Vagueness and non-Transitivity in Epistemic Logic (II)

Paul Égré

Institut Jean-Nicod, CNRS

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Denis Bonnay

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- How does it do this? double-indexing
- What else is it good for?

Reminder: CS

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2. $M, (w, w') \models_{CS} \neg\phi$ iff $M, (w, w') \not\models_{CS} \phi$
3. $M, (w, w') \models_{CS} (\phi \wedge \psi)$ iff $M, (w, w') \models_{CS} \phi$ and $M, (w, w') \models_{CS} \psi$.
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Proposition 2: $\models_{CMS} \phi$ *iff* $\vdash_{S5} \phi$

$\Rightarrow$ CMS as a logic of introspective knowledge

$\Rightarrow$ K45 and S5 are not logics of exact knowledge per se, since we can now work with non-transitive and non-euclidian models.

Proof sketch

Lemma

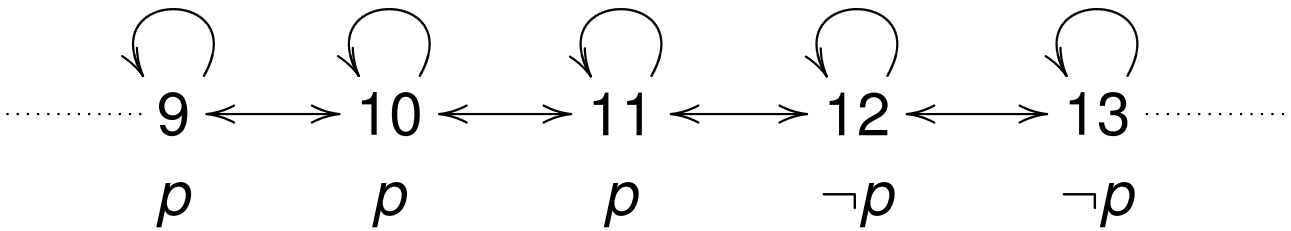
$M, w \models \phi$ iff $M, w \vDash_{CS} \phi$ for every transitive euclidian model M.

Furthermore: $K45 \vdash \phi$ iff for every transitive euclidian model M, $M \models \phi$ (completeness).

Suppose $\vDash_{CS} \phi$, yet $K45 \nvdash \phi$. Then there is a transitive euclidian model M, such that $M \not\models \phi$. By the lemma, $M, w \not\vDash_{CS} \phi$: contradiction.

Back to luminosity

Luminosity-without-triviality: $\models_{cs} \phi \rightarrow \Box\phi \not\Rightarrow \models_{cs} \phi$ or $\models_{cs} \neg\phi$



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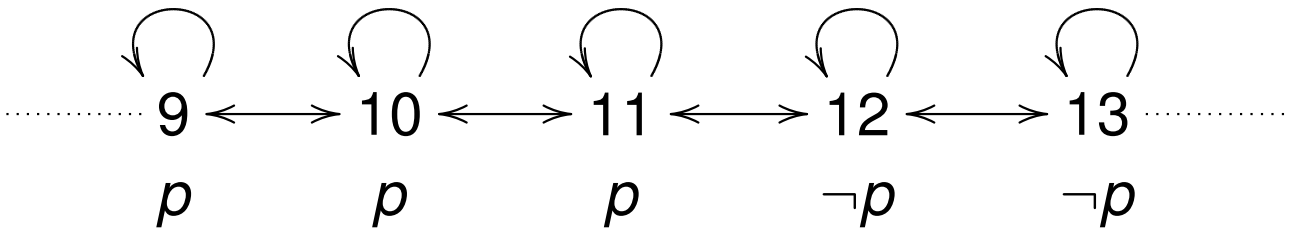

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Consequence: we can agree with Williamson that not every mental state is luminous, or even that most of our mental states are not luminous, and still disagree about knowledge (seen as a mental state).

Beyond CS: outline

1. Token semantics
2. Common Knowledge and Almost Common Knowledge
3. Multi-agent Token Semantics
4. The Email Game

Generalizing Centered Semantics

Key ideas

- CS allows one to visit only a subpart of the initial model: worlds 1 step away. What about relaxing this constraint to worlds that are 2 steps away, 3 steps,... *n* steps?

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- Motivation: think of n as the number of nestings of knowledge operators that require checking
- This number n is materialized by means of a parameter: tokens
- Enlargement of the supervenience basis of higher-order knowledge (relative to CS)

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- For each non-trivial move in a model (box or diamond), a token is spent, so not for reflexive moves, which come at no cost.
- When all tokens have been spent, get a token back, backtrack to the previous position in the model, and continue (loop).

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Evaluation of sentences with respect to a sequence of worlds (and a token):

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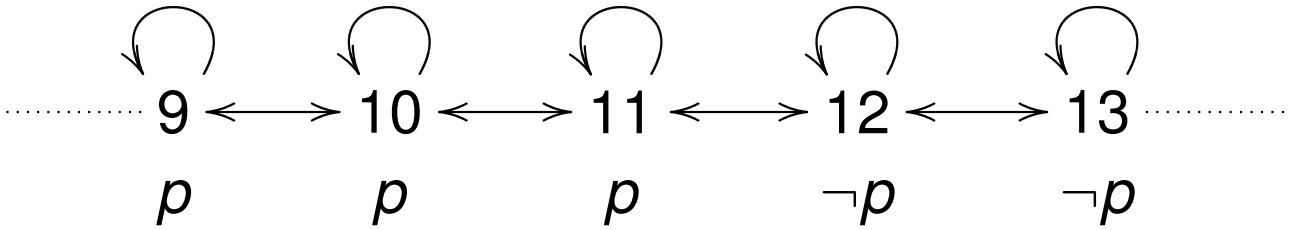

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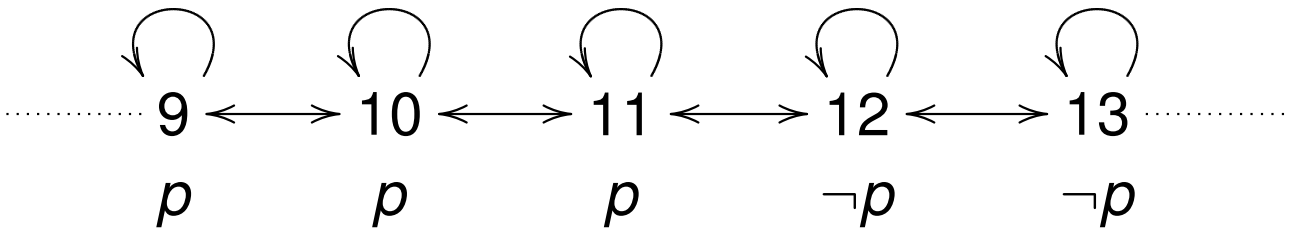

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$\Leftrightarrow (10, x) \models_{TS} \Box p$ [0] for $x = 9, 11$, and $(10, x) \models_{TS} \Box p$ [1] for $x = 10$.

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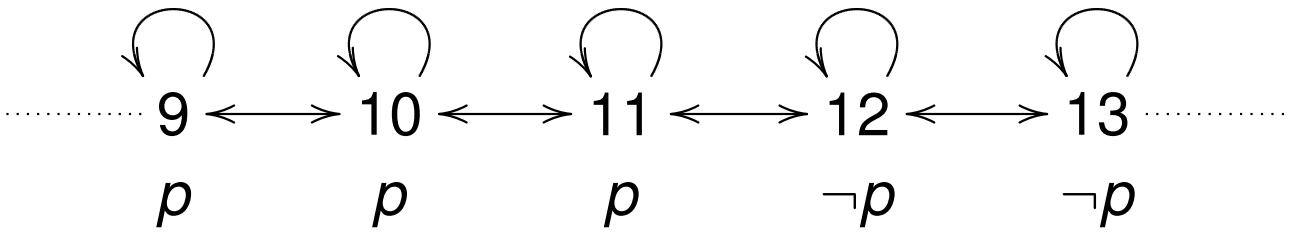

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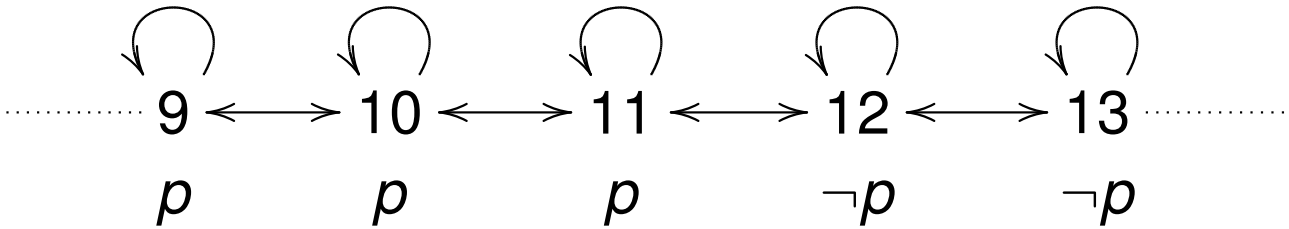
Example Cont'd



However, $10 \not\models_{TS} \Box\Box p$ [2]

otherwise we would have $10, 11 \models_{TS} \Box p$ [1]

and, $10, 11, 12 \models_{TS} p$ [0]: not so.

A spectrum of semantics

$TS(1)$	aka Centered Semantics, validates positive and negative introspection over arbitrary structures
$TS(\omega)$	aka Kripke Semantics, no introspection principles are validated
$TS(n)$	$1 < n < \omega$, aka Token Semantics, weakened versions of the introspection principles

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- The resulting logics are intermediate in strength between K45 and K

ex: $TS(2) \models \Box\Box p \rightarrow \Box\Box\Box p$
ex: $TS(3) \models \Box^3 p \rightarrow \Box\Box^3 p$
...

Example: TS(2)

(4.2'). $(\neg p_1 \wedge \Diamond(p_1 \wedge \Diamond\Diamond r)) \rightarrow \Diamond(p_1 \wedge \Diamond r)$

(5.2'). $(\neg p_1 \wedge \Diamond(p_1 \wedge \Diamond r) \rightarrow \Diamond(p_1 \wedge \Box\Diamond r)$

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- A more fine-grained control of iterations
- Interest for the multi-agent case

Common knowledge

- Shared knowledge: everyone knows that p
- Common knowledge: everyone knows that p, everyone knows that everyone knows that p, everyone knows that everyone knows that everyone knows that p, ...

Multi-agent Epistemic logic

- $\Box_a \phi$: a knows/believes ϕ
- $E_{a,b}\phi \equiv \Box_a \phi \wedge \Box_b \phi$
- $C_{a,b}\phi \equiv E_{a,b}\phi \wedge E_{a,b}E_{a,b}\phi \wedge \ldots$

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- $M, w \models E_{a,b}\phi$ iff for every w' in $(R_a \cup R_b)(w)$, $M, w' \models \phi$
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- Coordinated attack problem (dynamic, hard): 2 generals communicate sequentially; *a* send a message to *b* to say he will attack at dawn; *b* replies to *a* to confirm reception of the message; *a* replies to *b* to say he got *b*’s confirmation,...

Consecutive numbers

Kooi, van Ditmarsh, van der Hoek 2003

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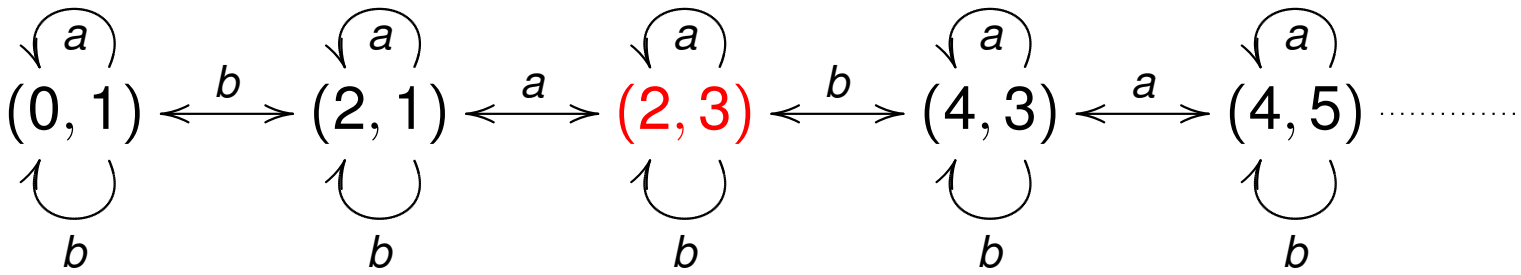

A puzzle about common knowledge

$\phi_{\leq 10000}$:= "*a* and *b*'s numbers are less than 10000"
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Common knowledge about the size of the numbers is never attained, however large the number.

Non-transitivity

- Individual accessibility relations R_a and R_b are transitive
- Not so for for $R_a \cup R_b$
- Similarity with our initial example for a single agent

Two intuitions

Step by step reasoning:

a if *b* holds a 3
he may think I hold a 4 ($\Diamond_b 4_a$)
and think that [if I hold a 4] I think he holds a 5 ($\Diamond_b \Diamond_a 5_b$)
and think I think that [if he holds a 5] he may think I hold a 6 ($\Diamond_b \Diamond_a \Diamond_b 6_a$)

Spontaneous intuition: *a* and *b* both know that both numbers are less than 100000. Each of them believes that the other believes it, and so on / that it is common knowledge

Almost common knowledge

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- In the game of consecutive numbers, the agents have almost common knowledge that the numbers are less than, say, 1000, or even 100

Proposal

- Account for situations of this kind by generalizing tokens to several agents
- Show that (almost) common knowledge can then be reduced to a finite level of shared knowledge

Multi-agent Token Semantics (2 agents)

- Main idea: use as many token registers as there are agents

$$M, qw \models_{\mathrm{MTS}} \phi \; [m_a, m_b]$$

- The semantics, informally: same as the one-agent case, but when $m_i = 0$ and $\Box_i$ is to be evaluated:

  (i) backtrack to the closest antecedent world v reached by an i-move

  (ii) pick up and reassign all tokens that were spent along the way, including for other agents.

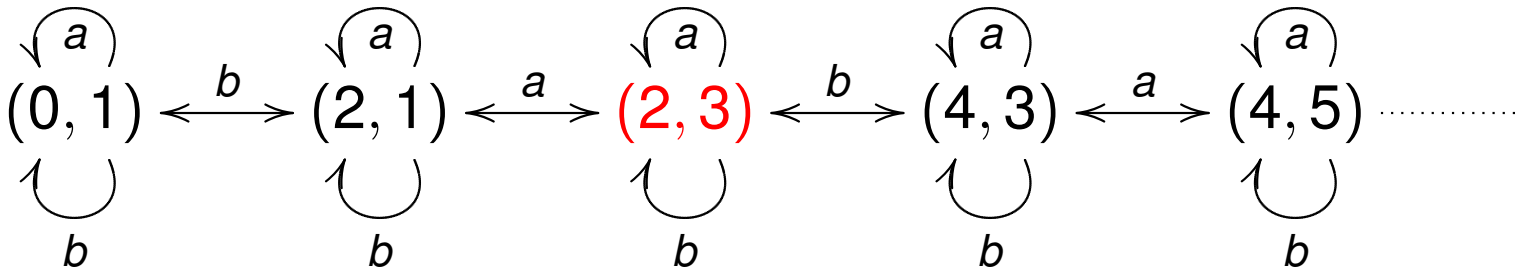
  (iii) continue.

Common Knowledge Trivialized

Theorem (trivialization)

$\models_{\mathrm{MTS}} (E_{a,b})^{\leq n+n}\phi \leftrightarrow C_{a,b}\,\phi\,[n,n]$

Example: $M,(2,3) \models_{\mathrm{MTS}} C_{a,b}\,\phi_{\leq 5}\,[1,1]$



Interpretation

How legitimate is it to equate common knowledge with some finite amount of shared knowledge?

In principle, the use of TS is neutral between two interpretations:

- Illusion of common knowledge as a side-effect of bounded rationality (agents are lazy in their computations)
  or
- Common knowledge actually reached on a finite amount of shared knowledge.

Problem: how can we tease apart the two interpretations?

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5. $C\,(n(a) \neq k+1)$

An objection against TS

If it is CK that a's number is less than $k+1$, then it is CK that it is less than k [from an. rev.]

1. $C\,(n(a) \leq k+1)$ (assumption)
2. $n(a) = k+1 \rightarrow \neg\Box_a\Box_b\,(n(a) \leq k+1)$ (structure of the game)
3. $C\,(n(a) = k+1) \rightarrow \neg\Box_a\Box_b\,(n(a) \leq k+1)$ (CK of the structure of the game)
4. $C\Box_a\Box_b\,(n(a) \leq k+1)$ (from the def. of C)
5. $C\,(n(a) \neq k+1)$

However: *TS* does not validate the inference from 2 to 3: a property can be true everywhere in a game without being CK.

Interpretation

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- The concept of CK described using TS is most likely a common illusion of common knowledge, rather than real common knowledge.
- This does not mean that such a notion is not operational for practical decisions.

The Electronic Mail Game

Rubinstein 1989

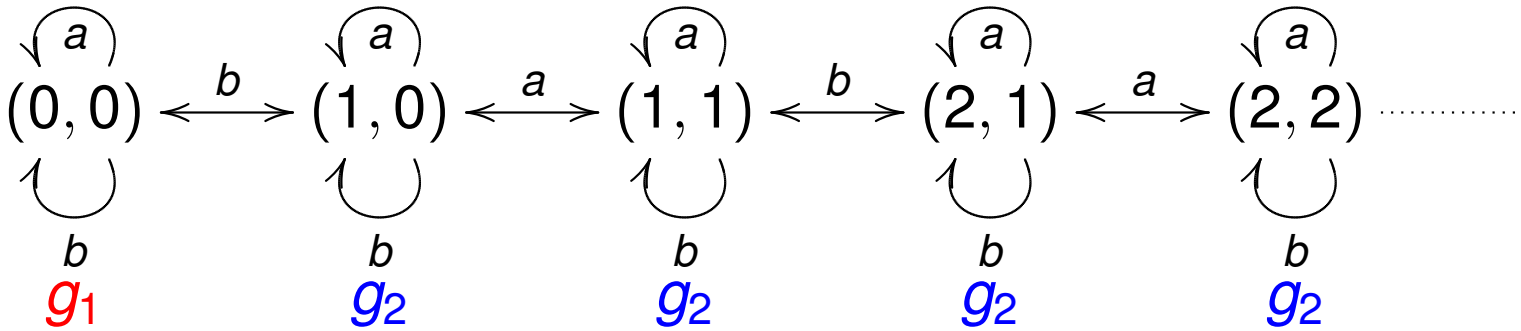
Bayesian game: Agents a and b have the choice between two actions A and B. The game is either g_1 or g_2, depending on the state of nature, which only a can observe. a sends an email to b only if the game is g_2; b's machine sends an automatic response in that case, and likewise for a. Both machines have the same probability of transmission failure ε. Each agent sees on his screen the number of messages he sent at the end of the communication process, but not the other's number.

g_1	A	B
A	10,10	0, -5
B	-5,0	0,0

g_2	A	B
A	0,0	0, -5
B	-5,0	10,10

g_1	A	B
A	10,10	0, -5
B	-5,0	0,0

g_2	A	B
A	0,0	0, -5
B	-5,0	10,10



Theorem (Rubinstein)

The email game has a unique Nash Equilibrium, in which both players always choose A.

Main ingredients of the proof:

- Induction, with base case the fact that action A is strictly dominant for *a* in the state (0,0) (when the game is g_1)
- Bayesian hypotheses in order to compute *b*'s best action in that case and in the following.

Diagnosis

Rubinstein: “the source of the discrepancy lies in the fact that mathematical induction is not part of the reasoning process of human beings”.

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- the induction proof rests crucially on the fact that the state (0,0) is a relevant epistemic alternative for at least one player
- However, it is relevant only when the numbers are sufficiently small. When the numbers are high, agents simply fail to compute knowledge iterations that would lead them too far from their respective context.

Towards a solution

- Suppose the real state of the world is (17,16), namely *a*'s last message failed. p_1= the game is G_1, and p_2= the game is G_2.
- Suppose that each agent has 2 tokens

$$(17, 16) \models_{\text{MTS}} C_{a,b} p_2 \ [2, 2]$$

Work in Progress

- Can (B,B) be derived as an interesting outcome (equilibrium?) of the game, if one makes use of the revised concept of common knowledge?
- Idea: consider the first state (m, n) from $(0, 0)$ such that it becomes CK (in MTS) that the game played is g_2. Can we prove that below (m, n), (A,A) is the equilibrium, and that from (m, n) onward, (B,B) becomes the equilibrium?

The Vagueness problem

- Arbitrariness of the number of tokens assigned to the agents: below 4 or 5 messages exchanged, agents are likely to consider (0,0) as a relevant alternative, while above 50 messages exchanged, (0,0) certainly is no longer considered relevant.
- Experimental data by Camerer & al. 2003: when the Email Game is repeated a number of times, agents gradually learn to play A after experiencing a loss on unsuccessful play of B.

Summary and conclusion

- TS: logics for introspection, bridging K and K45
- MTS: Literal implementation of the idea of bounded rationality

Perspectives

- Further applications of TS: higher-order vagueness (Egré & Bonnay forthcoming)
- Applications in game theory to work out
- Work in progress on dynamic centered semantics and learning

MTS

Evaluation relative to sequences (w, k) of ordered pairs $k = 0$ if no token is spent, $k = i$ if i spends one token.

i) $M, q(w,k) \models_{\text{MTS}} \phi\ [m_a, m_b]$ iff $w \in V(p)$.

(ii) $M, q(w,k) \models_{\text{MTS}} \neg\phi\ [m_a, m_b]$ iff $M, q(w,k) \not\models_{MTS} \phi\ [m_a, m_b]$.

(iii) $M, q(w,k) \models_{\text{MTS}} (\phi \wedge \psi)\ [m_a, m_b]$ iff $M, q(w,k) \models_{\text{MTS}} \phi$ and $M, q(w,k) \models_{\text{MTS}} \psi\ [m_a, m_b]$.

(iv) $M, q(w,k) \models_{\text{MTS}} \Box_a\psi\ [m_a, m_b]$ iff

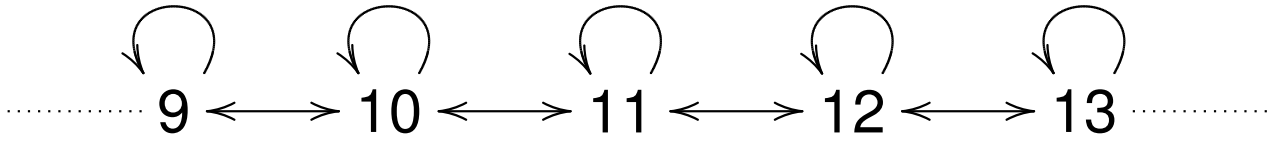
- $m_a \neq 0$ and for all w' such that wR_aw', $M, q(w,k)(w', l) \models_{\text{MTS}} \psi\ [m_a - s, m_b]$ where $(l, s) = (1, i)$ for non reflexive moves, $s = l = 0$ otherwise.
- Or $m_a = 0$ and $M, q' \models_{\text{MTS}} \Box_a\psi\ [m_a + r_a, m_b + r_b]$ with r_i=number of tokens picked up along the path to reach q' where q' is the longest initial segment of $q(w,k)$ such that (v, i) belongs to $q(w,k)$ but not q

Model-necessitation and CS

The rule of necessitation: if ϕ, then $\Box\phi$

- is standardly valid over frames and over models, namely $M \models \phi$ implies $M \models \Box\phi$ for Kripke semantics.
- is not model-valid relative to CS, although frame-valid

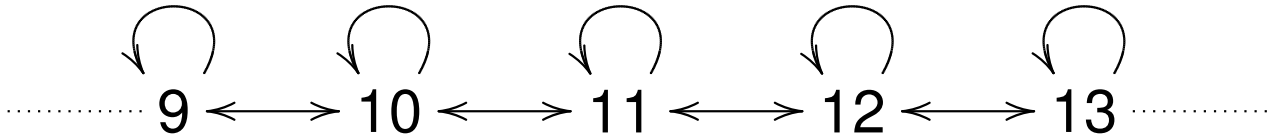
Model validity and CS



- $M \models_{CS} \Box\neg(i+1) \rightarrow \neg i$ (for $i \in \mathcal{N}$)
- but $M \not\models_{CS} \Box(\Box\neg(i+1) \rightarrow \neg i)$

Example:

Model validity and CS



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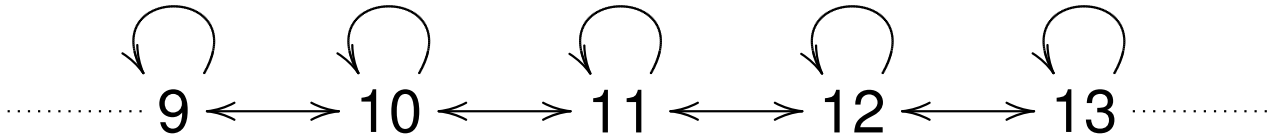
Example:

- $10 \models_{CS} \Box\neg 12 \rightarrow \neg 11$

Model validity and CS



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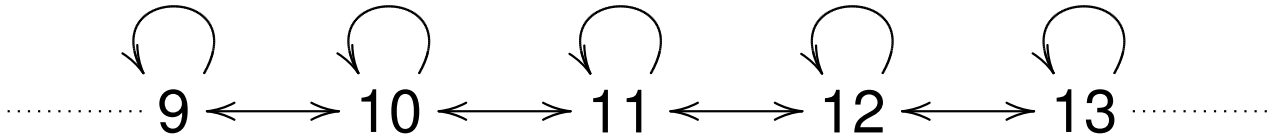

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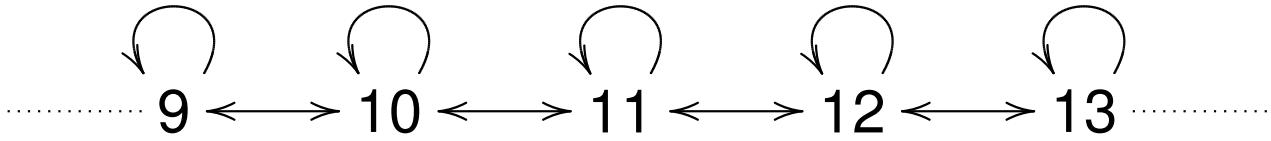
Model validity and CS



- $M \models_{CS} \Box\neg(i+1) \rightarrow \neg i$ (for $i \in \mathcal{N}$)
- but $M \not\models_{CS} \Box(\Box\neg(i+1) \rightarrow \neg i)$

Example:

- $10 \models_{CS} \Box\neg 12 \rightarrow \neg 11$
- but $10 \not\models_{CS} \Box(\Box\neg 12 \rightarrow \neg 11)$
- because $\Rightarrow 10, 11 \models_{CS} \Box\neg 12 \rightarrow \neg 11$
- yet $10, 11 \models_{CS} \Box\neg 12$, but $10, 11 \not\models_{CS} \neg 11$.

Further specificities

The correspondence properties of TS are not the expected ones. Aspects worth emphasizing concern:

- The rationale for having reflexive arrows at no cost
- the case of symmetry and the B axiom
- Frame definability properties will also shift with the new semantics (transitivity, enclideanness).