Uncertainty in Knowledge Representation : Beyond probabilistic Reasoning

Didier Dubois
Institut de Recherche en Informatique de Toulouse (IRIT)
CNRS and Universite P. Sabatier
www.irit.fr/~Didier.Dubois

1:Incompleteness and gradualness

- Typology of imperfect information
- Incomplete information : reasoning with certainty and possibility
- Paradoxes of truth-functionality
- Gradualness and many-valued logics.

INFORMATION

- Any organized collection of symbols or signs produced
  - either by **observing** natural or artificial phenomena,
  - or by the **cognitive activity of agents**
- useful for
  - *understanding our world*
  - *support decision-making*
  - *communicate with other agents*
- ***Knowledge Representation and Reasoning:*** Theories and methods whose aim is to exploit all types of available information useful for problem solving and communication using intelligent machines

A TYPOLOGY OF INFORMATION

- **NATURE**: BELIEFS VS. PREFERENCES
  - *Here, only information on how the world is, not how it should be*
- **ORIGIN**: OBJECTIVE VS. SUBJECTIVE
- **FORM:** SYMBOLIC VS NUMERICAL
- **SCOPE**: SINGULAR EVIDENCE VS. GENERIC KNOWLEDGE
- IMPERFECTIONS OF INFORMATION
  - Incompleteness
  - Indistinguishability
  - Inconsistency
  - Vagueness
  - Graduality
  - Variability
  - Uncertainty

OBJECTIVE VS. SUBJECTIVE INFORMATION

- WHERE DOES INFORMATION COME FROM?
- **Objective =** measurements from sensors (data) or other automatic observation devices and stored in databases for instance.
  - Nowadays, huge uncontrollable amounts.
  - Often numerical, but can be complex objects.
- **Subjective =** information supplied by humans
  - Testimonies, « perceptions »
  - Pieces of knowledge, text
  - Often in natural language

ELEMENTARY FORMS OF INFORMATION

- NUMERICAL (integers, real numbers, intervals, functions,…)
- QUALITATIVE (ordinal, finite ordered value scales)
- SYMBOLIC (Boolean, sets of symbols, words in natural language)
  - *Note that words in natural language may refer to continuous numerical scale and are not always Boolean, plus they may be tainted with vagueness*.
- MORE COMPLEX FORMS INVOLVING ELEMENTARY COMPONENTS:
  - Structures: graphs (Bayes nets, semantic nets, ontologies…)
  - Strings
  - Arrays….

MOTIVATION OF UNCERTAINTY FORMALISMS

- REPRESENT, MERGE AND INFER WITH VARIOUS FORMS AND TYPES OF INFORMATION, possibly:
  - **IMPRECISE** (incomplete, ill-perceived, approximate, summarized)
  - **UNRELIABLE** (hence conflicting)
  - **GRADUAL** (flexibility of natural language words, implicit preferences)
- A unified setting for representing imperfection in symbolic or numerical information, on continuous or finite (logical) universes

INCOMPLETE/IMPRECISE INFORMATION

- **= Not** sufficient to answer questions of interest **because information is lacking,** for instance
- Boolean (Logic) : A set of propositional formulas $\mathcal{B}$ (beliefs) with more than one model
  - expressed by a disjunction of models
- Numerical : The ill-known value of some quantity « x ∈ A » (interval, set)
  - Age(Paul) ∈ [20, 25]
- *What is imprecise is the content of information.*
- **Incompleteness is always modelled by sets and disjunctions** [20, 25] = 20∨21∨22∨23∨24∨25

DISJUNCTIVE SETS

Let S be the set of states of affairs, possible worlds, etc.

- **A disjunctive set** is a subset of states one *of which is the real one*.
- It may represent information possessed by an agent knowing that $x \in A$.
  - In probability theory: « events » (*the die outcome is odd*)
  - In propositional logic: sets of models of « propositions » encoded as wffs are disjunctive (e.g disjunction of interpretations)
- *Poorly expressive for generic information*

Set-valued information expresses incompleteness

- A precisely known multivalued attribute is expressed by a *conjunctive set:*
  - sisters(Paul) = Mary$\wedge$Susan

- An imprecisely known single-valued attribute is expressed by a *disjunctive set:*
  - THE-ONLY-sister(Paul): Mary$\vee$ Susan

- An imprecisely known multivalued attribute is expressed by a *disjunction of conjunctive sets:*
  - sisters(Paul) = (Mary$\wedge$Susan)$\vee$(Mary$\wedge$Ann)

Set-valued representations of incomplete information

- **The two main settings : intervals and propositional logic.**
- **Numerical:** Intervals on the real line
  - **Typical Problem** : if agent believes $x_i \in A_i$ $i = 1,\ldots n$ and wishes to know about $f(x_1 \ldots x_n)$, compute $Y = \{f(x_1 \ldots x_n), x_i \in A_i\ i = 1,\ldots n\}$

    (constraint propagation)
- **Symbolic (Boolean): propositional logic**
  - Infer new beliefs from a set $\mathcal{B}$ of wffs representing beliefs or knowledge of an agent about the world.

BOOLEAN POSSIBILITY THEORY

If all we know is that $x \in E$ then

- Event A is possible if $A \cap E \neq \emptyset$

(logical consistency)

$\Pi(A) = 1$, and 0 otherwise

- Event A is sure if $E \subseteq A$

(logical deduction)

$N(A) = 1$, and 0 otherwise

This is a simple modal logic (KD45)

THE LOGICAL SETTING FOR BELIEFS

- *Information encoded in propositional logic*
- Let $\mathcal{L}$ be propositional language based on a set of Boolean attributes: a proposition p is true (1) of false (0).
- Let $S(\mathcal{L})$ be the set of interpretations (each attribute is given a value 0 or 1)
- Let $\mathcal{B}$ be a *consistent* set of propositions the agent believes (or knows) as true.
  - If $p \in \mathcal{L}$, $A = [p]$ is the subset of states where p is true
  - $[\mathcal{B}] = \cap \{ [p], p \in \mathcal{B}\}$ is the *non-empty* disjunctive set representing the incomplete information described by $\mathcal{B}$ (the models of $\mathcal{B}$)

REASONING WITH INCOMPLETE INFORMATION :
What does an agent believe on the basis of a consistent belief set $\mathcal{B}$ under logical omniscience?
Boolean belief = 3 possible situations:

Belief state	**Logical encoding**	**Set view**	**Possible truth-values**
p is believed	$\mathcal{B}$ *implies p*	$[\mathcal{B}] \subseteq [p]$	$\{\mathbf{1}\}$
¬p is believed	$\mathcal{B}$ *implies ¬p*	$[\mathcal{B}] \subseteq [p]^c$	$\{\mathbf{0}\}$
neither p nor ¬p is believed	$\mathcal{B}$ *implies neither ¬p nor p*	$[\mathcal{B}] \cap [p] \neq \emptyset$ $[\mathcal{B}] \cap [p]^c \neq \emptyset$	$\{\mathbf{0, 1}\}$ disjunctive set

In the last line, p and ¬p are consistent with $\mathcal{B}$ *:*
ignorance state

TRUTH VS. BELIEF

- **A belief state can be encoded as a non-empty disjunctive subset of truth-values**
  - *Belief representation refers to the notion of* ***validity or provability*** *of p in the face of* $\mathcal{B}$*, not to the notion of truth of* ***p :***
  - NOT(p is believed) ≠ ¬p is believed
  - *If an agent states that « p is true », (s)he means « I know » or « believe » that p. If* $\mathcal{B}$ *is a set of believed propositions, all propositions in* $\mathcal{B}$ *bear a (hidden) modality referring to belief.*
  - *No direct access to the real world : information is always possessed by an agent.*

A 3-VALUED LOGIC CANNOT HANDLE INCOMPLETENESS

- $\mathcal{T} = \{0, 1\}$ **Truth set** : propositional variables are Boolean
- $\mathcal{BS} = \{\{0\}, \{1\}, \{0, 1\}\}$ Belief states as disjunctive sets of truth-values

- **A temptation**: $\mathcal{BS}$ is an enlarged truth-set $\{\mathbf{0}, \mathbf{1}, U\}$

where U = unknown, and $\mathbf{1} > U > \mathbf{0}$.

  - Let t be a 3-valued truth assignment to $\{\mathbf{0}, \mathbf{1}, U\}$ s. t.
  - $t(\neg p) = 1 - t(p)$,
  - $t(p \vee q) = \max(t(p), t(q))$,
  - $t(p \wedge q) = \min(t(p), t(q))$
  - (*) If p is a classical tautology then $t(p) = \mathbf{1}$

- THEN **the system collapses to classical logic.**
- (For all p, t(p) = **1** or **0**, t(p) = **1** iff t(¬p) = **0**.)
- ***Proof***:
  - *Just note that t(p∧¬p) =0 and t(p∨¬p) = 1 ; then because of (*)*:
  - max(t(p),t(¬p)) = 1 min(t(p),t(¬p)) =0
- **The presence of incompleteness in Boolean information does not question the excluded middle and contradiction laws.**
  - Such laws are incompatible with the compositionality of belief in a Boolean setting.

BELIEF, an epistemic notion
≠ TRUTH, a convention

- De Finetti (1936) on Lukasiewicz 3d truth-value:
  - “*Even if, in itself, a proposition cannot be but true or false, it may occur that a given person does not know the answer, at least at a given moment. Hence for this person, there is a third attitude in front of a proposition. This third attitude does not correspond to a third truth-value distinct from yes or no, but to the doubt between the yes and the no (as people, who, due to incomplete or indecipherable information, appear as of "unknown sex" in a given statistics. They do not constitute a third sex. They only form the group of people whose sex is unknown”).*
  - Belief (as provability) lies at the meta-level with respect to truth: $\{\mathbf{0}\}$, $\{\mathbf{1}\}$, $\{\mathbf{0}, \mathbf{1}\}$ are not truth values w. r. t. propositions in $\mathcal{B}$

BELIEF REPRESENTATION IN MODAL LOGIC

- *Boolean belief is not compositional even in propositional logic*
  - NOT(p is believed) $\neq \neg$p is believed
  - *Expressing NOT(p is believed) in the object language requires a modal logic :* $\neg[]p$ in the epistemic logic *KD45, a logic that includes axiom*
  - $[]p \rightarrow <> p$, instead of $[]p \rightarrow p$
- "p $\wedge$ q" is believed $\Leftrightarrow$ "p is believed" and "q is believed" : $[](p \wedge q) \equiv []p \wedge []q$
  - (The intersection of deductively closed belief sets is deductively closed.
- *"p is believed" or "q is believed"* only implies *"p $\vee$ q is believed":* $[]p \vee []q \rightarrow [](p \vee q)$
    - The union of deductively closed belief sets is not deductively closed.

Casting belief sets in modal epistemic logic

Proposition *(Dubois, Hajek Prade, 2000) :* if $\mathcal{B}$ is a propositional belief base and $[]\mathcal{B} = \{[]p, p \in \mathcal{B}\}$, where [] is the **belief modality**

then *B implies p classically iff* $[]\mathcal{B}$ *implies []p in KD45*.

- Not the classical extension of propositional logic to modal logic.
  - *Modal logic is NOT viewed as propositional logic + modality-prefixed formulas*,
- *Propositional logic is isomorphic to the fragment []p of a modal logic*

SPECIFICITY of MANY-VALUED LOGICS

(Lukasiewicz, Goedel, product logic, Kleene…)

- *In MVLs many-valuedness does NOT refer to belief quantification and is rather a matter of convention.*
- De Finetti (1936)   (our translation from the French):
  - "*Propositions are assigned two values, true or false, and no other, not because there "exists" an a priori truth called "excluded middle law", but because we call "propositions" logical entities built in such a way that only a yes/no answer is possible… A logic, similar to the usual one, but leaving room for three or more [truth] values, cannot aim but at compressing several ordinary propositions into a single many-valued logical entity, which may well turn out to be very useful...*".
- De Finetti = forerunner of fuzzy sets ??? a collection of sets (level-cuts) representing a single proposition ????

Problems naturally addressed by MVLs

- How to express statements involving a many-valued, possibly infinite measurement scale (non-Boolean variables) in a concise, linguistically meaningful way(fuzzy propositions)
- How to describe real-valued functions by logical sentences involving logical connectives ? (See works of Mundici and al.)
- Conditionals (if-then rules) are 3-valued entities

Logical modelling of gradual statements

- A measured quantity *height* and a logical variable "*tall*"
- How to define the truth value of proposition "*John is tall*"

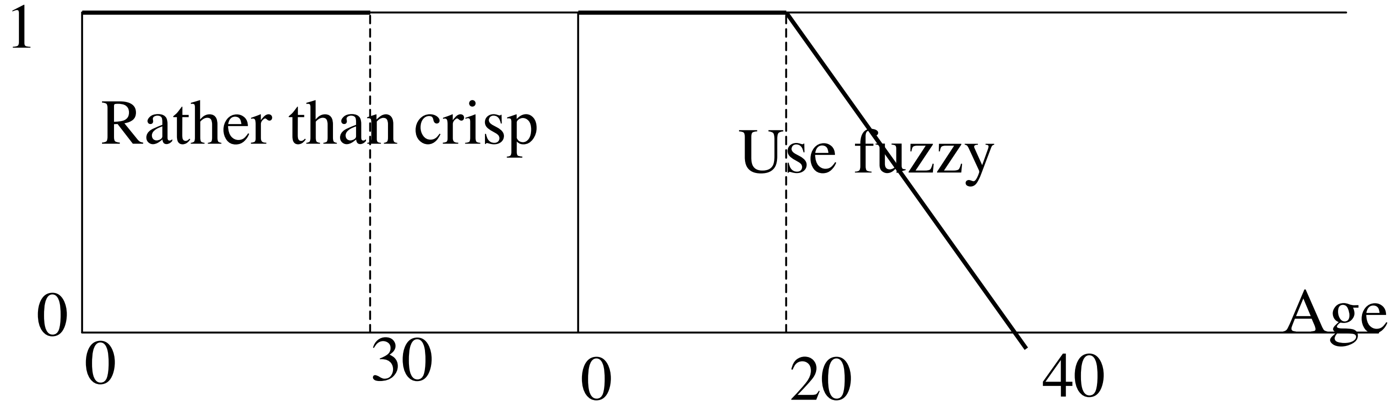
1. *Boolean Logic*: ht(John) $\in$ {tall, ¬tall} truth ("John is tall") $\in$ {0, 1}
   - Too coarse, as ht is NOT Boolean, usually
2. *More Boolean:* describe ht by many Boolean variables describing the range of ht, + logical constraints
   - Not concise, combinatorial, not linguistically plausible.
3. *Many-valued logic*: "Tall" is a single many-valued variable : The truth set reflects the RANGE of the logical variable.

LINGUISTIC GRADUAL INFORMATION

- Categories manipulated in natural language are not always all-or-nothing.
    - "Many' Americans are tall
    - Pierre and Paul have approximately the same age"
        - **PLAIN SETS ARE NOT ENOUGH**
            - » The set of YOUNG ages is ill-defined (even in a prescribed context!)
- GRADUAL (fuzzy) PREDICATES : **THEIR EXTENSIONS HAVE A NON-CRISP BOUNDARY.**
- ZADEH : USE **GRADES** OF MEMBERSHIP

Fuzzy sets

- Fuzzy set F on S : defined by a membership function $\mu_F : \forall\, s, \mu_F(s) \in [0,1]$
  - **Merits of a gradual representation** : preserving continuity makes the representation less sensitive to the choice of a threshold.
  - Example : F= YOUNG
- – **Note** : *the membership function may be ill-known: this is vagueness*



FUZZY SETS : DEFINITION

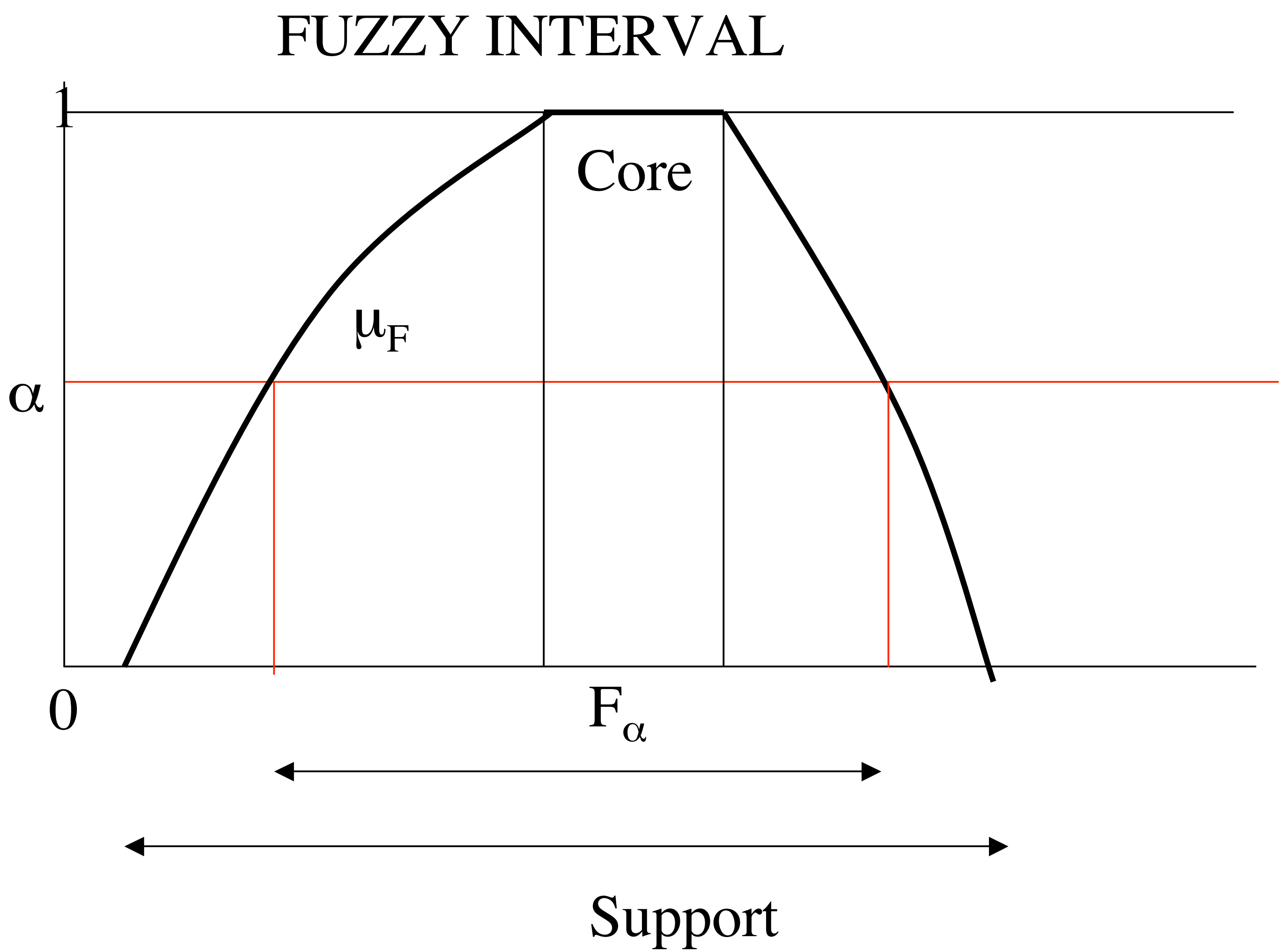
- A SET WITH GRADUAL BOUNDARIES
  - generalized characteristic function

    $\mu_F : U \rightarrow [0,1]$, (a lattice more generally)

- A WEIGHTED NESTED FAMILY OF SETS

  α-cuts $F_\alpha = \{u \mid \mu_F(u) \geq \alpha\}$

- CORE(F) $= \{u \mid \mu_F(u) = 1\}$ prototypes of F
- SUPPORT(F) $= \{u \mid \mu_F(u) > 0\}$ also includes less typical elements


FUZZY INTERVAL
1
Core
μ_F
α
0
F_α
Support

GRADUAL TRUTH

- ***A proposition involving a gradual predicate can be true to a degree between true and false***
  - (A bottle can be neither empty nor full ; a 50-year old person is old to some extent)
  - Truth(Old(Paul)) = $\mu_{Old}(55) \in (0, 1)$

- Degrees of truth can be linguistic : an individual can be « somewhat old», « rather old » « very old»
- The set of models of a gradual proposition is a fuzzy set

SORITES PARADOX

- Classical logic becomes paradoxical if gradual predicates are involved.
- Young(Paula, t)
  - *A young person remains young the next day*
  - *A baby is young*
  - *So any person is young regardless of her age*.
- **Solution** :
  - truth(Young(Paula, t)) $\in$ [0,1].
  - truth(Young(Paula, t+1)) = truth(Young(Paula), t) – ε.
- Idem : « Heap », « bald », etc…

FORMS OF GRADUALITY

The existence of gradual predicates is due to:

1. **Matching a continuous observable scale and a finite vocabulary**
   - [0, 200] cm → {short, medium, tall}(Italian person)
   - There is no infinitely precise height s* such that :

     if s > s* tall(s) is true; if s < s* tall(s) is false

- *It is not that this threshold is unknown : it simply does not exist*.
- **The truth-scale is continuous because the observable is continuous**

FORMS OF GRADUALITY

2. The notion of typicality

- Elements of a class of objects can be more or less typical of that class
  - For instance, Bird, Chair…
  - Sparrows are more typical birds than penguins
- Here: an ordinal **view of gradual membership.**
- A typicality relation $\mathbf{>_F}$ such that x $>_F$ y : x is more typically F than y

BASIC CONNECTIVES OF FUZZY LOGIC

- NEGATION : $t(\neg p) = n(t(p))$
  - $t(\neg p) = 1 - t(p)$ justified by the following axioms

  $n \circ n$ = identity ; n continuous decreasing

- CONJUNCTION (intersection):

  $$t(p \wedge q) = \mathbf{T}(t(p), t(q))$$

- DISJUNCTION (union): $t(p \vee q) = \mathbf{S}(t(p), t(q))$
- BOOLEAN ALGEBRA IS IMPOSSIBLE
  - ***There are more than one fuzzy logics !!!!***

FUZZY SET ALGEBRAS

- keep as many properties from Boolean algebras as possible:
- **If we only reject excluded middle** $p \vee q \neq \top$ **and contradiction law** $p \wedge q \neq \bot$: $\mathbf{T} = \min$, $\mathbf{S} = \max$
- if $p \vee q = \top$ and $p \wedge q = \bot$ then $p \wedge p \neq p \neq p \vee p$
  - It implies $\mathbf{T}(x, y) = \max(0, x + y - 1)$ and $\mathbf{S}(x, y) = \min(x + y, 1)$
  - Then : no mutual distributivity of $\wedge$ and $\vee$
- $\mathbf{T}(x, y) = x \cdot y$, $\mathbf{S}(x, y) = x + y - xy$: poorer structure.

Triangular Norms (t-norms) : fuzzy « And »

- They model many-valued conjunctions
- A function $\mathbf{T}$ [0,1] x [0,1] → [0,1] such that ∀ x, y, t ,z, in [0,1]:

1. $\mathbf{T}(x,y) = \mathbf{T}(y,x)$ *commutativity*
2. $\mathbf{T}(x, \mathbf{T}(y,z)) = \mathbf{T}(\mathbf{T}(x,y),z)$ *associativity*
3. $\mathbf{T}(x,y) \leq \mathbf{T}(z,t)$ if $x \leq z$ and $y \leq t$ *monotony*
4. $\mathbf{T}(x,1) = x$ *identity 1*
5. $\mathbf{T}(0,1) = 0$ *absorbing element 0*

Triangular Conorms (t-conorms) : fuzzy « Or »

- They model many-valued disjunctions
- A function S [0,1] x [0,1] → [0,1] such that ∀ x, y, t ,z, in [0,1]:
  1. **S**(x,y) = **S**(y,x) *commutativity*
  2. **S**(x, **S**(y,z)) = **S**(**S**(x,y),z) *associativity*
  3. **S**(x,y) ≤ **S**(z,t) if x ≤ z and y ≤ t *monotony*
  4. **S**(x,0) = x *identity 0*
  5. **S**(0,1) = 0 *absorbing element 0*
- Dual T-norms and T-conorms satisfy ***De Morgan law***: if **S** is a conorm then T(x,y))= n(**S**(n(s),n(y)) is a t-norm, for any negation n

Principal t-norms and t-conorms

t-norm ($\wedge$ = AND)	t-conorm ($\vee$ = OR)	negation	Name
min(x,y)	max(x,y)	1-x	Zadeh
x·y	x+y-xy	1-x	probabilistic
max (x+y-1,0)	min (x+y,1)	1-x	Lukasiewicz
x if y=1 y if x=1 else 0	x if y=0 y if x=0 else 1	1-x	drastic

Dual pairs of connectives

Properties of fuzzy connectives

- $\mathbb{T}_{dras}(x,y) \leq \mathbb{T}(x,y) \leq \min(x,y)$
- $\max(x,y) \leq \mathbb{S}(x,y) \leq \mathbb{S}_{dras}(x,y)$
- There exist parameterized families.
- **Continuous Archimedean T-norms**

  ($\mathbb{T}(x,x) < x$ for $x < 1$) are of the form

  $\mathbb{T}(x,y) = f^{-1}(\min(f(0), f(s) + f(y))$

where f is a continuous decreasing application from $[0, 1]$ to $[0, +\infty)$ with $f(1) = 0$.

SYNTACTIC FUZZY LOGICS

- *The old critique that fuzzy logics had no syntax is no longer valid since the works of P. Hajek and others*
- **Continuous t-norm/conorm logics have been axiomatized and subsume various multivalued logics :**
  - **Lukasiewicz logic**
  - **Goedel logic (max-min)**
  - **Product logic**
- There are proof systems that need efficient reasoning tools.

Hajek's axiomatic system for continuous t-norms

- *A strong conjunction & (t-norm) and its residuated implication →*
  - A1 : $(p \to q) \to ((q \to r) \to (p \to r))$ (transitivity)
  - A2 : $(p \,\&\, q) \to p$ (conjunction is strong)
  - A3 : $(p \,\&\, q) \to (q \,\&\, p)$ (commutativity of &)
  - A4 : $(p \,\&\, (p \to r)) \to (r \,\&\, (r \to p))$ (commutativity of $\wedge$)
  - A5 : $(p \to (q \to r)) \to ((p \,\&\, q) \to r)$; $((p \,\&\, q) \to r)$ $(p \to (q \to r))$
  - A6 : $((p \to q) \to r) \to (((q \to p) \to r) \to r)$. (cases)
  - A7 $\perp \to p$
- *The idempotent $p \wedge q$ is $p \,\&\, (p \to q)$*

UNCERTAINTY IN (MOST) MANY-VALUED LOGICS IS BOOLEAN

- In MVLs, p |– q means
  - « if t(p) = 1 then t(q) = 1 »
- Boolean Uncertainty = *ill-known truth-values*, e.g. All that is known is t∈ [a, b]
  - Cannot be accounted for by the basic MVL systems
- Need extension of the language to weighted systems (Pavelka)
  - Weighted formulas (p, [a, b]), etc…
- Cannot be handled by truth-functional interval-valued fuzzy sets.