A short introduction to the algebrization of logical systems and Gentzen calculi
Lecture 2: The algebraization of Intuitionistic Logic

Ramon Jansana

Department of Logic, History and Philosophy of Science
University of Barcelona

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Outline

- The logical system of intuitionistic logic: $\mathcal{IPL}$
- The algebraization of $\mathcal{IPL}$. Heyting algebras. Lindenbaum-Tarski method
- $\mathcal{IPL}$ as an algebraizable logical system
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- Filters and congruences in Heyting algebras: logical meaning
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- The preorder given by the logical filters
- The algebraization of the Gentzen calculus **LJ**
- The concept of algebraizable Gentzen calculus

The logical system of Intuitionistic Logic

The internal logical system $\langle L, \vdash^{in}_{\mathbf{LJ}}\rangle$ of **LJ** is the system $\mathcal{IPL}$ that we introduced throughout a Hilbert style calculus, that is $\vdash^{in}_{\mathbf{LJ}} = \vdash_{\mathcal{IPL}}$.

Some properties of $\vdash_{\mathcal{IPL}}$:

Theorem[Deduction theorem]
For every set of formulas Γ and formulas φ, ψ, and δ:

$$\text{if } \Gamma, \varphi \vdash_{\mathcal{IPL}} \psi, \quad \text{then} \quad \Gamma \vdash_{\mathcal{IPL}} (\varphi \to \psi).$$

Lemma
For any formulas φ, ψ, δ, $\varphi_1, \varphi_2, \psi_1, \psi_2$ the following formulas are theorems of $\mathcal{IPL}$:

1. $\vdash_{\mathcal{IPL}} \varphi \to \varphi$
2. $\{\varphi \to \psi, \psi \to \delta\} \vdash_{\mathcal{IPL}} \varphi \to \delta$
3. $\{\varphi_1 \to \varphi_2, \psi_1 \to \psi_2\} \vdash_{\mathcal{IPL}} (\varphi_1 \wedge \psi_1) \to (\varphi_2 \wedge \psi_2)$,
4. $\{\varphi_1 \to \varphi_2, \psi_1 \to \psi_2\} \vdash_{\mathcal{IPL}} (\varphi_1 \vee \psi_1) \to (\varphi_2 \vee \psi_2)$,
5. $\{\varphi_2 \to \varphi_1, \psi_1 \to \psi_2\} \vdash_{\mathcal{IPL}} (\varphi_1 \to \psi_1) \to (\varphi_2 \to \psi_2)$.

Lemma

The following formulas are theorems of $\mathcal{IPL}$:

1. $(\varphi \to \psi) \to ((\psi \to \delta) \to (\varphi \to \delta))$
2. $(\varphi \to \psi) \to (\varphi \to (\psi \vee \delta))$
3. $\varphi \to (\psi \to (\delta \vee \varphi))$
4. $(\varphi \to (\psi \to \delta)) \to (\psi \to (\varphi \to \delta)$
5. $(\varphi \to \delta) \to ((\psi \to \delta) \to ((\varphi \vee \psi) \to \delta))$
6. $(\varphi \to \psi) \to ((\psi \to \delta) \to (\varphi \to \delta))$
7. $(\varphi \to \psi) \to (\varphi \to (\psi \vee \delta))$
8. $\varphi \to (\psi \to (\delta \vee \varphi))$
9. $(\varphi \to (\psi \to \delta)) \to (\psi \to (\varphi \to \delta)$
10. $(\varphi \to \delta) \to ((\psi \to \delta) \to ((\varphi \vee \psi) \to \delta))$

The following list of theorems of $\mathcal{IPL}$ is helpful:

1. $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
2. $(\varphi \vee \psi) \to (\psi \vee \varphi)$
3. $(\varphi \wedge (\psi \wedge \delta)) \leftrightarrow ((\varphi \wedge \psi) \wedge \delta)$
4. $(\varphi \vee (\psi \vee \delta)) \leftrightarrow ((\varphi \vee \psi) \vee \delta)$
5. $(\varphi \wedge (\psi \vee \delta)) \leftrightarrow ((\varphi \wedge \psi) \vee (\varphi \wedge \delta))$
6. $(\varphi \vee (\psi \wedge \delta)) \leftrightarrow ((\varphi \vee \psi) \wedge (\varphi \vee \delta))$
7. $\varphi \to \neg\neg\varphi$
8. $(\varphi \to (\psi \to \delta)) \leftrightarrow ((\varphi \wedge \psi) \to \delta)$
9. $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$
10. $\varphi \to (\neg\varphi \to \psi)$
11. $\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$
12. $(\neg\varphi \vee \neg\psi) \to \neg(\varphi \wedge \psi)$
13. $\perp \leftrightarrow (\varphi \wedge \neg\varphi)$
14. $\neg\varphi \leftrightarrow \neg\neg\neg\varphi$
15. $(\varphi \wedge \neg\psi) \to \neg(\varphi \to \psi)$
16. $\neg\neg(\varphi \to \psi) \leftrightarrow (\neg\neg\varphi \to \neg\neg\psi)$
17. $\neg\neg(\varphi \wedge \psi) \leftrightarrow (\neg\neg\varphi \wedge \neg\neg\psi)$

The algebraization of $\mathcal{IPL}$

$\boxed{\textbf{Heyting algebras}\text{: } \textit{algebraic semantics} \text{ of Intuitionistic logic.}}$

$\boxed{\text{The tie is as strong as it is possible}}$

The algebraization of $\mathcal{IPL}$

Heyting algebras: *algebraic semantics* of Intuitionistic logic.

The tie is as strong as it is possible

Definition

An algebra $\mathbf{A} = \langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \rightarrow^{\mathbf{A}}, \top^{\mathbf{A}}, \bot^{\mathbf{A}} \rangle$ is a Heyting algebra if

1. $\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \top^{\mathbf{A}}, \bot^{\mathbf{A}} \rangle$ is a *bounded distributive lattice*, that is,
   1.1. $\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}} \rangle$ is a distributive lattice and
   1.2. $a \vee^{\mathbf{A}} \top^{\mathbf{A}} = \top^{\mathbf{A}}$ and $a \wedge^{\mathbf{A}} \bot^{\mathbf{A}} = \bot^{\mathbf{A}}$, for every $a \in A$.
2. for every $a, b, c \in A$,

$$(a \wedge^{\mathbf{A}} c) \leq b \quad \text{iff} \quad c \leq (a \rightarrow^{\mathbf{A}} b),$$

   where $\leq$ is the lattice ordering.
   The operation $\rightarrow$ is the residuum operation of $\wedge$.

Heyting algebras are also known as pseudo Boolean algebras.

A valuation on a Heyting algebra $\mathbf{A}$: map $v : Var \to A$.
We identify valuations on $\mathbf{A}$ with homomoprhisms from $\mathbf{Fm}$ to $\mathbf{A}$.
Thus,
$v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$ means: v is a valuation on $\mathbf{A}$.

Given a Heyting algebra $\mathbf{A}$ and a valuation v on $\mathbf{A}$,

- v satisfies a formula φ if $v(\varphi) = \top^{\mathbf{A}}$
- v satisfies a set of formulas Γ if it satisfies any one of its elements.

Let HA be the class of Heyting algebras.
The the relation $\models_{\mathsf{HA}}$ between subsets of Fm and elements of Fm is defined by

$$\Gamma \models_{\mathsf{HA}} \varphi \text{ iff } (\forall \mathbf{A} \in \mathsf{HA})(\forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}))(v \text{ satisfies } \Gamma \Rightarrow v \text{ satisfies } \varphi).$$

Proposition

$\models_{\mathsf{HA}}$ *is a finitary and substitution-invariant consequence relation.*

An algebraic completeness theorem

Theorem (Soundness of $\mathcal{IPL}$)

For any set of formulas Γ and any formula φ,

$$\text{if } \Gamma \vdash_{\mathcal{IPL}} \varphi \text{ then } \Gamma \models_{\mathsf{HA}} \varphi.$$

Exercise: Prove the soundness theorem by showing that for every derivable sequent $\Delta \rhd \varphi$ in **LJ** we have $\Delta \models_{\mathsf{HA}} \varphi$, taking Δ as a set.

To prove the converse of the soundness theorem, namely, the completeness theorem, it is enough to show that if $\Gamma \nvdash_{\mathcal{IPL}} \varphi$ then there is a Heyting algebra and a valuation that satisifies Γ but not φ.

To obtain an algebra with these properties we will use

$\boxed{\textbf{the technique of Lindenbaum-Tarski algebras}.}$

Lindenbaum-Tarski algebras for $\mathcal{IPL}$

Given a theory T of $\mathcal{IPL}$ we define the binary relation $\mathbf{\Omega}(T)$ on Fm by:

$$\langle \varphi, \psi \rangle \in \mathbf{\Omega}(T) \text{ iff } T \vdash_{\mathcal{IPL}} (\varphi \to \psi) \wedge (\psi \to \varphi).$$

We abbreviate $(\varphi \to \psi) \wedge (\psi \to \varphi)$ by $\varphi \leftrightarrow \psi$.

$\mathbf{\Omega}(T)$ has the properties:

1. $\mathbf{\Omega}(T)$ is a congruence relation on the formula algebra;
2. $\mathbf{\Omega}(T)$ is *compatible* with T, namely, for every φ and ψ

   $$\text{if } \langle \varphi, \psi \rangle \in \mathbf{\Omega}(T) \text{ and } \varphi \in T, \text{ then } \psi \in T;$$

3. $\mathbf{\Omega}(T)$ is the greatest congruence compatible with T, i.e. with the above property;
4. It holds that for any formula φ,

   $$\langle \varphi, \top \rangle \in \mathbf{\Omega}(T) \quad \text{iff} \quad \varphi \in T;$$

The proof of these properties goes as follows:

1. That $\mathbf{\Omega}(T)$ is an equivalence relation follows from the definition and Lemma 3. That it is a congruence relation follows from Lemma 3.

2. That $\mathbf{\Omega}(T)$ is compatible with T follows immediately from the definition, using Modus Ponens.

3. It is left as an exercise to prove that $\mathbf{\Omega}(T)$ is the greatest congruence compatible with T.

4. It is left as an exercise

The fact that $\mathbf{\Omega}(T)$ is a congruence of $\mathbf{Fm}_L$ allows to consider the quotient algebra

$$\mathbf{Fm}/\mathbf{\Omega}(T).$$

This algebra is called the Lindenbaum-Tarski algebra of T.
We have:

(5) $\mathbf{Fm}/\mathbf{\Omega}(T)$ is a Heyting algebra.

(6) The valuation v_T on $\mathbf{Fm}/\mathbf{\Omega}(T)$ defined by

$$v_T(p) = p/\mathbf{\Omega}(T),$$

for every $p \in Var$, has the property that

$$v_T(\varphi) = \varphi/\mathbf{\Omega}(T),$$

for every formula φ.

(6) and (4) above imply:

$$v_T(\varphi) = \top^{\mathbf{Fm}/\mathbf{\Omega}(T)} \quad \text{iff} \quad \varphi \in T$$

for every formula φ.

Then the completeness theorem follows.

Theorem (Completeness of $\mathcal{IPL}$)

For any set of formulas Γ *and any formula* φ*,*

$$\Gamma \vdash_{\mathcal{IPL}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathsf{HA}} \varphi.$$

Proof.

Assume that $\Gamma \nvdash_{\mathcal{IPL}} \varphi$.

Let T be the theory generated by Γ.

Consider its Lindenbaum-Tarski algebra $\mathbf{Fm}/\boldsymbol{\Omega}(T)$ and the valuation v_T.

By the above properties, $\mathbf{Fm}/\boldsymbol{\Omega}(T)$ is a Heyting algebra.

Moreover, for every $\psi \in \Gamma$, $v_T(\psi) = \top^{\mathbf{Fm}/\boldsymbol{\Omega}(T)}$ and $v_T(\varphi) \neq \top^{\mathbf{Fm}/\boldsymbol{\Omega}(T)}$.

Thus, $\Gamma \not\models_{\mathsf{HA}} \varphi$. □

Remark

In the proof of the Completeness Theorem we have used properties (1), (2) and (4) above.
Property (3) gives a characterization of $\mathbf{\Omega}(T)$ which does not use its definition by means of $\leftrightarrow$. Thus,
given an arbitray logical system $\mathcal{L}$ we can always take (3) as a definition of a congruence associated with a theory of $\mathcal{L}$.

Remark

The relation $\mathbf{\Omega}(T)$ has an important property in Intuitionistic logic, that follows from the deduction theorem:

(7) $\mathbf{\Omega}(T)$ *is the interderivability relation modulo* T*, that is:*

$$\langle\varphi,\psi\rangle \in \mathbf{\Omega}(T) \quad \textit{iff} \quad \langle\varphi,\psi\rangle \in \Lambda_{IPC}(T) \quad \textit{iff} \quad T,\varphi \vdash_{\mathcal{IPL}} \psi \text{ and } T,\psi \vdash_{\mathcal{IPL}} \varphi.$$

When we apply the Lindenbaum-Tarski method to other logical systems it could happen that properties (1)-(4) hold but (7) doesn't. This happens in the modal logical system gK.

If in general, we would like to adopt Condition (7) as a definition of the congruence associated with a theory we will run into trouble. There are logical systems where the relation defined by (7) is not a congruence relation, gK *for instance.*

The greatest congruence which is included in the interderivability relation $\Lambda_{\mathcal{L}}(T)$ *of an arbitrary theory of* $\mathcal{L}$ *always exists: This relation plays an important role in the theory of the algebraization of the logical systems.*

Exploring the connection between Heyting algebras and $\mathcal{IPL}$ further

Heyting algebras and intuitionistic logic are deeply related.

Their relation is stronger that the one established in the Completeness Theorem.

Heyting algebras equational consequence relation

Let L be the type of Heyting algebras. Recall:

- the L-terms over Var are the L-formulas over Var.

We associate with HA an equational consequence relation $\models_{\mathsf{HA}}$ between sets of L-equations and L-equations as follows:
Let Δ be a set of equations and let $\varphi \approx \psi$ be an equation.
We define:

$$\Delta \models_{\mathsf{HA}} \varphi \approx \psi \text{ iff } \text{ for every } \mathbf{B} \in \mathsf{HA} \text{ and every valuation } v \text{ on } \mathbf{B},$$

$$\text{if } \mathbf{B} \models \Delta[v], \text{ then } \mathbf{B} \models \varphi \approx \psi[v].$$

Note that:

A valuation v on a Heyting algebra **B** satisfies a formula φ iff $v(\varphi) = v(\top)$.

Therefore, the Completeness Theorem can be restated as:

Theorem

For any set of formulas Γ *and any formula* φ,

$$\Gamma \vdash_{\mathcal{IPL}} \varphi \ \textit{ iff } \ \{\psi \approx \top : \psi \in \Gamma\} \models_{\mathsf{HA}} \varphi \approx \top.$$

$\mathcal{IPL}$ as an algebraizable logical system

In addition to the above theorem, if $\mathbf{B}$ is a Heyting algebra and $v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{B})$, then for all formulas φ and ψ,

$$\mathbf{B} \models \varphi \approx \psi[v] \text{ iff } v(\varphi) = v(\psi) \text{ iff } v(\varphi \leftrightarrow \psi) = \top^{\mathbf{B}} = v(\top) \text{ iff } \mathbf{B} \models (\varphi \leftrightarrow \psi) \approx \top[v].$$

This observation can be restated in the following form:

$$\varphi \approx \psi \models_{\mathsf{HA}} (\varphi \leftrightarrow \psi) \approx \top \quad \text{and} \quad (\varphi \leftrightarrow \psi) \approx \top \models_{\mathsf{HA}} \varphi \approx \psi.$$

We have:

Theorem

For any set $\{\psi_i \approx \varphi_i : i \in I\}$ of equations and any equation $\varphi \approx \psi$,

$$\{\psi_i \approx \varphi_i : i \in I\} \models_{\mathsf{HA}} \varphi \approx \psi \quad \textit{iff} \quad \{(\varphi_i \leftrightarrow \psi_i) : i \in I\} \vdash_{\mathcal{IPL}} (\varphi \leftrightarrow \psi).$$

Moreover,

$$\varphi \vdash_{\mathcal{IPL}} \varphi \leftrightarrow \top \quad \text{and} \quad \varphi \leftrightarrow \top \vdash_{\mathcal{IPL}} \varphi.$$

The above facts show that there are two translations,

- **translation** τ: **formulas** $\longrightarrow$ **equations**: $\quad \varphi \longmapsto \varphi \approx \top$.
- **translation** ρ: **equations** $\longrightarrow$ **formulas**: $\quad \varphi \approx \psi \longmapsto \varphi \leftrightarrow \psi$.

such that Intuitionistic logic can be faithfully interpreted into $\models_{\mathsf{HA}}$ and conversely, namely:

1. for any set of formulas Γ and any formula φ,
$$\Gamma \vdash_{\mathcal{IPL}} \varphi \text{ iff } \tau[\Gamma] \models_{\mathsf{HA}} \tau(\varphi),$$
2. for any set of equations E and any equation $\varphi \approx \psi$,
$$E \models_{\mathsf{HA}} \varphi \approx \psi \quad \text{iff} \quad \rho[E] \vdash_{\mathcal{IPL}} \rho(\varphi \approx \psi).$$

And that the translations are inverse to one another:

3. for every equation $\varphi \approx \psi$
$$\varphi \approx \psi \models_{\mathsf{HA}} \tau(\rho(\varphi \approx \psi)) \quad \text{and} \quad \tau(\rho(\varphi \approx \psi)) \models_{\mathsf{HA}} \varphi \approx \psi,$$
4. for every formula φ,
$$\varphi \models_{\mathsf{HA}} \rho(\tau(\varphi)) \quad \text{and} \quad \rho(\tau(\varphi)) \models_{\mathsf{HA}} \varphi.$$

Algebraizable logical systems

Definition (Blok-Pigozzi)

A logical system $\mathcal{L} = \langle L, \vdash_{\mathcal{L}} \rangle$ is algebraizable if there are a class of L-algebras K, a translation τ of formulas into finite sets of equations:

$$\varphi \longmapsto \tau(\varphi) \subseteq Eq(L)$$

and a translation ρ of equations into finite sets of formulas:

$$\varphi \approx \psi \longmapsto \rho(\varphi \approx \psi) \subseteq Fm_L$$

such that:

1. $\Gamma \vdash_{\mathcal{L}} \varphi$ iff $\tau[\Gamma] \models_{\mathsf{K}} \tau(\varphi)$,
2. $E \models_{\mathsf{K}} \varphi \approx \psi$ iff $\rho[E] \vdash_{\mathcal{L}} \rho(\varphi \approx \psi)$,
3. $\varphi \approx \psi \models_{\mathsf{K}} \tau(\rho(\varphi \approx))$ and $\tau(\rho(\varphi \approx)) \models_{\mathsf{K}} \varphi \approx \psi$,
4. $\varphi \models_{\mathsf{K}} \rho(\tau(\varphi))$ and $\rho(\tau(\varphi)) \models_{\mathsf{K}} \varphi$,

The class of algebras K is called an equivalent algebraic semantics for L.

If L is algebraizable, there is a greatest class of algebras which is an equivalent algebraic semantics for L. This class is always a quasivariety. It is called the equivalent algebraic semantics for L.

Filters and congruences in Heyting algebras: logical meaning

Filters, Implicative filters and Logical filters

Any Heyting algebra has a lattice reduct, thus we have the algebraic notion of (lattice-) filter.

Let $\mathbf{A} = \langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \rightarrow^{\mathbf{A}}, \top^{\mathbf{A}}, \bot^{\mathbf{A}} \rangle$ be a Heyting algebra.

A set $F \subseteq A$ is a filter if

1. $\top^{\mathbf{A}} \in F$;
2. if $a, b \in F$, then $a \wedge b \in F$;
3. if $a \in F$ and $a \leq b$, then $b \in F$.

The collection of all filters of a Heyting algebra $\mathbf{A}$ is closed under arbitrary intersections of non-empty families and contains A. It is a complete lattice, whose order is the inclusion relation.

There is an apparently different notion of filter for Heyting algebras of a more logical nature:

Let $\mathbf{A}$ be a Heyting algebra.

A non-empty subset F of A is an implicative filter if

1. $1 \in F$;
2. if $x, x \to y \in F$, then $y \in F$ (Detachment property)

The algebraic notion of filter is not apparently related to Intuitionistic logic.

A notion of filter defined purely in logical terms is that of **logical filter** of $\mathcal{IPL}$, or $\mathcal{IPL}$-filter.

Definition

Let $\mathbf{A}$ be a Heyting algebra.
$F \subseteq A$ is an *$\mathcal{IPL}$-filter* if for every valuation v on $\mathbf{A}$, every set of formulas Γ and every formula φ, if $\Gamma \vdash_{\mathcal{IPL}} \varphi$ and $v[\Gamma] \subseteq F$, then $v(\varphi) \in F$.

The $\mathcal{IPL}$-filters of a Lindenbaum-Tarski algebra $\mathbf{Fm}/\boldsymbol{\Omega}(T)$ correspond to the theories of $\mathcal{IPL}$:

$$T \mapsto \{\varphi/\boldsymbol{\Omega}(T) : \varphi \in T\} \qquad F \mapsto \{\varphi : \varphi/\boldsymbol{\Omega}(T) \in F\}$$

Interesting facts about the notion of logical filter:

1. It can have a *purely algebraic characterization*, because: For any Heyting algebra $\mathbf{A}$, the following are equivalent:
    1. F is an $\mathcal{IPL}$-filter,
    2. F is a filter,
    3. F is an implicative filter.
2. It *can be defined for every algebra*. Let $\mathbf{A}$ be an algebra. A set $F \subseteq A$ is an $\mathcal{IPL}$*-filter* if for every valuation v on $\mathbf{A}$, every set of formulas Γ and every formula φ, if $\Gamma \vdash_{\mathcal{IPL}} \varphi$ and $v[\Gamma] \subseteq F$, then $v(\varphi) \in F$.

Moreover,

- this notion **generalizes to every logical system**.

Definition

Let $\mathcal{L} = \langle L, \vdash_{\mathcal{L}} \rangle$ be a logical system and $\mathbf{A}$ and L-algebra. A set $F \subseteq A$ is a $\mathcal{L}$*-filter* if for every valuation v on $\mathbf{A}$, every set of formulas Γ and every formula φ, if $\Gamma \vdash \varphi$ and $v[\Gamma] \subseteq F$, then $v(\varphi) \in F$.

The theories of $\mathcal{L}$ are the logical filters of $\mathbf{Fm}_L$

Congruences associated with filters

In a similar way that we defined a congruence relation for every $\mathcal{IPL}$-theory, given an algebra $\mathbf{A}$ we can define a congruence for every $\mathcal{IPL}$-filter of $\mathbf{A}$.

An equivalence relation R on the carrier of $\mathbf{A}$ is said to be compatible with $F \subseteq A$ if for every $a, b \in A$ such that $\langle a, b \rangle \in R$ and $a \in F$ it holds that $b \in F$.

The Leibniz congruence of an $\mathcal{IPL}$-filter F of $\mathbf{A}$ is the greatest congruence of $\mathbf{A}$ that is compatible with F (which always exists) and it is denoted by $\boldsymbol{\Omega}^{\mathbf{A}} F$.

Theorem

Let $\mathbf{A}$ *be an algebra.*

1. *For every* $\mathcal{IPL}$*-filter* F *of* $\mathbf{A}$*,*
    1. $\mathbf{\Omega}^{\mathbf{A}}F = \{\langle a, b\rangle \in A \times A : a \to b, b \to a \in F\}$.
    2. $F = 1/\mathbf{\Omega}^{\mathbf{A}}F$.
2. *For every congruence* θ *of* $\mathbf{A}$ *such that* $\mathbf{A}/\theta$ *is a Heyting algebra,*
    1. $1/\theta$ *is a* $\mathcal{IPL}$*-filter of* $\mathbf{A}$*,*
    2. $\mathbf{\Omega}^{\mathbf{A}}(1/\theta) = \theta$.
3. *The operator* $\mathbf{\Omega}^{\mathbf{A}}(.)$ *gives an isomorphism between the lattice of* $\mathcal{IPL}$*-filters of* $\mathbf{A}$ *and the lattice of congruences* θ *of* $\mathbf{A}$ *such that* $\mathbf{A}/\theta \in \mathsf{HA}$.

As a consequence we have:

> *in any Heyting algebra* $\mathbf{A}$ *the Leibniz operator is an isomorphism between the lattice of filters of* $\mathbf{A}$ *and the lattice of congruences of* $\mathbf{A}$.

This is a known algebraic result, now with a clear logical meaning.

The class of Heyting algebras is

$$\mathsf{HA} := \{\mathbf{A}/\boldsymbol{\Omega}_{\mathbf{A}}F : F \text{ is an } \mathcal{IPL}\text{-filter of } \mathbf{A}\}.$$

and it is the equivalent algebraic semantics of $\mathcal{IPL}$.

An important property of $\mathcal{IPL}$.

Recall that for every $\mathcal{IPL}$-theory T, $\boldsymbol{\Omega}(T)$ is the interderivability relation modulo T. Similarly:

For every algebra $\mathbf{A}$ and every $\mathcal{IPL}$-filter F of $\mathbf{A}$,

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}}(F) \quad \text{iff} \quad (\forall G \in \mathrm{Fi}_{\mathcal{IPL}}\mathbf{A})(F \subseteq G \Rightarrow (a \in G \Leftrightarrow b \in G)).$$

Given an arbitrary logical system $\mathcal{L}$, an algebra $\mathbf{A}$ and an $\mathcal{L}$-filter F of $\mathbf{A}$, the clause

$$(1) \qquad (\forall G \in \mathrm{Fi}_{\mathcal{L}}\mathbf{A})(F \subseteq G \Rightarrow (a \in G \Leftrightarrow b \in G))$$

does not necessarily define a congruence relation. But there always exists the greatest congruence included in the relation defined by (1).

This relation is called the Suszko $\mathcal{L}$-congruence of F on $\mathbf{A}$. It is denoted by $\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}_{\mathcal{L}}(F)$.

Proposition

If $\mathcal{L}$ is an algebraizable logical system, then for every $\mathbf{A}$ and every $F \in \mathrm{Fi}_{\mathcal{L}}\mathbf{A}$, $\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}_{\mathcal{L}}(F) = \boldsymbol{\Omega}^{\mathbf{A}}(F)$.

There are logical systems where $\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}_{\mathcal{L}}(F)$ and $\boldsymbol{\Omega}^{\mathbf{A}}(F)$ can be different for some algebra $\mathbf{A}$ and some $F \in \mathrm{Fi}_{\mathcal{L}}\mathbf{A}$.

The class of algebras of a logical system

The Suszko congruences give the way to canonically associate a class of algebras with every logical system.
The class of algebras canonically associated with $\mathcal{L}$ is

$$\mathbf{Alg}(\mathcal{L}) = \{\mathbf{A}/\widetilde{\mathbf{\Omega}}^{\mathbf{A}}_{\mathcal{L}}(F) : F \in \mathrm{Fi}_{\mathcal{L}}\mathbf{A}\}.$$

Thus, for $\mathcal{IPL}$,

$$\mathbf{Alg}(\mathcal{IPL}) = \mathsf{HA}$$

Algebraizable logical systems II

Let $\mathcal{L}$ be a logical system. The *Leibniz congruence* of an $\mathcal{L}$-filter F of an algebra $\mathbf{A}$ is the greatest congruence of $\mathbf{A}$ that is compatible with F (which always exists) and is denoted by $\boldsymbol{\Omega}^{\mathbf{A}}F$.

If $\mathcal{L}$ is algebraizable, then

1. the greatest equivalent semantics of $\mathcal{L}$ is the class of algebras

$$\mathbf{Alg}^*(\mathcal{L}) = \{\mathbf{A}/\boldsymbol{\Omega}_{\mathbf{A}}F : F \text{ is an } \mathcal{L}\text{-filter of } \mathbf{A}\}.$$

   And

$$\mathbf{Alg}(\mathcal{L}) = \mathbf{Alg}^*(\mathcal{L}).$$

2. for every algebra $\mathbf{A}$, the operator $\boldsymbol{\Omega}^{\mathbf{A}}(.)$ gives an isomorphism between the lattice of $\mathcal{L}$-filters of $\mathbf{A}$ and the lattice of congruences θ of $\mathbf{A}$ such that $\mathbf{A}/\theta \in \mathbf{Alg}^*(\mathcal{L}) = \mathbf{Alg}(\mathcal{L})$.

The preorder given by the logical filters

Given a logical system and an algebra, the logical filters of the algebra induce a quasi-order (or preorder) on the algebra.

Let $\mathcal{L}$ be a logical system and $\mathbf{A}$ an L-algebra. The relation $\leq_{\mathcal{L}}$ on A is defined as follows:

$$a \leq_{\mathcal{L}} b \quad \text{iff} \quad (\forall F \in \mathrm{Fi}_{\mathcal{L}}\mathbf{A})(a \in F \Rightarrow b \in F)$$

This relation is always transitive and reflexive, hence a quasi-order on A. The naturally associated equivalence relation given by

$$a \sim_{\mathcal{L}}^{\mathbf{A}} b \quad \text{iff} \quad a \leq_{\mathcal{L}}^{\mathbf{A}} b \text{ and } b \leq_{\mathcal{L}}^{\mathbf{A}} a.$$

plays an important role. If $\mathbf{A} = \mathbf{Fm}_L$, then $\sim_{\mathcal{L}}^{\mathbf{Fm}_{\mathcal{L}}} = \Lambda_{\mathcal{L}}$.

For some logical systems $\mathcal{L}$, on every algebra $\mathbf{A}$, $\sim_{\mathcal{L}}^{\mathbf{A}}$ is a congruence, but for others need not be.

If $\mathbf{A}$ is a Heyting algebra, then $\leq_{\mathcal{IPL}}^{\mathbf{A}}$ is a partial order and it is the lattice order of $\mathbf{A}$. We know that in the Tarski-Lindenbaum algebra $\mathbf{Fm}/\mathbf{\Omega}(T)$, for every formulas φ, ψ,

$$\varphi/\mathbf{\Omega}(T) \leq \psi/\mathbf{\Omega}(T) \quad \text{iff} \quad T \vdash_{\mathcal{IPL}} \varphi \to \psi \quad \text{iff} \quad T, \varphi \vdash_{\mathcal{IPL}} \psi.$$

Thus,

$$\varphi/\mathbf{\Omega}(T) \leq_{IPC} \psi/\mathbf{\Omega}(T) \quad \text{iff} \quad T, \varphi \vdash_{\mathcal{IPL}} \psi.$$

The algebraization of **LJ**

Recall: the notion of derivation in **LJ** from a set of sequents defines a relation between sets of sequents and sequents, that we denote by $\vdash_{\mathbf{LJ}}$ and is defined as follows. Let $\mathbf{S}$ be a set of sequents and let $\Gamma \rhd \Delta$ be a sequent.

$$\mathbf{S} \vdash_{\mathbf{LJ}} \Gamma \rhd \Delta \quad \text{iff} \quad \text{there is derivation of } \Gamma \rhd \Delta \text{ from } \mathbf{S}.$$

If $\mathbf{S}$ and $\mathbf{S}'$ are sets of sequents, $\mathbf{S} \vdash_{\mathbf{LJ}} \mathbf{S'}$ means that for every sequent $\Gamma \rhd \Delta \in \mathbf{S}'$, $\mathbf{S} \vdash_{\mathbf{LJ}} \Gamma \rhd \Delta$.

We translate sequents into inequalities.

Obvious how to *translate* a binary sequent $\varphi \rhd \psi$ into an inequality:

$$\eta(\varphi \rhd \psi) := \varphi \leq \psi.$$

This suggest how to translate a sequent $\Delta \rhd \Gamma$ into an inequality.

$$\tau(\Delta \rhd \Gamma) := \eta(t(\Delta \rhd \Gamma).$$

Thus,

- $\tau(\emptyset \rhd \varphi) = \top \leq \varphi$
- $\tau(\varphi \rhd \emptyset) = \varphi \leq \bot$
- $\tau(\langle \psi_0, \dots \psi_{n-1} \rangle \rhd \varphi) = \psi_0 \wedge \dots \wedge \psi_{n-1} \leq \varphi$.
- $\tau(\langle \psi_0, \dots \psi_{n-1} \rangle \rhd \emptyset) = \psi_0 \wedge \dots \wedge \psi_{n-1} \leq \bot$.

Let us write $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$. With this convention, $\tau(\Gamma \rhd \Delta) = \bigwedge \Gamma \leq \bigvee \Delta$.

Since an inequality $\varphi \leq \psi$ can be expressed by the equation $\varphi \wedge \psi \approx \varphi$, we assume that inequalities "are" equations.

Theorem (Extended completeness theorem)

Let **S** *be a set of sequents and let* $\Delta \rhd \varphi$ *be a sequent. The following are equivalent:*

1. $\mathbf{S} \vdash_{\mathbf{LJ}} \Gamma \rhd \Delta$,
2. $t(\mathbf{S}) \vdash_{\mathbf{LJ}} t(\Gamma \rhd \Delta)$
3. $\tau(\mathbf{S}) \models_{\mathrm{HA}} \tau(\Gamma \rhd \Delta)$.

Proof:

We know that (1) and (2) are equivalent.

The implication from (1) to (3) can be proved by induction on the derivations in **LJ** from **S**.

To prove the implication from (3) to (2) we will use a **new version** of Lindenbaum-Tarski's method.

Suppose $t(\mathbf{S}) \nvdash_{\mathbf{LJ}} t(\Delta \rhd \varphi)$. Let us consider the set of binary sequents

$$Th_{\mathbf{LJ}}(t(\mathbf{S})) = \{\varphi \rhd \psi : t(\mathbf{S}) \vdash_{\mathbf{LJ}} \varphi \rhd \psi\}.$$

Let

$$\boldsymbol{\Omega}(\mathbf{S}) = \{\langle \varphi, \psi \rangle : \varphi \rhd \psi, \psi \rhd \varphi \in Th_{\mathbf{LJ}}(t(\mathbf{S}))\}.$$

Then

1. $\boldsymbol{\Omega}(\mathbf{S})$ is a congruence of **Fm** which is compatible with $Th_{\mathbf{LJ}}(t(\mathbf{S}))$, that is, such that

$$\text{if } \langle \varphi, \varphi' \rangle, \langle \psi, \psi' \rangle \in \boldsymbol{\Omega}(\mathbf{S}) \text{ and } \varphi \rhd \psi \in Th_{\mathbf{LJ}}, \text{then } \varphi' \rhd \psi' \in Th_{\mathbf{LJ}}.$$

2. $\boldsymbol{\Omega}(\mathbf{S})$ is the greatest congruence of **Fm** which is compatible with $Th_{\mathbf{LJ}}(t(\mathbf{S}))$.

Consider the algebra $\mathbf{Fm}/\mathbf{\Omega}(\mathbf{S})$ and the valuation v on it that sends every propositional variable p to $[p] = \{\varphi : \langle p, \varphi\rangle \in \mathbf{\Omega}(\mathbf{S})\}$. Then

1. $\mathbf{Fm}/\mathbf{\Omega}(\mathbf{S})$ is a Heyting algebra.
2. for every sequent $\Gamma \rhd \Delta$,

$$\mathbf{Fm}/\mathbf{\Omega}(\mathbf{S}) \models \tau(\Gamma \rhd \Delta)[v] \quad \text{iff} \quad t(\Gamma \rhd \Delta) \in Th_{\mathbf{LJ}}(t(\mathbf{S})).$$

The proof of (1) is left to the reader. To prove (2) we proceed as follows

$$\begin{array}{lll}
\mathbf{Fm}/\mathbf{\Omega}(\mathbf{S}) \models \tau(\Gamma \rhd \Delta)[v] & \text{iff} & v(\bigwedge \Gamma) \leq v(\bigvee \Delta) \\
& \text{iff} & [\bigwedge \Gamma] \leq [\bigvee \Delta] \\
& \text{iff} & [\bigvee \Delta] \wedge [\bigwedge \Gamma] = [\bigwedge \Gamma] \\
& \text{iff} & \langle \bigvee \Delta \wedge \bigwedge \Gamma, \bigwedge \Gamma\rangle \in \mathbf{\Omega}(\mathbf{S}) \\
& \text{iff} & \bigvee \Delta \wedge \bigwedge \Gamma \rhd \bigwedge \Gamma, \\
& & \bigwedge \Gamma \rhd \bigvee \Delta \wedge \bigwedge \Gamma \in Th_{\mathbf{LJ}}(t(\mathbf{S})) \\
& \text{iff} & \bigwedge \Gamma \rhd \bigvee \Delta \in Th_{\mathbf{LJ}}(t(\mathbf{S})) \\
& \text{iff} & t(\Gamma \rhd \Delta) \in Th_{\mathbf{LJ}}(t(\mathbf{S}))
\end{array}$$

It follows that $\tau(\mathbf{S}) \not\models_{\mathrm{HA}} \tau(\Delta \rhd \varphi)$.

Any equation $\varphi \approx \psi$ can be turned into an equivalent pair of inequalities $\varphi \leq \psi$ and $\psi \leq \varphi$ and translated into a pair of sequents $\varphi \rhd \psi$ and $\psi \rhd \varphi$.
We let

$$\rho(\varphi \approx \psi) = \{\varphi \rhd \psi, \psi \rhd \varphi\}.$$

Theorem

For any equation $\varphi \approx \psi$,

$$\varphi \approx \psi \models_{\mathsf{HA}} \tau(\rho(\varphi \approx \psi)) \text{ and } \tau(\rho(\varphi \approx \psi)) \models_{\mathsf{HA}} \varphi \approx \psi.$$

As a consequence:

Theorem

For any sequent $\Delta \rhd \varphi$,

$$\{\Delta \rhd \varphi\} \vdash_{\mathbf{LJ}} \rho(\tau(\Delta \rhd \varphi) \quad \text{and} \quad \rho(\tau(\Delta \rhd \varphi) \vdash_{\mathbf{LJ}} \varphi\Delta \rhd \varphi.$$

Theorem

For any set of equations $\{\varphi_i \approx \psi_i : i \in I\}$ and any equation $\varphi \approx \psi$,

$$\{\varphi_i \approx \psi_i : i \in I\} \models_{\mathsf{HA}} \varphi \approx \psi \quad \text{iff} \quad \bigcup \rho(\varphi_i \approx \psi_i) \vdash_{\mathbf{LJ}} \rho(\varphi \approx \psi).$$

The concept of algebraizable Gentzen calculus

A Gentzen calculus **G** is algebraizable if there is a class of algebras K, a translation τ from sequents to sets of equations and a translation ρ from equations to sets of sequents such that

1. If **S** is a set of sequents and $\Delta \rhd \Gamma$ is a sequent, then,

$$\mathbf{S} \vdash_{\mathbf{G}} \Delta \rhd \Gamma \quad \text{iff} \quad \bigcup\{\tau(\Gamma \rhd \Sigma) : \Gamma \rhd \Sigma \in \mathbf{S}\} \models_{\mathsf{K}} \tau(\Delta \rhd \Gamma).$$

2. For any equation $\varphi \approx \psi$,

$$\varphi \approx \psi \models_{\mathsf{K}} \tau(\rho(\varphi \approx \psi)) \text{ and } \tau(\rho(\varphi \approx \psi)) \models_{\mathsf{K}} \varphi \approx \psi.$$

(1) and (2) imply:

(3) For any set of equations $\{\varphi_i \approx \psi_i : i \in I\}$ and any equation $\varphi \approx \psi$,

$$\{\varphi_i \approx \psi_i : i \in I\} \models_{\mathsf{K}} \varphi \approx \psi \quad \text{iff} \quad \bigcup_{i \in I} \rho(\varphi_i \approx \psi_i) \vdash_{\mathbf{G}} \rho(\varphi \approx \psi).$$

(4) For any sequent $\Delta \rhd \Gamma$,

$$\{\Delta \rhd \Gamma\} \vdash_{\mathbf{G}} \rho(\tau(\Delta \rhd \Gamma) \quad \text{and} \quad \rho(\tau(\Delta \rhd \Gamma) \vdash_{\mathbf{G}} \Delta \rhd \Gamma.$$

LJ-filters

Let $\mathbf{A}$ be an algebra and v be a valuation on $\mathbf{A}$.
Then we set

$$v(\varphi \rhd \psi) = \langle v(\varphi), v(\psi)\rangle,$$

for every binary sequent $\varphi \rhd \psi$.

A binary relation R on A is an LJ-filter of $\mathbf{A}$ if for every valuation v on $\mathbf{A}$, every set of binary sequents $\mathbf{S}$ and every binary sequent $\varphi \rhd \psi$, if $\mathbf{S} \vdash_{\mathbf{LJ}} \varphi \rhd \psi$ and $v[\mathbf{S}] \subseteq R$, then $v(\varphi \rhd \psi) \in R$.

Every LJ-filter of $\mathbf{A}$ is a reflexive and transitive relation. If $\mathbf{A}$ is a Heyting algebra, it is a partial order.

The greatest congruence which is compatible with an LJ-filter R is

$$\mathbf{\Omega}^{\mathbf{A}}(R) = \{\langle a, b\rangle : \langle a, b\rangle, \langle b, a\rangle \in R\} = R \cap R^{-1}.$$

In fact the operator $\mathbf{\Omega}^{\mathbf{A}}(.)$ gives an isomorphism between the lattice of LJ-filters of $\mathbf{A}$ and the lattice of congruences θ of $\mathbf{A}$ such that $\mathbf{A}/\theta$ is a Heyting algebra.

There is a strong relation between LJ-filters and $\mathcal{IPL}$-filters.

Proposition

Let $\mathbf{A}$ *be an algebra. If* R *is an LJ-filter then* $F_R = \{a \in A : \langle \top, a \rangle \in R\}$ *is an* $\mathcal{IPL}$*-filter and* $\boldsymbol{\Omega}_{\mathbf{A}}(F_R) = \boldsymbol{\Omega}_{\mathbf{A}}(R)$*.*

Proposition

If F *is an* $\mathcal{IPL}$*-filter on* $\mathbf{A}$*, the LJ-filter* R_F *on* $\mathbf{A}$ *generated by* $\{\langle \top, a \rangle : a \in F\}$ *is such that* $F = F_{R_F}$*.*

Theorem

For every algebra $\mathbf{A}$ *the lattices of* $\mathcal{IPL}$*-ilters,* **LJ***-filters and congruences* θ *such that* $\mathbf{A}/\theta \in \mathrm{HA}$ *are isomorphic.*