrow \alpha, \Theta \quad \Pi, \beta, \Sigma \Rightarrow \Delta}{\Pi, \alpha \to \beta, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\ (\to\Rightarrow) \qquad\qquad \frac{\Gamma, \alpha \Rightarrow \beta, \Theta}{\Gamma \Rightarrow \alpha \to \beta, \Theta}\ (\Rightarrow\to)$$

$$\frac{\Gamma, \alpha, \Sigma \Rightarrow \Delta}{\Gamma, \alpha \wedge \beta, \Sigma \Rightarrow \Delta}\ (\wedge 1 \Rightarrow) \qquad\qquad \frac{\Gamma, \beta, \Sigma \Rightarrow \Delta}{\Gamma, \alpha \wedge \beta, \Sigma \Rightarrow \Delta}\ (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Lambda, \alpha, \Theta \quad \Gamma \Rightarrow \Lambda, \beta, \Theta}{\Gamma \Rightarrow \Lambda, \alpha \wedge \beta, \Theta}\ (\Rightarrow \wedge)$$

$$\frac{\Gamma, \alpha, \Sigma \Rightarrow \Delta \quad \Gamma, \beta, \Sigma \Rightarrow \Delta}{\Gamma, \alpha \vee \beta, \Sigma \Rightarrow \Delta} \ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Lambda, \alpha, \Theta}{\Gamma \Rightarrow \Lambda, \alpha \vee \beta, \Theta} \ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \Lambda, \beta, \Theta}{\Gamma \Rightarrow \Lambda, \alpha \vee \beta, \Theta} \ (\Rightarrow \vee 2)$$

$$\frac{\Gamma \Rightarrow \alpha, \Theta}{\neg\alpha, \Gamma \Rightarrow \Theta} \ (\neg \Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow \Theta}{\Gamma \Rightarrow \neg\alpha, \Theta} \ (\Rightarrow \neg)$$

Proofs and the provability of formulas in **LK** are defined in the usual way. In standard Hilbert-style formulations, "implication" plays usually a special role, different from other logical connectives. This comes from the fact that *modus ponens* is a single rule of inferences in the standard formulation, which is of the form "*from* α *and* $\alpha \to \beta$, *infer* β". On the other hand, in sequent systems as above, none of the logical connectives have particular roles. We can see indeed that for a given logical connective $\sharp$, rules for $\sharp$ are expressed by using only $\sharp$, without using other connectives. Such a formulation becomes possible only with the help of two metalogical symbols, arrow $\Rightarrow$ and comma. Roles of comma in relation to the arrow are described in the form of structural rules. Since rules for logical connectives are described separately from each other, we can get important syntactic results such as cut elimination theorems, the subformula property and so on.

Sequents of the sequent system **LJ** for intuitionistic logic are expressions of the form $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$, where $m \geq 0$ and β may be empty. Initial sequents and rules of inference of **LJ** are obtained from those of **LK** given above, by deleting first both $(\Rightarrow c)$ and $(\Rightarrow e)$, and then assuming that both Λ and Θ are empty and that Δ consists of at most one formula.

2.2. Roles of the structural rules

To understand the roles of the structural rules, we will give here an example of a proof of the distributive law in **LJ**, in which both contraction and weakening rules are used in an essential way:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha, \beta \Rightarrow \alpha}\;(w \Rightarrow) \quad \dfrac{\beta \Rightarrow \beta}{\alpha, \beta \Rightarrow \beta}\;(w \Rightarrow)}{\alpha, \beta \Rightarrow \alpha \wedge \beta}}{\alpha, \beta \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)} \quad \dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha, \gamma \Rightarrow \alpha}\;(w \Rightarrow) \quad \dfrac{\gamma \Rightarrow \gamma}{\alpha, \gamma \Rightarrow \gamma}\;(w \Rightarrow)}{\alpha, \gamma \Rightarrow \alpha \wedge \gamma}}{\alpha, \gamma \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}}{\alpha, \beta \vee \gamma \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}}{\alpha \wedge (\beta \vee \gamma), \beta \vee \gamma \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}}{\alpha \wedge (\beta \vee \gamma), \alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}}{\alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}\;(c \Rightarrow)
$$

A careful inspection of some parts of the above proof tells us that any sequent of the form $\delta, \varphi \Rightarrow \delta \wedge \varphi$ can be derived by using the weakening rule (see the upper parts), and also that any sequent of the form $\delta \wedge \varphi \Rightarrow \psi$ can be derived from a sequent $\delta, \varphi \Rightarrow \psi$ by using the contraction rule (see the lower four lines, replacing α by δ, $\beta \vee \gamma$ by φ and $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ by ψ). Therefore, in both **LK** and **LJ**, a sequent $\delta, \varphi \Rightarrow \psi$ is provable if and only if $\delta \wedge \varphi \Rightarrow \psi$ is provable. In fact, the if-part is obtained in the following way:

$$
\dfrac{\delta, \varphi \Rightarrow \delta \wedge \varphi \quad \delta \wedge \varphi \Rightarrow \psi}{\delta, \varphi \Rightarrow \psi}\;(cut).
$$

By generalizing this argument (and considering the dual argument between disjunctions and commas in the right-hand side of sequents, in case of **LK**), we have the following proposition. It says that in both **LK** and **LJ**, where we have both weakening and contraction rules, *commas in the left-hand side of a sequent mean conjunctions*, and moreover in **LK** *commas in the right-hand side mean disjunctions.*

PROPOSITION 2.1. *A sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is provable in* **LK** *if and only if the sequent $\alpha_1 \wedge \ldots \wedge \alpha_m \Rightarrow \beta_1 \vee \ldots \vee \beta_n$ is provable in it. This holds also for* **LJ** *if $n \leq 1$.*

Then, the following questions will come naturally to mind:

- What do commas mean in a sequent system lacking either or both of the weakening and contraction rules?
- In general, how and in which respects does the existence of structural rules affect logical properties?

To answer these questions, we need to examine the roles of each of the left structural rules again.

1) Exchange rule ($e \Rightarrow$):

$$\frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \Delta}{\Gamma, \beta, \alpha, \Sigma \Rightarrow \Delta}$$

The exchange rule ($e \Rightarrow$) allows us to use assumptions, i.e. formulas in the left-hand side, in an arbitrary order.

2) Weakening rule ($w \Rightarrow$):

$$\frac{\Gamma, \Sigma \Rightarrow \Delta}{\Gamma, \alpha, \Sigma \Rightarrow \Delta}$$

The weakening rule ($w \Rightarrow$) allows us to add any redundant assumption. In other words, when a sequent $\Pi \Rightarrow \Theta$ is proved in a system that has no weakening rule, every assumption (i.e. every formula in Π) must be used *at least once* in a proof of $\Pi \Rightarrow \Theta$.

3) Contraction rule ($w \Rightarrow$):

$$\frac{\Gamma, \alpha, \alpha, \Sigma \Rightarrow \Delta}{\Gamma, \alpha, \Sigma \Rightarrow \Delta}$$

The contraction rule allows us to use each assumption more than once. Thus, when a sequent is proved in a system lacking the contraction rule, each of its assumptions is used *at most once* in its proof.

Therefore, in a sequent system with all of these structural rules, if a given sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable, it means that β can be derived from $\alpha_1, \ldots, \alpha_m$ by using them in arbitrary order and an arbitrary number of times including none.

Roughly speaking, *substructural logics* are logics lacking some or all of these structural rules, when they are formulated as sequent systems. The above explanation suggests that they are logics *sensitive* to the number and order of occurrences of assumptions. By this reason, they are sometimes called *resource-sensitive logics*. In particular, when a sequent system under consideration has the exchange rule, but neither the weakening nor the contraction rule, every assumption must be used once and only once to derive a conclusion. This is a basic idea of *linear logic* (without exponentials), introduced by J.-Y. Girard, which is obtained from **LK** by deleting both weakening and contraction rules. Also, in relevant logics, where weakening rules are not allowed, every assumption must be used at least once in derivations, and thus no redundant assumptions, i.e. no *irrelevant* assumptions, are used.

2.3. Comma, fusion and implication

The above argument suggests that commas on the left-hand side of sequents do not behave like conjunctions when some of structural rules are lacking, while they are identified with conjunctions in **LK** and **LJ** as shown in Proposition 2.1. Then, what does each comma mean in such a situation? To see this in a more explicit way, let us introduce a new logical connective $*$ (or $\otimes$) which *represents* a comma in substructural logics. This connective $*$ is sometimes called the *fusion* or the *multiplicative conjunction*, while the usual conjunction is called the *additive* conjunction, to distinguish it from the multiplicative one.[1] We assume the following rules for $*$.

$$\frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \Delta}{\Gamma, \alpha * \beta, \Sigma \Rightarrow \Delta}\ (* \Rightarrow) \qquad\qquad \frac{\Gamma \Rightarrow \alpha, \Lambda \quad \Sigma \Rightarrow \beta, \Theta}{\Gamma, \Sigma \Rightarrow \alpha * \beta, \Lambda, \Theta}\ (\Rightarrow *)$$

Then, as one might expect, the following holds.

PROPOSITION 2.2. *In the sequent system having only rules for $*$ and cut rule, a sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable if and only if $\alpha_1 * \ldots * \alpha_m \Rightarrow \beta$ is provable.*

We show moreover an important relationship between fusion and implication.

LEMMA 2.3. *A sequent $\alpha * \beta \Rightarrow \gamma$ is provable if and only if $\alpha \Rightarrow \beta \to \gamma$ is provable.*

Note that to show this, it is not necessary to use any structural rule except the cut rule. Suppose that $\alpha * \beta \Rightarrow \gamma$ is provable. Then $\alpha, \beta \Rightarrow \gamma$ is also provable by Proposition 2.2. Using $(\Rightarrow\to)$ we have that $\alpha \Rightarrow \beta \to \gamma$ is provable. Conversely, suppose that $\alpha \Rightarrow \beta \to \gamma$ is provable. Then $\alpha, \beta \Rightarrow \gamma$ is provable as the following proof shows, and hence $\alpha * \beta \Rightarrow \gamma$ is provable.

[1] Similarly, commas in the right-hand side of sequents when they are allowed, will be denoted by $+$ and called *multiplicative disjunction*. We will consider in this subsection only sequents with a single formula in the conclusion, and thus consider only $*$. This is partly for the sake of simplicity, but also because we don't know exactly how to attach adequate and comprehensible meaning to $+$.

$$\dfrac{\alpha \Rightarrow \beta \to \gamma \quad \dfrac{\beta \Rightarrow \beta \quad \gamma \Rightarrow \gamma}{\beta \to \gamma, \beta \Rightarrow \gamma}}{\alpha, \beta \Rightarrow \gamma}\ (cut)$$

Lemma 2.3 with Proposition 2.2 says that deletion or addition of structural rules has a significant effect on the "meaning" of implication. In algebraic terms, what is shown in the above lemma means that implication is the *residual* of fusion, or $*$ and $\to$ form a *residuated pair*. This fact will be the basis of our algebraic study of substructural logics in Sections 4 and 5.

2.4. Propositional constants

Sometimes it is convenient to add propositional constants when introducing formal systems. For instance, we use propositional constants $\top$ and $\bot$ to denote the constantly *true* and *false* propositions, respectively. When we introduce them, we need to add the following initial sequents for $\top$ and $\bot$:

1. $\Gamma \Rightarrow \top$,
2. $\Gamma, \bot, \Sigma \Rightarrow \Delta$.

Here, Γ, Σ and Δ may be empty. When a system under consideration has weakening rules, they can be replaced by weaker initial sequents $\Rightarrow \top$ and $\bot \Rightarrow$, respectively. On the other hand, if a system doesn't have them, constants defined by these weaker initial sequents behave in a different way. Now let us introduce additional new propositional constants, denoted by 1 and 0. We assume the following initial sequents and rules of inference for them:

3. $\Rightarrow 1$,
4. $0 \Rightarrow$,

$$\frac{\Gamma, \Sigma \Rightarrow \Delta}{\Gamma, 1, \Sigma \Rightarrow \Delta}\ (1w) \qquad\qquad \frac{\Gamma \Rightarrow \Lambda, \Theta}{\Gamma \Rightarrow \Lambda, 0, \Theta}\ (0w)$$

Intuitively, constants 1 and 0 denote the "empty sequence of formulas" in the left-hand side of an arrow and in the right-hand side, respectively. Also we can see that 1 (0) is the *weakest* (*strongest*) proposition among provable formulas (contradictory formulas, respectively). Here, by a contradictory formula, we mean a formula α such that $\alpha \Rightarrow$ is provable. When we have the constant 0, we may define the negation $\neg\alpha$ of a formula α by $\alpha \to 0$, and can dispense with rules for $\neg$. Note here that $\top$ (0) is logically equivalent to $\neg \bot$ ($\neg 1$, respectively). These four propositional constants are used in the standard formulation of linear logic and relevant logics.[2] Using weakening rules, we can show that $\top$ ($\bot$) is logically equivalent to 1 (0, respectively). Conversely, if $\top$ is equal to 1, then by using the initial sequent 1, the rule $(1w)$ and the cut rule, we can derive the weakening rule $(w \Rightarrow)$.

2.5. Basic substructural logics

We introduce here sequent systems for several basic substructural logics. They are obtained from either **LK** or **LJ** by deleting some or all of the structural rules.

Let **FL** be the sequent system obtained from **LJ** by deleting all of the structural rules and then adding rules for $*$. The name **FL** comes from *full Lambek calculus*, which is an extension of the sequent system for categorial grammar, introduced by J. Lambek [26]. Here, we need a comment regarding the definition of implication. Since we don't assume exchange rule in **FL**, it is more natural to introduce two kinds of implication / and \, which are called in a more suitable way, left and right *residuals*, respectively. More precisely, rules for left residuals / and right residuals \ are given as follows.

$$\frac{\Gamma \Rightarrow \alpha \quad \Pi, \beta, \Sigma \Rightarrow \delta}{\Pi, \beta/\alpha, \Gamma, \Sigma \Rightarrow \delta}\ (/ \Rightarrow) \qquad\qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (\Rightarrow /)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Pi, \beta, \Sigma \Rightarrow \delta}{\Pi, \Gamma, \alpha\backslash\beta, \Sigma \Rightarrow \delta}\ (\backslash \Rightarrow) \qquad\qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\Rightarrow /)$$

It is obvious that if a system has exchange rule then we can show that β/α and $\alpha\backslash\beta$ are equivalent. In such a case, we denote it as $\alpha \to \beta$. Corresponding to Lemma 2.3, we have the following.

LEMMA 2.4. *In* **FL**, *the following three conditions are mutually equivalent. For all formulas* α, β *and* γ,
1. $\alpha * \beta \Rightarrow \gamma$ *is provable,*
2. $\alpha \Rightarrow \gamma/\beta$ *is provable,*
3. $\beta \Rightarrow \alpha\backslash\gamma$ *is provable.*

It is natural to introduce two kinds of negations $0/\alpha$ and $\alpha\backslash 0$ of a formula α in **FL**, whose rules are given as follows:

[2] The reader should be cautious of the usage of symbols for propositional constants in the literature, as the same symbols sometimes denote different constants in other papers.

$$\frac{\Gamma \Rightarrow \alpha}{0/\alpha, \Gamma \Rightarrow}\ (\neg 1 \Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow}{\Gamma \Rightarrow 0/\alpha}\ (\Rightarrow \neg 1)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma, \alpha\backslash 0 \Rightarrow}\ (\neg 2 \Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \alpha\backslash 0}\ (\Rightarrow \neg 2)$$

We introduce several sequent systems for substructural logics that are extensions of **FL**. Let e, w and c denote the exchange rule $(e \Rightarrow)$, the weakening rule $(w \Rightarrow)$ and the contraction rule $(c \Rightarrow)$, respectively. We denote sequent systems obtained from **FL** by adding some of these *left* structural rules, by attaching corresponding subscripts e, w, c to **FL**. For example, $\mathbf{FL_e}$ is **FL** with the exchange rule, which is equal to intuitionistic (multiplicative, additive) linear logic, and $\mathbf{FL_{ew}}$ is **FL** with both exchange and weakening rules. Extensions of $\mathbf{FL_{ew}}$ are discussed extensively in [37, 22] and [36]. See also [16], in which $\mathbf{FL_{ew}}$ is called the monoidal logic. Important classes of extensions of $\mathbf{FL_{ew}}$ are Hájek's fuzzy logics and Łukasiewicz's many-valued logics as discussed in §5. The following is easily shown.

LEMMA 2.5. *For all formulas α and β, the sequent $\alpha * \beta \Rightarrow \alpha \wedge \beta$ is provable in $\mathbf{FL_w}$, and also the sequent $\alpha \wedge \beta \Rightarrow \alpha * \beta$ is provable in $\mathbf{FL_c}$.*

Therefore, in $\mathbf{FL_{cw}}$ $\alpha * \beta$ and $\alpha \wedge \beta$ are equivalent. Since $\alpha \wedge \beta \Rightarrow \beta \wedge \alpha$ is provable, $\alpha * \beta \Rightarrow \beta * \alpha$ is also provable, from which it follows that the exchange rule can be derived in $\mathbf{FL_{cw}}$. Thus, $\mathbf{FL_{cw}}$ is a sequent system for intuitionistic logic.

In the same way as above, we can introduce some of sequent systems obtained from **LK** by restricting the structural rules. When a system lacks the exchange rules, there are many possibilities of introducing rules for fusions and implications, but we have no reasonable criteria of making a choice of proper ones. Thus, we consider in the following only sequent systems with exchange rules. Let $\mathbf{CFL_e}$ be the sequent system obtained from **LK** by deleting both weakening and contraction rules. It has both left and right exchange rules. (The letter C of $\mathbf{CFL_e}$ comes from "classical type".) It is essentially equivalent to the multiplicative, additive linear logic **MALL** introduced by Girard [13]. The sequent systems $\mathbf{CFL_{ew}}$ and $\mathbf{CFL_{ec}}$ are obtained from $\mathbf{CFL_e}$ by adding (left and right) weakening rules and contraction rules, respectively.

In the following, we often identify a sequent system with the *logic* determined by it, i.e. the set of all formulas provable in it. Here, we say that a formula α is provable in a sequent system **L** if the sequent $\Rightarrow \alpha$ is provable in it. Also, we call those logics determined by various sequent systems introduced in this subsection, including **LK** and **LJ**, *basic substructural logics.*[3] The system $\mathbf{CFL_{ew}}$ and a set theory based on it were already studied by V. Grišin from the mid 1970s (see e.g. [14]). Also the system $\mathbf{CFL_{ec}}$ is essentially the same as the system **LR** studied by [27], which is known as the relevant logic **R** without the distributive law. It is easy to see the following.

LEMMA 2.6. *The system* $\mathbf{CFL_{e}}$ *is equivalent to the system* $\mathbf{FL_{e}}$ *with initial sequents of the form* $\neg\neg\alpha \Rightarrow \alpha$. *Precisely speaking, for each formula* β, β *is provable in* $\mathbf{CFL_{e}}$ *if and only if it is provable in* $\mathbf{FL_{e}}$ *by using any sequent of the form* $\neg\neg\alpha \Rightarrow \alpha$ *as additional initial sequents. The same relation holds between* $\mathbf{CFL_{ew}}$ *and* $\mathbf{FL_{ew}}$, $\mathbf{CFL_{ec}}$ *and* $\mathbf{FL_{ec}}$, *and* **LK** *and* **LJ**.

3. Proof theory of Substructual Logics

In this section, we will discuss the cut elimination theorem for substructural logics and its logical consequences, including decidability results. It will be clarified how weakening and contraction rules play key roles in them. Topics touched in this section are discussed in [33] in full details (see also [32] for more information on decision problems).

3.1. Cut elimination theorems

A proof containing no applications of the cut rule is called a *cut-free* proof. The cut elimination theorem for a given sequent system **L** says that any sequent which is provable in **L** always has a cut-free proof in **L**. To get a sequent system for which the cut elimination theorem holds is quite important, since many important results follow as its consequences. To be precise, from the cut elimination theorem the *subformula property* follows in most cases, which says that for any sequent $\Gamma \Rightarrow \Delta$ if it is provable then it has such a proof that every formula appearing in it is a subformula of some formula in $\Gamma \Rightarrow \Delta$. Then, from the subformula property important results like decidability and Craig's interpolation theorem follow. In other words, even if we have the cut elimination theorem for a given propositional sequent system **L**, **L** may be undecidable because of the lack of the subformula property.

[3]One of earliest attempts of considering all of these basic substructural logics is made in [30]. The nomenclature of basic substructural logics taken here is proposed in the paper, and then is modified into the present form in [31]. The name "substructural logics" was suggested by K. Došen at the first conference of the topics held at Tübingen in 1990 (see [10]).

Also it may happen that while the cut elimination theorem doesn't hold, the subformula property holds and therefore both decidability and Craig's interpolation theorem follow from it. An example of the latter is a sequent system for modal logic **S5** introduced by Ohnishi and Matsumoto. Though the cut elimination theorem doesn't hold for it, M. Takano proved that every application of cut rule can be restricted in such a way that the cut formula is a subformula of formulas in the lower sequent. Using this we can derive the subformula property. For details, see e.g. [33].

THEOREM 3.1. *The cut elimination theorem holds for all sequent systems of basic propositional logics except* $\mathbf{FL_c}$, *i.e. for* $\mathbf{FL}$, $\mathbf{FL_e}$, $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$, $\mathbf{FL_{ec}}$, $\mathbf{LJ}$, $\mathbf{CFL_e}$, $\mathbf{CFL_{ew}}$, $\mathbf{CFL_{ec}}$, *and* $\mathbf{LK}$.

The proof of the cut elimination theorem for these sequent systems goes essentially in the same way as Gentzen's original proof. But we make some remarks on how the presence or the lack of structural rules has an effect on the proof.

We start from Gentzen's original proof of the cut elimination theorem for **LJ**, which has all structural rules. The cut rule of **LJ** can be formulated in the following form:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Pi \Rightarrow \delta}{\Gamma, \Pi \Rightarrow \delta}$$

To prove the cut elimination theorem, we replace all occurrences of applications of the cut rule in a given proof by the following *mix rule*, which is a generalized form of the cut rule:

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma \Rightarrow \delta}{\Gamma, \Sigma_\alpha \Rightarrow \delta}$$

where Σ contains at least one occurrence of the formula α, and Σ_α denotes the sequence of formulas obtained from Σ by deleting *all* occurrences of α. It is easy to see that any application of the cut rule can be replaced by an application of the mix rule, and vice versa, with the help of structural rules. Therefore, the new sequent system, obtained from **LJ** by replacing the cut rule by the mix rule, is also a system for intuitionistic logic. Thus, it is enough to prove the mix elimination theorem (for the new system), instead of showing the cut elimination theorem for **LJ**. This is proved by using double induction on the *grade* of the mix formulas and the *rank* of a given proof.

But, why is it necessary to replace the cut rule by the mix rule? The reason is found in the presence of contraction rules. The basic idea of eliminating cut rules is either to *push up* each application of the cut rule by exchanging the order of applications of rules, or to replace cut formulas by simpler ones. This idea works well for all cases except the case shown below, where we want to push the cut rule up so that the left contraction rule will be applied after the cut rule:

$$\dfrac{\Gamma \Rightarrow \alpha \quad \dfrac{\alpha, \alpha, \Pi \Rightarrow \delta}{\alpha, \Pi \Rightarrow \delta}\,(c\Rightarrow)}{\Gamma, \Pi \Rightarrow \delta}\,(cut)$$

But this is impossible, since the lower application of the cut rule in the following proof is not *simpler* than the one in the above.

$$\dfrac{\dfrac{\Gamma \Rightarrow \alpha \quad \dfrac{\Gamma \Rightarrow \alpha \quad \alpha, \alpha, \Pi \Rightarrow \delta}{\alpha, \Gamma, \Pi \Rightarrow \delta}\,(cut)}{\Gamma, \Gamma, \Pi \Rightarrow \delta}\,(cut)}{\Gamma, \Pi \Rightarrow \delta}\,(e\Rightarrow)(c\Rightarrow)$$

To resolve this difficulty, we need to introduce the mix rule. This observation shows that when a system lacks the contraction rule, e.g. systems like $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$, our basic idea mentioned above works fully and hence we can prove the cut elimination theorem directly. (As a matter of fact, in such a system the cut rule cannot be replaced by the mix rule, because of the lack of contraction rules.)

On the other hand, to show the cut elimination theorem for a sequent system which has both exchange and contraction rules but doesn't have weakening rules, e.g. systems like $\mathbf{FL_{ec}}$ and $\mathbf{CFL_{ec}}$, we need to take the following generalized form of the mix rule instead of the original mix rule:

$$\dfrac{\Gamma \Rightarrow \alpha \quad \Sigma \Rightarrow \delta}{\Gamma, \tilde{\Sigma}_\alpha \Rightarrow \delta}$$

where Σ contains at least one occurrence of the formula α, and $\tilde{\Sigma}_\alpha$ denotes a sequence of formulas obtained from Σ by deleting an arbitrary number, but *at least one* of occurrences of α. For more details of the cut elimination theorem for these basic substructural logics, see [33]. An algebraic proof of the cut elimination theorem for them is given in [5]. As a collorary of Theorem 3.1, we have the following. We note that Craig's interpolation theorem can be shown by using Maehara's method. See e.g. [33] for the details.

THEOREM 3.2. *Both the subformula property and Craig's interpolation theorem hold for* **FL**, $\mathbf{FL_e}$, $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$, $\mathbf{FL_{ec}}$, **LJ**, $\mathbf{CFL_e}$, $\mathbf{CFL_{ew}}$, $\mathbf{CFL_{ec}}$, *and* **LK**.

3.2. Disjunction property and variable sharing property

A logic **L** has the *disjunction property* when for all α and β if $\alpha \vee \beta$ is provable in **L** then either α or β is provable in it. While classical logic doesn't have the disjunction property, intuitionistic logic does. This is usually attributed to constructive features of the latter. In fact the disjunction property of intuitionistic logic is an immediate consequence of the cut elimination theorem for **LJ**. The proof goes as follows. Suppose that the sequent $\Rightarrow \alpha \vee \beta$ is provable in **LJ**. Consider any cut-free proof of it. Since the sequent is not an initial sequent, it is obtained by applying some rule I. Then it is easily seen that I must be either $(\Rightarrow \vee 1)$ or $(\Rightarrow \vee 2)$. Therefore, the upper sequent must be either $\Rightarrow \alpha$ or $\Rightarrow \beta$.

This argument doesn't work for **LK**, since the last rule may be the right contraction rule $(\Rightarrow c)$ as the following example shows:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\Rightarrow p, \neg p}}{\Rightarrow p \vee \neg p, \neg p}}{\Rightarrow p \vee \neg p, p \vee \neg p}}{\Rightarrow p \vee \neg p}\ (\Rightarrow c)$$

From this observation the following theorem can be easily derived.

THEOREM 3.3. *A basic substructural logic except* $\mathbf{FL_c}$ *has the disjunction property if and only if its sequent system doesn't have the right contraction rule.*

As we mentioned before, weakening rules allow us to introduce redundant assumptions and conclusions. For instance, using the weakening rule, we can show that the sequent $p * \neg p \Rightarrow q$ is provable in $\mathbf{FL_{ew}}$ though in this sequent there is no relation between the antecedent and the succedent. In fact, there are no propositional variables common to formulas in the left-hand side and the right-hand side of the sequent. We say that a logic **L** has the *variable sharing property*, if such a case never happens. More precisely, **L** has the variable sharing property, if for any formula $\alpha \to \beta$ containing no propositional constants, $\alpha \to \beta$ is never provable in **L** whenever formulas α and β have no propositional variables in common. Then, we have the following.

THEOREM 3.4. *A basic substructural logic, except* $\mathbf{FL_c}$*, has the variable sharing property if and only if its standard sequent system doesn't have the weakening rules.*

It is trivial that a basic substructural logic doesn't have the variable sharing property whenever its sequent system has the weakening rules. On the other hand, we can show that $\mathbf{CFL_{ec}}$ has the variable sharing property. Since it is the strongest among basic substructural logics without weakening rules, it follows that any of $\mathbf{FL}$, $\mathbf{FL_e}$, $\mathbf{FL_{ec}}$ and $\mathbf{CFL_e}$ has the property too (see [33] for the proof). Both the disjunction property and the variable sharing property for $\mathbf{FL_c}$ remain open.

3.3. Decision problems

Another important consequence of the cut elimination theorem is the decidability. Let us take first classical propositional logic, and see how Gentzen in [12] derived the decidability as a consequence of the cut elimination theorem for **LK**. To get the decidability, we introduce a proof-search procedure for any given sequent. This procedure tries to find a proof of a given sequent $\Gamma \Rightarrow \Delta$ when it is provable. Moreover, it terminates the search for proofs in a finite number of steps and tells us that the sequent is not provable when it fails to find a proof.

Suppose that a given sequent $\Gamma \Rightarrow \Delta$ is provable. By the cut elimination theorem and the subformula property, it has a cut-free proof in which every sequent is composed only of subformulas of formulas in $\Gamma \Rightarrow \Delta$. But if we count proofs with redundancies, the total number of proofs will be infinite. Here, we say that a proof contains a redundancy, if there exists a *branch* in the proof such that sequents of the same form appear more than once. So, we need to exclude proofs with redundancies.

Therefore, to show that a given sequent $\Gamma \Rightarrow \Delta$ is provable, it suffices to find a cut-free proof of $\Gamma \Rightarrow \Delta$ with no redundancies (in which every sequent is composed only of subformulas of formulas in $\Gamma \Rightarrow \Delta$). When we fail to find such a proof by an exhaustive search among all *possible proofs*, we can say that $\Gamma \Rightarrow \Delta$ is not provable. Here, by a possible proof we mean a proof-like figure (containing neither applications of the cut rule nor redundancies) in which every application of a rule is carried out in a correct way but some of the top sequents may not be initial ones.

Then, is the total number of all possible proofs finite? Unfortunately not, because of contraction rules. For example, suppose that $\alpha, \Sigma \Rightarrow \Pi$ is a top sequent in a possible proof P. As this top sequent may be obtained by applying contraction rule, the figure obtained from P by putting the sequent $\alpha, \alpha, \Sigma \Rightarrow \Pi$ over $\alpha, \Sigma \Rightarrow \Pi$ will be another possible proof. As this argument can be repeated, infinitely many possible proofs will be produced.

To settle this problem, Gentzen introduced the notion of *reduced* sequents. For $k > 0$, a sequent $\Sigma \Rightarrow \Pi$ is *k-reduced* if each formula in Σ (Π) occurs at most k times in Σ (and Π, respectively). A sequent $\Sigma' \Rightarrow \Pi'$ is a *1-reduced contraction* of a sequent $\Sigma \Rightarrow \Pi$ if $\Sigma' \Rightarrow \Pi'$ is 1-reduced, and Σ' and Π' are equal to Σ and Π, respectively, as sets of formulas. For instance, $\alpha, \beta \Rightarrow \gamma$ is a 1-reduced contraction of $\alpha, \beta, \alpha, \alpha \Rightarrow \gamma, \gamma$. With the help of the structural rules, we can easily show that a sequent $\Sigma \Rightarrow \Pi$ is provable if and only if any one of its 1-reduced contractions is provable. Therefore, to see whether a given sequent is provable or not, it is enough to take any one of its 1-reduced contractions and check whether it is provable or not. Now, by observing the form of each rule of **LK** and using the argument mentioned above, we can show the following lemma by using the length of proofs.

LEMMA 3.5. *For a given sequent $\Gamma \Rightarrow \Delta$, let $\Gamma' \Rightarrow \Delta'$ be any of its 1-reduced contraction. If $\Gamma \Rightarrow \Delta$ is provable in* **LK** *then $\Gamma' \Rightarrow \Delta'$ has a cut-free proof with no redundancies, in which only 3-reduced sequents appear.*

It is easily seen this time that for a given 1-reduced sequent $\Gamma' \Rightarrow \Delta'$, the total number of its possible proofs in which every sequent is 3-reduced and is composed only of subformulas of formulas in $\Gamma' \Rightarrow \Delta'$ is finite. Therefore, we have a proof-search procedure which searches for only such a proof.

As the above outline shows, the proof of the decidability relies much on structural rules. But, the necessity of the notion of reduced sequents comes from the contraction rules. In other words, if a sequent system under consideration has no contraction rules, then we can get a much simpler decision procedure. Let us consider $\mathbf{FL_{ew}}$ for example. In this case it is easily seen that in each rule except the cut rule, (each of) the upper sequent(s) is always simpler than the lower sequent. Thus, the proof-search procedure terminates always. This idea of proving the decidability can be extended also to sequent systems for basic substructural *predicate logics with function symbols*. Thus we have the following. (See e.g. [20] for the details. See also [32].)[4]

THEOREM 3.6. *Any of the sequent systems for basic substructural predicate logics without contraction rules, i.e. any of* $\mathbf{FL}$, $\mathbf{FL_e}$, $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$, $\mathbf{CFL_e}$ *and* $\mathbf{CFL_{ew}}$, *is decidable even when the language contains function symbols.*

[4] H. Wang noticed this fact already in his book [43] published in 1963.

On the other hand, the decision procedure becomes complicated for $\mathbf{FL_{ec}}$ and $\mathbf{CFL_{ec}}$, i.e. substructural logics with contraction rules but without weakening rules. To show the termination of the procedure, we need some combinatorial result like Kripke's lemma or Higman's theorem. The original proof is given essentially by Kripke in [23] of 1959. (See e.g. [33] for the details.) This complication leads us to the undecidability of their predicate extensions (see [20]).

THEOREM 3.7. *Both substructural propositional logics* $\mathbf{FL_{ec}}$ *and* $\mathbf{CFL_{ec}}$ *are decidable, while their predicate extensions are undecidable.*

The relevant logic $\mathbf{R}$ is obtained from $\mathbf{CFL_{ec}}$ by adding the distributive law (see §2.2). Urquhart proved the following quite nontrivial and important result in [42].

THEOREM 3.8. *The propositional logic* $\mathbf{R}$ *is undecidable.*

4. Residuated Lattices

In this section, we will introduce algebraic structures which serve us with semantics suitable for substructural logics. A key notion here is *residuation*, and the algebraic structures for substructural logics introduced here are *residuated lattices*, which have been already studied by algebraists in 1930s, e.g. Krull [24], and Ward and Dilworth [44] (see also [3]).

A residuated lattice consists of a lattice and a partially-ordered semigroup with residuation. By imposing some additional conditions on the lattice-part and/or the semigroup-part, we can get various interesting subclasses of residuated lattices, e.g. the class of MV-algebras and the class of BL-algebras. They correspond to Łukasiewicz's many-valued logics and extensions of the basic logic **BL** introduced by P. Hájek, respectively. For more information on recent results of residuated lattices, see [19] and also [22].

4.1. Residuated structures and residuation theory

In the following, to denote algebraic operations and constants, we use the same symbols as logical connectives and propositional constants corresponding to them, when no confusions may occur. A single exception is to use $\cdot$ for a semigroup operation, while we use $*$ for fusion.

A structure $\mathbf{P} = \langle P, \cdot, \leq \rangle$ is a *partially-ordered semigroup* if $\langle P, \cdot \rangle$ is a semigroup and $\leq$ is a partial order on P such that $\cdot$ is monotone increasing; i.e.

$$x \leq x' \text{ and } y \leq y' \text{ imply } x \cdot y \leq x' \cdot y'.$$

An algebra $\mathbf{P} = \langle P, \cdot, \backslash, /, \leq \rangle$ is a *residuated* partially-ordered semigroup $\mathbf{P}$ if $\langle P, \cdot \rangle$ is a semigroup, $\leq$ a partial order on P and moreover the following condition is satisfied by $\cdot, \backslash$ and $/$: For all $x, y, z \in P$

$$x \cdot y \leq z \text{ if and only if } y \leq x \backslash z \text{ if and only if } x \leq z/y.$$

This condition is sometimes called *the law of residuation*, and $\backslash$ and $/$ are called the right and left *residual* of $\cdot$, respectively. Any residuated partially-ordered semigroup is in fact a partially-ordered semigroup, though we don't assume the monotonicity of $\cdot$ in its definition. This can be shown by using the law of residuation as follows:

Suppose that both $x \leq x'$ and $y \leq y'$ hold. Since $x' \cdot y' \leq x' \cdot y'$, $y \leq y' \leq x' \backslash (x' \cdot y')$. Therefore $x' \cdot y \leq x' \cdot y'$. Thus, $x \leq x' \leq (x' \cdot y')/y$ and hence $x \cdot y \leq x' \cdot y'$. (This argument works in a more general setting. See Lemma 4.1.)

Any residuated partially-ordered semigroup $\mathbf{P}$ such that $\langle P, \leq \rangle$ forms a lattice and $\langle P, \cdot \rangle$ has a unit is called a *residuated lattice*. More precisely, an algebra $\mathbf{P} = \langle P, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$ is a residuated lattice if

1. $\langle P, \wedge, \vee \rangle$ is a lattice,
2. $\langle P, \cdot, 1 \rangle$ is a monoid such that $\backslash$ and $/$ are the right and left residual of $\cdot$, respectively.

When $\cdot$ is commutative, we call $\mathbf{P}$ a commutative residuated lattice. In any commutative residuated lattice, $x \backslash y = y/x$ holds for all x, y. In such a case, we use the symbol $\rightarrow$ and write $x \rightarrow y$ instead of $x \backslash y$ (and of y/x). Also $\mathbf{P}$ is denoted by a sextuple $\langle P, \wedge, \vee, \cdot, \rightarrow, 1 \rangle$.

As an example, we show that any ℓ-group forms a residuated lattice. Recall that an algebra $\mathbf{G} = \langle G, \wedge, \vee, \cdot, ^{-1}, e \rangle$ is a lattice-ordered group (or an ℓ-group, for short) if

1. $\langle G, \wedge, \vee \rangle$ is a lattice with the lattice order $\leq$,
2. $\langle G, \cdot, ^{-1}, e \rangle$ is a group,

3. $\cdot$ is monotone with respect to $\leq$, i.e. $\langle G, \cdot, \leq \rangle$ is a partially-ordered semigroup.

Then $\cdot$ has both residuals $\backslash$ and $/$ that are defined by $x \backslash y = x^{-1} \cdot y$ and $y/x = y \cdot x^{-1}$, respectively.

Some of the basic properties of residuated lattices hold in a more general setting, and as *residuation theory* tells us, they come from consequences of residuated structures in general. To see this, we now take a brief look at some of basic facts in residuation theory. For general information on residuation theory, see [7].

Let $\mathbf{P}$ and $\mathbf{Q}$ be posets. A map $f \colon \mathbf{P} \to \mathbf{Q}$ is *residuated* if there exists a map $g \colon \mathbf{Q} \to \mathbf{P}$ such that the following holds for any $p \in P$ and any $q \in Q$:

$$f(p) \leq q \text{ if and only if } p \leq g(q).$$

If the above holds, we say that f and g form a *residuated pair*, and that g is a *residual* of f. We can show the following.

Lemma 4.1. *If f and g form a residuated pair, both f and g are monotone increasing.*

It is easy to see that if f is residuated then its residual is determined uniquely by f. Hereafter, f^* denotes the residual of a map f when f is residuated. We can show conversely that f is determined uniquely by f^* when they form a residuated pair. In fact, these uniqueness results come from the following formulas:

1. $f^*(q) = \max\{p \in P : f(p) \leq q\}$,
2. $f(p) = \min\{q \in Q : p \leq f^*(q)\}$.

Proposition 4.2. *Suppose that f and f^* form a residuated pair between posets $\mathbf{P}$ and $\mathbf{Q}$, and that $X \subseteq P$ and $Y \subseteq Q$. If the supremum $\bigvee X$ of X exists then $\bigvee f(X)$ $(= \bigvee\{f(p) : p \in X\})$ exists and $f(\bigvee X) = \bigvee f(X)$. Also if the infimum $\bigwedge Y$ of Y exists then $\bigwedge f^*(Y)$ exists and $f^*(\bigwedge Y) = \bigwedge f^*(Y)$.*

When both $\mathbf{P}$ and $\mathbf{Q}$ are lattices in the above proposition, we have the following.

Corollary 4.3. *Suppose that f and f^* form a residuated pair between lattices $\mathbf{P}$ and $\mathbf{Q}$. Then $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in \mathbf{P}$, and $f^*(u \wedge v) = f^*(u) \wedge f^*(v)$ for all $u, v \in \mathbf{Q}$.*

If both $\mathbf{P}$ and $\mathbf{Q}$ are moreover complete, the converse of Proposition 4.2 is true. More precisely, the following holds.

PROPOSITION 4.4. *Suppose that both $\mathbf{P}$ and $\mathbf{Q}$ are complete lattices and that f is a map from $\mathbf{P}$ to $\mathbf{Q}$. Then, f is residuated if and only if f preserves all (possibly infinite) joins. Dually, a map $f^* : \mathbf{Q} \to \mathbf{P}$ is the residual of a map $f : \mathbf{P} \to \mathbf{Q}$ if and only if f^* preserves all (possibly infinite) meets.*

Now let $\mathbf{P}$ be a partially-ordered semigroup. For each fixed $u \in P$, define maps g_u and h_u from $\mathbf{P}$ to $\mathbf{P}$ by $g_u(x) = u \cdot x$ and $h_u(x) = x \cdot u$, respectively, for each $x \in P$. Then, $\mathbf{P}$ is a residuated partially-ordered semigroup if and only if both g_u and h_u are residuated maps for every $u \in P$. In this case, the residuals ${g_u}^*$ and ${h_u}^*$ are given by ${g_u}^*(y) = u\backslash y$ and ${h_u}^*(y) = y/u$, respectively.

Thus, applying Corollary 4.3 and Proposition 4.4 to residuated lattices, we have the following results.

COROLLARY 4.5. *The following equations hold in any residuated lattice. For all x, y, z,*
1. $(x \vee y) \cdot z = (x \cdot z) \vee (y \cdot z)$,
2. $z \cdot (x \vee y) = (z \cdot x) \vee (z \cdot y)$,
3. $(x \vee y)\backslash z = (x\backslash z) \wedge (y\backslash z)$,
4. $z/(x \vee y) = (z/x) \wedge (z/y)$,
5. $z\backslash(x \wedge y) = (z\backslash x) \wedge (z\backslash y)$,
6. $(x \wedge y)/z = (x/z) \wedge (y/z)$.

COROLLARY 4.6. *In any complete residuated lattice, i.e. any residuated lattice which is complete as a lattice, the following equations hold:*
1. $x\backslash z = \max\{y : x \cdot y \leq z\}$,
2. $z/y = \max\{x : x \cdot y \leq z\}$.

4.2. Residuated lattices as algebras for substructural logics

For each basic substructural logic $\mathbf{L}$, there exists a class of residuated lattices which *characterizes* the logic $\mathbf{L}$. To show this, we will introduce some classes of residuated lattices. A residuated lattice is *integral* if the unit 1 of the monoid is equal to the greatest element, denoted by $\top$, of the lattice. In any integral residuated lattice, both $x \cdot y \leq x$ and $x \cdot y \leq y$ hold, since by the monotonicity $x \cdot y \leq x \cdot \top = x \cdot 1 = x$ and similarly for the latter. A residuated lattice $\mathbf{P}$ is *increasing idempotent* if $x \leq x \cdot x$ for any $x \in P$. It is easy to see that a residuated lattice is both integral and increasing idempotent if and only if $x \cdot y = x \wedge y$ for all x, y.

By using residuated lattices, we can introduce an algebraic interpretation of each formula. To give an interpretation of negation in a given residuated lattice $\mathbf{P}$, we need to introduce a element 0 which is an arbitrary element of P. Sometimes, $x \to 0$ is denoted as $\neg x$ in any commutative residuated lattice $\mathbf{P}$ with 0. If our language for describing formulas contains propositional constants $\top$ and $\bot$, we need to assume that $\mathbf{P}$ is bounded, i.e. it has a greatest element $\top$ and a least element $\bot$.

To simplify the naming of classes of residuated lattices corresponding to basic substructural logics and to avoid unnecessary complications, we will take the following approach.

Let us call any residuated lattice with a fixed element 0, an **FL**-algebra. A commutative (increasing idempotent, and commutative, increasing idempotent) **FL**-algebra is called an $\mathbf{FL_e}$- ($\mathbf{FL_c}$-, $\mathbf{FL_{ec}}$-, respectively) algebra. An integral **FL**-algebra *whose least element is 0*, which therefore is bounded, is called an $\mathbf{FL_w}$-algebra. Also, any commutative $\mathbf{FL_w}$-algebra is called an $\mathbf{FL_{ew}}$-algebra. If an $\mathbf{FL_e}$-algebra $\mathbf{P}$ satisfies $\neg\neg x \le x$ for any x, it is called a $\mathbf{CFL_e}$-algebra. Similarly, we can define $\mathbf{CFL_{ec}}$- and $\mathbf{CFL_{ew}}$-algebras. Note that the equation $x \cdot y = \neg(x \to \neg y)$ holds always in any $\mathbf{CFL_e}$-algebra.

It is well-known that Heyting algebras are algebraic structures for intuitionistic logic. Recall that an algebra $\mathbf{P} = \langle P, \wedge, \vee, \to, 0\rangle$ is a *Heyting algebra* if

1. $\langle P, \wedge, \vee, 0\rangle$ is a lattice with least element 0, and
2. for all $x, y, z \in P$, $x \wedge y \le z$ iff $x \le (y \to z)$.

It is easy to see that any Heyting algebra is a bounded commutative residuated lattice with the greatest element 1 defined by $1 = 0 \to 0$, whose monoid operation is the meet $\wedge$, and in which $\wedge$ and $\to$ form a residuated pair. Therefore each Heyting algebra is moreover increasing idempotent and integral. Conversely, any increasing idempotent, integral commutative residuated lattice with least element 0 is a Heyting algebra. By Corollary 4.5, every Heyting algebra is distributive, i.e. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ holds in it. Conversely, every finite distributive lattice is a Heyting algebra, by Proposition 4.4. When a Heyting algebra is complete, it satisfies the following infinite distributive law by Proposition 4.4

$$(\bigvee_i x_i) \wedge y = \bigvee_i (x_i \wedge y),$$

and conversely, any complete lattice with least element 0 satisfying the above law is a reduct of a Heyting algebra (see §4.3).

Also, residuated lattices related to relevant logics have been studied since the mid 1960s. The special feature of these residuated lattices is that they are increasing idempotent residuated lattices satisfying the distributive law between $\wedge$ and $\vee$. Both increasing idempotent, commutative residuated lattices with the distributive law and $\mathbf{CFL_{ec}}$-algebras with the distributive law are basic algebraic structures for relevant logics, which are called *Dunn monoids* and *De Morgan monoids*, respectively. See [1].

In the usual way, we can give an interpretation of formulas in an **FL**-algebra. For a given **FL**-algebra **P**, a *valuation* v on **P** is any mapping from the set of all propositional variables to the set P. Each valuation v is extended to a mapping from the set of all formulas to P inductively as follows. Here, recall that we use the same symbols for logical connectives (and constants) as those for corresponding algebraic operations (and constants, respectively).

1. $v(1) = 1$ and $v(0) = 0$,
2. $v(\top) = \top$ and $v(\bot) = \bot$, when the language contains $\top$ and $\bot$, and **P** is bounded,
3. $v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta)$,
4. $v(\alpha \vee \beta) = v(\alpha) \vee v(\beta)$,
5. $v(\alpha * \beta) = v(\alpha) \cdot v(\beta)$,
6. $v(\alpha\backslash\beta) = v(\alpha)\backslash v(\beta)$,
7. $v(\alpha/\beta) = v(\alpha)/v(\beta)$.

A formula α is *valid* in **P** if $v(\alpha) \geq 1$ holds for any valuation v on **P**. Also, a given sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is said to be valid in **P** if and only if the formula $(\alpha_1 * \cdots * \alpha_m) \to \beta$ is valid in it, or equivalently, $v(\alpha_1) \cdots v(\alpha_m) \leq v(\beta)$ holds for any valuation v on **P**. Here, we assume $v(\alpha_1) \cdots v(\alpha_m) = 1$ when $m = 0$, and $v(\beta) = 0$ when β is empty. For a commutative **P**, a sequent of the form $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is valid in **P** if and only if $v(\alpha_1) \cdots v(\alpha_m) \leq v(\beta_1) + \ldots + v(\beta_n)$ holds for any valuation v on it, where $x + y$ is defined by $x + y = \neg(\neg x \cdot \neg y)$.

We can show the following completeness theorem for basic substructural logics, by using the standard argument on Lindembaum algebras.

THEOREM 4.7. *For any sequent $\mathcal{S}$, $\mathcal{S}$ is provable in* **FL** *if and only if it is valid in all* **FL**-*algebras. This holds also for other basic substructural logics and corresponding classes of* **FL**-*algebras.*

Note that when the language contains constants $\top$ and $\bot$ it is necessary to replace the word "**FL**-algebras" in the above theorem by "bounded **FL**-algebras".

Until now, we were mainly concerned with basic substructural logics that are defined by using sequent systems. We go a step further now and introduce the notion of substructural logics in a general sense. Let **L** be a set of formulas. We say that **L** is a *substructural logic* (over **FL**) if

1. every formula provable in **FL** belongs to **L**,
2. for all formulas α and β, if both α and $\alpha\backslash\beta$ are in **L** then β is also in it,
3. for all formulas α and β, if both α and β are in **L** then $\alpha \wedge \beta$ is also in it,
4. for all formulas α and β, if α is in **L** then both $(\alpha\backslash\beta)\backslash\beta$ and $\beta/(\beta/\alpha)$ are also in it.

Here are some comments. First, every substructural logic **L** is closed under uniform substitutions. Second, for any substructural logic **L**, the following holds:

> for all formulas α and β, if both α and β/α are in **L** then β is also in it.

In fact, suppose that α belongs to **L**. Then $(\beta/\alpha)\backslash\beta$ belongs also to **L**, since $\alpha\backslash((\beta/\alpha)\backslash\beta)$ is provable and hence belongs to **L**. Thus, if moreover β/α is in **L** then β belongs to it by the second condition. Lastly, we can replace the fourth condition by the following two: For all formulas α, β and γ,

- if both α and $\beta\backslash(\alpha\backslash\gamma)$ are in **L** then $\beta\backslash\gamma$ is also in it,
- if both α and $(\gamma/\alpha)/\beta$ are in **L** then γ/β is also in it.

Note that the third condition becomes redundant when $(\alpha * \beta)\backslash(\alpha \wedge \beta) \in \mathbf{L}$ for all α, β, e.g. when the weakening rule holds in it. Also, the fourth condition becomes redundant when $(\alpha * \beta)\backslash(\beta * \alpha) \in \mathbf{L}$ for all α, β, i.e. when the exchange rule holds in **L**.

It is obvious that every basic substructural logic (as a set of formulas) is a substructural logic in our sense. We say that a substructural logic $\mathbf{L}$ is *characterized* by a set $\{\mathbf{P}_i : i \in I\}$ of $\mathbf{FL}$-algebras if $\mathbf{L} = \bigcap\{\mathbf{L}(\mathbf{P}_i) : i \in I\}$ holds. In particular, when $\mathbf{L}$ is characterized by a singleton set $\{\mathbf{P}\}$, we say simply that $\mathbf{L}$ is characterized by $\mathbf{P}$. It is easily seen that for any set $\{\mathbf{P}_i : i \in I\}$ of $\mathbf{FL}$-algebras, the set of formulas $\bigcap\{\mathbf{L}(\mathbf{P}_i) : i \in I\}$ is a substructural logic. We can show the following.

Theorem 4.8. *Any substructural logic* $\mathbf{L}$ *is characterized by a single* $\mathbf{FL}$-*algebra.*

Here is an outline of the proof. Let Ψ be the set of all formulas. Define a binary relation $\approx$ on Ψ by

$\alpha \approx \beta$ if and only if both $\alpha\backslash\beta$ and $\beta\backslash\alpha$ are in $\mathbf{L}$.

Then, we can show that $\approx$ is a congruence relation on Ψ. Moreover the quotient set $\Psi/\approx$ forms a $\mathbf{FL}$-algebra, say $\mathbf{P}$, and $\mathbf{L} = \mathbf{L}(\mathbf{P})$ holds.

4.3. Quantales and completions of residuated lattices

A structure $\mathbf{Q} = \langle Q, \wedge, \vee, \cdot\rangle$ is a *quantale* if

1. $\langle Q, \wedge, \vee\rangle$ is a complete lattice (and hence is bounded),
2. $\langle Q, \cdot\rangle$ is a semigroup which satisfies that for $x_i, y \in Q$
   a) $(\bigvee_i x_i) \cdot y = \bigvee_i (x_i \cdot y)$,
   b) $y \cdot (\bigvee_i x_i) = \bigvee_i (y \cdot x_i)$.

By Proposition 4.4, $\mathbf{Q}$ is a residuated lattice whose residuals are defined by the equations in Corollary 4.6. Conversely, any complete residuated lattice, i.e. a residuated lattice whose lattice reduct is complete, is a quantale. Therefore, quantales are essentially equal to complete residuated lattices. For general information on quantales, see [40]. When $\langle Q, \cdot\rangle$ is moreover a commutative monoid, $\mathbf{Q}$ becomes a complete commutative residuated lattices. By adding 0 to it, we get a complete $\mathbf{FL_e}$-algebra, which sometimes is called an *intuitionistic phase structure.*

Theorem 4.7 says that for any basic substructural logic $\mathbf{L}$, a formula α is provable in $\mathbf{L}$ if and only if it is valid in all $\mathbf{L}$-algebras. We can strengthen this in the following way.

Theorem 4.9. *Let* $\mathbf{L}$ *be any one of the basic substructural logics. Then, for any formula* α*,* α *is provable in* $\mathbf{L}$ *if and only if it is valid in all complete* $\mathbf{L}$-*algebras.*

To show this, it suffices to prove the if-part by Theorem 4.7. This is shown as follows. Let α be a formula not provable in $\mathbf{L}$. By Theorem 4.7 α is not valid in a $\mathbf{L}$-algebra $\mathbf{P}$. We use the following lemma.

LEMMA 4.10. *Let* $\mathbf{L}$ *be any one of the basic substructural logics. Each* $\mathbf{L}$-*algebra can be embedded into a complete* $\mathbf{L}$-*algebra.*

By using this, $\mathbf{P}$ is embedded into a complete $\mathbf{L}$-algebra $\mathbf{P}^\dagger$, and it is obvious that α is not valid in $\mathbf{P}^\dagger$. To get such $\mathbf{P}^\dagger$, it is enough to take the *Dedekind-MacNeille completion* of $\mathbf{P}$. For the details of the proof of this completeness result, see [31]. For more information on completions of residuated lattices and complete embeddings, see [34] and [35].

5. A Prelude to Algebraic Study of Substructural Logics

The algebraic study of substructural logics is a rapidly growing research field, which attracts both logicians and algebraists. In this section we will touch some topics of the study briefly. These results tell us that the algebraic approach to substructural logics is quite useful and promising. For more information on the study in this direction, see [18].

5.1. Varieties of residuated lattices

We discuss further relations between substructural logics (in our sense) and classes of residuated lattices, using some basic notions and results from universal algebra. In the previous section, we have shown that each substructural logic is characterized by an $\mathbf{FL}$-algebra and vice versa. Slightly changing our viewpoint, instead of taking a single $\mathbf{FL}$-algebra we consider a class $\mathcal{V}(\mathbf{L})$ of $\mathbf{FL}$-algebras for each substructural logic $\mathbf{L}$, which is defined by:

$$\mathcal{V}(\mathbf{L}) = \{\mathbf{P} : \mathbf{L} \subseteq \mathbf{L}(\mathbf{P})\}.$$

Then, we can show that the class $\mathcal{V}(\mathbf{L})$ is closed under homomorphic images, subalgebras and direct products. In universal algebra, such a class is called a *variety*. By a fundamental result due to G. Birkhoff, a class $\mathcal{V}$ of algebras is a variety if and only if it is an *equational class*, i.e. it is defined by a set of equations. More precisely, a class $\mathcal{V}$ of algebras is an equational class if there exists a set Σ of equations such that

$$\mathcal{V} = \{\mathbf{P} : s = t \text{ holds in } \mathbf{P} \text{ for any } s = t \in \Sigma\}.$$

We explain here how the class of all residuated lattices is defined by a set of equations. It is well-known that both the class of (bounded) lattices and the class of monoids are defined by certain sets of equations. So it is enough to express the law of residuation by equations. Here, we consider the following.

$x \cdot y \leq z$ if and only if $y \leq x\backslash z$

We note first that each inequality $s \leq t$ can be expressed in a lattice by the equation $s \wedge t = s$. Thus, it suffices to show that the above condition can be expressed by using inequalities. In fact, it is expressed by the following two inequalities.

1. $y \leq x\backslash((x \cdot y) \vee z)$,
2. $x \cdot (y \wedge (x\backslash z)) \leq z$.

THEOREM 5.1. *The class $\mathcal{V}(\mathbf{L})$ of* **FL***-algebras is a variety and hence is an equational class, for each substructural logic* **L***.*

Conversely, suppose that an equational class $\mathcal{V}$ of **FL**-algebras is given. We assume here that a set Σ of equations defines $\mathcal{V}$. Now let $\mathbf{L} = \{\alpha :$ the inequality $t_\alpha \geq 1$ follows from equations in $\Sigma\}$, where t_α is the term expression of a given formula α. In other words, $\mathbf{L} = \{\alpha :$ the inequality $t_\alpha \geq 1$ holds in every $\mathbf{P}$ in $\mathcal{V}\}$. Then it is easily seen that $\mathbf{L}$ is a substructural logic. Let us denote this $\mathbf{L}$ as $\mathbf{L}_{\mathcal{V}}$.

THEOREM 5.2. *For each variety $\mathcal{V}$ of* **FL***-algebras, the set $\mathbf{L}_{\mathcal{V}}$ of formulas is a substructural logic. Moreover, $\mathcal{V}(\mathbf{L}_{\mathcal{V}}) = \mathcal{V}$ holds.*

Let $\mathcal{V}$ be a variety of **FL**-algebras, and W be a nonempty subclass of $\mathcal{V}$. We say that W *generates* the variety $\mathcal{V}$ if $\mathcal{V}$ is the smallest variety containing W. We can verify easily that a substructural logic $\mathbf{L}$ is characterized by a set $\{\mathbf{P}_i : i \in I\}$ of **FL**-algebras if and only if it generates the variety $\mathcal{V}(\mathbf{L})$.

Let us denote the variety of all **FL**-algebras by $\mathcal{FL}$, which is equal to $\mathcal{V}(\mathbf{FL})$. It is shown that all subvarieties of $\mathcal{FL}$ as well as all substructural logics over **FL** form a complete lattice. The above two theorems say that there exists a one-to-one correspondence φ between the lattice of all substructural logics and the lattice of all subvarieties of $\mathcal{FL}$, where φ is defined by $\varphi(\mathbf{L}) = \mathcal{V}(\mathbf{L})$. This φ is indeed a dual complete lattice isomorphism. This implies that we can understand the lattice structure of all substructural logics by studying the lattice of all subvarieties of $\mathcal{FL}$, instead. It follows also that for a given substructural logic $\mathbf{L}$, the restriction of φ to the lattice of all extensions of $\mathbf{L}$ is a dual complete lattice isomorphism from it to the lattice of all subvarieties of $\mathcal{V}(\mathbf{L})$. Based on this dual lattice isomorphism and using methods and results from universal algebra, we have many important results on the lattice structure of extensions of the logic $\mathbf{FL_{ew}}$.

Since the variety $\mathcal{HA}$ of Heyting algebras, which is one of the subvarieties of $\mathcal{V}(\mathbf{FL_{ew}})$, has been studied quite well, it is interesting to make a comparison of the lattice structure of subvarieties of $\mathcal{HA}$ with that of $\mathcal{V}(\mathbf{FL_{ew}})$. In fact, there is a big difference between them as the following example shows.

It is easy to see that there is only one *minimal* variety $\mathcal{BA}$ of $\mathbf{FL_{ew}}$-algebras, which is in fact the variety of Boolean algebras, since any nontrivial $\mathbf{FL_{ew}}$-algebra has the two-element Boolaen algebra as a subalgebra. Also, among subvarieties of $\mathcal{HA}$, there is only one *almost minimal variety*, i.e. the variety which covers $\mathcal{BA}$ in the lattice of subvarieties of $\mathcal{HA}$. This is the variety generated by the three-element Heyting algebra. On the other hand, T. Kowalski and M. Ueda obtained the following. See [22] and [18] for the details.

THEOREM 5.3. *There exist uncountably many almost minimal varieties of* $\mathbf{FL_{ew}}$*-algebras.*

For more information on extensions of the logic $\mathbf{FL_{ew}}$ and on subvarieties of the variety of $\mathbf{FL_{ew}}$-algebras, see e.g. [37], [22] and [36].

5.2. Algebras for many-valued logics and fuzzy logics

We will take here special subvarieties of the variety of $\mathbf{FL_{ew}}$-algebras that correspond to Łukasiewicz's many-valued logics and fuzzy logics in the narrow sense (see [9] and [15]).

Algebraic structures for Łukasiewicz's many-valued logics are originally introduced as follows. Let R be either the set $\{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$ (for $(n+1)$-valued logic) or the unit interval $[0,1]$ (for infinite-valued logic). In either case, R is totally ordered by the natural order on the reals. Define two operations $\rightarrow$ and $\cdot$ on R by

- $x \rightarrow y = \min\{1, 1 - x + y\}$,
- $x \cdot y = \max\{0, x + y - 1\}$.

Then the algebra $\mathbf{R} = \langle R, \min, \max, \cdot, \rightarrow, 1\rangle$ forms a bounded commutative residuated lattice with least element 0, and is in fact an $\mathbf{FL_{ew}}$-algebra. We note that when R denotes a set of the form $\{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$, $\mathbf{R}$ is isomorphic to an $\mathbf{FL_{ew}}$-algebra with the underlying set $\{a^n, \ldots, a^2, a, a^0\}$ such that $a^n = 0$, whose total order $<$ satisfies $a^{i+1} < a^i$ for each $i = 0, \ldots, n-1$. Here, a^j is defined by $a^0 = 1$ and $a^j = a * a^{j-1}$ for each $j > 0$, where $*$ is a semigroup operation on R. In this case, the residual $\rightarrow$ is defined by $a^k \rightarrow a^m = a^{\max\{0, m-k\}}$.

We can show that the following holds always in any algebra $\mathbf{R}$ under consideration:

$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x \quad \text{for all } x, y \qquad (1).$$

Now, let us generalize this. We say that a $\mathbf{FL_{ew}}$-algebra $\mathbf{P}$ is an *MV-algebra* if it satisfies the above condition (1). It is known that they serve as algebraic structures for many-valued logics.[5] We note that any MV-algebra is in fact a $\mathbf{CFL_{ew}}$-algebra, and moreover the following two equations hold in it: for all x, y $x \vee y = (x \rightarrow y) \rightarrow y$ and $x \wedge y = \neg(\neg x \wedge \neg y)$. The following completeness theorem holds. For further information on many-valued logics, see e.g. [9].

THEOREM 5.4. *The following three conditions are mutually equivalent for any formula α.*
1. α is provable in the logic obtained from $\mathbf{FL_{ew}}$ by adding $((\beta \rightarrow \gamma) \rightarrow \gamma) \rightarrow ((\gamma \rightarrow \beta) \rightarrow \beta)$,
2. α is valid in the Łukasiewicz's infinite-valued algebra, i.e. the MV-algebra determined by the unit interval $[0, 1]$,
3. α is valid in any MV-algebra.

A map T from $[0, 1]^2$ to $[0, 1]$ is a *triangular norm* (or, simply, a t-norm) if $\langle [0, 1], \circ, 1\rangle$ is a partially ordered commutative monoid where $\circ$ is defined by $x \circ y = T(x, y)$. It is obvious that $\langle [0, 1], \min, \max, 0, 1\rangle$ forms a complete lattice. Typical examples of t-norms are $\min\{x, y\}$, $x \times y$ (the multiplication of real numbers), and $\max\{x + y - 1, 0\}$. The last one is the monoid operation used in Łukasiewicz's many-valued algebra.

Now, a question is when these partially ordered commutative monoids become residuated. Proposition 4.4 tells us that a partially ordered monoid defined by a t-norm T is residuated if and only if $\circ$ satisfies

$$(\bigvee x_i) \circ y = \bigvee (x_i \circ y).$$

[5] Usually, MV-algebras are defined in a different way. But, the definition given here determines essentially the same algebras as those defined in the standard way.

Since $\circ$ is defined on an interval $[0, 1]$ of reals, this can be expressed also as

$$T(x, y) = \lim_{z \to x-0} T(z, y) = \lim_{w \to y-0} T(x, w),$$

which means that a map T is *left-continuous*. Thus, we can see that every left-continuous t-norm determines a commutative, integral residuated lattice, and then this residuated lattice characterizes a substructural logic over $\mathbf{FL_{ew}}$. For instance, t-norms $\min\{x, y\}$, $x \times y$ and $\max\{x + y - 1, 0\}$ determine a superintuitionistic logic called Dummett-Gödel logic, the *product* logic introduced by Hájek (see [15]), and Łukasiewicz's infinite-valued logic, respectively.

When a left-continuous t-norm $\circ$ satisfies moreover that

$$(\bigwedge x_i) \circ y = \bigwedge (x_i \circ y),$$

it is *continuous*. We introduce here two extensions of $\mathbf{FL_{ew}}$ in an axiomatic way. Define **MTL** (monoidal t-norm logic) to be the logic obtained from $\mathbf{FL_{ew}}$ by adding $(\alpha \to \beta) \vee (\beta \to \alpha)$ as the axiom, and **BL** (basic logic) to be the logic obtained from **MTL** by adding $(\alpha \wedge \beta) \to (\alpha * (\alpha \to \beta))$ as the additional axiom. They are introduced by Esteva and Godo in [11] and Hájek in [15], respectively. Then the following results, called the *standard completeness theorems* for **MTL** and **BL**, are obtained in [17] and [8], respectively. Note here that Łukasiewicz's infinite-valued logic is shown to be equal also to the logic obtained from **BL** by adding $\neg\neg\alpha \to \alpha$ as the axiom.

THEOREM 5.5. *1. The logic* **MTL** *is complete with respect to the class of all* $\mathbf{FL_{ew}}$*-algebras determined by left-continuous t-norms.*
2. The logic **BL** *is complete with respect to the class of all* $\mathbf{FL_{ew}}$*-algebras determined by continuous t-norms.*

5.3. Finite model property and finite embeddability property

A substructural logic $\mathbf{L}$ has the *finite model propery* if there exists a set $\{\mathbf{P}_i : i \in I\}$ of finite **FL**-algebras such that $\mathbf{L} = \bigcap\{\mathbf{L}(\mathbf{P}_i) : i \in I\}$ holds. This is equivalent to saying that the variety $\mathcal{V}(\mathbf{L})$ is generated by its finite members. The finite model property is a useful property in showing the decidability of a given logic, as Harrop's theorem says that a logic is decidable if it has the finite model property and is finitely axiomatizable.

In modal logic, proving the finite model property with respect to Kripke models is a standard and most powerful technique of showing decidability. On the other hand, since Kripke frames for substructural logics, e.g. those introduced in [37], are hardly manageable, the finite model property of basic substructural logics had remained open until R.K. Meyer and the present author gave a positive answer to the implicational fragments of both $\mathbf{FL_{ew}}$ and $\mathbf{FL_{ec}}$ (see [28]). This makes an interesting contrast with the fact that the decidability of basic substructural logics except $\mathbf{FL_{c}}$ is an easy consequence of the cut elimination theorem, as discussed already in §3. Then, Y. Lafont [25] succeeded to show the finite model property of both $\mathbf{CFL_{ew}}$ and $\mathbf{CFL_{ec}}$. The methods used there were then extended and elaborated in the paper by M. Okada and K. Terui [29], who succeeded to show the finite model property of other basic substructural logics. Interestingly enough, in [25] and [29] the cut elimination theorem is used in an essential way in proving the finite model property of some of logics. An algebraic presentation of the method used in them is given in [5].

Extending an idea of [29], Blok and van Alten [6] have developed a method of proving the finite model property using the *finite embeddability property.* The method can be regarded as an algebraic substitute of the filtration method for Kripke frames.

We say that the class $\mathcal{K}$ of algebras has the *finite embeddability property* when for a given finite *partial* subalgebra $\mathbf{R}$ of an algebra $\mathbf{P}$ in $\mathcal{K}$, there exists a *finite* algebra $\mathbf{Q}$ in $\mathcal{K}$ into which $\mathbf{R}$ can be embedded. It is well-known that the class $\mathcal{HA}$ of Heyting algebras has the finite embeddability property. For, if $\mathbf{R}$ is a finite partial subalgebra of a Heyting algebra $\mathbf{P}$, then the sublattice $\mathbf{Q}$ generated by the underlying set of $\mathbf{R}$ becomes a finite distributive lattice and hence is a finite Heyting algebra, into which $\mathbf{R}$ is embedded.

Suppose that a formula α is not provable in $\mathbf{L}$. Then there exist an algebra $\mathbf{P}$ in $\mathcal{V}(\mathbf{L})$ and a valuation v of $\mathbf{P}$ such that $v(\alpha) \geq 1$ doesn't hold. The set $\{v(\delta) : \delta \text{ is a subformula of } \alpha\} \cup \{0, 1\}$ forms a finite partial subalgebra $\mathbf{R}$ of $\mathbf{P}$. We assume now that the variety $\mathcal{V}(\mathbf{L})$ has the finite embeddability property. Then $\mathbf{R}$ is embedded into a finite algebra $\mathbf{Q}$ in $\mathcal{V}(\mathbf{L})$. Therefore, α is not valid in $\mathbf{Q}$. Hence we have the finite model property of $\mathbf{L}$. Thus the finite embeddability property of $\mathcal{V}(\mathbf{L})$ implies the finite model property of $\mathbf{L}$.

For the class of $\mathbf{FL_{ew}}$-algebras, the proof of the finite embeddability property becomes much more complicated than that for $\mathcal{HA}$. We will give here a brief outline of the proof. Suppose that $\mathbf{R}$ is a partial subalgebra of a $\mathbf{FL_{ew}}$-algebra $\mathbf{P}$. Let $\mathbf{M} = \langle M, \cdot, 1, \leq \rangle$ be the partially-ordered submonoid generated by the domain R of $\mathbf{R}$. The set M is not necessarily finite even if R is finite. For each $u \in M$ and $r \in R$, we define a subset $(u \rightsquigarrow r]$ of M

by $(u \rightsquigarrow r] = \{w \in M : uw \leq r\}$. Let $\overline{D}$ be the set of all subsets of M of the form $(u \rightsquigarrow r]$ with $u \in M$ and $r \in R$. For each subset X of M, define $CX = \bigcap\{Z \in \overline{D} : X \subseteq Z\}$. Then C determines a closure operation on $\wp(\mathbf{M})$ and it is shown that the collection of all C-closed subsets of M forms a $\mathbf{FL_{ew}}$-algebra $\mathbf{Q}$, into which $\mathbf{R}$ can be embedded. Moreover, we can show that $\mathbf{Q}$ is finite when R is finite. Thus, we have the following (see [6]).

THEOREM 5.6. *The class of* $\mathbf{FL_{ew}}$*-algebras has the finite embeddability property. Therefore, the logic* $\mathbf{FL_{ew}}$ *has the finite model property. This holds also for* $\mathbf{FL_{w}}$*.*

We note that though $\mathbf{FL_{e}}$ has the finite model property, the class of $\mathbf{FL_{e}}$-algebras doesn't have the finite embeddability property. Thus, the finite model property of $\mathbf{L}$ doesn't always imply the finite embeddability property of $\mathcal{V}(\mathbf{L})$.

6. Final Remarks

More than ten years have already passed since a study of nonclassical logics under the name "substructural logics" started. The study seems to be quite promising, as so many different kinds of nonclassical logics studied so far can be discussed within this framework. On the other hand, it seems to be not clear yet why it works so well, or more precisely why restrictions on "structural rules" play such a key role when logics are formalized as sequent systems. Here we will try to give some explanations for this by using the notion of residuation.

Most of people will agree that the "implication" is the most important logical connective. The meanings and purpose of implication have been argued from various philosophical and mathematical points of view. Let us suppose here that our implication satisfies the following relation in a given sequent system $\mathbf{L}$:

> $\Gamma, \alpha \Rightarrow \beta$ is provable in $\mathbf{L}$ if and only if $\Gamma \Rightarrow \alpha \to \beta$ is provable in it.

This relation can be shown by the help of the cut rule as long as we have the standard sequent rules for $\to$. We introduce now an auxiliary logical connective $*$, called fusion, to represent commas (in the left-hand side) of sequents, as we have done in §2. Then the above relation, which we call the *residuation relation*, expresses the fact that implication is the residual of fusion. Fusion is regarded as a monoid operation from the mathematical point of view, that is much more manageable than implication. Thus, once we admit the residuation relation between them, we can shift our attention from implication to fusion, and study the latter in order to know properties of the former. This shift is justified by residuation theory, which tells us that each fusion determines uniquely an implication and vice versa.

Some of the basic properties of fusions as monoid operations can be described by structural rules in sequent systems, such as the commutativity of fusions is described by exchange rules, for instance. This means that by controling structural rules we can deal with logics having various kinds of implications. This, we think, will explain the reason why the framework of substructural logics works well.

Though there is already much literature on substructural logics, we have no common understanding of the definition of substructural logics. In the present paper, we have given a definition of substructural logics *over* $\mathbf{FL_{ew}}$. But there are still many other logics that should be counted substructural logics. For example, let us consider the implicational logic **BCK** which is the implicational fragment of $\mathbf{FL_{ew}}$. This logic is not an extension of **FL**, but should be regarded as a substructural logic. Then, what is a suitable definition of substructural logics in general? Based on discussions in the present paper, it will be quite reasonable, we think, to understand substructural logics in the following way:

Substructural logics are logics of residuated structures.

We need to add a comment on the above proposal. While **BCK** has no fusion, it can be extended conservatively to a logic having the law of residuation, in fact to the logic $\mathbf{FL_{ew}}$. Because of this, we may regard **BCK** as a substructural logic in our sense. Thus, by a "logic of residuated structures" we mean a logic which can be extended conservatively to a logic in which the law of residuation holds.

If we adopt this definition, logics like the "basic logic" **B** which was introduced in [4] and its extensions without the *deductive face*, i.e. logics without the law of residuation, will not be within the scope of substructural logics in our sense, though they are of their own importance and interest.

As a matter of fact, most nonclassical logics except linear logic and relevant logics don't contain fusion explicitly as a logical connective. On the other hand, since each sequent system contains usually comma as an auxiliary symbol, which behaves as a monoid operation, we may say that it includes already fusion in an implicit way. For example, consider a cut-free sequent system of **BCK** introduced by Y. Komori in [21]. By simply adding rules for fusion to it, we get a cut-free sequent system for an extension of **BCK** with fusion, in which the residuation relation holds between fusion and implication. Note that the fact that the extended sequent system is cut-free ensures that it is a conservative extension of the original **BCK**. This shows that sequent systems are quite useful in extracting residuation relations in logics. In other words, such residuation relations will hardly be observed as long as we take Hilbert-style formalization. These observations can be summarized as follows.

> *Formalizing logics in cut-free sequent systems will reveal hidden residuation relations in them. Therefore, sequent systems are suitable for describing residuated structures.*

The structure of the partially-ordered monoid in a given residuated lattice corresponds to structural rules while the lattice structure corresponds to rules for conjunction and disjunction. These two parts are combined by implication, through the law of residuation.

As we have shown, algebraic study of substructural logics is important not only from a technical point of view but also from a conceptual one.

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