Substructural Logics - Algebraic Approach

Hiroakira Ono

RCIS, Japan Advanced Institute of Science and Technology

Calcutta Logic Circle, September 2009

Algebraic approach to logic

- Boole, de Morgan, Schroeder
- Łukasiewicz, Tarski, Lindenbaum, Rasiowa, Sikorski: Polish school
- Birkhoff, Stone, Tarski, Jónsson, Mal'cev: universal algebra
- Blok, Pigozzi, Czelakowski: abstract algebraic logic

Algebraic interpretation

Let **A** be an algebra of a suitable type for substructural logics.

A sequent $\alpha_1, \alpha_2, \ldots, \alpha_m \Rightarrow \beta$ is valid in **A** iff $f(\alpha_1 \cdot \alpha_2 \cdots \alpha_m) \leq f(\beta)$ holds for every assignment f on **A**, in symbol

$$\mathbf{A} \models \alpha_1 \cdot \alpha_2 \cdots \alpha_m \leq \beta$$

In particular, a formula β is valid in **A** iff $\mathbf{A} \models 1 \leq \beta$.

Then, what kind of algebras are suitable for substructural logics? They must be partially ordered monoids.

Residuated structures

A p.o. monoid is a structure $\langle L; \cdot, 1; \leq \rangle$ such that

- $\langle L; \leq \rangle$ is a p.o. set,
- $\langle L; \cdot, 1 \rangle$ is a monoid such that

$$x \leq y \ \Rightarrow \ xz \leq yz \text{ and } zx \leq zy.$$

A p.o. monoid is residuated if there exist division operations $\backslash$ and $/$ such that

$$xy \leq z \ \Leftrightarrow \ x \leq z/y \ \Leftrightarrow \ y \leq x\backslash z$$

Residuated lattices

Moreover, when $\langle L; \leq \rangle$ forms a lattice in a given residuated p.o. monoid , the algebra $\langle L; \wedge, \vee, \cdot, 1, \backslash, / \rangle$ is called a residuated lattice. In commutative residuated lattices, $x \backslash y = y / x$ holds always. In this case, residuals are denoted as $x \rightarrow y$.

Note that residuated lattices are equationally definable. In particular, the law of residuation is expressed by equations;

$$x(x \backslash z \wedge y) \leq z \text{ and } y \leq x \backslash (xy \vee z), \text{ etc.}$$

An **FL**-algebra is a residuated lattice with a fixed element 0. Using 0, we can introduce two negations by defining $\sim x = x \backslash 0$ and $-x = 0/x$.

Important RLs

- Lattice ordered groups: $x\backslash y = x^{-1}y, \; y/x = yx^{-1}$
- Heyting algebras: commutative residuated lattices with a least element 0 such that $x \cdot y = x \wedge y$ holds. 1 is the greatest element.
- Boolean algebras: involutive Heyting algebras, i.e. HAs with $x = --x$, where $-x = x \to 0$.

- Unital quantales: complete lattices with a monoidal operation $\cdot$ such that
$$(\bigvee x_i) \cdot y = \bigvee(x_i \cdot y) \text{ and } y \cdot (\bigvee x_i) = \bigvee(y \cdot x_i).$$
$x \backslash y$ is defined by $\bigvee\{z; x \cdot z \leq y\}$. (Similarly, for y/x.)

- RLs determined by t-norms: Each left-continuous t-norm over the unit interval [0,1] with the unit 1 is in particular a unital quantale, and thus a commutative residuated lattice. They are exactly models of fuzzy logics.

From logics to algebras

By a standard argument using Lindenbaum algebras, we can show

- a sequent is provable in **FL** iff it is valid in every **FL**-algebra.

This result can be easily generalized as follows (algebraic completeness).

For each substructural logic **L** there exists a class $\mathcal{K}$ of **FL**-algebras such that

- a sequent is provable in **L** iff it is valid in every **FL**-algebra in $\mathcal{K}$.

Varieties and equational classes

A class of algebras $\mathcal{K}$ is a variety iff it is closed under H (homomorphic images), S (subalgebras) and P (direct products).

For a given set of equations Σ, let $\mathrm{Mod}(\Sigma)$ be the class of algebras $\mathbf{A}$ such that $\mathbf{A} \models s = t$ for all $s = t$ in Σ. A class of algebras $\mathcal{K}$ is an equational class iff $\mathcal{K} = \mathrm{Mod}(\Sigma)$ for some Σ.

Birkhoff showed:

$$\text{varieties} \quad = \quad \text{equational classes}$$

Important subvarieties

For instance, the class $\mathcal{RL}$ of all residuated lattices and the class $\mathcal{FL}$ of all **FL**-algebras are varieties.

The classes of all Boolean algebras and of all Heyting algebras are subvarieties of $\mathcal{FL}$.

Each of the following equations determine important subvarieties of the variety of $\mathcal{FL}$ (cf. structural rules).

- commutativity: $x \cdot y = y \cdot x$ (or equivalently, $x \backslash y = y / x$)
- square-increasingness: $x \leq x^2$,
- integrality: $x \leq 1$,
- minimality of 0: $0 \leq x$.

From algebras to logics

Correspondences between equations and formulas

terms: $s, t, u, \ldots \quad \longmapsto \quad$ formulas: $\alpha, \beta, \ldots$

- $s = t \qquad \Longrightarrow \qquad s \leftrightarrow t$, i.e. $(s \backslash t) \wedge (t \backslash s)$
- $1 \leq \alpha$, i.e. $\alpha \wedge 1 = 1 \quad \Longleftarrow \quad \alpha$

- the variety of Boolean algebras $\longrightarrow$ classical logic
- the variety of Heyting algebras $\longrightarrow$ intuitionistic logic
- subvarieties of $\mathcal{RL}$ $(\mathcal{FL})$ $\longrightarrow$?

Logics vs algebras

Algebraization a la Lindenbaum

1. For each subvariety $\mathcal{V}$ of $\mathcal{FL}$, the set $\mathbf{L}(\mathcal{V}) = \{\alpha; \mathcal{V} \models 1 \leq \alpha\}$ forms a substructural logic.
2. Conversely, for each substructural logic $\mathbf{L}$, the set of equations $\{s \approx t; (s\backslash t) \wedge (t\backslash s) \in \mathbf{L}\}$ determines a subvariety $\mathcal{V}(\mathbf{L})$ of $\mathcal{FL}$ (completeness).
3. Moreover, these two maps $\mathbf{L}$ and $\mathcal{V}$ are dual lattice-isomorphisms.

Thus,

substructural logics are exactly logics of residuated lattices (more precisely, **FL**-algebras).

Equational consequences

The equational consequence $\{u_i = v_i; i \in I\} \models_V s = t$ of a subvariety $\mathcal{V}$ of $\mathcal{FL}$ is defined for a set of equations $\{u_i = v_i; i \in I\} \cup \{s = t\}$ by

> for each algebra $\mathbf{A}$ in $\mathcal{V}$ and each assignment f on A, $f(s) = f(t)$ holds whenever $f(u_i) = f(v_i)$ holds for all $i \in I$.

In particular, $\{u_i = v_i; 1 \leq i \leq m\} \models_V s = t$ is equivalent to the validity of the following quasi-equation in every $\mathbf{A}$ in $\mathcal{V}$.

- $(u_1 = v_1 \textit{ and } \ldots \textit{ and } u_m = v_m) \textit{ implies } s = t$.

Algebraization a la Blok-Pigozzi

The deducibility relation corresponds exactly to the equational consequence.

1. For each subvariety $\mathcal{V}$ of $\mathcal{FL}$, $\{u_i = v_i; i \in I\} \models_V s = t$ iff $\{u_i \backslash v_i \wedge v_i \backslash u_i; i \in I\} \vdash_{L(V)} s \backslash t \wedge t \backslash s$,
2. Conversely, for each substructural logic **L**, $\{\beta_j; j \in J\} \vdash_L \alpha$ iff $\{1 \leq \beta_j; j \in J\} \models_{V(L)} 1 \leq \alpha$,
3. Moreover, they are mutually inverse transformations.

In abstract algebraic logic, we say this as:

for each substructural logic **L**, $\vdash_L$ *is algebraizable and* $\mathcal{V}(\mathbf{L})$ *is an equivalent algebraic semantics for it*.

Decision problems on (quasi-) equational theories

The deducibility relation for a logic **L** is decidable iff

there is an effective procedure of deciding whether or not $\Sigma \vdash_{\mathbf{L}} \alpha$ holds for each finite set of formulas Σ and each formula α.

Algebraization theorem implies that the decision problem of the deducibility relation for a logic **L** is equivalent to the decision problem of quasi-equational theory of the corresponding variety $V(\mathbf{L})$.

The equational theory of residuated lattices is decidable.

On the other hand, since the deducibility relation for $\mathbf{FL_e}$ (without 0) is undecidable, we have

the quasi-equational theory of commutative RLs is undecidable.

Buszkowski showed that this holds even for the $\{\rightarrow, \cdot, \wedge\}$ fragment (by personal communication).

Using the undecidability of the quasi-equational theory of semigroups, Jipsen-Tsinakis showed the quasi-equational theory of $\mathcal{RL}$ is undecidable. Thus, (conversely)

> the deducibility relation of **RL** (**FL**) is undecidable.

Why sequent systems?

Why sequent systems and their structural rules play critical roles in substructural logics?

- Implication is admittedly the most important logical connective.
- In sequent formulation, a monoid operation is always introduced explicitly introduced as comma, and moreover
- implication(s) behaves exactly as its residual(s).
- Thus, different behaviors of commas, expressed usually by structural rules, will affect directly those of implications, and vice versa.

In this way, the theory of implications (divisions) can be transferred faithfully into the theory of monoids (multiplications).

"Algebra and Logic"

- Algebra and Substructural Logics: JAIST (1999, 2002), Kraków (2006)
- Patras (2004), Sorrento (2006)
- Algebraic and topological methods in non-classical logics: Barcelona (2005), Oxford (2007), Amsterdam TACL (2009)
- Order, Algebra, and Logics: Nashville (2007)
- Buenos Aires, Siena (2008)

Appendix: Kripke semantics for substructural logics

- Kripke semantics using ternary relations for relevant logics – Routley-Meyer, Fine (early 70s)
- Kripke semantics using semilattice-ordered monoids – HO-Komori (1985), Došen
- Kripke semantics for linear logics – Allwein-Dunn (1993), Hartonas
- Kripke semantics for **MTL**$\forall$ – Montagna-HO (2002)

More on algebraic studies

- Algebraic approach to cut elimination: Belardinelli-Jipsen-O, Jipsen-Tsinakis etc.
- Finte model property: Blok-van Alten, Okada-Terui
- Glivenko theorems: Galatos-O
- Interpolation properties and amalgamation properties: Kihara-O (cf. Maksimova)

Further intrinsic, strong linkages between *logic* and *algebra* will be discovered, which surely lead us deeper understanding of both logic and algebra .

Algebraic cut elimination

- Syntactic proofs of cut elimination are quite informative, as they analyze structures of proofs directly.
- On the other hand, several attempts have been made to show cut elimination in an algebraic way, e.g. Maehara, Okada, Jipsen-Tsinakis and Belardinelli-Jipsen-Ono (BJO), etc..
- Sometimes the algebraic proofs will be more flexible and may provide a wider view.

Algebraic proof of cut elimination by BJO.

- For each sequent system $\mathcal{S}_L$ of a logic **L**, partial structures, called Gentzen matrices for $\mathcal{S}_L$, are introduced.
- It is shown that each Gentzen matrix **Q** for $\mathcal{S}_L$ is quasi-embeddable into a complete algebra **B** for **L**, called a quasi completion of **Q**.

Completion & cut elimination

Our algebraic approach works well for many of known "standard" sequent systems for modal and substructural logics.

- Every algebra **A** for **L** can be regarded as a special Gentzen matrix for $\mathcal{S}_L$. In this case, its quasi completion is isomorphic to the MacNeille completion of **A**.
- Thus, when our approach works for a sequent system for **L**, the corresponding variety must be closed under the MacNeille completion.

Note that only three varieties of Heyting algebras are closed under the MacNeille completion.

Finite generation

A subvariety $\mathcal{V}$ of $\mathcal{RL}$ ($\mathcal{FL}$) is finitely generated if it is generated by its finite members. That is, any equation which fails in $\mathcal{V}$ fails also in a finite algebra in $\mathcal{V}$.

The corresponding notion in logic is the finite model property (FMP). A logic **L** has the FMP, if any formula α which is not provable in **L** there exists a finite algebra of **L** in which α fails.

Finite model property

Harrop: If a finitely axiomatizable logic has the FMP, it is decidable.

Different from modal logics, it is not easy to show the FMP of substructural logics. Thus, the FMP is not so powerful in proving the decidability.

In fact, the situation is quite twisted.

Lafont, Okada: If every proof search ends in finitely many steps in a cut-free system for a logic **L** (and hence **L** is in fact decidable), then it has the FMP.

As a collorary,

- The substructural logic **RL** (**FL**, $\mathbf{FL_e}$) has the FMP.
- The variety $\mathcal{RL}$ ($\mathcal{FL}$, $\mathcal{FL}_e$) is finitely generated.

Finite embeddability property

Another way of showing the finite generation of a variety is to prove the FEP.

A class $\mathcal{K}$ of algebras has the FEP if every finite partial algebra $\mathbf{A}$ of an algebra $\mathbf{B}$ in $\mathcal{K}$ can be embedded into a finite algebra $\mathbf{C}$ in $\mathcal{K}$.

$\mathbf{A}$ is a partial algebra of $\mathbf{B}$ if for any n-ary operation f and $a_1, \ldots, a_n \in A$;

$f^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{B}}(a_1, \ldots, a_n)$, whenever $f^{\mathbf{A}}(a_1, \ldots, a_n)$ is defined.

FEP vs FMP

The following relations hold.

- If $\mathcal{K}$ is locally finite, it has the FEP.
- If $\mathcal{K}$ has the FEP then $\mathbf{L}(\mathcal{K})$ has the FMP.
- In fact, $\mathcal{K}$ has the FEP iff every quasi-equation which fails in $\mathcal{K}$ fails in a finite algebra in $\mathcal{K}$ (SFMP) (as long as $\mathcal{K}$ is of finite type).

Moreover, a class $\mathcal{K}$ has the FEP and is finitely axiomatizable, its universal theory is decidable. Thus,

The variety $\mathcal{FL}_e$ does not have the FEP.

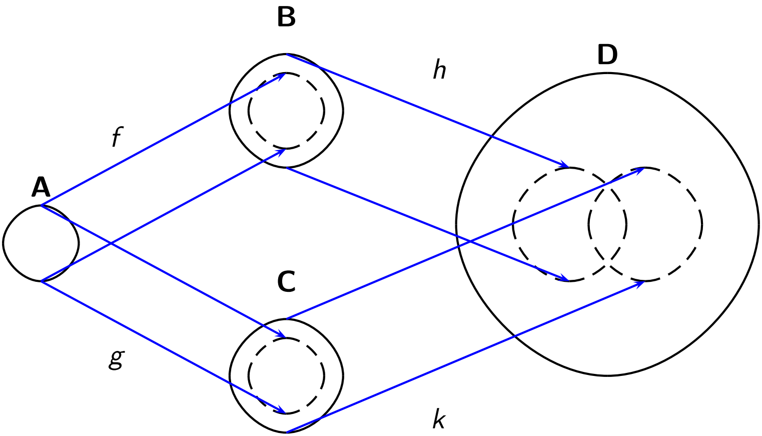
Amalgamation property

A class $\mathcal{K}$ of algebras has the amalgamation property (AP), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathcal{K}$ and for all embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ there exist an algebra $\mathbf{D}$ in $\mathcal{K}$ and embeddings $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ such that

- $h \circ f = k \circ g$.

In addition, when the following holds always for such algebras and embeddings, $\mathcal{K}$ is said to have the super AP:

if $h(y) \leq_D k(z)$ for $y \in B$ and $z \in C$ then there exists $x \in A$ such that $y \leq_B f(x)$ and $g(x) \leq_C z$.



$$h \circ f = k \circ g$$

Craig interpolation property

We introduce two types of interpolation properties.

A substructural logic **L** has the Craig interpolation property (CIP), if for any formulas ϕ and ψ, if $\vdash_{\mathbf{L}} \phi\backslash\psi$ holds then there exists a formula δ such that

- $\vdash_{\mathbf{L}} \phi\backslash\delta$ and $\vdash_{\mathbf{L}} \delta\backslash\psi$,
- $Var(\delta) \subseteq Var(\phi) \cap Var(\psi)$,

where $Var(\gamma)$ denotes the set of all propositional variables in a formula γ.

Deductive interpolation property

A substructural logic **L** has the deductive interpolation propety (DIP), if for any set of formulas $\Gamma \cup \{\psi\}$, if $\Gamma \vdash_{\mathbf{L}} \psi$ holds then there exists a formula δ such that

- $\Gamma \vdash_{\mathbf{L}} \delta$ and $\delta \vdash_{\mathbf{L}} \psi$,
- $Var(\delta) \subseteq Var(\Gamma) \cap Var(\psi)$.

Robinson property

A substructural logic **L** has the Robinson property (RP), if the following holds:

Let X, Y and Z are sets of variables such that $X = Y \cap Z$, and let $Var(\Gamma) \subseteq Y$ and $Var(\Sigma) \subseteq Z$. Moreover, suppose that for each α such that $Var(\alpha) \subseteq X$

- $\Gamma \vdash_{\mathbf{L}} \alpha$ iff $\Sigma \vdash_{\mathbf{L}} \alpha$.

Then, for any formula ψ such that $Var(\psi) \subseteq Z$,

- $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ implies $\Sigma \vdash_{\mathbf{L}} \psi$.

What are relations among them?

In general,

H.O.: RP for a logic **L** is equivalent to AP for the corresponding variety $V(\mathbf{L})$.

For logics over $\mathbf{FL_e}$ (in fact, by usinglocal deduction theorem);

- CIP implies DIP,
- DIP is equivalent to RP,
- Hence, CIP for a logic **L** implies AP for $V(\mathbf{L})$.

AP for $\mathcal{FL}_e$

S. Maehara introduced a way of showing the CIP from cut elimination. By using Maehara's method for $\mathbf{FL_e}$, we have the following.

- $\mathbf{FL_e}$ has the CIP. Hence, $\mathcal{FL}_e$ has the AP.

- Interpolation and Robinson properties in general.

$$\begin{array}{ccccc} \text{SCIP} & \Rightarrow & \text{SRP} & \Rightarrow & \text{CIP} \\ \Downarrow & & \Downarrow & & \\ \text{SDIP} & \Rightarrow & \text{RP} & \Rightarrow & \text{DIP} \end{array}$$

SCIP (SRP, SDIP) is a strong form of CIP (RP, DIP resp.)

- Interpolation and Robinson properties for logics over $\mathbf{FL_e}$.

$$\begin{array}{|ccccc|}\hline \text{SCIP} & \Leftrightarrow & \text{SRP} & \Leftrightarrow & \text{CIP} \\ \hline \end{array}$$

$$\Downarrow$$

$$\begin{array}{|ccccc|}\hline \text{SDIP} & \Leftrightarrow & \text{RP} & \Leftrightarrow & \text{DIP} \\ \hline \end{array}$$

- Interpolation and amalgamation for logics over $\mathbf{FL_e}$

$$
\begin{array}{ccccccccc}
\boxed{\text{SCIP} \quad \Leftrightarrow \quad \text{SRP} \quad \Leftrightarrow \quad \text{CIP}} & \Leftrightarrow & \text{superAP} \\
\Downarrow & & \Downarrow \\
\boxed{\text{SDIP} \quad \Leftrightarrow \quad \text{RP} \quad \Leftrightarrow \quad \text{DIP}} & \Leftrightarrow & \text{AP}
\end{array}
$$

- Connections between *algebraic* methods and *proof-theoretic* methods might be much closer than what we expected.
- Stronger linkages between *algebra* and *logic* can be discovered in future, which surely lead us deeper understanding of both algebra and logic.