Algebraic Logic: practice for classes on 16 Jan 2006

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1 Basic notions

Exercise 1.1. A relation $\vdash\ \subseteq\ \mathcal{P}(\mathbf{Fm}) \times \mathbf{Fm}$ is a *consequence relation* if for every $\Delta, \Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$:
(C1) $\varphi \in \Gamma$ implies $\Gamma \vdash \varphi$.
(C2) If $\Delta \vdash \psi$ for every $\psi \in \Gamma$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.
Show that (C1) and (C2) imply the following *monotonicity* condition:
(C3) If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$.

Exercise 1.2. For every logical system $\mathcal{L}$, the *interderivability relation* $\Lambda_{\mathcal{L}} \subseteq \mathbf{Fm} \times \mathbf{Fm}$ is defined as follows:

$$\langle \varphi, \psi \rangle \in \Lambda_{\mathcal{L}} \quad \text{iff} \quad \varphi \vdash \psi \ \text{ and } \ \psi \vdash \varphi.$$

Show that $\Lambda_{\mathcal{L}} = \{\langle \varphi, \psi \rangle \mid \text{ for every } \mathcal{L}\text{-theory } T,\ \varphi \in T \ \text{ iff } \ \psi \in T\}$.

Exercise 1.3. Consider the Gentzen system LK for classical propositional logic. Show that for every $\varphi, \varphi_0, \ldots, \varphi_n, \psi_0, \ldots, \psi_m \in \mathbf{Fm}$,
(1) $\varphi_0, \ldots, \varphi_n \rhd \psi_0, \ldots, \psi_m \quad \dashv\vdash_{\mathsf{LK}} \quad \varphi_0 \wedge \ldots \wedge \varphi_n \rhd \psi_0 \vee \ldots \vee \psi_m$.
(2) $\emptyset \rhd \varphi \quad \dashv\vdash_{\mathsf{LK}} \quad \top \rhd \varphi$.
(3) $\varphi \rhd \emptyset \quad \dashv\vdash_{\mathsf{LK}} \quad \varphi \rhd \bot$.

2 Intuitionistic propositional logic

Exercise 2.1. Consider the Hilbert-style presentation for the Intuitionistic propositional logic $\mathcal{IPL}$. Show that:
(1) $\vdash_{\mathcal{IPL}} \varphi \to \varphi$.
(2) $\{\varphi \to \psi, \psi \to \delta\} \vdash_{\mathcal{IPL}} \varphi \to \delta$.
(3) $\{\varphi_1 \to \varphi_2, \psi_1 \to \psi_2\} \vdash_{\mathcal{IPL}} (\varphi_1 \wedge \psi_1) \to (\varphi_2 \wedge \psi_2)$.
(4) $\{\varphi_1 \to \varphi_2, \psi_1 \to \psi_2\} \vdash_{\mathcal{IPL}} (\varphi_1 \vee \psi_1) \to (\varphi_2 \vee \psi_2)$.
(5) $\{\varphi_2 \to \varphi_1, \psi_1 \to \psi_2\} \vdash_{\mathcal{IPL}} (\varphi_1 \to \psi_1) \to (\varphi_2 \to \psi_2)$.

Exercise 2.2. Prove the soundness of $\mathcal{IPL}$ w.r.t. the class HA of Heyting algebras by showing that for every sequent $\Delta \rhd \varphi$, if $\Delta \rhd \varphi$ is derivable in LJ, then $\Delta \models_{\mathsf{HA}} \varphi$, taking Δ as a set.

Exercise 2.3. For every $\mathcal{IPL}$-theory T, the *Leibniz congruence* associated with T is the relation $\Omega(T) \subseteq \mathbf{Fm} \times \mathbf{Fm}$ defined as follows:

$$\langle \varphi, \psi \rangle \in \Omega(T) \quad \text{iff} \quad T \vdash_{\mathcal{IPL}} (\varphi \to \psi) \wedge (\psi \to \varphi).$$

Show that:

(1) $\Omega(T)$ is a congruence of $\mathbf{Fm}$, and moreover $\Omega(T)$ is *compatible* with T, namely that if $\langle \varphi, \psi \rangle \in \Omega(T)$ and $\varphi \in T$, then $\psi \in T$.

(2) $\Omega(T)$ is the greatest congruence of $\mathbf{Fm}$ which is compatible with T, namely, if θ is a congruence of $\mathbf{Fm}$ and for every $\varphi, \psi \in \mathbf{Fm}$, $\langle \varphi, \psi \rangle \in \theta$ and $\varphi \in T$ imply $\psi \in T$, then $\theta \subseteq \Omega(T)$.

(3) For every $\varphi \in \mathbf{Fm}$, $\langle \varphi, \top \rangle \in \Omega(T)$ iff $\varphi \in T$.

Exercise 2.4. Prove the crucial lemma for the strong completeness of $\mathcal{IPL}$ w.r.t. HA, namely, that for every $\mathcal{IPL}$-theory T, $\mathbf{Fm}/\Omega(T)$ is a Heyting algebra. Suggestion: give priority to showing that the operations $\wedge$ and $\to$ of $\mathbf{Fm}/\Omega(T)$ are residuated.

Exercise 2.5. Show that for every $\mathcal{IPL}$-theory T and every $\varphi, \psi \in \mathbf{Fm}$,

$$\varphi/\Omega(T) \leq \psi/\Omega(T) \quad \text{iff} \quad \top \vdash_{\mathcal{IPL}} \varphi \to \psi,$$

$\leq$ being the lattice order in the Heyting algebra $\mathbf{Fm}/\Omega(T)$.

Exercise 2.6. Show that the following are equivalent for every Heyting algebra $\boldsymbol{A}$ and every $F \subseteq \boldsymbol{A}$:

(a) F is a lattice filter of $\boldsymbol{A}$.

(b) F is an implicative filter of $\boldsymbol{A}$.

(c) F is an $\mathcal{IPL}$-filter of $\boldsymbol{A}$.

Exercise 2.7. Let T be an $\mathcal{IPL}$-theory. Show that:

(1) for every $\mathcal{IPL}$-theory T' such that $T \subseteq T'$, the set $T'/\Omega(T) = \{\varphi/\Omega(T) \mid \varphi \in T'\}$ is a lattice filter of $\mathbf{Fm}/\Omega(T)$.

(2) If F is a lattice filter of $\mathbf{Fm}/\Omega(T)$, then $T' = \{\varphi \mid \varphi/\Omega(T) \in F\}$ is an $\mathcal{IPL}$-theory and $T \subseteq T'$.

(3) The correspondences defined in the items above define order-isomorphisms between the set of $\mathcal{IPL}$-theories that include T and the set of filters of $\mathbf{Fm}/\Omega(T)$, both ordered by inclusion.

Exercise 2.8. Show that for every Heyting algebra $\boldsymbol{A}$, every $\mathcal{IPL}$-filter of $\boldsymbol{A}$ and every congruence θ of $\boldsymbol{A}$,

(1) $\Omega^{\boldsymbol{A}}F = \{\langle a, b\rangle \mid a \to b, b \to a \in F\}$.

(2) $F = 1/\Omega^{\boldsymbol{A}}F$.

(3) $\top/\theta$ is an $\mathcal{IPL}$-filter of $\boldsymbol{A}$.

(4) $\Omega^{\boldsymbol{A}}(\top/\theta) = \theta$.

Moreover, show that the correspondences $F \mapsto \Omega^{\boldsymbol{A}}F$ and $\theta \mapsto \top/\theta$ define order-isomorphisms between the sets $Fi_{\mathcal{IPL}}\boldsymbol{A}$ and $Con(\boldsymbol{A})$ of filters and congruences of $\boldsymbol{A}$ ordered by inclusion.

Exercise 2.9. Show that for every Heyting algebra $\boldsymbol{A}$,

(1) if $R \subseteq \boldsymbol{A} \times \boldsymbol{A}$ is an LJ-filter of $\boldsymbol{A}$, then $F_R = \{a \in \boldsymbol{A} \mid \langle\top, a\rangle \in R\}$ is an $\mathcal{IPL}$-filter of $\boldsymbol{A}$ and $\Omega^{\boldsymbol{A}}F_R = \Omega^{\boldsymbol{A}}R = R \cap R^{-1}$.

(2) If F is an $\mathcal{IPL}$-filter of $\boldsymbol{A}$, the LJ-filter R_F of $\boldsymbol{A}$ generated by $\{\langle\top, a\rangle \mid a \in F\}$ is such that $F = F_{R_F}$.