A short introduction to the algebrization of logical systems and Gentzen calculi. Lecture 4: The algebraization of Substructural logics

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Algebraization of Substructural Logics

The class of algebras that corresponds to **FL** is the class of FL-algebras, closely related to *residuated lattices*.

Definition

A *residuated lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, *, \rightarrow, \leftarrow, 1\rangle$ such that

1. $\langle A, \wedge, \vee\rangle$ is a lattice,
2. $\langle A, *, 1\rangle$ is a monoid,
3. $a * b \leq c$ iff $b \leq a \rightarrow c$ iff $a \leq b \leftarrow c$, for all $a, b, c \in A$

Definition

An FL-*algebra* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, *, \rightarrow, \leftarrow, 1, 0\rangle$ such that $\langle A, \wedge, \vee, *, \rightarrow, \leftarrow, 1\rangle$ is a residuated lattice and 0 is an arbitrary element of A.

We will translate an LJ-like sequent $\Gamma \rhd \Delta$ into an inequality using a different method than the one we used in the **LJ** case, but following a similar pattern.

First we translate the sequent into a binary sequent in the way we explained in the first lecture, and then we translate the binary sequent into an inequality as we did in the cases of **LJ** and $\mathbf{LK}^K$.

We obtain the following translation τ:

$$\tau(\psi_0, \ldots \psi_{n-1} \rhd \varphi) = \psi_0 * \ldots * \psi_{n-1} \leq \varphi,$$

$$\tau(\emptyset \rhd \varphi) = 1 \leq \varphi,$$

$$\tau(\psi_0, \ldots \psi_{n-1} \rhd \varphi \rhd \emptyset) = \psi_0 * \ldots * \psi_{n-1} \leq 0.$$

Since in any *FL*-algebra an inequality $\varphi \leq \psi$ can be expressed by the equation $\varphi \wedge \psi \approx \varphi$, we can assume that inequalities "are" equations.

Theorem (Extended completeness theorem)

Let $\mathbf{S}$ *be a set of sequents and let* $\Delta \rhd \varphi$ *be a sequent. The following are equivalent:*

1. $\mathbf{S} \vdash_{\mathbf{FL}} \Gamma \rhd \Delta$,
2. $t(\mathbf{S}) \vdash_{\mathbf{FL}} t(\Gamma \rhd \Delta)$
3. $t(\mathbf{S}) \models_{\mathsf{FLA}} t(\Gamma \rhd \Delta)$.

Proof:

(1) and (2) are equivalent.

The implication from (1) to (3) can be proved by induction on the derivations in **FL** from **S**.

To prove the implication from (3) to (2) we use the Lindenbaum-Tarski's method, extended to Gentzen calculi.

Suppose $t(\mathbf{S}) \nvdash_{\mathbf{FL}} t(\Gamma \rhd \Delta)$.

We consider the set of binary sequents

$$Th_{\mathbf{LJ}}(t(\mathbf{S})) = \{\varphi \rhd \psi : t(\mathbf{S}) \vdash_{\mathbf{FL}} \varphi \rhd \psi\}.$$

Let $\mathbf{\Omega}(\mathbf{S})$ be the greatest congruence of $\mathbf{Fm}$ which is compatible with $Th_{\mathbf{LJ}}(t(\mathbf{S}))$, that is, with the property

$$\langle\varphi,\varphi'\rangle,\langle\psi,\psi'\rangle \in \mathbf{\Omega}(\mathbf{S}) \text{ and } \varphi \rhd \psi \in Th_{\mathbf{LJ}}(t(\mathbf{S})) \text{ implies } \varphi' \rhd \psi' \in Th_{\mathbf{LJ}}(t(\mathbf{S})).$$

In fact,

$$\mathbf{\Omega}(\mathbf{S}) = \{\langle\varphi,\psi\rangle : \varphi \rhd \psi, \psi \rhd \varphi \in Th_{\mathbf{LJ}}(t(\mathbf{S}))\}.$$

On the one hand, $\mathbf{\Omega}(\mathbf{S}) \subseteq \{\langle\varphi,\psi\rangle : \varphi \rhd \psi, \psi \rhd \varphi \in Th_{\mathbf{LJ}}(t(\mathbf{S}))\}$, because if $\langle\varphi,\varphi'\rangle \in \mathbf{\Omega}(\mathbf{S})$, since $\varphi \rhd \varphi, \varphi' \rhd \varphi' \in Th_{\mathbf{LJ}}(t(\mathbf{S}))$, then $\varphi' \rhd \varphi, \varphi \rhd \varphi' \in Th_{\mathbf{LJ}}(t(\mathbf{S}))$.

On the other hand, it is not difficult to show that

$$\{\langle\varphi,\psi\rangle : \varphi \rhd \psi, \psi \rhd \varphi \in Th_{\mathbf{LJ}}(t(\mathbf{S}))\}$$

is a congruence which is compatible with $Th_{\mathbf{LJ}}(t(\mathbf{S}))$.

We consider the algebra $\mathbf{Fm}/\boldsymbol{\Omega}(\mathbf{S})$. Then

1. $\mathbf{Fm}/\boldsymbol{\Omega}(\mathbf{S})$ is an FL-algebra.
2. Let v be the valuation given by:

$$v(p) = \{\varphi : \langle p, \varphi \rangle \in \boldsymbol{\Omega}(\mathbf{S})\}.$$

   Then for every sequent $\Gamma \rhd \Delta$,

$$\mathbf{Fm}/\boldsymbol{\Omega}(\mathbf{S}) \models t(\Gamma \rhd \Delta)[v] \text{ iff } t(\Gamma \rhd \Delta) \in Th_{\mathbf{LJ}}(t(\mathbf{S})).$$

To show (2) we proceed as follows:

a) If $\Gamma = \varphi_0, \ldots, \varphi_n$ and $\Delta = \varphi$

$$\begin{array}{lll}
v(\varphi_0 * \ldots * \varphi_n) \leq v(\varphi) & \text{iff} & [\varphi_0 * \ldots * \varphi_n] \leq [\varphi] \\
& \text{iff} & [\varphi_0 * \ldots * \varphi_n] \wedge [\varphi] = \\
& & [\varphi_0 * \ldots * \varphi_n] \\
& \text{iff} & [(\varphi_0 * \ldots * \varphi_n) \wedge \varphi] = \\
& & [\varphi_0 * \ldots * \varphi_n] \\
& \text{iff} & \langle (\varphi_0 * \ldots * \varphi_n) \wedge \varphi, \varphi_0 * \ldots * \varphi_n) \rangle \in \boldsymbol{\Omega}(\mathbf{S}) \\
& \text{iff} & (\varphi_0 * \ldots * \varphi_n) \wedge \varphi \rhd \varphi_0 * \ldots * \varphi_n, \\
& & \varphi_0 * \ldots * \varphi_n \rhd (\varphi_0 * \ldots * \varphi_n) \wedge \varphi \in Th_{\mathbf{LJ}}(t(\mathbf{S})) \\
& \text{iff} & \varphi_0 * \ldots * \varphi_n \rhd \varphi \in Th_{\mathbf{LJ}}(t(\mathbf{S})).
\end{array}$$

b) If $\Gamma = \emptyset$ and $\Delta = \varphi$

$$\begin{array}{lll}
1 \leq v(\varphi) & \text{iff} & [1] \leq [\varphi] \\
 & \text{iff} & [1] \wedge [\varphi] = [\varphi] \\
 & \text{iff} & [1 \wedge \varphi] = [\varphi] \\
 & \text{iff} & \langle 1 \wedge \varphi, \varphi \rangle \in \mathbf{\Omega}(\mathbf{S}) \\
 & \text{iff} & 1 \wedge \varphi \rhd \varphi, \varphi \rhd 1 \wedge \varphi \in Th_{\mathbf{LJ}}(t(\mathbf{S})) \\
 & \text{iff} & 1 \rhd \varphi \in Th_{\mathbf{LJ}}(t(\mathbf{S})).
\end{array}$$

c) The case where $\Gamma = \varphi_0, \ldots, \varphi_n$ and $\Delta = \emptyset$ is left as an exercise.

It follows that $\mathbf{Fm}/\mathbf{\Omega}(\mathbf{S}) \models t(\mathbf{S})[v]$ and $\mathbf{Fm}/\mathbf{\Omega}(\mathbf{S}) \not\models t(\Gamma \rhd \psi)[v]$.

As a corollary to the Extended completeness theorem we have:

Theorem

Let $\mathbf{S}$ *be a set of sequents and* $\Gamma \rhd \Delta$ *be a sequent. Then,*

$$\mathbf{S} \vdash_{\mathbf{FL}} \Gamma \rhd \Delta \quad \textit{iff} \quad \bigcup\{\tau(\Sigma \rhd \Sigma') : \Sigma \rhd \Sigma' \in \mathbf{S}\} \models_{\mathsf{FLA}} \tau(\Gamma \rhd \Delta).$$

Recall that any equation $\varphi \approx \psi$ can be turned into an equivalent pair of inequalities $\varphi \leq \psi$ and $\psi \leq \varphi$ and can be translated into a pair of sequents $\varphi \rhd \psi$ and $\psi \rhd \varphi$. Thus we translate an equation $\varphi \approx \psi$ into the set of sequents

$$\rho(\varphi \approx \psi) = \{\varphi \rhd \psi, \psi \rhd \varphi\}.$$

Theorem

For any equation $\varphi \approx \psi$*,*

$$\varphi \approx \psi \models_{\mathsf{FLA}} \tau(\rho(\varphi \approx \psi)) \textit{ and } \tau(\rho(\varphi \approx \psi)) \models_{\mathsf{FLA}} \varphi \approx \psi.$$

As a consequence:

Theorem

For any sequent $\Gamma \rhd \Delta$,

$$\{\Gamma \rhd \Delta\} \vdash_{\mathbf{FL}} \rho(\tau(\Gamma \rhd \Delta) \quad \textit{and} \quad \rho(\tau(\Gamma \rhd \Delta) \vdash_{\mathbf{FL}} \varphi\Gamma \rhd \Delta.$$

Theorem

For any set of equations $\{\varphi_i \approx \psi_i : i \in I\}$ *and any equation* $\varphi \approx \psi$,

$$\{\varphi_i \approx \psi_i : i \in I\} \models_{\mathsf{FLA}} \varphi \approx \psi \quad \textit{iff} \quad \bigcup_{i \in I} \rho(\varphi_i \approx \psi_i) \vdash_{\mathbf{LJ}} \rho(\varphi \approx \psi).$$

We obtained:

Theorem

LF *is algebraizable. The class of FL-algebras is its equivalent algebraic semantics and the translations are the maps* τ *and* ρ *given above.*

The extended completeness theorem implies:

Theorem

The following are equivalent:

1. $\{\varphi_i : i \in I\} \vdash^{ex}_{\mathbf{FL}} \varphi$
2. $\{\varphi_i : i \in I\} \models_{\mathsf{FLA}} \varphi$
3. $\{1 \wedge \varphi_i \approx \varphi_i : i \in I\} \models_{\mathsf{FLA}} 1 \wedge \varphi \approx \varphi$

Proof:

$$\begin{aligned} \{\varphi_i : i \in I\} \vdash^{ex}_{\mathbf{FL}} \varphi \quad & \text{iff} \quad \{\emptyset \rhd \varphi_i : i \in I\} \models_{\mathsf{FLA}} \emptyset \rhd \varphi \\ & \text{iff} \quad \{1 \wedge \varphi_i \approx \varphi_i : i \in I\} \models_{\mathsf{FLA}} 1 \wedge \varphi \approx \varphi \\ & \text{iff} \quad \{\varphi_i : i \in I\} \models_{\mathsf{FLA}} \varphi. \end{aligned}$$

Similarly to the case of the logical system gK of modal logic we obtain:

Theorem

The external logical system of **FL** *is algebraizable and the class of FL-algebras is its equivalent algebraic semantics.*

FL-filters

Every **LF**-filter of an algebra **A** is a reflexive and transitive relation.

If **A** is a FL-algebra, the least **LF**-filter of **A** is a partial order and coincides with the lattice order of **A**.

The greatest congruence which is compatible with an **LF**-filter R is

$$\boldsymbol{\Omega}_{\mathbf{A}}(R) = \{\langle a, b\rangle : \langle a, b\rangle, \langle b, a\rangle \in R\} = R \cap R^{-1}.$$

Proposition

If R is an **LF***-filter of* **A***, then* $F_R = \{a \in A : \langle 1, a\rangle \in R\}$ *is an* FL^{ex}*-filter and* $\boldsymbol{\Omega}_{\mathbf{A}}(F_R) = \boldsymbol{\Omega}_{\mathbf{A}}(R)$.

Proposition

If F is an FL^{ex}*-filter on* **A***, the* **LF***-filter* R_F *on* **A** *generated by* $\{\langle 1, a\rangle : a \in F\}$ *is such that* $F = F_{R_F}$.