A short introduction to the algebrization of logical systems and Gentzen calculi. Lecture 3: The algebraization of modal logics

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Outline

- Algebraization of $\mathbf{LK}^K$
- The class of algebras of a Gentzen system
- gK is an algebraizable logical system
- Algebraization of lK. It is not an algebraizable logical system
- Comparison between lK and gK
- Weakly congruential, congruential logical systems

Algebraization of $\mathbf{LK}^K$

The algebraic semantics of $\mathbf{LK}^K$ is the class of modal algebras.

Definition

A modal algebra is an L-algebra $\mathbf{A} = \langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \to^{\mathbf{A}}, \Box^{\mathbf{A}}, \top^{\mathbf{A}}, \bot^{\mathbf{A}} \rangle$ such that $\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \to^{\mathbf{A}}, \top^{\mathbf{A}}, \bot^{\mathbf{A}} \rangle$ is a Boolean algebra and $\Box^{\mathbf{A}}$ is a map from A to A such that

1) $\Box^{\mathbf{A}} \top^{\mathbf{A}} = \top^{\mathbf{A}}$,
2) $\Box^{\mathbf{A}}(a \wedge^{\mathbf{A}} b) = \Box^{\mathbf{A}} a \wedge^{\mathbf{A}} \Box^{\mathbf{A}} b$.

MA: denotes the class of modal algebras.

Lemma

In any normal modal algebra $\mathbf{A}$,

1. $\Box(a \to b) \leq \Box a \to \Box b$
2. $\Box(a \leftrightarrow b) \leq \Box a \leftrightarrow \Box b$
3. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$
4. $\Diamond 0 = 0$
5. $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$
6. $\Box(a \vee b) \leq \Box a \vee \Diamond b$
7. *if* $a \leq b$, *then* $\Box a \leq \Box b$ *and* $\Diamond a \leq \Diamond b$

Recall that the notion of derivation in $\mathbf{LK}^K$ from a set of sequents defines a relation between sets of sequents and sequents.

$$\mathbf{S} \vdash_{\mathbf{LK}^K} \Gamma \rhd \Delta \quad \text{iff} \quad \text{there is derivation of } \Gamma \rhd \Delta \text{ from } \mathbf{S}.$$

We translate a sequent $\Gamma \rhd \Delta$ into an inequality as we did in $\mathbf{LK}$, that is:

$$\tau(\Gamma \rhd \Delta) = \bigwedge \Gamma \leq \bigvee \Delta.$$

Theorem (Extended completeness theorem)

Let $\mathbf{S}$ *be a set of sequents and let* $\Gamma \rhd \Delta$ *be a sequent. The following are equivalent:*

1. $\mathbf{S} \vdash_{\mathbf{LK}^{\kappa}} \Gamma \rhd \Delta$,
2. $t(\mathbf{S}) \vdash_{\mathbf{LK}^{\kappa}} t(\Gamma \rhd \Delta)$
3. $\tau(\mathbf{S}) \models_{\mathsf{MA}} \tau(\Gamma \rhd \Delta)$.

Proof.

Similar to the case of **LJ**. □

As in the **LJ** case, any equation $\varphi \approx \psi$ can be turned into an equivalent pair of inequalities $\varphi \leq \psi$ and $\psi \leq \varphi$ and translated into a pair of sequents $\varphi \rhd \psi$ and $\psi \rhd \varphi$. Thus we translate an equation $\varphi \approx \psi$ into the set of sequents

$$\rho(\varphi \approx \psi) = \{\varphi \rhd \psi, \psi \rhd \varphi\}.$$

Theorem

For any equation $\varphi \approx \psi$,

$$\varphi \approx \psi \models_{\mathsf{MA}} \tau(\rho(\varphi \approx \psi)) \textit{ and } \tau(\rho(\varphi \approx \psi)) \models_{\mathsf{MA}} \varphi \approx \psi.$$

Theorem

For any sequent $\Delta \rhd \varphi$,

$$\{\Delta \rhd \varphi\} \vdash_{\mathbf{LK}^\kappa} \rho(\tau(\Delta \rhd \varphi) \quad \textit{and} \quad \rho(\tau(\Delta \rhd \varphi) \vdash_{\mathbf{LK}^\kappa} \varphi\Delta \rhd \varphi.$$

Theorem

For any set of equations $\{\varphi_i \approx \psi_i : i \in I\}$ and any equation $\varphi \approx \psi$,

$$\{\varphi_i \approx \psi_i : i \in I\} \models_{\mathsf{MA}} \varphi \approx \psi \quad \textit{iff} \quad \bigcup_{i \in I} \rho(\varphi_i \approx \psi_i) \vdash_{\mathbf{LK}^\kappa} \rho(\varphi \approx \psi).$$

We obtained:

Theorem

$\mathbf{LK}^K$ *is algebraizable. The variety* MA *is its equivalent algebraic semantics and the translations are the maps* τ *and* ρ *given above.*

Let $\mathbf{A}$ be an algebra and v a valuation on $\mathbf{A}$. For every binary sequent $\varphi \rhd \psi$ we set $v(\varphi \rhd \psi) = \langle v(\varphi), v(\psi)\rangle$.

A binary relation R on A is an $\mathbf{LK}^K$-filter of $\mathbf{A}$ if for every valuation v on $\mathbf{A}$, every set of binary sequents $\mathbf{S}$ and every binary sequent $\varphi \rhd \psi$, if $\mathbf{S} \vdash_{\mathbf{LK}^K} \varphi \rhd \psi$ and $v[\mathbf{S}] \subseteq R$, then $v(\varphi \rhd \psi) \in R$.

Every $\mathbf{LK}^K$-filter of $\mathbf{A}$ is a reflexive and transitive relation.

Given a $\mathbf{LK}^K$-filter R on $\mathbf{A}$, let

$$\mathbf{\Omega}_{\mathbf{A}}(R) = \{\langle a, b\rangle : \langle a, b\rangle, \langle b, a\rangle \in R\} = R \cap R^{-1}.$$

This relation is the greatest congruence which is compatible with R.
Moreover,

Theorem

$\mathbf{\Omega}^{\mathbf{A}}(.)$ *gives an isomorphism between the lattice of* $\mathbf{LK}^K$*-filters of* $\mathbf{A}$ *and the lattice of congruences of* $\mathbf{A}$.

Theorem

$$\mathrm{MA} = \{\mathbf{A}/\mathbf{\Omega}^{\mathbf{A}}(R) : R \textit{ is an } \mathbf{LK}^K\textit{-filter of } \mathbf{A}\}$$

The class of algebras of a Gentzen system

Let **G** be a Gentzen calculus equivalent to its restriction to binary L-sequents.

Let **A** be an L-algebra. Let v be a valuation on **A** and $\varphi \rhd \psi$ a binary sequent. Then we set $v(\varphi \rhd \psi) = \langle v(\varphi), v(\psi)\rangle$.

A binary relation R on A is an **G**-filter of **A** if for every valuation v on **A**, every set of binary sequents **S** and every binary sequent $\varphi \rhd \psi$,

$$\text{if } \mathbf{S} \vdash_{\mathbf{G}} \varphi \rhd \psi \text{ and } v[\mathbf{S}] \subseteq R, \text{ then } v(\varphi \rhd \psi) \in R.$$

If **G** satisfies Identity and Cut for binary sequents, every **G**-filter of **A** is a reflexive and transitive relation.

A congruence θ of **A** is compatible with a **G**-filter R if

$$a\theta a', b\theta b', aRb \Rightarrow a'Rb'.$$

Given a **G**-filter R on **A**, the greatest congruence which is compatible with R always exist. It is the greatest congruence included in $R \cap R^{-1} = \{\langle a, b\rangle : \langle a, b\rangle, \langle b, a\rangle \in R\}$. We denote it by $\mathbf{\Omega}^{\mathbf{A}}(R)$.

If **G** has Cut for binary sequents, the class of algebras of **G** is

$$\mathbf{Alg}(\mathbf{G}) = \{\mathbf{A}/\mathbf{\Omega}^{\mathbf{A}}(R) : R \text{ is a } \mathbf{G}\text{-filter of } \mathbf{A}\}.$$

gK is algebraizable

The extended completeness theorem implies:

Theorem

The following are equivalent:

1. $\{\varphi_i : i \in I\} \vdash_{gK} \varphi$
2. $\{\varphi_i : i \in I\} \models_{\mathsf{MA}} \varphi$
3. $\{\top \approx \varphi_i : i \in I\} \models_{\mathsf{MA}} \top \approx \varphi$

Moreover, if we translate an equation $\varphi \approx \psi$ by $\rho(\varphi \approx \psi) = \varphi \leftrightarrow \psi$ and a formula φ by $\tau(\varphi) = \top \approx \varphi$, then for every equation $\varphi \approx \psi$

$$\varphi \approx \psi \models_{\mathsf{MA}} \tau(\rho(\varphi \approx \psi)) \quad \text{and} \quad \tau(\rho(\varphi \approx \psi)) \models_{\mathsf{HA}} \varphi \approx \psi.$$

Therefore,

Theorem

gK is algebraizable. Its equivalent algebraic semantics is MA *and translations are given by the maps:* $\quad \varphi \mapsto \top \approx \varphi \quad$ *and* $\quad \varphi \approx \psi \mapsto \varphi \leftrightarrow \psi$

Proposition

Let $\mathbf{A}$ *be an algebra. If* R *is an* $\mathbf{LK}^K$*-filter then*

1. $F_R = \{a \in A : \langle \top, a \rangle \in R\}$ *is a gK-filter,*
2. $\boldsymbol{\Omega}_{\mathbf{A}}(F_R) = \boldsymbol{\Omega}_{\mathbf{A}}(R)$.

Proposition

If F *is an sK-filter on* $\mathbf{A}$*, the* $\mathbf{LK}^K$*-filter* R_F *on* $\mathbf{A}$ *generated by* $\{\langle \top, a \rangle : a \in F\}$ *is such that* $F = F_{R_F}$.

Theorem

$\boldsymbol{\Omega}^{\mathbf{A}}(.)$ *gives an isomorphism between the lattice of gK-filters of* $\mathbf{A}$ *and the lattice of congruences of* $\mathbf{A}$.

Algebraizing IK

Let us try to apply the Lindenbaum-Tarski method to IK.
Given a theory T of IK we define the following binary relation $\theta(T)$ on Fm in the same way that we defined $\mathbf{\Omega}(T)$ in the case of Intuitionistic logic:

$$\langle \varphi, \psi \rangle \in \theta(T) \text{ iff } T \vdash_{IK} \varphi \leftrightarrow \psi.$$

This relation does not need to be a congruence relation. It could happen that

$$T \vdash_{IK} \varphi \leftrightarrow \psi \quad \text{but} \quad T \nvdash_{IK} \Box\varphi \leftrightarrow \Box\psi.$$

Example: Let T be the theory generated by p. Then $T \vdash_{IK} \top \leftrightarrow p$, but $T \vdash_{IK} \Box\top \leftrightarrow \Box p$, for if this was the case, then since $\vdash_{IK} \Box\top$, we would have $p \vdash_{IK} \Box p$, which is not the case.

1. $\theta(T)$ is *compatible* with T, namely, for every φ and ψ

$$\text{if } \langle\varphi,\psi\rangle \in \theta(T) \text{ and } \varphi \in T, \text{ then } \psi \in T;$$

2. $\theta(T)$ is the greatest **equivalence** relation on Fm which is compatible with T,
3. For every formula φ,

$$\langle\varphi,\top\rangle \in \theta(T) \quad \text{iff} \quad \varphi \in T;$$

4. $\theta(T)$ is *the interderivability relation* modulo T, that is:

$$\langle\varphi,\psi\rangle \in \theta(T) \quad \text{iff} \quad \langle\varphi,\psi\rangle \in \Lambda_{IK}(T) \quad \text{iff} \quad T,\varphi \vdash_{IK} \psi \text{ and } T,\psi \vdash_{IK} \varphi.$$

Thus in IK, $\Lambda_{IK}(T)$ is not a congruence for every IK-theory T.
Recall that the greatest congruence included in $\Lambda_{IK}(T)$ is called the Suszko congruence of T and is denoted by $\widetilde{\mathbf{\Omega}}_{IK}(T)$.
We say that $\mathbf{Fm}/\widetilde{\mathbf{\Omega}}_{IK}(T)$ is the *Lindenbaum-Tarski algebra* of T.

Proposition

The Lindenbaum-Tarski algebra $\mathbf{Fm}/\widetilde{\mathbf{\Omega}}_{IK}(T)$ *of* T *is a modal algebra.*

Comparing lK and gK

Let T be a theory of gK. The relation $\mathbf{\Omega}(T)$ on Fm defined by

$$\langle \varphi, \psi \rangle \in \mathbf{\Omega}(T) \text{ iff } T \vdash_{gK} \varphi \leftrightarrow \psi.$$

is a congruence relation on **Fm** (like for Intuitionistic logic).

1. $\mathbf{\Omega}(T)$ is *compatible* with T, namely, for every φ and ψ

$$\text{if } \langle \varphi, \psi \rangle \in \mathbf{\Omega}(T) \text{ and } \varphi \in T, \text{ then } \psi \in T;$$

2. $\mathbf{\Omega}(T)$ is the greatest congruence relation on Fm which is compatible with T,
3. For any formula φ,

$$\langle \varphi, \top \rangle \in \mathbf{\Omega}(T) \quad \text{iff} \quad \varphi \in T.$$

But

1. $\mathbf{\Omega}(T)$ *is not the interderivability relation* $\Lambda_{gK} T$ in gK relative to T. It could happen that

$$T, \varphi \vdash_{gK} \psi \text{ and } T, \psi \vdash_{gK} \varphi, \text{ but } T \nvdash_{gK} \varphi \leftrightarrow \psi.$$

We already saw $\mathbf{\Omega}(T)$ is the greatest congruence relation which is compatible with the $\mathbf{LK}^K$ filter $\{\langle\varphi,\psi\rangle : \varphi \to \psi \in T\}$.
By the usual technique of Lindenbaum-Tarski algebras we can also obtain the algebraic completeness theorem:

Theorem (Completeness)

For any set of formulas Γ *and any formula* φ,

$$\Gamma \vdash_{gK} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathsf{MA}} \varphi.$$

Let now T be a theory of IK.

The greatest theory of gK which is included in T always exists and is the set of formulas

$$T^s = \{\varphi : (\forall n)\ \Box^n\varphi \in T\}.$$

Then $\mathbf{\Omega}(T^s)$ is the greatest congruence which is included in $\Lambda_{IK}(T)$.

Therefore, $\mathbf{\Omega}(T^s) = \widetilde{\mathbf{\Omega}}_{IK}(T)$.

Similarly, given an algebra $\mathbf{A}$ and a IK-filter F of $\mathbf{A}$, the greatest gK filter of $\mathbf{A}$ which is included in F always exists, and its Leibniz congruence is the IK-Suszko congruence of F.

Weakly Congruential and Congruential logics

Let $\mathcal{L} = \langle L, \vdash_{\mathcal{L}} \rangle$ be a logical system. We denote the interderivability relation $(\varphi \dashv\vdash \psi)$ by $\Lambda_{\mathcal{L}}$.

If $\Lambda_{\mathcal{L}}$ is a congruence of $\mathbf{Fm}_L$ we say that $\mathcal{L}$ is *weakly congruential*.

If for every $\mathcal{L}$-theory T $\Lambda_{\mathcal{L}}(T)$ is a congruence of $\mathbf{Fm}_L$ we say that $\mathcal{L}$ is *Fregean*.

Note that $\Lambda_{\mathcal{L}} = \Lambda_{\mathcal{L}}(C_{\vdash_{\mathcal{L}}}(\emptyset))$. Hence every Fregean logic is weakly congruential.

The logic lK is weakly congruential. The logic gK is not weakly congruential.

Recall that given a logical system $\mathcal{L}$ and an algebra $\mathbf{A}$, the $\mathcal{L}$-filters of $\mathbf{A}$ induce the quasi-order (or preorder) given by

$$a \leq_{\mathcal{L}}^{\mathbf{A}} b \quad \text{iff} \quad (\forall F \in \mathrm{Fi}_{\mathcal{L}}\mathbf{A})(a \in F \Rightarrow b \in F)$$

whose naturally associated equivalence relation we denote by $\sim_{\mathcal{L}}^{\mathbf{A}}$.

If $\mathbf{A}$ is an algebra of the type of modal logic, then the relation $\sim_{IK}^{\mathbf{A}}$ is a congruence of $\mathbf{A}$ and $\mathbf{A}/{\sim_{IK}}$ is a modal algebra. Moreover, if $\mathbf{A}$ is a modal algebra, then $\sim_{IK}^{\mathbf{A}}$ is the identity and $\leq_{IK}^{\mathbf{A}}$ is the semi-lattice order of $\mathbf{A}$.

A logical system $\mathcal{L}$ is *congruential* if for every algebra $\mathbf{A}$, the relation $\sim_{\mathcal{L}}^{\mathbf{A}}$ is a congruence. Thus IK is congruential.