A short introduction to the algebrization of logical systems and Gentzen calculi.
Lecture 1: Basic concepts

Ramon Jansana

Department of Logic, History and Philosophy of Science
University of Barcelona

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Introduction

We will study the algebraization process of logical systems and Gentzen calculi throughout examples.
The method is a generalization of Lindenbaum-Tarski method.

We will arrive at defining:

- the class of algebras associated with a logical system.
- the class of algebras associated with a Gentzen calculus.
- the concept of algebraizable logical system.
- the concept of algebraizable Gentzen calculus.

In the process we will study the role played by

- the logical filters of an algebra
- the congruence associated with a logical filter
- their algebraic characterizations.

Ouline

- Languages, formulas.
- Algebras
- Logical systems. Examples
- Gentzen calculi. Examples
- The internal logical system of a Gentzen calculus
- The external logical system of a Gentzen calculus

A logic, whatever formal definition we take, consists of at least two parts:

1. the language (symbols and formulas),
2. the means to obtain,
    1. a set of formulas: the **theorems** or logical validities,
    2. a relation between sets of formulas and formulas: the **consequence relation**.

The stress in the literature is frequently put on (a).

(b) always gives in a unique way (a): the theorems are the consequences of the emptyset.

$\boxed{\text{The consequence relation is central to the general theory of logics.}}$

In this course

$\boxed{\text{logics will essentially be consequence relations.}}$

Some examples

1. Classical logic

   - Language: $\wedge, \vee, \rightarrow, \bot, \top$
     (negation can be defined as $\neg\varphi := \varphi \rightarrow \bot$),
   - Set of tautologies
   - Consequence relation.

Obtained either by means of a calculus (Gentzen-style, natural deduction, Hilbert-style, etc.) or from the semantics provided by the assignments of truth-values to the propositional variables and the truth-tables.

An assignment of truth values is a map $v : Var \rightarrow \{0, 1\}$.

Every assignment is extended to a map from Fm_L to $\{0, 1\}$ in the usual way ($\bot$ is sent to 0 and $\top$ to 1) using the truth tables for the connectives.

An assignment v satisfies φ if $v(\varphi) = 1$.

An assignment v satisfies a set of formulas Σ if v satisfies every $\varphi \in \Sigma$.

A formula φ is a *consequence* of a set of formulas Σ, $\Sigma \models_{CPL} \varphi$, if every assignment of truth values v that satisfies Σ also satisfies φ.

2. Intuitionistic logic

- Language: $\wedge, \vee, \rightarrow, \bot, \top$ (negation defined as $\neg\varphi := \varphi \rightarrow \bot$),
- Set of theorems
- Consequence relation

Usually defined by a Hilbert style calculus, by a Gentzen calculus, or a natural deduction calculus.

Axioms:

A0. $\top$
A1. $\varphi \rightarrow (\psi \rightarrow \varphi)$
A2. $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$
A3. $(\varphi \wedge \psi) \rightarrow \varphi$
A4. $(\varphi \wedge \psi) \rightarrow \psi$
A5. $\varphi \rightarrow (\varphi \vee \psi)$
A6. $\psi \rightarrow (\varphi \vee \psi)$
A7. $(\varphi \vee \psi) \rightarrow ((\varphi \rightarrow \delta) \rightarrow ((\psi \rightarrow \delta) \rightarrow \delta))$
A8. $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow (\psi \rightarrow \delta)) \rightarrow (\varphi \rightarrow \delta))$
A9. $\bot \rightarrow \varphi$

Rule of inference: Modus Ponens

$$\frac{\varphi, \varphi \rightarrow \psi}{\psi}$$

If Γ is a set of formulas, a proof from Γ is a finite sequence of formulas such that each formula in the sequence is

- an element of Γ or
- an axiom or
- is obtained from previous formulas in the sequence by an application of Modus ponens.

A proof is a proof of its last component.

The relation $\vdash_{\mathcal{IPL}}$ is defined by:

$$\Gamma \vdash_{\mathcal{IPL}} \varphi \quad \text{iff} \quad \text{there is a proof of } \varphi \text{ from } \Gamma$$

3. Normal modal logics K, $S4$, etc.

- Language: $\wedge, \vee, \rightarrow, \perp, \top, \Box$
  (we define $\neg\varphi := \varphi \rightarrow \perp$ and $\Diamond\varphi := \neg\Box\neg\varphi$),
- Set of valid formulas.

Tehy can be obtained throughout the semantics of Kripke models.
Usually two consequence relations are considered: the local and the global.
Standard definition in the modal logic literature:

A normal modal logic is a **set** of formulas $\mathbb{L}$ such that

1. every substitution instance of a tautology of CPL belongs to $\mathbb{L}$,
2. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi) \in \mathbb{L}$, for all φ, ψ
3. if $\varphi \in \mathbb{L}$, every substitution instance of φ belongs to $\mathbb{L}$,
4. if $\varphi, \varphi \rightarrow \psi \in \mathbb{L}$, $\psi \in \mathbb{L}$,
5. if $\varphi \in \mathbb{L}$, then $\Box\varphi \in \mathbb{L}$.
6. $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi \in \mathbb{L}$.

The least normal modal logic is known as K, in honor to Saul Kripke.

The local consequence of $\mathbb{L}$, is the relation $\vdash_{l\mathbb{L}}$:

$$\Sigma \vdash_{l\mathbb{L}} \varphi \text{ iff } \varphi \in \mathbb{L} \text{ or there are } \varphi_0, \ldots, \varphi_n \in \Sigma \text{ s.t. } \varphi_0 \wedge \ldots \wedge \varphi_n \to \varphi \in \mathbb{L}.$$

The global consequence of $\mathbb{L}$, is the relation $\vdash_{g\mathbb{L}}$:

$$\Sigma \vdash_{g\mathbb{L}} \varphi \quad \text{iff} \quad \Box^{<\omega}\Sigma \vdash_{l\mathbb{L}} \varphi.$$

where

$$\Box^{<\omega}\Sigma = \{\Box^n\varphi : \varphi \in \Sigma\},$$

and

$$\Box^0\varphi := \varphi \qquad \Box^{n+1}\varphi := \Box\Box^n\varphi.$$

Remark

For almost all $\mathbb{L}$ the relations $\vdash_{l\mathbb{L}}$ and $\vdash_{g\mathbb{L}}$ are different.

Propositional languages. Formulas

A propositional language is a set of *connectives*.

Each connective has an *arity*: a natural number.

The 0-ary connectives are called *propositional constants*.

In the most familiar languages the arities are $0, 1, 2$.

Let L be a set of connectives.

The formulas of L (or L-formulas) are defined inductively starting from a set *Var* of propositional variables.

We assume

- *Var* is disjoint from L.
- the cardinality of *Var* is the cardinality of $\mathbb{N}$
- *Var* enumerated in a one-to-one way: $p_0, p_1, p_2, \ldots$.

Definition of L-formula:

1. Every propositional variable p is an L-formula,
2. every 0-ary connective, or propositional constant, is an L-formula,
3. if $\star$ is an n-ary connective (with $n > 0$) and $\varphi_1, \dots \varphi_n$ are L-formulas, then $\star\varphi_1 \dots \varphi_n$ is an L-formula.

If $\star$ is binary we use infix notation $(\varphi \star \psi)$ instead of $\star\varphi\psi$.

Notation: Fm_L: set of L-formulas

If L is countable, then Fm_L has the cardinality of $\mathbb{N}$.
We assume a fixed one-to-one enumeration of the L-formulas.

A substitution is a map $\sigma : Var \to Fm_L$.
If $\varphi \in Fm_L$ and σ is a substitution, $\sigma(\varphi)$ is the formula obtained by replacing simultaneously the variables that occur in φ, say $p_0, \dots, p_n$, by $\sigma(p_0), \dots, \sigma(p_n)$, respectively.

A substitution instance of φ is any formula $\sigma(\varphi)$ for a substitution σ.

Algebras

Let L be a set of connectives.

From an algebraic point of view L is a **set of function symbols**.

Then

1. the L-terms are identified with the L-formulas
2. we have a notion of L-algebra
3. we have a notion of L-equation
4. we have a notion of satisfaction of an L-equation on an L-algebra under a valuation.

An L-algebra is a tuple $\mathbf{A} = \langle A, \langle f^{\mathbf{A}} \rangle_{f \in L} \rangle$ where

1. A is a nonempty set, the *domain* or *carrier* of $\mathbf{A}$,
2. for each $f \in L$, if n is its arity, then $f^{\mathbf{A}} : A^n \to A$.

An L-equation is a pair $\langle \varphi, \psi \rangle$ of L-formulas, usually written as $\varphi \approx \psi$.

The set of L-formulas can be turned into the L-algebra, $\mathbf{Fm}_L$, of formulas:

1. The domain is Fm_L,
2. For every connective $\star \in L$, $\star^{\mathbf{Fm}_L}$ is defined by:

$$\star^{\mathbf{Fm}_L}(\varphi_1, \ldots, \varphi_n) = \star\varphi_1 \ldots \varphi_n, \text{ if } n \text{ is the arity of } \star \text{ and } n > 0$$

and by

$$\star^{\mathbf{Fm}_L} = \star, \text{ if } \star \text{ is 0-ary.}$$

This algebra is the absoultey free L-algebra generated by Var,

A valuation on $\mathbf{A}$ is a map $v : Var \to A$.
Due to the universal mapping property of $\mathbf{Fm}_L$, v is extended to every L-formula in a unique way, giving an homomorphism from $\mathbf{Fm}_L$ to $\mathbf{A}$.

We use $\mathrm{Hom}(\mathbf{Fm}_L, \mathbf{A})$ to denote the set of valuations on $\mathbf{A}$.

$v(\varphi)$ denotes the value of φ under (the extension of) the valuation v on $\mathbf{A}$.

We say that $\mathbf{A}$ satisfies $\varphi \approx \psi$ under a valuation v, if $v(\varphi) = v(\psi)$.
We write $\mathbf{A} \models \varphi \approx \psi[v]$.
We say that $\mathbf{A}$ satisfies a set of equations Π under a valuation v if it satisfies every $\varphi \approx \psi \in \Pi$ under v. We write $\mathbf{A} \models \Pi[v]$.

Logical systems

A consequence relation on a propositional language L is a relation $\vdash$ between sets of L-formulas and formulas (a subset of $\mathcal{P}(Fm_L) \times L$) such that:

(C1) if $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$,

(C2) if for all $\psi \in \Gamma$, $\Delta \vdash \psi$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.

(C1) and (C2) imply:

(C3) if $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$.

$\vdash$ is finitary when

(C4) if $\Gamma \vdash \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \varphi$.

A consequence relation $\vdash$ on L is substitution-invariant when

(C5) if $\Gamma \vdash \varphi$, then for every substitution σ $\sigma[\Gamma] \vdash \sigma(\varphi)$.

Definition

A logical system is a pair $\mathcal{L} = \langle L, \vdash \rangle$ where

- L is a propositional language,
- $\vdash$ a finitary and substitution-invariant consequence relation on L.

Let $\mathcal{L} = \langle L, \vdash \rangle$ be a logical system.

A theory of $\mathcal{L}$, or *$\mathcal{L}$-theory*, is a set of formulas Γ closed under $\vdash$, that is, such that

$$\Gamma = \{\varphi : \Gamma \vdash \varphi\}.$$

- $\mathrm{Th}\mathcal{L}$ denotes the set of $\mathcal{L}$-theories.
- $\mathrm{Th}\mathcal{L}$ is closed under arbitrary intersections. Therefore it is a complete lattice.

Note that:

$$\varphi \vdash_{\mathcal{L}} \psi \quad \text{iff} \quad (\forall T \in \mathrm{Th}\mathcal{L})(\varphi \in T \Rightarrow \psi \in T).$$

The *interderivability relation* $\Lambda_{\mathcal{L}}$ of $\mathcal{L}$ is defined as follows:

$$\langle \varphi, \psi \rangle \in \Lambda_{\mathcal{L}} \quad \text{iff} \quad \varphi \dashv_{\mathcal{L}}\vdash \psi$$

Given an $\mathcal{L}$-theory, *the interderivability relation $\Lambda_{\mathcal{L}} T$ of $\mathcal{L}$ relative to T* is defined by

$$\langle \varphi, \psi \rangle \in \Lambda_{\mathcal{L}} T \quad \text{iff} \quad T, \varphi \dashv_{\mathcal{L}}\vdash \psi, T$$

Note that

$$\langle \varphi, \psi \rangle \in \Lambda_{\mathcal{L}} \qquad \text{iff} \qquad (\forall T \in \mathrm{Th}\mathcal{L})\ (\varphi \in T \Leftrightarrow \psi \in T).$$

and

$$\langle \varphi, \psi \rangle \in \Lambda_{\mathcal{L}} T \qquad \text{iff} \qquad (\forall T' \in \mathrm{Th}\mathcal{L})(T \subseteq T' \Rightarrow (\varphi \in T \Leftrightarrow \psi \in T)).$$

Examples of logical systems

CLASSICAL LOGIC:

$L = \{\wedge, \vee, \rightarrow, \bot, \top\}$.

The relation $\models_{CPL}$ is a finitary and substitution-invariant consequence relation on Fm_L.

The logical system $\mathcal{CPL} = \langle L, \models_{CPL} \rangle$ is classical propositional logic.

NORMAL MODAL LOGICS AS LOGICAL SYSTEMS:

$L = \{\wedge, \vee, \rightarrow, \bot, \top, \Diamond, \Box\}$.

Let $\mathbb{L}$ a normal modal logic.

The local logical system of $\mathbb{L}$ is $l\mathbb{L} = \langle L, \vdash_{l\mathbb{L}} \rangle$

The global logical system of $\mathbb{L}$ is $g\mathbb{L} = \langle L, \vdash_{g\mathbb{L}} \rangle$

INTUITIONISTIC LOGIC:

$L = \{\wedge, \vee, \rightarrow, \bot, \top\}$.

The relation $\vdash_{\mathcal{IPL}}$ is a finitary and substitution-invariant consequence relation.

$\mathcal{IPL} = \langle L, \vdash_{\mathcal{IPL}} \rangle$ is the a logical system of Intuitionistic logic.

Gentzen calculi. Sequents

Let L be a propositional language.

An L-sequent is a pair $\langle \Gamma, \Delta \rangle$, that we write $\Gamma \rhd \Delta$, where Γ and Δ are finite, and possibly empty, sequences of L-formulas.

We use the notations

$\rhd\varphi$ and $\rhd\Gamma$ instead of $\emptyset \rhd \varphi$ and $\emptyset \rhd \Gamma$

$\varphi\rhd$ and $\Gamma\rhd$ instead of $\varphi \rhd \emptyset$ and $\Gamma \rhd \emptyset$.

Capital greek letters $\Gamma, \Delta, \Pi, \Sigma$ will be used as variables for sequences.

Concatenation of sequences will we signaled by the comma; thus:

Γ, Δ is the sequence obtained by concatenating Γ and Δ in this order.

Γ, φ, Δ is the sequence $\Gamma, \langle \varphi \rangle, \Delta$.

The sequents of the form $\Gamma \rhd \varphi$ and $\Gamma \rhd \emptyset$ are called **LJ**-like sequents.

The sequents of the form $\varphi \rhd \psi$ are called binary sequents.

A Gentzen-style rule is a pair $\langle \mathbf{S}, \Gamma \rhd \Delta \rangle$ where $\mathbf{S}$ is a (possibly empty) finite set of sequents and $\Gamma \rhd \Delta$ is a sequent.

A Gentzen-style rule $\langle \mathbf{S}, \Gamma \rhd \Delta \rangle$ is *initial*, or *axiomatic*, if $\mathbf{S}$ is empty.

We will use the standard notation for Gentzen-style rules, namely

$$\frac{\Gamma_0 \rhd \Delta_0, \ldots, \Gamma_{n-1} \rhd \Delta_{n-1}}{\Gamma \rhd \Delta}$$

Examples of Gentzen rules are:

$$\frac{}{p \rhd p} \qquad\qquad \frac{}{p \rhd \Box\top}$$

which are axiomatic, and

$$\frac{\Gamma \rhd \Sigma, \varphi \qquad \psi, \Pi \rhd \Delta}{\varphi \to \psi, \Gamma, \Pi \rhd \Sigma, \Delta}\ (\to \mathsf{L}) \qquad\qquad \frac{\varphi, \Gamma \rhd \Delta, \psi}{\Gamma \rhd \Delta, \varphi \to \psi}\ (\to \mathsf{R})$$

which are not axiomatic.

A *Gentzen calculus* **G** has two components,

- a set of sequents $\mathrm{Seq}(\mathbf{G})$,
- a set of Gentzen-style rules.

We will consider Gentzen calculi **G** such that $\mathrm{Seq}(\mathbf{G})$ will be

- the set of all sequents,
- the set of all **LJ**-like sequents
- the set of all binary sequents.

G is binary if $\mathrm{Seq}(\mathbf{G})$ is the set of all binary sequents.

Let **G** be a Gentzen calculus.

A *proof in* **G** *from* a set of sequents $\mathbf{S} \subseteq \mathrm{Seq}(\mathbf{G})$ is a finite sequence of sequents in $\mathrm{Seq}(\mathbf{G})$ each one of whose elements is

- a substitution-instance of an initial rule of **G** or
- a sequent in **S** or
- is obtained by applying a substitution-instance of a rule of **G** to previous elements in the sequence.

Definition

Let **G** be a Gentzen calculus.

A sequent $\Gamma \rhd \Delta \in \mathrm{Seq}(\mathbf{G})$ is derivable in **G** from a set of sequents $\mathbf{S} \subseteq \mathrm{Seq}(\mathbf{G})$ if there is a proof in **G** from **S** whose last sequent is $\Gamma \rhd \Delta$; in this situation we write $\mathbf{S} \vdash_{\mathbf{G}} \Gamma \rhd \Delta$.

A derivable sequent of **G** is a sequent derivable in **G** from the emptyset of sequents, i.e. a sequent $\Gamma \rhd \varphi$ such that $\emptyset \vdash_{\mathbf{G}} \Gamma \rhd \Delta$.

A Gentzen rule $\langle \mathbf{S}, \Gamma \rhd \Delta \rangle$ is a derived rule of **G** if $\mathbf{S} \vdash_{\mathbf{G}} \Gamma \rhd \Delta$.

Given a Gentzen calculus **G**, the relation $\vdash_{\mathbf{G}}$ is

- a consequence relation on $\mathrm{Seq}(\mathbf{G})$,
- is finitary
- for every substitution σ, if $\mathbf{S} \vdash_{\mathbf{G}} \Gamma \rhd \Delta$, then $\sigma[\mathbf{S}] \vdash_{\mathbf{G}} \sigma[\Gamma] \rhd \sigma[\Delta]$.

The pairs $\langle \mathbf{Seq}, \vdash \rangle$ where **Seq** is one of the three sets of *L*-sequents and $\vdash$ is consequence relations on it with the three properties above are known as Gentzen systems.

Structural rules

Identity axiom

$$\frac{}{\varphi \rhd \varphi}$$

Exchange

$$\frac{\Gamma, \varphi, \psi, \Delta \rhd \Pi}{\Gamma, \psi, \varphi, \Delta \rhd \Pi} \text{ (EL)} \qquad \frac{\Gamma \rhd \Pi, \varphi, \psi, \Delta}{\Gamma \rhd \Pi, \varphi, \psi, \Delta} \text{ (ER)}$$

Weakening

$$\frac{\Gamma \rhd \Delta}{\varphi, \Gamma \rhd \Delta} \text{ (WL)} \qquad \frac{\Gamma \rhd \Delta}{\Gamma \rhd \Delta, \varphi} \text{ (WR)}$$

Contraction

$$\frac{\varphi, \varphi, \Gamma \rhd \Delta}{\varphi, \Gamma \rhd \Delta} \text{ (CL)} \qquad \frac{\Gamma \rhd \Delta, \varphi, \varphi}{\Gamma \rhd \Delta, \varphi} \text{ (CR)}$$

Cut

$$\frac{\Gamma \rhd \Delta, \varphi \qquad \varphi, \Pi \rhd \Sigma}{\Gamma, \Pi \rhd \Delta, \Sigma} \text{ (Cut)}$$

The Gentzen calculus **LK**

The Gentzen calculus **LK** for classical logic introduced by Gentzen in 1935.

Set of sequents of **LK**: all L-sequents.

The rules of the calculus **LK** are the structural rules plus the operational rules:

$$\frac{}{\bot, \Gamma \rhd \Delta}\ \text{(Bot)} \qquad \frac{}{\Gamma \rhd \top, \Delta}\ \text{(Top)}$$

$$\frac{\varphi, \Gamma \rhd \Delta}{\varphi \wedge \psi, \Gamma \rhd \Delta} \quad \frac{\psi, \Gamma \rhd \Delta}{\varphi \wedge \psi, \Gamma \rhd \Delta}\ (\wedge\ \mathsf{L}) \qquad \frac{\Gamma \rhd \Delta, \varphi \quad \Gamma \rhd \Delta, \psi}{\Gamma \rhd \Delta, \varphi \wedge \psi}\ (\wedge\ \mathsf{R})$$

$$\frac{\varphi, \Gamma \rhd \Delta \quad \psi, \Gamma \rhd \Delta}{\varphi \vee \psi, \Gamma \rhd \Delta}\ (\vee\ \mathsf{L}) \qquad \frac{\Gamma \rhd \Delta, \varphi}{\Gamma \rhd \Delta, \varphi \vee \psi} \quad \frac{\Gamma \rhd \Delta, \psi}{\Gamma \rhd \Delta, \varphi \vee \psi}\ (\vee\ \mathsf{R})$$

$$\frac{\Gamma \rhd \Sigma, \varphi \quad \psi, \Pi \rhd \Delta}{\varphi \to \psi, \Gamma, \Pi \rhd \Sigma, \Delta}\ (\to \mathsf{L}) \qquad \frac{\varphi, \Gamma \rhd \Delta, \psi}{\Gamma \rhd \Delta, \varphi \to \psi}\ (\to \mathsf{R})$$

The *derived rules for negation* (defined, recall, by $\varphi \to \bot$) are:

$$\frac{\varphi, \Gamma \rhd \Delta}{\Gamma \rhd \Delta, \neg\varphi} \qquad \frac{\Gamma \rhd \Delta, \varphi}{\neg\varphi, \Gamma \rhd \Delta}$$

Theorem (Soundness and Completeness of **LK**)

The following are equivalent

1. $\varphi_0, \ldots, \varphi_n \rhd \psi_0, \ldots, \psi_m$ *is derivable in LK*
2. $\{\varphi_0, \ldots, \varphi_n\} \models_{CPL} \psi_0 \vee \ldots \vee \psi_m$
3. $\varphi_0 \wedge \ldots \wedge \varphi_n \models_{CPL} \psi_0 \vee \ldots \vee \psi_m$

The notion of derivable sequent of **LK** serves to obtain a finitary and substitution-invariant consequence relation by

$\Gamma \vdash_{LK} \varphi$ iff there is a finite $\Delta \subseteq \Gamma$ such that $\overline{\Delta} \rhd \varphi$ is derivable in **LK** .

We use the convention of turning a finite set of formulas Δ into a sequence $\overline{\Delta}$.

Theorem

For every set of formulas Γ *and every formula* φ*,*

$$\Gamma \vdash_{LK} \varphi \quad \textit{iff} \quad \Gamma \models_{CPL} \varphi.$$

The equivalence of **LK** with a binary Gentzen system

In **LK** we have the following properties

1. $\varphi_0, \ldots, \varphi_n \rhd \psi_0, \ldots, \psi_m \dashv\vdash_{\mathbf{LK}} \varphi_0 \wedge \ldots \wedge \varphi_n \rhd \psi_0 \vee \ldots \vee \psi_m$
2. $\emptyset \rhd \psi_0, \ldots, \psi_m \dashv\vdash_{\mathbf{LK}} \top \rhd \psi_0 \vee \ldots \vee \psi_m$
3. $\varphi_0, \ldots, \varphi_n \rhd \emptyset \dashv\vdash_{\mathbf{LK}} \varphi_0 \wedge \ldots \wedge \varphi_n \rhd \bot$

We define a translation from the set of sequents of **LK** into the set of binary sequents by

- $t(\varphi_0, \ldots, \varphi_n \rhd \psi_0, \ldots, \psi_m) = \varphi_0 \wedge \ldots \wedge \varphi_n \rhd \psi_0 \vee \ldots \vee \psi_m$
- $t(\emptyset \rhd \psi_0, \ldots, \psi_m) = \top \rhd \psi_0 \vee \ldots \vee \psi_m$
- $t(\varphi_0, \ldots, \varphi_n \rhd \emptyset) = \varphi_0 \wedge \ldots \wedge \varphi_n \rhd \bot$

Then

$$\{\Gamma_i \rhd \Delta_i : i \in I\} \vdash_{\mathbf{LK}} \Gamma \rhd \Delta \quad \text{iff} \quad \{t(\Gamma_i \rhd \Delta_i) : i \in I\} \vdash_{\mathbf{LK}} t(\Gamma \rhd \Delta).$$

A Gentzen calculus for Modal Logic

Let $\mathbf{LK}^K$ be the Gentzen calculus obtained by adding the following rules to $\mathbf{LK}$:

$$\frac{\Sigma, \varphi \rhd \Delta}{\Box\Sigma, \Diamond\varphi \rhd \Diamond\Delta} \ (M1) \qquad\qquad \frac{\Sigma \rhd \Delta, \varphi}{\Box\Sigma \rhd \Diamond\Delta, \Box\varphi} \ (M2)$$

From $\mathbf{LK}^K$ we can obtain the two logical systems (lK and gK) associated with the normal modal logic K

An analogous translation from sequents to binary sequents to the above translation t shows that $\mathbf{LK}^K$ is equivalent to a binary Gentzen calculus.

The Gentzen calculus **LJ**

LJ has the same rules as the sequent calculus **LK** but applied only to **LJ**-like sequents: of the form $\Gamma \rhd \varphi$ or $\Gamma \rhd \emptyset$.

The operational rules of **LJ** take then the following form.

$$\frac{}{\bot, \Gamma \rhd \Delta} \text{ (Bot)} \qquad \frac{}{\Gamma \rhd \top} \text{ (Top)}$$

$$\frac{\varphi, \Gamma \rhd \Delta}{\varphi \wedge \psi, \Gamma \rhd \Delta} \quad \frac{\psi, \Gamma \rhd \Delta}{\varphi \wedge \psi, \Gamma \rhd \Delta} \ (\wedge \text{ L}) \qquad \frac{\Gamma \rhd \varphi \quad \Gamma \rhd \psi}{\Gamma \rhd \varphi \wedge \psi} \ (\wedge \text{ R})$$

$$\frac{\varphi, \Gamma \rhd \Delta \quad \psi, \Gamma \rhd \Delta}{\varphi \vee \psi, \Gamma \rhd \Delta} \ (\vee \text{ L}) \qquad \frac{\Gamma \rhd \varphi}{\Gamma \rhd \varphi \vee \psi} \quad \frac{\Gamma \rhd \psi}{\Gamma \rhd \varphi \vee \psi} \ (\vee \text{ R})$$

$$\frac{\Gamma \rhd \varphi \quad \psi, \Pi \rhd \Delta}{\varphi \to \psi, \Gamma, \Pi \rhd \Delta} \ (\to \text{ L}) \qquad \frac{\varphi, \Gamma \rhd \psi}{\Gamma \rhd \varphi \to \psi} \ (\to \text{ R})$$

The equivalence of **LJ** with a binary Gentzen system

Similarly to **LK**, in **LJ** we have

1. $\varphi_0, \ldots, \varphi_n \rhd \varphi \dashv\vdash_{\mathbf{LJ}} \varphi_0 \wedge \ldots \wedge \varphi_n \rhd \varphi$
2. $\emptyset \rhd \varphi \dashv\vdash_{\mathbf{LJ}} \top \rhd \varphi$
3. $\varphi_0, \ldots, \varphi_n \rhd \emptyset \dashv\vdash_{\mathbf{LJ}} \varphi_0 \wedge \ldots \wedge \varphi_n \rhd \bot$

For the restriction of the translation t to LJ-like sequents, that gives

- $t(\varphi_0, \ldots, \varphi_n \rhd \varphi) = \varphi_0 \wedge \ldots \wedge \varphi_n \rhd \varphi$
- $t(\varphi_0, \ldots, \varphi_n \rhd \emptyset) = \varphi_0 \wedge \ldots \wedge \varphi_n \rhd \bot$
- $t(\varphi \rhd \emptyset) = \varphi \rhd \bot$

we have

$$\{\Gamma_i \rhd \Delta_i : i \in I\} \vdash_{\mathbf{LJ}} \Gamma \rhd \Delta \quad \text{iff} \quad \{t(\Gamma_i \rhd \Delta_i) : i \in I\} \vdash_{\mathbf{LJ}} t(\Gamma \rhd \Delta).$$

Full Lambek calculus **FL**

Language $L = \{\wedge, *, \vee, \to, \leftarrow 0, 1\}$.
Two negations: $\neg\varphi := \varphi \to 0 \qquad -\varphi := 0 \leftarrow \varphi$.

The set of sequents for **FL** is the set of all *LJ*-like sequents.

The structural rules are:

Identity axiom

$$\frac{}{\varphi \rhd \varphi}$$

Cut

$$\frac{\Gamma \rhd \varphi \qquad \varphi, \Pi \rhd \delta}{\Gamma, \Pi \rhd \delta}\ (\mathrm{Cut})$$

Operational rules:

$$\frac{\Gamma, \varphi, \Delta \rhd \delta}{\Gamma \varphi \wedge \psi, \Delta \rhd \delta} \quad \frac{\Gamma, \psi, \Delta \rhd \delta}{\Gamma, \varphi \wedge \psi, \Delta \rhd \delta} \ (\wedge \ \mathrm{L}) \qquad \frac{\Gamma \rhd \varphi \quad \Gamma \rhd \psi}{\Gamma \rhd \varphi \wedge \psi} \ (\wedge \ \mathrm{R})$$

$$\frac{\Gamma, \varphi, \psi, \Delta \rhd \delta}{\Gamma, \varphi * \psi, \Delta \rhd \delta} \ (* \ \mathrm{L}) \quad \frac{\Gamma \rhd \varphi \quad \Gamma' \rhd \psi}{\Gamma, \Gamma' \rhd \varphi * \psi} \ (* \ \mathrm{R})$$

$$\frac{\Gamma, \varphi, \Delta \rhd \delta \quad \Gamma, \psi, \Delta \rhd \delta}{\Gamma, \varphi \vee \psi, \Delta \rhd \delta} \ (\vee \ \mathrm{L}) \qquad \frac{\Gamma \rhd \varphi}{\Gamma \rhd \varphi \vee \psi} \quad \frac{\Gamma \rhd \psi}{\Gamma \rhd \varphi \vee \psi} \ (\vee \ \mathrm{R})$$

$$\frac{\Gamma \rhd \varphi \quad \Pi, \psi, \Delta \rhd \delta}{\Gamma, \Pi, \varphi \to \psi, \Delta \rhd \delta} \ (\to \ \mathrm{L}) \qquad \frac{\varphi, \Gamma \rhd \psi}{\Gamma \rhd \varphi \to \psi} \ (\to \ \mathrm{R})$$

$$\frac{\Gamma \rhd \varphi \quad \Pi, \psi, \Delta \rhd \delta}{\Pi, \varphi \leftarrow \psi, \Gamma, \Delta \rhd \delta} \ (\leftarrow \ \mathrm{L}) \qquad \frac{\Gamma, \varphi \rhd \psi}{\Gamma \rhd \varphi \leftarrow \psi} \ (\leftarrow \ \mathrm{R})$$

$$\frac{\Gamma, \Delta \rhd \varphi}{\Gamma, 1, \Delta \rhd \varphi} \qquad \frac{\Gamma \rhd \emptyset}{\Gamma \rhd 0}$$

$$\frac{}{\emptyset \rhd 1} \qquad \frac{}{0 \rhd \emptyset}$$

The equivalence of **FL** with a binary Gentzen system

Similarly to **LJ**, in **FL** we have

1. $\varphi_0, \dots, \varphi_n \rhd \varphi \dashv\vdash_{\mathbf{FL}} \varphi_0 * \dots * \varphi_n \rhd \varphi$
2. $\emptyset \rhd \varphi \dashv\vdash_{\mathbf{FL}} 1 \rhd \varphi$
3. $\varphi_0, \dots, \varphi_n \rhd \emptyset \dashv\vdash_{\mathbf{FL}} \varphi_0 * \dots * \varphi_n \rhd 0$

We translate LJ-like sequents, to binary sequents by

- $t(\varphi_0, \dots, \varphi_n \rhd \varphi) = \varphi_0 \star \dots \star \varphi_n \rhd \varphi$
- $t(\varphi_0, \dots, \varphi_n \rhd \emptyset) = \varphi_0 \star \dots \star \varphi_n \rhd 0$
- $t(\emptyset \rhd \varphi) = 1 \rhd \varphi$
- $t(\varphi \rhd \emptyset) = \varphi \rhd 0.$

It holds:

$$\{\Gamma_i \rhd \Delta_i : i \in I\} \vdash_{\mathbf{FL}} \Gamma \rhd \Delta \quad \text{iff} \quad \{t(\Gamma_i \rhd \Delta_i) : i \in I\} \vdash_{\mathbf{FL}} t(\Gamma \rhd \Delta).$$

Extensions of Full Lambek calculus

Extensions of **FL** are obtained by adding combinations of the structural rules (EL), (CL), (WL). For example:

$\mathbf{FL}_{ew} = \mathbf{FL} + (EL) + (WL)$.

$\mathbf{FL}_{c} = \mathbf{FL} + (CL)$.

$\mathbf{FL}_{ecw} = \mathbf{FL} + (EL) + (CL) + (WL)$ is equivalent to **LJ**.

Other extensions are obtained by adding axiomatic Gentzen rules.

$$\mathbf{FL} \; + \; \frac{}{-\neg\varphi \rhd \varphi} \quad \frac{}{\varphi \rhd -\neg\varphi} \quad \frac{}{\neg - \varphi \rhd \varphi} \quad \frac{}{\varphi \rhd \neg - \varphi}$$

is the involutive extension **InFL** of **FL**

If we expand the language with $\perp$ and $\top$, then

$$\mathbf{FL}_{\perp} = \mathbf{FL} \; + \; \frac{}{\Gamma \rhd \top} \quad \frac{}{\Gamma, \perp, \Delta \rhd \top}$$

$$\mathbf{FL}_{e\perp} = \mathbf{FL} + \; + \frac{}{\Gamma \rhd \top} \quad \frac{}{\Gamma, \perp, \Delta \rhd \top}$$

$\mathbf{FL}_{e\perp}$ gives multiplicative-additive intuitionistic linear logic, known as IMALL.

The internal system of a Gentzen calculus

Let **G** be a Gentzen calculus. We define the relation $\vdash^{in}_{G}$ by

$$\Sigma \vdash^{in}_{G} \varphi \quad \text{iff} \quad \text{there is a finite } \Delta \subseteq \Sigma \text{ s.t. } \overline{\Delta} \rhd \varphi \text{ is derivable in } \mathbf{G}.$$

where $\overline{\Delta}$ is the sequence of the formulas in Δ given by a fixed enumeration of the formulas.

If **G** has all the structural rules, then $\vdash_{G}$ is a finitary and substitution-invariant consequence relation on Fm_L.

The logical system $\langle L, \vdash^{in}_{G} \rangle$ is the internal logical system of **G**.

- The internal logical system of **LK** is classical logic.
- The internal logical system of **LJ** is intuitionistic logic
- The internal logical system of $\mathbf{LK}^K$ is the logical system IK.

The internal relation of **FL** is not a consequence relation: **FL** lacks Exchange, Weakening and Contraction.

The external logical system of a Gentzen calculus

Let **G** be a Gentzen calculus on L, the relation $\vdash_G^{ex}$ defined by

$$\Sigma \vdash_G^{ex} \varphi \quad \text{iff} \quad \{\emptyset \rhd \psi : \psi \in \Sigma\} \vdash_{\mathbf{G}} \emptyset \rhd \varphi$$

is a finitary and substitution-invariant consequence relation on Fm_L.

This holds for any Gentzen calculus.

The logical system $\langle L, \vdash_G^{ex} \rangle$ is the external logical system of **G**.

- The external logical system of **LK** is classical logic.
- The external logical system of **LJ** is intuitionisitc logic.
- The external logical system of $\mathbf{LK}^K$ is the logical system gK
- The external logical system of **FL** is the substructural logical system known as $\mathcal{FL}$.

Remark

The external and the internal logical systems of a Gentzen calculus may be different.

If one takes the perspective of logical systems, then substructural logics over *FL* are defined as axiomatic extensions of $\mathcal{FL}$. And if one takes the perspective of a logic as a set of formulas substructural logics they are defined as follows (in analogy with normal modal logics as sets of formulas):

A substructural logic over *FL* is a sets of formulas $\mathbb{L}$ such that

1. Every theorem of *FL* belongs to $\mathbb{L}$
2. if $\varphi, \varphi \to \psi \in \mathbb{L}$, then $\psi \in \mathbb{L}$
3. if $\varphi, \psi \in \mathbb{L}$, then $\varphi \wedge \psi \in \mathbb{L}$
4. if $\varphi \in \mathbb{L}$, then for every ψ, $\psi \to (\varphi * \psi), \psi \leftarrow (\psi * \varphi) \in \mathbb{L}$
5. if $\varphi \in \mathbb{L}$, then every substitution instance of φ belongs to $\mathbb{L}$.